Tôhoku Math. J.
46 (1994), 393–401

THE TRACE THEOREM ON ANISOTROPIC SOBOLEV SPACES

Dedicated to Professor Takesi Kotake on his sixtieth birthday

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(Received April 28, 1993, revised June 10, 1993)

Abstract. The trace theorem on anisotropic Sobolev spaces is proved. These function spaces which can be regarded as weighted Sobolev spaces are particularly important when we discuss the regularity of solutions of the characteristic initial boundary value problem for linear symmetric hyperbolic systems.

In this note, we give the trace theorem on anisotropic Sobolev spaces which appear in the study of the initial boundary value problem for linear symmetric hyperbolic systems with characteristic boundary. The function spaces with which we concern ourselves will be denoted by $H^m_*(\Omega)$, Ω being an open set in $\boldsymbol{R}^n$ lying on one side of its boundary. Denoting the usual Sobolev space by $H^m(\Omega)$, we have the continuous embedding $H^m(\Omega) \hookrightarrow H^m_*(\Omega)$ for $m=0, 1, \dots$. $H^m_*(\Omega)$ is anisotropic in the sense that the tangential derivatives and the normal derivatives are treated in different ways in this space. In contrast with the case where the boundary is non-characteristic, the solution of the characteristic initial boundary value problem for symmetric hyperbolic systems lies in general in $H^m_*(\Omega)$, not in $H^m(\Omega)$. The trace theorem on $H^m_*(\Omega)$ is needed especially when we consider the compatibility condition. This is the motivation for the present work.

For simplicity, we suppose that Ω is a half-space in $\boldsymbol{R}^n$. Let

$$\boldsymbol{R}^n_+ = \{(t, y) \mid t>0,\, y \in \boldsymbol{R}^{n-1}\} .$$

Let $\rho \in C^\infty(\overline{\boldsymbol{R}_+})$ be a monotone increasing function such that $\rho(t)=t$ in a neighborhood of the origin and $\rho(t)=1$ for any t large enough. By means of this function, we define the differential operator in the tangential directions

$$\partial^{(r,\alpha)}_{\tan} = (\rho(t)\partial_t)^r \partial_1^{\alpha_1} \cdots \partial_{n-1}^{\alpha_{n-1}} ,$$

where $\alpha=(\alpha_1, \dots, \alpha_{n-1})$, $\partial_i = \partial/\partial y_i$, $1 \le i \le n-1$. The differential operator in the normal direction is ∂_t^k. We fix a nonnegative integer m. Let $u \in L^2(\boldsymbol{R}^n_+)$ satisfy

$$\|u\|^2_{m,*} = \sum_{r+|\alpha|+2k \le m} \|\partial^{(r,\alpha)}_{\tan} \partial_t^k u\|^2 < \infty ,$$

1991 *Mathematics Subject Classification*. Primary 46E35.

the derivatives being considered in the distribution sense. The set of such functions is denoted by $H^m_*(\boldsymbol{R}^n_+)$. The norm in this space is given by $\|\cdot\|_{m,*}$. It is easily seen that $H^m_*(\boldsymbol{R}^n_+)$ is a Hilbert space for $m=0, 1, \ldots$. In particular, $H^0_*(\boldsymbol{R}^n_+)=L^2(\boldsymbol{R}^n_+)$. $H^1_*(\boldsymbol{R}^n_+)$ coincides with $H^1_{\tan}(\boldsymbol{R}^n_+)$ defined by Bardos and Rauch [1]. For $m\geq 2$, we have $H^m_*(\boldsymbol{R}^n_+)\subsetneqq H^m_{\tan}(\boldsymbol{R}^n_+)$. It follows directly from the definition that $H^{2m}_*(\boldsymbol{R}^n_+)\hookrightarrow H^m(\boldsymbol{R}^n_+)$. We note that the norm in $H^m_*(\boldsymbol{R}^n_+)$ is equivalent to the norm defined by

$$\left\{\sum_{r+|\alpha|+2k\leq m}\|\partial^{(r,\alpha)}_*\partial^k_t u\|^2\right\}^{1/2},$$

where $\partial^{(r,\alpha)}_*=\rho(t)^r\partial^r_t\partial^\alpha_y$. This observation is useful in the following computations. We denote by $[p/2]$ the largest integer not exceeding $p/2$.

THEOREM 1. *Let $p\geq 2$ be an integer. Then the mapping*

$$C^\infty_0(\overline{\boldsymbol{R}^n_+})\ni u\mapsto\{\partial^i_t u(0, y)\,|\,i=0, \ldots, [p/2]-1\}\in\underbrace{C^\infty_0(\boldsymbol{R}^{n-1}_y)\times\cdots\times C^\infty_0(\boldsymbol{R}^{n-1}_y)}_{[p/2]\ times}$$

extends by continuity to a continuous linear mapping of

$$H^p_*(\boldsymbol{R}^n_+)\to\prod_{j=0}^{[p/2]-1}H^{p-2j-1}(\boldsymbol{R}^{n-1}_y).$$

This mapping is surjective and there exists a continuous linear right inverse

$$(h_0, \ldots, h_{[p/2]-1})\mapsto\mathscr{R}(h_0, \ldots, h_{[p/2]-1})$$

of

$$\prod_{j=0}^{[p/2]-1}H^{p-2j-1}(\boldsymbol{R}^{n-1}_y)\to H^p_*(\boldsymbol{R}^n_+)$$

such that

$$\partial^j_t\mathscr{R}(h_0, \ldots, h_{[p/2]-1})(0, y)=h_j(y), \qquad 0\leq j\leq[p/2]-1.$$

PROOF. Let $\kappa=[p/2]-1$. We note that $C^\infty_{(0)}(\overline{\boldsymbol{R}^n_+})$ is dense in $H^p_*(\boldsymbol{R}^n_+)$. (For the proof of this fact, see [2, Appendix B, Lemma B.1].) We show that

$$\|\partial^j_t u(0, y)\|_{H^{p-2j-1}(\boldsymbol{R}^{n-1}_y)}\leq C\|u\|_{p,*}, \qquad j=0, 1, \ldots, \kappa, \tag{1}$$

for $u\in C^\infty_{(0)}(\overline{\boldsymbol{R}^n_+})$, where C is a positive constant. Let $0\leq j\leq\kappa$. Then

$$\|\partial^j_t u(0, y)\|^2_{H^{p-2j-1}(\boldsymbol{R}^{n-1}_y)}=\int_{\boldsymbol{R}^{n-1}}|\langle\eta\rangle^{p-2j-1}\partial^j_t\hat{u}(0, \eta)|^2 d\eta. \tag{2}$$

Here $\hat{u}$ denotes the partial Fourier transform of u defined by

$$\hat{u}(t, \eta)=\int_{\boldsymbol{R}_y^{n-1}} e^{-iy\cdot\eta}u(t, y)dy .$$

We write $\langle\eta\rangle=\sqrt{1+|\eta|^2}$ and $đ\eta=(2\pi)^{-(n-1)}d\eta$. The right hand side of (2) is equal to

$$\begin{aligned}
&-\int_{\boldsymbol{R}^{n-1}} \langle\eta\rangle^{2(p-2j-1)}\left(\int_0^\infty \partial_t|\partial_t^j\hat{u}(t, \eta)|^2dt\right)đ\eta \\
&=-\int_{\boldsymbol{R}^{n-1}} \langle\eta\rangle^{2(p-2j-1)}\left(2\,\mathrm{Re}\int_0^\infty \partial_t^{j+1}\hat{u}(t, \eta)\overline{\partial_t^j\hat{u}(t, \eta)}dt\right)đ\eta \\
&=-2\,\mathrm{Re}\int_0^\infty\int_{\boldsymbol{R}^{n-1}} (\langle\eta\rangle^{p-2j-2}\partial_t^{j+1}\hat{u}(t, \eta))(\langle\eta\rangle^{p-2j}\overline{\partial_t^j\hat{u}(t, \eta)})dtđ\eta \\
&\le 2\left(\int_0^\infty\int_{\boldsymbol{R}^{n-1}} |\langle\eta\rangle^{p-2j-2}\partial_t^{j+1}\hat{u}(t, \eta)|^2dtđ\eta\right)^{1/2} \\
&\quad\times\left(\int_0^\infty\int_{\boldsymbol{R}^{n-1}} |\langle\eta\rangle^{p-2j}\partial_t^j\hat{u}(t, \eta)|^2dtđ\eta\right)^{1/2} \\
&\le 2\left(\sum_{|\alpha|+2(k+1)\le p}\|\partial_y^\alpha\partial_t^{k+1}u\|^2\right)^{1/2}\left(\sum_{|\alpha|+2k\le p}\|\partial_y^\alpha\partial_t^k u\|^2\right)^{1/2} \\
&\le 2\|u\|_{p,*}^2 .
\end{aligned}$$

This proves the first assertion of Theorem 1. Next we define the mapping $\mathscr{R}$. Let $h_j\in C_0^\infty(\boldsymbol{R}^{n-1})$, $j=0, 1, \dots, \kappa$. We set

$$g_0(y)=h_0(y) .$$

We define g_j successively by

$$\hat{g}_j(\eta)=\hat{h}_j(\eta)-\sum_{i=0}^{j-1}\partial_t^j w_i(0, \eta) , \qquad 1\le j\le\kappa \tag{3}$$

where

$$w_i(t, \eta)=e^{-t\langle\eta\rangle^2}\frac{t^i}{i!}\hat{g}_i(\eta) . \tag{4}$$

Let $\varphi\in C_{(0)}^\infty(\overline{\boldsymbol{R}_+})$ be a nonnegative decreasing function. Suppose that $\varphi(t)=1$ in a neighborhood of the origin. We fix φ once and for all. Let $u\in L^2(\boldsymbol{R}_+^n)$ be a function such that

$$\hat{u}(t, \eta)=\varphi(t)\sum_{j=0}^{\kappa} w_j(t, \eta) .$$

We define the mapping $\mathscr{R}$ by $(h_0, \dots, h_{[p/2]-1})\mapsto u$. Then we have $\partial_t^j u(0, y)=h_j(y)$, $0\le j\le\kappa$. This is seen from

$$\partial_t^j \hat{u}(0,\eta)=\sum_{i=0}^{\kappa}\partial_t^j w_i(0,\eta)=\sum_{i=0}^{j}\partial_t^j w_i(0,\eta)=\partial_t^j w_j(0,\eta)+\sum_{i=0}^{j-1}\partial_t^j w_i(0,\eta)$$
$$=\hat{g}_j(\eta)+(\hat{h}_j(\eta)-\hat{g}_j(\eta))=\hat{h}_j(\eta)\,,$$

for $1\leq j\leq\kappa$. When $j=0$, we have $u(0,y)=g_0(y)=h_0(y)$ by definition.

We prove in turn the continuity of $\mathscr{R}$ as a mapping of $\prod_{j=0}^{[p/2]-1}H^{p-2j-1}(\boldsymbol{R}_y^{n-1})\to H_*^p(\boldsymbol{R}_+^n)$. We note that

$$\|u\|_{p,*}^2=\sum_{r+|\alpha|+2k\leq p}\|\rho(t)^r\partial_t^r\partial_y^\alpha\partial_t^k u\|^2=\sum_{(r+k)+|\alpha|+k\leq p}\|\rho(t)^r\partial_t^{r+k}\partial_y^\alpha u\|^2$$
$$=\sum_{\substack{s+|\alpha|+k\leq p\\ s\geq k}}\|\rho(t)^{s-k}\partial_t^s\partial_y^\alpha u\|^2\,.$$

Each term on the right hand side of the last equality is at most

$$\int_0^\infty\int_{\boldsymbol{R}^{n-1}}|t^{s-k}\langle\eta\rangle^{|\alpha|}\partial_t^s\hat{u}(t,\eta)|^2 dt d\eta\,.$$

This in turn is not greater than a constant multiple of

$$\sum_{j=0}^{\kappa}\sum_{r=0}^{s}\sum_{q=(r-j)\vee 0}^{r}\int_0^\infty\int_{\boldsymbol{R}^{n-1}}\left|t^{s-k}\langle\eta\rangle^{|\alpha|}\binom{s}{r}\right.$$
$$\left.\times\varphi^{(s-r)}(t)\binom{r}{q}(-\langle\eta\rangle^2)^q e^{-t\langle\eta\rangle^2}\frac{t^{j-(r-q)}}{(j-(r-q))!}\hat{g}_j(\eta)\right|^2 dt d\eta$$
$$\leq C\sum_{j=0}^{\kappa}\sum_{r=0}^{s}\sum_{q=(r-j)\vee 0}^{r}\int_0^\infty\int_{\boldsymbol{R}^{n-1}}|t^{s-k+q-r+j}\langle\eta\rangle^{|\alpha|+2q}e^{-t\langle\eta\rangle^2}\hat{g}_j(\eta)|^2 dt d\eta\,.$$

To compute each term on the right hand side of this inequality, we change the variable of integration from t to z by setting $z=t\langle\eta\rangle^2$. Then

$$\int_0^\infty\int_{\boldsymbol{R}^{n-1}}|t^{s-k+q-r+j}\langle\eta\rangle^{|\alpha|+2q}e^{-t\langle\eta\rangle^2}\hat{g}_j(\eta)|^2 dt d\eta \tag{5}$$
$$=\int_0^\infty z^{2(s-k+q-r+j)}e^{-2z}dz\int_{\boldsymbol{R}^{n-1}}\langle\eta\rangle^{2|\alpha|-4(s-k-r+j)-2}|\hat{g}_j(\eta)|^2 d\eta\,.$$

Since $s\geq k$ and $q\geq r-j$, we have

$$\int_0^\infty z^{2(s-k+q-r+j)}e^{-2z}dz\leq C\,.$$

Hence the right hand side of (5) does not exceed

$$(6) \qquad C\int_{\mathbf{R}^{n-1}} \langle\eta\rangle^{2|\alpha|-4(s-k-r+j)-2}|\hat{g}_j(\eta)|^2 d\eta\,.$$

We notice that

$$(7) \qquad \hat{g}_j(\eta)=\sum_{l=0}^{j}\binom{j}{l}\langle\eta\rangle^{2(j-l)}\hat{h}_l(\eta)\,, \qquad j=0,\dots,\kappa\,.$$

This equality will be proved later as a lemma. The integral (6) is estimated by using (7) as follows.

$$\begin{aligned}
&\int_{\mathbf{R}^{n-1}} \langle\eta\rangle^{2|\alpha|-4(s-k-r+j)-2}|\hat{g}_j(\eta)|^2 d\eta \\
&\quad \le C\sum_{l=0}^{j}\int_{\mathbf{R}^{n-1}} \langle\eta\rangle^{2|\alpha|-4(s-k-r+j)-2+4(j-l)}|\hat{h}_l(\eta)|^2 d\eta \\
&\quad = C\sum_{l=0}^{j}\|\langle\eta\rangle^{|\alpha|-2s+2k+2r-2l-1}\hat{h}_l(\eta)\|^2_{L^2(\mathbf{R}^{n-1}_\eta)}\,.
\end{aligned}$$

If $|\alpha|-2s+2k+2r-2l-1\ge 0$, then each term on the right hand side is estimated by $\|h_l\|^2_{H^{|\alpha|-2s+2k+2r-2l-1}(\mathbf{R}^{n-1})}$. Otherwise the corresponding term is bounded by $\|h_l\|^2_{L^2(\mathbf{R}^{n-1})}$. Combining these estimates, we obtain finally

$$\begin{aligned}
\|\rho(t)^{s-k}\partial_t^s\partial_y^\alpha u\|^2 &\le C\sum_{j=0}^{\kappa}\sum_{r=0}^{s}\sum_{q=(r-j)\vee 0}^{r}\sum_{l=0}^{j}\|h_l\|^2_{H^{(|\alpha|+2k-2s+2r-2l-1)\vee 0}(\mathbf{R}^{n-1})} \\
&\le C\sum_{j=0}^{\kappa}\|h_j\|^2_{H^{|\alpha|+2k-2j-1}(\mathbf{R}^{n-1})}\,,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u\|^2_{p,*} &\le C\sum_{\substack{s+|\alpha|+k\le p\\ s\ge k}}\|\rho(t)^{s-k}\partial_t^s\partial_y^\alpha u\|^2 \le C\sum_{|\alpha|+2k\le p}\sum_{j=0}^{\kappa}\|h_j\|^2_{H^{|\alpha|+2k-2j-1}(\mathbf{R}^{n-1})} \\
&\le C\sum_{j=0}^{\kappa}\|h_j\|^2_{H^{p-2j-1}(\mathbf{R}^{n-1})}\,.
\end{aligned}$$

Since $C_0^\infty(\mathbf{R}^{n-1})\times\cdots\times C_0^\infty(\mathbf{R}^{n-1})$ is dense in $H^{p-1}(\mathbf{R}^{n-1})\times\cdots\times H^{p-2[p/2]+1}(\mathbf{R}^{n-1})$, the mapping $(h_0,\dots,h_{[p/2]-1})\mapsto u$ extends by continuity to a mapping of $\prod_{j=0}^{[p/2]-1}H^{p-2j-1}(\mathbf{R}^{n-1}_y)\to H^p_*(\mathbf{R}^n_+)$. This completes the proof of Theorem 1.

LEMMA. *For $j=0,\dots,\kappa$, we have*

$$(8) \qquad \frac{\hat{g}_j(\eta)}{\langle\eta\rangle^{2j}}=\sum_{l=0}^{j}\binom{j}{l}\frac{\hat{h}_l(\eta)}{\langle\eta\rangle^{2l}}\,.$$

PROOF. First we observe that by (4)

$$\partial_t^j w_i(0, \eta) = \partial_t^j(e^{-t\langle\eta\rangle^2} \frac{t^i}{i!} \hat{g}_i(\eta))|_{t=0}$$

$$= \sum_{q=0}^{j} \binom{j}{q} \frac{t^{i-q}}{(i-q)!} (-\langle\eta\rangle^2)^{j-q} e^{-t\langle\eta\rangle^2} \hat{g}_i(\eta)|_{t=0} = \binom{j}{i} (-\langle\eta\rangle^2)^{j-i} \hat{g}_i(\eta)$$

for $0 \le i \le j-1$, $1 \le j \le \kappa$. Substitution of this into (3) and division of both sides by $(-\langle\eta\rangle^2)^j$ yield

$$\frac{\hat{g}_j(\eta)}{(-\langle\eta\rangle^2)^j} = \frac{\hat{h}_j(\eta)}{(-\langle\eta\rangle^2)^j} - \sum_{i=0}^{j-1} \binom{j}{i} \frac{\hat{g}_i(\eta)}{(-\langle\eta\rangle^2)^i} \tag{9}$$

for $j=1, \ldots, \kappa$. Let $x_j = \hat{g}_j(\eta)/\langle\eta\rangle^{2j}$ and let $y_j = \hat{h}_j(\eta)/(-\langle\eta\rangle^2)^j$, $j=0, \ldots, \kappa$. Then we have by (9)

$$y_j = \sum_{l=0}^{j} (-1)^l \binom{j}{l} x_l, \qquad 0 \le j \le \kappa.$$

This implies that

$$x_j = \sum_{l=0}^{j} (-1)^l \binom{j}{l} y_l, \qquad 0 \le j \le \kappa.$$

From this follows (8) at once. We end the proof of the Lemma.

THEOREM 2. *Let p and q be integers with $[p/2] > [q/2] \ge 1$. Then the mapping $\mathscr{R}$ whose existence is guaranteed by Theorem 1 can be chosen so that the mapping*

$$\prod_{j=0}^{[q/2]-1} H^{p-2j-1}(\boldsymbol{R}_y^{n-1}) \ni (h_0, \ldots, h_{[q/2]-1}) \mapsto \mathscr{R}(h_0, \ldots, h_{[q/2]-1}, \underbrace{0, \ldots, 0}_{[p/2]-[q/2] \text{ times}}) \in H_*^p(\boldsymbol{R}_+^n)$$

extends by continuity to a continuous linear mapping of

$$\prod_{j=0}^{[q/2]-1} H^{q-2j-1}(\boldsymbol{R}_y^{n-1}) \to H_*^q(\boldsymbol{R}_+^n).$$

PROOF. Let p and q be integers such that $[p/2] > [q/2] \ge 1$. Let $h_j \in H^{p-2j-1}(\boldsymbol{R}^{n-1})$, $j=0, 1, \ldots, [q/2]-1$. We set

$$\tilde{h}_j = \begin{cases} h_j, & j=0, 1, \ldots, [q/2]-1, \\ 0, & j=[q/2], \ldots, [p/2]-1. \end{cases}$$

Then the mapping

$$(h_0, \ldots, h_{[q/2]-1}) \mapsto (\tilde{h}_0, \ldots, \tilde{h}_{[p/2]-1})$$

defines a continuous linear mapping $\mathscr{T}$ of

$$\prod_{j=0}^{[q/2]-1} H^{p-2j-1}(\boldsymbol{R}_y^{n-1}) \to \prod_{j=0}^{[p/2]-1} H^{p-2j-1}(\boldsymbol{R}_y^{n-1}) .$$

Let $\mathscr{R}$ be a continuous linear mapping of

$$\prod_{j=0}^{[p/2]-1} H^{p-2j-1}(\boldsymbol{R}_y^{n-1}) \to H_*^p(\boldsymbol{R}_+^n)$$

described in the proof of Theorem 1. It is obvious that $\mathscr{R}\mathscr{T}$ is a continuous linear mapping of

$$\prod_{j=0}^{[q/2]-1} H^{p-2j-1}(\boldsymbol{R}_y^{n-1}) \to H_*^p(\boldsymbol{R}_+^n) .$$

Now let

$$u = \mathscr{R}\mathscr{T}(h_0, \dots, h_{[q/2]-1}) .$$

Then we have

$$\|u\|_{q,*}^2 \le C \sum_{\substack{s+|\alpha|+k\le q \\ s\ge k}} \|\rho(t)^{s-k}\partial_t^s \partial_y^\alpha u\|^2 \tag{10}$$

$$\le C \sum_{|\alpha|+2k\le q} \sum_{j=0}^{[q/2]-1} \|\tilde{h}_j\|^2_{H^{|\alpha|+2k-2j-1}(\boldsymbol{R}^{n-1})} .$$

The computation can be carried through in much the same way as in the proof of Theorem 1. It follows from (10) that

$$\|u\|_{q,*}^2 \le C \sum_{j=0}^{[q/2]-1} \|h_j\|^2_{H^{q-2j-1}(\boldsymbol{R}^{n-1})} . \tag{11}$$

This implies that $\mathscr{R}\mathscr{T}$ extends by continuity to a continuous linear mapping of

$$\prod_{j=0}^{[q/2]-1} H^{q-2j-1}(\boldsymbol{R}_y^{n-1}) \to H_*^q(\boldsymbol{R}_+^n) .$$

The proof of Theorem 2 is thus complete.

Now we consider anisotropic Sobolev spaces defined on a general domain. Let Γ be a smooth compact hypersurface of dimension $n-1$ in $\boldsymbol{R}^n$ which does not interesct itself. Let Ω be the interior or the exterior of Γ. A vector field Λ on Ω is said to belong to $\mathscr{B}^\infty(\bar{\Omega}; \boldsymbol{C}^n)$ if each component of Λ is a bounded C^∞-function on $\bar{\Omega}$ and the derivatives of any order are also bounded on $\bar{\Omega}$. We say that a vector field Λ is tangential if $\langle \Lambda(x), \nu(x) \rangle = 0$ for all $x \in \Gamma$. Here $\nu = (\nu_1, \dots, \nu_n)$ denotes the unit outward normal to Γ. The function space $H_*^m(\Omega)$, $m \ge 1$, is defined as the set of functions satisfying the following properties:

(1) $u \in L^2(\Omega)$.

(2) Let $\Lambda_1, \dots, \Lambda_j \in \mathscr{B}^\infty(\bar{\Omega}: \boldsymbol{C}^n)$ be tangential vector fields and let $\Lambda'_1, \dots, \Lambda'_k \in \mathscr{B}^\infty(\bar{\Omega}: \boldsymbol{C}^n)$ be any vector fields. Then $\Lambda_1 \cdots \Lambda_j \Lambda'_1 \cdots \Lambda'_k u \in L^2(\Omega)$, if $j+2k \le m$.

Let us define a norm on $H^m_*(\Omega)$ by making use of the partition of unity. We choose a finite open covering $\{\mathcal{O}_j\}_{j=1}^N$ of Γ. Then, for each j, there exists a diffeomorphism τ_j from $\mathcal{O}_j \cap \bar{\Omega}$ to a semi-ball $\{(t, y) \mid t \ge 0, t^2 + |y|^2 < a_j^2\}$ such that $\tau_j(\Gamma \cap \mathcal{O}_j) \subset \{t=0\}$ and that the normal vector field ∂_ν corresponds to $-\partial_t$ there. We see that any vector field in $\mathcal{O}_j$ which is tangential to Γ can be represented in the semi-ball as a linear combination of $t\partial_t, \partial_1, \dots, \partial_{n-1}$ with C^∞-coefficients. Let $\mathcal{O}_0 = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) > \delta\}$ for a small positive number δ. Then, for each j $(0 \le j \le N)$, there exists a function χ_j of class C^∞ with support in $\mathcal{O}_j$ such that $\sum_{j=0}^N \chi_j^2 = 1$ on $\bar{\Omega}$. A function u on Ω belongs to $H^m_*(\Omega)$ if $u^{(j)} = (\chi_j u) \circ \tau_j^{-1} \in H^m_*(\boldsymbol{R}^n_+)$ for $1 \le j \le N$ and if $\chi_0 u \in H^m(\boldsymbol{R}^n)$. The space $H^m_*(\Omega)$ is endowed with the norm

$$\|u\|_{m,*}^2 = \|\chi_0 u\|_m^2 + \sum_{j=1}^N \|\chi_j u\|_{m,*}^2 ,$$

$$\|\chi_j u\|_{m,*}^2 = \sum_{r+|\alpha|+2k \le m} \|\partial_{\tan}^{(r,\alpha)} \partial_1^k u^{(j)}\|_{L^2(\boldsymbol{R}^n_+)}^2 ,$$

where $\|\cdot\|_m$ denotes the usual Sobolev norm, $\alpha = (\alpha_1, \dots, \alpha_n)$ and

$$\partial_{\tan}^{(r,\alpha)} = (\rho(t)\partial_t)^r \partial_1^{\alpha_1} \cdots \partial_{n-1}^{\alpha_{n-1}} .$$

The norms arising from different choices of $\mathcal{O}_j$, τ_j, χ_j are equivalent norms.

The trace theorem on $H^m_*(\Omega)$ can be derived from Theorem 1 by using a partition of unity and local coordinate changes. Theorem 2 also has its counterpart in $H^m_*(\Omega)$.

REMARK 1. We recall the definition of the norm given earlier which is equivalent to the original norm of $H^m_*(\Omega)$. Then it turns out that $H^m_*(\Omega)$ can be regarded as a weighted Sobolev space. In fact, we have

$$\|u\|_{m,*}^2 = \sum_{s+|\alpha| \le m} \int_\Omega |\partial_t^s \partial_y^\alpha u(t, y)|^2 \sigma_{s,\alpha}(t) dt dy$$

with

$$\sigma_{s,\alpha}(t) = \sum_{(2s+|\alpha|-m)_+ \le l \le s} \rho(t)^{2l} ,$$

when $\Omega = \boldsymbol{R}^n_+$. Notice that the weight depends explicitly on the multi-index (s, α). We refer the reader to Triebel [4] for the general weighted Sobolev spaces.

REMARK 2. We have no trace theorem on $H^1_*(\Omega)$. There is a function that belongs to $H^1_*(\Omega)$ and has no trace. We mention here a weighted Sobolev space which is similar to $H^1_*(\Omega)$ but is used for different problems. Let $W^m_k(\Omega)$ denote the set of functions such that $u \in H^{m-k}(\Omega)$ and $\rho^{m-k} u \in H^m(\Omega)$, where $0 \le k \le m$. Then $H^1_*(\Omega) \subsetneq W^1_1(\Omega)$. The function spaces $W_k^m(\Omega)$ are suitable for the study of the degenerate elliptic operators. The reader is referred to Shimakura [3].

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