TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 11, November 1997, Pages 4385–4399
S 0002-9947(97)01867-9

THE TRANSFER AND SYMPLECTIC COBORDISM

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ABSTRACT. The main result of this paper is the nilpotency fomula $\phi_i^4 = 0$, $\forall i \geq 1$ for N. Ray classes ϕ_i in the torsion of the symplectic bordism ring MSp_*

INTRODUCTION

This paper is organised as follows. Section 1 is devoted to calculation of the transfer homomorphism in the symplectic cobordism theory [D], [BG]. In particular, using the results of [BM], [Fe], [Sn] we calculate the transfer homomorphism for projective bundles associated with universal $Spin(m)$ bundles, $m = 3, 4, 5$. This section includes the following corollary in the case $m = 3$:

Let N be the normalizer of the torus $U(1)$ in $Sp(1)$; $\zeta \to BSp(1)$ be the universal $Sp(1)$ bundle and Λ be the universal $Spin(3)$ bundle over $BSpin(3) = BSp(1)$. Then the bundle $p : BN \to BSp(1)$ is the projective bundle associated with Λ. Let

$$x = pf_1(\zeta);$$

$$y = pf_1(p^*(\zeta));$$

$$e = pf_3(p^*(\Lambda \otimes_R H))$$

be the Conner-Floyd symplectic Pontryagin classes and

$$\tau_p^* : MSp^*(BN) \to MSp^*(BSp(1))$$

be the transfer homomorphism. Then τ_p^* satisfies the relations

$$\tau_p^*(1) = 1; \tag{1}$$

$$\tau_p^*(e) = 0. \tag{2}$$

In Section 2 we establish a connection of the Euler class e with the classes ϕ_i defined as follows:

Recall from [R] the classes θ_i arising from the expansion

$$pf_1((\eta - R) \otimes_R (\zeta - H)) = s \sum_{i \geq 1} \theta_i pf_1^i(\zeta) = s \sum_{i \geq 1} \theta_i x^i$$

in $MSp^4(S^1 \wedge BSp(1))$, where s is the generator of $MSp^1(S^1)$, $\eta \to S^1$ is the nontrivial real line bundle and ζ is as above. Also recall the relabelling $\theta_{2i} = \phi_i$ in

Received by the editors July 31, 1995.
1991 *Mathematics Subject Classification.* Primary 55N22, 55R12.
Key words and phrases. Symplectic cobordism, Euler class, classifying space, transfer.
The research described in this publication was made possible in part by Grant RVJ 000 from the International Science Foundation.

MSp_{8i-3}, and from [Ro] that $\theta_{2i-1} = 0$ for $i > 1$. As proved in [R], each ϕ_i is an indecomposable torsion element of order 2.

It is shown in [Na] that the homomorphism π^* induced by $\pi : BU(1) \to BSp(1)$ is not a monomorphism in the symplectic cobordism theory. In particular (see Section 2)

$$\pi^*(\theta_1 x + \sum_{i\geq 1} \phi_i x^{2i}) = 0.$$

Using this observation and the results of [G], [GR], we state that in $MSp^*(BN)$

$$e = \sum_{i\geq 1} \phi_i^4 y^{8i}(1 + \sum_{j\geq 1} \alpha_j y^j) \tag{3}$$

for some coefficients $\alpha_j \in MSp_*$.

Applying (1), (2), (3) we have

$$\tau_p^*(e) = 0$$

by (2),

$$= \tau_p^*(\sum_{i\geq 1} \phi_i^4 y^{8i}(1 + \sum_{j\geq 1} \alpha_j y^j))$$

by (3),

$$= \sum_{i\geq 1} \phi_i^4 x^{8i}(1 + \sum_{j\geq 1} \alpha_j x^j)\tau_p^*(1)$$

by the transfer property,

$$= \sum_{i\geq 1} \phi_i^4 x^{8i}(1 + \sum_{j\geq 1} \alpha_j x^j)$$

by (1).

Thus we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = 0$$

in $MSp^*(BSp(1)) = MSp_*[[x]]$.

This proves

Theorem. $\phi_i^4 = 0$, $\forall i \geq 1$.

We cannot use a reasoning similar to that of Section 2 for the self-conjugate cobordism, since in this theory it is impossible to construct characteristic classes with the required properties. Namely, as proved in [BaNa], for arbitrary natural classes

$$P_i(\xi^n) \in SC^{2i}(X)$$

in the self-conjugate cobordism theory

$$P(\xi^n) = 1 + P_1(\xi^n) + ... + P_n(\xi^n),$$

where $\xi^n \to X$ is the SC-vector bundle, the following conditions are contradictory:

1. $P_n(\xi^n)$ is the Euler class (normalization);
2. $P(\xi^n + \xi^m) = P(\xi^n)P(\xi^m)$ (the Whitney formula).

That is why in Section 3 we calculate the transfer homomorphism for the bundle of flags of the bundle Λ. As a corollary we obtain a new proof of the nilpotency formula for the N. Ray classes in the self-conjugate cobordism, which was proved for the first time in [Na].

As is known from [Mo] and [V], various three-fold products of N. Ray's family are nontrivial. In Section 4 we shall prove

Proposition 4.1. *All four-fold products of the N. Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.*

I would especially like to thank Professor Mark Mahowald for his encouragement and advice. My thanks are also due to the referee for many useful suggestions.

1. Calculation with Transfer

The result of this section is

Proposition 1. *Let $G_m = Spin(m)$ and $\xi^m \to BG_m$ be the universal $Spin(m)$ bundle, $m = 3, 4, 5$. Let*

$$p_m : P(\xi^m) \to BG_m$$

be the associated projective bundle with fibre RP^{m-1}, and $\lambda_m \to P(\xi^m)$ be the canonical real line bundle. Then the transfer homomorphism

$$\tau_m^* : MSp^*(P(\xi^m)) \to MSp^*(BG_m)$$

satisfies the relations

$$\tau_m^*(c_m^n) = 0, \tag{1.1}$$

for all $n \geq 1$, where $c_m = pf_1(\lambda_m \otimes_R H)$ is the first Conner-Floyd symplectic Pontryagin class;

$$\tau_m^*(1) = \chi(RP^{m-1}), \tag{1.2}$$

where $\chi(RP^{m-1})$ is the Euler characteristic of RP^{m-1} and hence is equal to 1 if $m = 3, 5$, and to 0 if $m = 4$;

$$\tau_m^*(e_m) = 0, \tag{1.3}$$

where $e_m = e(p_m^*(\xi^m \otimes_R H))$ is the Euler class.

For the proof we need the following facts.

1.4. $Spin(m)$ **bundles.** It is well known that the groups

$$Spin(2), Spin(3), Spin(4), Spin(5), Spin(6)$$

are isomorphic to

$$S^1 = U(1), S^3 = Sp(1) = SU(2), Sp(1)^2, Sp(2), SU(4).$$

The inclusions $Spin(i) \to Spin(i+1)$ up to an isomorphism are described as follows:

$Spin(2) \to Spin(3)$ is the standard $U(1) \to Sp(1)$;
$Spin(3) \to Spin(4)$ is the diagonal homomorphism $Sp(1) \to Sp(1)^2$;
$Spin(4) \to Spin(5)$ is the embedding $Sp(1)^2 \to Sp(2)$ of diagonal matrices.
$Spin(5) \to Spin(6)$ is the embedding of matrices A for which $A^T JA = J$, where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Denote $Spin(m)$ by G_m and consider N_m, the normalizer of G_m in G_{m+1}. Then

N_2 consists of $U(1)$ and $jU(1)$, where j is the quaternionic unit;

N_3 consists of matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, a is the quaternion, $a\bar{a} = 1$;

N_4 consists of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} o & a \\ b & 0 \end{pmatrix}$, where a and b are quaternions, $a\bar{a} = b\bar{b} = 1$.

The universal $Spin(m)$ bundles ξ^m have the following description.

Case $m = 5$. $G_5 = Spin(5) = Sp(2)$ acts by conjugation on the 5-dimensional vector space of 2×2 quaternionic Hermitian matrices of zero trace. These matrices are of the form $\begin{pmatrix} a_0 & b \\ \bar{b} & -a_0 \end{pmatrix}$, where a_0 is real, b is a quaternion and $(b, a_0) \in R^5$.

Let $E \to BG_5$ be the principal $Spin(5)$ bundle. Then the above action of G_5 on R^5 defines the sphere bundle of ξ^5,

$$BG_4 = E \times_{G_5} S^4 \to BG_5;$$

and the projective bundle of ξ^5,

$$BN_4 = E \times_{G_5} RP^4 \to BG_5.$$

Case $m = 4$. The universal $Spin(4)$ bundle ξ^4 is

$$\zeta_1 \otimes_H \zeta_2^* \to BSp(1)^2,$$

where ζ_1, ζ_2 are the canonical symplectic line bundles, ζ_2^* is the symplectic conjugate of ζ_2 and $(q_1, q_2) \in Sp(1)^2 = G_4$ acts on $R^4 \cong H$ by $v \to q_1 v q_2^{-1}$.

This defines the sphere bundle and the projective bundle of ξ^4:

$$BG_3 = E \times_{G_4} S^3 \to BG_4,$$

$$BN_3 = E \times_{G_4} RP^3 \to BG_4.$$

Case $m = 3$. The universal $Spin(3)$ bundle ξ^3 is

$$\Lambda \to BSp(1),$$

where $1 + \Lambda = \zeta \otimes_H \zeta^*$. $G_3 = Sp(1)$ acts on R^3 as conjugation on the pure quaternion.

This defines the sphere bundle and the projective bundle of ξ^3:

$$BG_2 = E \times_{G_3} S^2 \to BG_3,$$

$$BN_2 = E \times_{G_3} RP^2 \to BG_3.$$

Consider now the standard inclusion $RP^3 \to RP^4$. This is G_4 equivariant, where G_4 acts on RP^3 as above and on RP^4 as a subgroup of G_5.

This defines the inclusion of the projective bundle $P(\xi^4)$ in $P(\xi^4 + 1)$:

$$l : BN_3 = P(\xi^4) = E \times_{G_4} RP^3 \to E \times_{G_4} RP^4 = P(\xi^4 + 1).$$

The inclusion $RP^2 \to RP^3$, induced by the embedding of the pure quaternions into $H \cong R^4$ is G_3-equivariant. Here G_3 acts on RP^2 as above and on RP^3 as a subgroup of G_4.

This defines the inclusion

$$m : BN_2 = P(\xi^3) = E \times_{G_3} RP^2 \to E \times_{G_3} RP^3 = P(1 + \xi^3).$$

Let

$$\lambda_3 \to P(\xi^3), \quad \lambda_4 \to P(\xi^4),$$

$$\tilde{\lambda_4} \to P(1+\xi^3), \quad \tilde{\lambda_5} \to P(\xi^4+1)$$

be the canonical real line bundles. Then it is easy to see

Lemma 1.5. $l^!(\tilde{\lambda_4}) = \lambda_3$, $m^!(\tilde{\lambda_5}) = \lambda_4$.

1.6. Double coset formula. Let G be a compact Lie group and H and K closed subgroups.

Recall that the bundle $\rho(H,G) : BH \to BG$ has the fibre G/H and structure group G. Consider the pullback of BH to BK,

$$\begin{tikzcd} \Gamma = \times_K(G/K) \arrow[r] \arrow[d, "\gamma"] & BH \arrow[d, "\rho(G,H)"] \\ BK \arrow[r, "\rho(K,G)"'] & BG \end{tikzcd}$$

$\gamma : \Gamma \to BK$ has the fibre G/H and structure group K.

Let $\tau^*(H,G)$ be the transfer homomorphsm associated to $\rho(H,G)$. Then there is a formula for calculation of $\rho^*(K,G)\tau^*(H,G)$.

Theorem [Fe]**.** *Let $\gamma : \Gamma \to BK$ be the fibre bundle with fibre $F = G/H$ and structure group K acting on the left on F. Let $\{M\}$ be the set of orbit-type manifold components of the orbit space $K|F$, and let q be any K-orbit in M. Let $\tilde{q}$ be the subbundle of γ corresponding to q. Let $k : \tilde{q} \to \gamma$ be the inclusion and $\chi^*(M) = \chi(\bar{M} - M)$. Then*

$$\tau^*_\gamma = \sum \chi^*(M)\tau^*_{\tilde{q}} k^*,$$

where the sum is over all the orbit-type manifold componemts $\{M\}$.

1.7. Calculations with transfer for sphere bundles. For the proof of Proposition 1 we need the following

Lemma 1.8. *Let $G_m = Spin(m)$ and $\rho(G_{m-1}, G_m) : BG_{m-1} \to BG_m$ be the sphere bundle of the universal $Spin(m)$ bundle ξ^m. Then*

$$\tau^*(G_{m-1}, G_m)(1) = \chi(S^{m-1}), \qquad m = 3, 4, 5,$$

in symplectic cobordism. Here $\chi(S^{m-1})$ is the Euler characterstic and because of this is equal to 2 if $m = 3, 5$, and 0 if $m = 4$.

Proof. Case $m = 4$. For the diagonal map $\rho = \rho(Sp(1), Sp(1)^2)$ we have $\rho(x_1) = \rho(x_2)$, where $x_i = cf_1(\zeta_i)$; ζ_1, ζ_2 are the canonical line symplectic bundles.

By the transfer property for $\tau^* = \tau^*(Sp(1), Sp(1)^2)$ we have

$$\tau^*(a)(x_1 - x_2) = \tau^*(\rho^*(x_1 - x_2)a) = 0$$

$\forall a \in MSp^*(BSp(1))$. Since $MSp^*(BSp(1)^2) = MSp^*[[x_1, x_2]]$, this proves that τ^* is the trivial homomorphism.

Case $m = 3$. Using the double coset formula for $\rho^*\tau^*$, we see that the double coset space $Sp(1)|Sp(1)^2|Sp(1)$ is the line segment, with isotropy group $Sp(1)$ at the endpoints and conjugate group of $U(1)$ in $Sp(1)$ in the interior. Taking into account the case $m = 4$, we have

$$0 = \tau^*(G_3, G_4)(1) = 2\tau^*(G_3, G_3)(1) - \tau^*(U(1), G_3)(1) = 2 - \tau^*(U(1), G_3)(1).$$

Since $\rho(U(1), G_3)$ is the sphere bundle of ξ^3, this proves the case $m = 3$.

Case $m = 5$. The sphere bundle of ξ^5 agrees with $\rho(Sp(1)^2, Sp(2))$. On the other hand this bundle is the quaternionic projective bundle associated to the universal symplectic plane bundle, and the statement is known from [D, p.235]. One may prove this case by the method we will use in the following section.

Proof of Propositions 1.1 and 1.2. Case $m = 5$. It is shown in [Sn, ch.1] that the following diagram of the stable maps is commutative (see also Remark 1.11):

$$\begin{tikzcd} BG_4 = BSp(1)^2 \arrow[r] \arrow[d] & BSp(2) = BG_5 \arrow[dl, "\tau"] \\ BZ_2 \wr Sp(1) = BN_4 & \end{tikzcd}$$

where $BSp(1)^2 \to BSp(2)$ is induced by $\rho(Sp(1)^2, Sp(2))$ and $BSp(1)^2 \to BZ_2 \wr Sp(1)$ by $\rho(Sp(1)^2, Z_2 \wr Sp(1))$.

Since, as it is well known, $\rho^*(Sp(1)^2, Sp(2))$ is a monomorphism, this proves the case $m = 5$.

Proof of Propositions 1.1 and 1.2. Case $m = 4$. The following lemma immediately follows from the definitions of 1.4.

Lemma 1.9. *The double coset space $G_4|G_5|N_4$ is a line segment. One endpoint corresponds to an orbit consisting of one point $(0, \pm 1) \in RP^4$, where*

$$G_5|N_4 = RP^4 = \{\pm(v, w) | v \textit{ is a quaternion}, \ w \textit{ is a real}, \ vv^* + w^2 = 1\}.$$

The point $(0, \pm 1)$ is a fixed point. The other endpoint corresponds to RP^3, consisting of points $(\pm v, 0) \in RP^4$. The isotropy groups for these points are conjugate groups of N_3 in G_4. The open interval corresponds to orbits S^3 consisting of points $\pm(v, w)$, $0 < vv^* < 1$. The isotropy groups for these points are conjugate groups of G_3 in G_4.

Proof. For the point $(0, \pm 1) \in RP^4$ the isotropy group is obviously $Sp(1)^2$. For the points $(\pm v, 0) \in RP^4$, the isotropy group K_v for the given $(\pm v, 0)$ consists of elements (vqv^{-1}, q) and $(-vqv^{-1}, q)$ from the group $Sp(1)^2$. Hence

$$K_v = gN_3g^{-1}, \ \ g = (v, 1).$$

For the points $(\pm v, \pm w) \in RP^4$, $0 < vv^* < 1$, we have

$$(q_1, q_2)(\pm v, \pm w) = (\pm q_1 v q_2^{-1}, \pm w);$$

$$v = q_1 v q_2^{-1};$$

$$q_1 = v q_2 v^{-1}.$$

So for the given $(\pm v, \pm w)$ the isotropy group is the conjugate group of $Sp(1)$ in $Sp(1)^2$.

Combining Lemma 1.9 and the double coset formula for $\rho^*(G_4, G_5)T(N_4, G_5)$, we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(1) = 1 - \tau^*(G_3, G_4)(1) + \tau^*(N_3, G_4)(1).$$

Since $\tau^*(N_4, G_5)(1) = 1$ and $\tau^*(G_3, G_4)(1) = 0$, this proves $\tau^*(N_3, G_4)(1) = 0$.

Consider now $\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^n)$. Again using the double coset formula above, this is decomposed into three summands. Of these, the two summands corresponding to the subbundles identity $BG_4 \to BG_4$ and $BG_3 \to BG_4$ are zero since there are no nontrivial real line bundles over BG_3 and BG_4. As for the third summand, it coincides with $\tau^*(N_3, G_4)(c_4^n)$ by Lemma 1.5.

Hence we have

$$\rho^*(G_4, G_5)\tau^*(N_4, G_5)(c_5^n) = 0$$

by the case $m = 5$,

$$= 0 - 0 + \tau^*(N_3, G_4)(c_4^n).$$

This proves the case $m = 4$.

Proof of Propositions 1.1, 1.2. Case $m = 3$**.** Consider now the double coset formula for $\rho^*(G_3, , G_4)\tau^*(N_3, G_4)$.

Recall, from 1.4, that the homogeneous space G_4/N_3 is the projective space

$$RP^3 = \{\pm h, hh^* = 1, h \in H\}.$$

It is easy to see the following.

Lemma 1.10. *The double coset space $G_3|G_4|N_3$ is a line segment. One endpoint corresponds to an orbit consisting of one point $(\pm 1) \in RP^3$. This point is fixed. The other endpoint corresponds to RP^2, consisting of points $\{\pm h, h$ pure quaternion, $hh* = 1\}$. The isotropy group for the given $(\pm h)$ is the conjugate group of N_2 in $Sp(1)$. The open interval corresponds to orbits S^2, consisting of points $(\pm h)$, whose real parts differ from 0 and ± 1. The isotropy groups for these points are the conjugate groups of $U(1)$ in $Sp(1)$.*

Using now the double coset formula, we obtain

$$0 = \rho * (G_3, G_4)\tau^*(N_3, G_4)(1)$$

by the case $m = 4$,

$$= 1 - \tau^*(U(1), Sp(1))(1) + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.10,

$$= 1 - 2 + \tau^*(N_2, Sp(1))(1)$$

by Lemma 1.8.

This proves that $\tau^*(N_2, Sp(1))(1) = 1$.

In the same spirit we obtain

$$0 = \rho * (G_3, G_4)\tau^*(N_3, G_4)(c_4^n)$$

by the case $m = 4$,

$$= 0 - 0 + \tau^*(N_2, Sp(1))(c_3^n)$$

by Lemma 1.10 and Lemma 1.5.

This proves the case $m = 3$.

Proof of Proposition 1.3. For $m = 3$ formula (1.3) coincides with (2) from the Introduction, which is the case we need to prove.

The projectivisation $p : BN \to BSp(1)$ of the bundle $\Lambda = \xi^3$ defines the canonical splitting over BN

$$p^*(\Lambda) = \mu + \lambda,$$

where μ and λ are a plane and a linear real bundle respectively.

Then we have the splitting

$$p^*(\Lambda \otimes_R H) = \mu \otimes_R H + \lambda \otimes_R H.$$

Apply now the Whitney formula to express the symplectic characteristic classes of the bundle $p^*(\Lambda \otimes_R H)$ in terms of the classes $\mu \otimes_R H$ and $\lambda \otimes_R H$. We obtain the equations

$$pf_1(p^*(\Lambda \otimes_R H)) = pf_1(\mu \otimes_R H) + pf_1(\lambda \otimes_R H);$$

$$pf_2(p^*(\Lambda \otimes_R H)) = pf_2(\mu \otimes_R H) + pf_1(\mu \otimes_R H)pf_1(\lambda \otimes_R H);$$

$$e = pf_3(p^*(\Lambda \otimes_R H) = pf_2(\mu \otimes_R H)pf_1(\lambda \otimes_R H).$$

Let $c = pf_1(\lambda \otimes_R H)$. Then the above equations give an exposition of e in terms of c and $pf_i(p^*(\Lambda \otimes_R H))$, $i = 1, 2$:

$$e = pf_2(p^*(\mu \otimes_R H))c$$

$$= [pf_2(p^*(\Lambda \otimes_R H)) - pf_1(\mu \otimes_R H)c]c$$

$$= pf_2(p^*(\Lambda \otimes_R H))c - [pf_1(p^*(\Lambda \otimes_R H)) - c]c^2$$

$$= pf_2(p^*(\Lambda \otimes_R H))c - pf_1(p^*(\Lambda \otimes_R H))c^2 + c^3.$$

Now apply the transfer homomorphism τ_p^* to this equation:

$$\tau_p^*(e) = \tau_p^*[p^*(pf_2(\Lambda \otimes_R H))c] - \tau_p^*[p^*(pf_1(\Lambda \otimes_R H))c^2] + \tau_p^*(c^3).$$

Taking into account the transfer property $\tau_p^*(p^*(t)) = t\tau_p^*(1)$, we obtain

$$\tau_p^*(e) = pf_2(\Lambda \otimes_R H)\tau_p^*(c) - pf_1(\Lambda \otimes_R H)\tau_p^*(c^2) + \tau_p^*(c^3).$$

But by virtue of Proposition 1.2 we have $\tau_p^*(c) = \tau_p^*(c^2) = \tau_p^*(c^3) = 0$. Therefore $\tau_p^*(e) = 0$.

The proofs of the cases $m = 4, 5$ are quite analogous. However the case $m = 4$ also follows from Proposition 1.1, namely, from the equality $\tau_4^*(1) = 0$:

$$\tau_4^*(e_4) = \tau_4^*(p_4^*(pf_4(\xi^4 \otimes_R H))) = pf_4(\xi^4 \otimes_R H)\tau_4^*(1) = pf_4(\xi^4 \otimes_R H) = 0.$$

Then, as proved in [GR], every $Spin(5)$ bundle and, in particular, ξ^4, is MSp-orientable and has zero Euler class. Thus $pf_5(\xi^5 \otimes_R H) = 0$, so we have nothing to prove in the case $m = 5$.

1.11. Remark on Propositions 1.1 and 1.2. Case $m = 5$. The commutativity of above diagram is stated by the method of equivariant vector fields on the homogeneus spaces [BM]. Namely there is [Sn, Example 1.13] an $Sp(1)^2$ equivariant vector field on $Sp(2)/Z_2 \wr Sp(1)$ with one singular point. Using this field we shall see here that in the case of the projective bundle $P(\xi^4 + 1)$ the transfer map is stably homotopic to the section of this bundle defined by the direct summand 1.

We need a simple particular case of [BM, Corollary 2.11]. Namely let $\pi : E \to B$ be the fiber bundle with fiber F. Suppose that F admits a G equivariant vector field with one singular point (fixed under the action G) and the Euler characteristic $\chi(F) = 1$. This fixed point obviously defines a section $i : B \to E$. Then i suspends to the transfer map $\tau(\pi)$, that is, $i^+ = \tau(\pi)$ in the track group $\{B^+, E^+\}$.

Taking into account Lemma 1.9, we see that the projective bundle $P(\xi^4 + 1)$, that is, the pullback of $BN_4 \to BG_5$ to BG_4, has section defined by the fixed point $(0, \pm 1) \in RP^4$ under the action of G_4. This section agrees with the section of $P(\xi^4 + 1)$ defined by the direct summand 1.

Lemma 1.12. *The above section of the projective bundle $P(\xi^4+1)$ suspends to the transfer map.*

Proof. Following [BM] we construct a $G_4 = Sp(1)^2$ equivariant vector field on $RP^4 = G_5/N_4$ with one zero point. It is easy to see that

$$G_5/N_4 = GL_2(H)/Z_2 \wr B(H)$$

where H is the quaternions, $GL_2(H)$ is the full linear group of 2×2 matrices, $B(H)$ are the all upper triangular matrices and the generator of Z_2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This follows from the fact, that $GL_2(H)$ acts on $G_5/N_4 = S^4$, that is, on the manifold of flags $F_1 \subset F_2 = H^2$, with the isotropy group $B(H)$.

Now let v be a vector from the Lie algebra of $GL_2(H)$, for which

$$\omega = exp(v) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

where x, y are real numbers and $x \neq y$.

Consider now the field φ_v on $GL_2(H)$ defined by the right translations:

$$\varphi_v(g) = dR_g(v), \qquad g \in GL_2(H).$$

This field induces the field ϱ_v on $GL_2(H)/Z_2 \wr B(H)$. The field ϱ_v is $Sp(1)^2$ equivariant, since $Sp(1)$ is a subgroup of the centralizer of ω . For the zero points of ϱ_v note that coset of g is the zero point if and only if $g^{-1}\omega g \in Z_2 \wr B(H)$, that is, $g \in Z_2 \wr B(H)$. Thus ϱ_v has one zero point. This proves Lemma 1.12.

The above lemma proves the analog of Proposition 1 for the projective bundle $P(\xi^4 + 1) \to BSp(1)^2$. But since this bundle is the pullback of $P(\xi^5) \to BG_5$ to BG_4 and the homomorphism induced by $BSp(1)^2 = BG_4 \to BG_5 = BSp(2)$ is a monomorphism, this proves the case $m = 5$.

2. Proof of (3) from the introduction

We need the following fact.

Propostion [Na]. *In $MSp^*(BU(1))$*

$$\theta_1 z + \sum_{k\geq 1} \phi_k z^{2k} = 0,$$

where $z = pf_1(\xi + \bar{\xi})$; ξ is the canonical complex line bundle; θ_1, ϕ_i are the Ray classes.

This follows immediately from the bundle relation

$$\eta \otimes_R (\xi + \bar{\xi}) = \xi + \bar{\xi}$$

in $KSp^0(S^1 \times BU(1))$ and from the definition of Ray classes.

Then, as it is known, any $Spin(4)$ bundle is MSp^* orientable. This follows from the isomorphism $KO^4 = KSp^0$: For the given KO orientation class of $Spin(4)$ bundle this isomorphism determines the symplectic bundle over the corresponding Thom space, and the first Conner-Floyd symplectic Pontryagin class of this symplectic bundle will be taken as the symplectic orientation class. So the $Spin(4)$ bundle $\zeta \otimes_H \zeta^* = 1 + \Lambda$, and because of this Λ is MSp^* orientable [RS].

By using these results and the fact that the bundle $BU(1) \to BSp(1)$ is the sphere bundle of Λ it is proved in [G] that the Thom class of the bundle Λ can be chosen in such a way that its restiction to the zero section $\tilde{e}(\Lambda)$ has the form

$$\tilde{e}(\Lambda) = \theta_1 x + \sum_{i\geq 1} \phi_i x^{2i},$$

where $x = pf_1(\zeta)$. For another proof, see [GR].

Since $2\theta_1 = 2\phi_i = 0$ [R] and $\theta_1^3 = 0$ [G], we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = (\tilde{e}(\Lambda))^4 = \tilde{e}(\Lambda \otimes_R H)$$

But $\tilde{e}(\Lambda \otimes_R H)$ agrees with the ordinary Euler class $e(\Lambda \otimes_R H)$ up to multiplication by a unit of $MSp^*(BSp(1)_+)$, and we obtain

$$\sum_{i\geq 1} \phi_i^4 x^{8i} = e(\Lambda \otimes_R H)(1 + \sum_{j\geq 1} \alpha_j x^j)^{-1}$$

for some coefficients $\alpha_j \in MSp^*$. This proves (3).

3. Nilpotency formula in self-conjugate cobordism

Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group, N the normalizer of S^1 in S^3 as above and Z_4 a cyclic group generated by j.

Recall that $\rho(N, S^3)$ is the projective bundle of the universal $Spin(3)$ bundle $\Lambda \to BS^3$, and we have the canonical splitting

$$\rho^*(N, S^3)(\Lambda) = \mu + \lambda.$$

Here μ is a plane and λ is a line real bundle.

It is easy to see that the bundle $\rho(Q, N)$ is the projective bundle of μ and the bundle $\rho(Q, S^3)$ is the bundle of flags of the bundle Λ.

This defines the splittings

$$\rho^*(Q, S^3)(\Lambda) = \chi_1 + \chi_2 + \chi_3;$$

$$\rho^*(Q, N)(\mu) = \chi_2 + \chi_3;$$

$$\rho^*(Q,N)(\lambda) = \chi_1.$$

Here $\chi_3 = \chi_1 \otimes_R \chi_2$.

Proposition. *The transfer homomorphism $\tau^*(Q,S^3)$ satisfies the following relations:*

$$e(\Lambda \otimes_R C) = -\tau^*(Q,S^3)(e^2(\chi_i \otimes_R C)e(\chi_j \otimes_R C)) = 0 \tag{3.1}$$

in the self-conjugate cobordism theory and

$$e(\Lambda \otimes_R H) = -\tau^*(Q,S^3)(e^2(\chi_i \otimes_R H)e(\chi_j \otimes_R H)) = 0 \tag{3.2}$$

in the symplectic cobordism theory, where $i,j = 1,2,3$; $i \neq j$.

Proof of (3.1). The double coset space $N|S^3|N$ is a line segment. The isotropy groups are N and Q at the endpoints and Z_4 (generated by j) in the interior.

By the double coset theorem and Proposition 1, case $m = 3$, we have

$$0 = e(\lambda \otimes_R C) + \tau^*(Q,N)(e(\chi_2 \otimes_R C)) - \tau^*(Z_4,N)(\rho^*(Z_4,N)(e(\lambda \otimes_R C))).$$

But

$$e(\rho^*(N,S^3)(\Lambda \otimes_R C)) = e(\lambda \otimes_R C)e(\mu \otimes_R C)$$

and $\rho^*(Z_4,N)(\mu)$ has the section. Hence by the above splittings and transfer properties we obtain

$$\begin{aligned} &e(\rho^*(N,S^3)(\Lambda \otimes_R C)) \\ &= -\tau^*(Q,N)(e(\chi_2 \otimes_R C))e(\mu \otimes_R C) \\ &= -\tau^*(Q,N)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)). \end{aligned}$$

Since $\tau^*(N,S^3)(1) = 1$ by the analogue of Proposition 1 for the self-conjugate cobordism, this proves

$$e(\Lambda \otimes_R C) = -\tau^*(Q,S^3)(e^2(\chi_2 \otimes_R C)e(\chi_3 \otimes_R C)).$$

We may prove relations analogous to (3.1) by changing N to its conjugate subgroup in S^3, but this follows also by symmetry.

Now

$$\begin{aligned} &\tau^*(Q,S^3)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C)) \\ &= \tau^*(N,S^3)(\tau^*(Q,N)(e^2(\chi_1 \otimes_R C)e(\chi_2 \otimes_R C))) \\ &= \tau^*(N,S^3)(e^2(\lambda \otimes_R C)(-e(\lambda \otimes_R C) + \tau^*(Z_4,N)(\rho^*(Z_4,N)(e(\lambda \otimes_R C))) \\ &= \tau^*(N,S^3)(-e^3(\lambda \otimes_R C)) + \tau^*(Z_4,S^3)(\rho^*(Z_4,N)(e^3(\lambda \otimes_R C))). \end{aligned}$$

The first summand is zero by Proposition 1 (by its analogue). The second summand is also zero. This follows immediate from the following theorem

Theorem [Fe]**.** *Assume $N_G(H)/H$ is not discrete, where $N_G(H)$ is the normalizer of H in G. Then $\tau^*(H,G) = 0$.*

The proof of (3.2) is analogous.

Now since (see Section 2) the symplectic Euler class of $\Lambda\otimes_R H$ (the Euler class of $\Lambda\otimes_R C$ in SC^*) coincides with $\sum_{i\geq 1}\phi_i^4 x^{8i}$ (with the image of $\sum_{i\geq 1}\phi_i^2 x^{4i}$ in SC^* theory) up to multiplication by a unit of $MSp^0(BS^3_+)$ (by a unit of $SC^0(BS^3_+)$), this proves

Corollary 3.3. *$\phi_i^4=0$, and the images of ϕ_i^2 in self-conjugate cobordism are zero.*

Remark 3.4. It follows from the relation between the transfer and the umkehr map [BG], [BO] that Proposition 1 is true also for $m=2$ and $m=6$.

4. On four-fold products of Ray classes

Here we improve the above method and obtain

Proposition 4.1. *All four-fold products of Ray classes are zero, and the images of double products of these classes in self-conjugate cobordism are zero.*

The proof is organized as follows:

Let N be the normalizer of the torus $U(1)$ in $Sp(1)$ as above. Consider again the bundle

$$p: BN\to BSp(1)$$

and the map

$$f: BN\to BZ_2$$

induced by projection of N on the Weil group Z_2. Let τ_p be the transfer map for p.

We have the following relations.

Proposition 4.2. *In $MSp^*(BSp(1)^4)=MSp^*[[x_1,x_2,x_3,x_4]]$ we have*

$$\sum_{i,j,k,l\geq 1}\phi_i\phi_j\phi_k\phi_l x_1^{2i}x_2^{2j}x_3^{2k}x_4^{2l}=\sum_{m,n,p,q\geq 0}\tau_p^*f^*(\gamma_{mnpq})x_1^m x_2^n x_3^p x_4^q,$$

where γ_{mnpq} are elements from $\tilde{MSp}^*(BZ_2)$.

Proposition 4.3. *In $SC^*(BSp(1)^2)=SC^*[[y_1,y_2]]$ we have*

$$\sum_{i,j\geq 1}\psi_i\psi_j y_1^{2i}y_2^{2j}=\sum_{m,n\geq 0}\tau_p^*f^*(\delta_{mn})y_1^m y_2^n,$$

where ψ_i is the image of ϕ_i in self-conjugate cobordism and the δ_{mn} are elements from $\tilde{SC}^*(BZ_2)$.

We shall see later that the map $f\tau_p$ induces trivial homomorphism for any generalized cohomology theory h^* .

Proposition 4.4.

$$\tau_p^*f^*(a)=0,\quad \forall a\in\tilde{h}^*(BZ_2);$$

$$\tau_p^*(1)=1.$$

Thus the right sides of the relations from 4.2 and 4.3 are zero. This proves Proposition 4.1

Proofs of 4.2 *and* 4.3. We need a simple lemma about orientable bundles, whose proof follows from the fact that $KO^4(X) = KSp^0(X)$.

Let $\eta \to BZ_2$ be the universal $O(1)$ bundle and ζ, ζ^*, Λ the bundles from the introduction.

Lemma 4.5. *i) The bundle* $\eta \otimes_R \zeta \otimes_H \zeta^* \to BZ_2 \times BSp(1)$ *is* MSp*-orientable.*

ii) The bundle $\eta \otimes_R \sum_{i=1}^4 \Lambda_i \to BZ_2 \times BSp(1)^4$ *is* MSp*-orientable.*

iii) The bundle $\eta \otimes_R \sum_{i=1}^2 \Lambda_i \to BZ_2 \times BSp(1)^2$ *is SC-orientable.*

Proof. i) This bundle is a $Spin(4)$ bundle and so is MSp-orientable.

ii) Since $\zeta_i \otimes_H \zeta_i^* = \Lambda_i + R^1$, the bundle ii) is MSp-orientable as a difference of two MSp-orientable bundles

$$\eta \otimes_R \sum_{i=1}^{4} \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R H.$$

iii) This bundle is a difference of SC-orientable bundles

$$\eta \otimes_R \sum_{i=1}^{2} \zeta_i \otimes_H \zeta_i^* - \eta \otimes_R C.$$

Recall from section 2 that

$$\tilde{e}(\Lambda) = \theta_1 + \sum_{i \geq 1} \phi_i x^{2i}, \qquad x = e(\zeta).$$

Any two orientation classes of the given orientable bundle agrees up to multiplication by an invertible element. So there is

$$\tilde{e} = \tilde{e}(\eta \otimes_R \sum_{i=1}^{4} \Lambda_i),$$

which as an element from

$$MSp^*(BZ_2 \times BSp(1)^4) = MSp^*(BZ_2)[[x_1, x_2, x_3, x_4]], \qquad x_i = e(\zeta_i),$$

has the form

$$\begin{aligned}\tilde{e} &= \prod_{s=1}^{4} (\theta_1 + \sum_{r \geq 1} \phi_r x_s^{2r}) + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q \\ &= \sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l x_1^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} \gamma_{mnpq} x_1^m x_2^n x_3^p x_4^q.\end{aligned}$$

Here we take into account the relation $\theta_1 \phi_i \phi_j = 0$ from [G].

Consider now the map

$$g = (f, p) \times 1 : BN \times BSp(1)^3 \to BZ_2 \times BSp(1) \times BSp(1)^3.$$

Lemma 4.6. $g^*(\tilde{e}) = 0$.

Proof. Recall from Section 3 that $p^*(\Lambda) = \mu + \lambda$. But $f^*(\eta) = \lambda$ and $\lambda^2 = 1$. Thus the bundle

$$g^*(\eta \otimes_R \sum_{i=1}^{4} \Lambda_i) = \lambda(\mu + \lambda + \Lambda_2 + \Lambda_3 + \Lambda_4)$$

has the section. This proves the lemma.

We now have in $MSp^*(BN \times BSp(1)^3)$ the relation

$$\sum_{i,j,k,l \geq 1} \phi_i \phi_j \phi_k \phi_l p^*(x_1)^{2i} x_2^{2j} x_3^{2k} x_4^{2l} + \sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq}) p^*(x_1)^m x_2^n x_3^p x_4^q = 0.$$

After application of the transfer homomorphism for the bundle

$$p \times 1 : BN \times BSp(1)^3 \to BSp(1)^4$$

we get Proposition 4.2.

The proof of 4.3 is analogous.

Proof of 4.4. In fact this is a particular case of Proposition 1, although we should rewrite it as follows:

Proposition 4.7. *Let $G_m = Spin(m)$, and let $\xi^m \to BG_m$ be the universal $Spin(m)$ bundle, $m = 2, 3, 4, 5$. Let $p_m : P(\xi^m) \to BG_m$ be the projective bundle associated to ξ^m and let*

$$f_m : P(\xi^m) \to BZ_2$$

be the classifying map for the canonical real line bundle $\lambda_m \to P(\xi^m)$. Then $\tau_m^*(1)$ is equal to 0 if $m = 2, 4$ and equal to 1 if $m = 3, 5$;

$$\tau_m^*(a) = 0, \quad \forall a \in M\tilde{S}p^*(BZ_2).$$

The case $m = 3$ gives Proposition 4.4.

We also remark that using [Bu] and Proposition 1 one can obtain a new proof of the relation $\theta_1 \theta_i \theta_j = 0$ proved in [GR]. Moreover, some relations between the θ_i's and the generators of the free part of the symplectic cobordism can be also derived. We plan to present the details in a future paper.

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