J. Math. Kyoto Univ. (JMKYAZ)
18-3 (1978) 435–451

The transfer homomorphism in equivariant generalized cohomology theories

By

Goro NISHIDA

Dedicated to Professor A. Komatu on his 70th birthday

(Received July 5, 1977)

§1. Introduction

In this note we shall study an equivariant version of the transfer homomorphism for fibre bundles defined by Becker and Gottlieb [4].

Let G be a compact Lie group. Let $\xi=(p: E \to X)$ be a fibre bundle with G-action in the sense of [6]; and let h_G^* be an $RO(G)$-graded generalized G-cohomology theory [9], where $RO(G)$ denotes the real representation ring of G. Then the transfer homomorphism

$$p_!: h_G^*(E) \longrightarrow h_G^*(X)$$

will be defined. The existence of the transfer seems to be an advantage of $RO(G)$-graded theories compairing to $\mathbf{Z}$-graded theories. Regarding $h_G^*(X)$ as a graded module over the stable G-cohomotopy theory $\pi_G^*(X)$, we shall prove (Theorem 4.6) that

$$p_! p^*(x) = w(\xi)x \in h_G^*(X)$$

for $x \in h_G^*(X)$, where $w(\xi)=p_!(1) \in \pi_G^0(X)$ and $1 \in \pi_G^0(E)$ denotes the unit.

For each closed subgroup H of G, usually we have also a generalized H-cohomology theory h_H^* such that $h_G^*(G/H)=h_H^*(\text{point})$, for $* \in Z$. In this case the transfer homomorphism for the bundle $p: G/H \to \text{point}$ gives an "induction" homomorphism

$$p_!: h_H^*(\text{point}) \longrightarrow h_G^*(\text{point}).$$

In particular let $K_G(X)=K_G^0(X)$ be the equivariant K-group [13], then we obtain a homomorphism

$$p_!: R(H) \longrightarrow R(G)$$

where $R(G)$ denotes the complex representation ring of G. It will be proved (Theorem 5.2) that this homomorphism coincides with the induction homomorphism defined by Segal [12].

Finally we shall give a proof of the Adams conjecture for complex vector bundles. It is a modification of that of Becker and Gottlieb [4]. The idea is just to use the fact that every element of $R(G)$ is a linear combination of representations induced from one dimensional representations [12], and to reduce the problem to line bundles using the naturality of the transfer homomorphism.

§2. Fibre bundles with group action

Let G and Γ be compact Lie groups. Suppose that there is given an action of G on Γ as smooth automorphisms, i.e., a homomorphism $\alpha: G \to \mathrm{Aut}\,\Gamma$ such that the adjoint of α, $\tilde{\alpha}: G \times \Gamma \to \Gamma$ is smooth. Let $\Gamma \times_\alpha G$ denote the semidirect product, that is the direct product $\Gamma \times G$ as a set and the multiplication is given by $(\gamma, g) \cdot (\gamma', g') = (\gamma \cdot \alpha(g)(\gamma'), gg')$. It is obvious that $\Gamma \times_\alpha G$ is a Lie group. According to tom Dieck [6], we now introduce the notion of fibre bundle with group action.

Definition 2.1. A principal Γ-bundle $\tilde{\xi} = (p: \tilde{E} \to X)$ (Γ acts on $\tilde{E}$ from the right) is called a principal (Γ, α, G)-bundle if

i) $\tilde{E}$ and X are left G-spaces and $p: \tilde{E} \to X$ is a G-map,

ii) actions of G and Γ are related as $g(x \cdot \gamma) = g(x) \cdot \alpha(g)(\gamma)$ for any $x \in \tilde{E}$, $g \in G$ and $\gamma \in \Gamma$.

Note that we changed the notation of G and Γ in [6].

Let F be a $\Gamma \times_\alpha G$-space. Then regarding F as a Γ $(\subset \Gamma \times_\alpha G)$-space, one can associate for $\tilde{\xi}$ a fibre bundle ξ

$$F \longrightarrow E \xrightarrow{p} X$$

with fibre F, which we call a (Γ, α, G)-bundle. It is obvious that the diagonal action of $\Gamma \times_\alpha G$ on $\tilde{E} \times F$ induces a G-action on $E = \tilde{E} \times_\Gamma F$ and that $p: E \to X$ is a G-map.

Now let $F \longrightarrow E \xrightarrow{p} X$ be a (Γ, α, G)-bundle, associated with a principal (Γ, α, G)-bundle $\tilde{\xi}$. Suppose that F is a closed smooth $\Gamma \times_\alpha G$-manifold and X is compact. We shall then associate a stable G-map $X_+ \to E_+$ as follows.

It is known [5] that there is a $\Gamma \times_\alpha G$-equivariant embedding $i: F \to W$ of F into a Euclidian $\Gamma \times_\alpha G$-space W. Let η denote the vector bundle with fibre W associated with $\tilde{\xi}$. This turns out to be a G-vector bundle in the sense of [13], and since the base space X is compact, there are a G-vector bundle $\eta^\perp$ and a G-vector bundle isomorphism

$$f: \eta \oplus \eta^\perp \cong B \times V$$

where V is a Euclidian G-space.

Let $\nu(F)$ denote the normal bundle of the embedding $F \subset W$. Then we obtain $\Gamma \times_\alpha G$-maps

$$k: \nu(F) \longrightarrow W$$

and

$$j\colon \nu(F) \longrightarrow \nu(F)\oplus\tau(F)\cong F\times W$$

where $j(v)=v\oplus 0$ and $\tau(F)$ is the tangent bundle of F. Clearly k is an embedding onto an open subspace of W, and j is a proper map of locally compact spaces. Consider G-maps

$$id\times_{\Gamma} k\colon \tilde{E}\times_{\Gamma}\nu(F) \longrightarrow \tilde{E}\times_{\Gamma} W$$

and

$$id\times_{\Gamma} j\colon \tilde{E}\times_{\Gamma}\nu(F) \longrightarrow \tilde{E}\times_{\Gamma}(F\times W).$$

These maps are clearly fibrewise regarding as fibre bundles over X.

For fibre bundles p_i; $E_i\to X$, $i=1, 2$, we define the "Whitney sum" $E_1\oplus E_2$ by the pull back diagram

$$\begin{array}{ccc} E_1\oplus E_2 & \longrightarrow & E_1\times E_2 \\ \downarrow & & \downarrow{\scriptstyle p_1\times p_2} \\ X & \xrightarrow[d]{} & X\times X \end{array}$$

where $d\colon X\to X\times X$ denotes the diagonal map. If $f_i\colon E_i\to E_i'$ $(i=1, 2)$ are fibrewise maps covering the identity map of X, then one can naturally construct a fibrewise map

$$f_1\oplus f_2\colon E_1\oplus E_2 \longrightarrow E_1'\oplus E_2'.$$

We apply this construction to fibrewise maps above and the identity of $\eta^{\perp}$. Then we obtain G-maps

$$(id\times_{\Gamma} k)\oplus id\colon (\tilde{E}\times_{\Gamma}\nu(F))\oplus\eta^{\perp} \longrightarrow (\tilde{E}\times_{\Gamma} W)\oplus\eta^{\perp}$$

and

$$(id\times_{\Gamma} j)\oplus id\colon (\tilde{E}\times_{\Gamma}\nu(F))\oplus\eta^{\perp} \longrightarrow (\tilde{E}\times_{\Gamma}(F\times W))\oplus\eta^{\perp}.$$

It is obvious from construction that $(id\times_{\Gamma} k)\oplus id$ is an embedding onto open subspace and $(id\times_{\Gamma} j)\oplus id$ is a proper map. By the definition, $(\tilde{E}\times_{\Gamma} W)\oplus\eta^{\perp}\cong X\times V$ and we have

Lemma 2.2. *$(\tilde{E}\times_{\Gamma}(F\times W))\oplus\eta^{\perp}$ is homomorphic to $E\times V\cong p^*(\eta\oplus\eta^{\perp})$.*

Proof. An element of $(\tilde{E}\times_{\Gamma}(F\times W))\oplus\eta^{\perp}$ can be written as $[\tilde{e}, (x, w)]\oplus v'$ where $\tilde{e}\in\tilde{E}$, $x\in F$, $w\in W$ and $v'\in\eta^{\perp}$. Define

$$u\colon (\tilde{E}\times_{\Gamma}(F\times W))\oplus\eta^{\perp} \longrightarrow p^*(\eta\oplus\eta^{\perp})$$

by $u([\tilde{e}, (x, w)]\oplus v')=([\tilde{e}, x], [\tilde{e}, w]\oplus v')$. Clearly u is a continuous G-map and the inverse of u is similarly defined. q.e.d.

For a locally compact space Y, let Y^c denote the one point compactification. Let U be an open subspace of Y, then by shrinking $Y-U$ to a one point, we obtain a map $Y^c\to U^c$. If $f\colon Y\to Z$ is a proper map, then we obtain $f^c\colon Y^c\to Z^c$. Then from G-maps $(id\times_\Gamma k)\oplus id$ and $(id\times_\Gamma j)\oplus id$, we obtain G-maps

$$(X\times V)^c \longrightarrow ((\tilde{E}\times_\Gamma \nu(F))\oplus\eta^\perp)^c \longrightarrow (E\times V)^c$$

and as the composite we have a G-map

$$t\colon (X\times V)^c \longrightarrow (E\times V)^c.$$

We call t a trace of the (Γ, α, G)-bundle $\xi=(p\colon E\to X)$.

We note that when the structure group Γ of a (Γ, α, G)-bundle ξ is reducible to a subgroup Γ' which is closed under G-action, then ξ is regarded as a (Γ', α, G)-bundle. In such a case, a trace $t\colon (X\times V)^c\to(E\times V)^c$ of the (Γ, α, G)-bundle ξ can be considered as a trace of the (Γ', α, G)-bundle ξ, i.e., a trace does not depend on a reduction of structure group. So in the following, a (Γ, α, G)-bundle with a closed smooth fibre and a compact base is called simply an admissible G-bundle.

Consider now a special case. Let M be a closed G-manifold. Then the unique map $p\colon M\to$point is an admissible G-bundle (with $\Gamma=e$). Let $i\colon M\to W$ be a G-equivariant embedding. In this case, $\eta=\text{point}\times W$ and we may take $V=W$. Then a trace of $p\colon M\to$point is given by the composition

$$V^c \xrightarrow{c} \nu(M)^c \xrightarrow{j^c} (\nu(M)\oplus\tau(M))^c \cong (M\times V)^c$$

where c is the Pontrjagin-Thom construction.

§3. *G*-cohomology theories

Let us first recall the definition of $RO(G)$-graded equivariant generalized cohomology theories ([14], for details also see [9]). Here $RO(G)$ denotes the real representation ring of a compact Lie group G.

A reduced generalized G-cohomology theory h_G^* consists of

i) a family $\tilde{h}_G^\alpha$, $\alpha\in RO(G)$, of contravariant functors from the category of compact based G-spaces to the category of abelian groups and

ii) a family $\sigma^{\alpha,V}$ ($\alpha\in RO(G)$ and V an irreducible representation of G) of natural transformations

$$\sigma^{\alpha,V}\colon \tilde{h}_G^\alpha(X) \longrightarrow \tilde{h}_G^{\alpha+V}(V^c\wedge X)$$

which is subject to the usual axioms. From a reduced theory, one can define an unreduced theory by

$$h_G^\alpha(X)=\tilde{h}_G^\alpha(X_+)$$

where $+$ means the disjoint base point.

Let h_G^* and k_G^* be generalized G-cohomology theories. A family $\varphi=\{\varphi^\alpha\}$

$$\varphi^\alpha;\ h_G^\alpha(X) \longrightarrow k_G^{\alpha+\beta}(X)$$

of natural transformations is called stable if φ^α commute with the suspension isomorphisms.

Some examples of G-cohomology theories are

Ex. 1. (*Stable cohomotopy*). Let $\alpha = V - W \in RO(G)$ where V and W are real representations of G. Define the stable G-cohomotopy group by

$$\tilde{\pi}_G^\alpha(X) = \varinjlim_U [(U \oplus W)^c \wedge X, (U \oplus V)^c]^G$$

where $[\ ,\ ]^G$ denotes the set of G-homotopy classes of G-maps, and the direct limit is taken over all real representations of G. It is shown (see e.g. [9]) that π_G^* is a generalized G-cohomology theory. Moreover we see that π_G^* is multiplicative. That is, by the smash product of stable G-maps, we have an associative and (anti-) commutative pairing

$$\tilde{\pi}_G^\alpha(X) \otimes \tilde{\pi}_G^\beta(Y) \longrightarrow \tilde{\pi}_G^{\alpha+\beta}(X \wedge Y)$$

Hence as non equivariant case, the unreduced group $\pi_G^*(X)$ is a $RO(G)$-graded ring with unit.

Let h_G^* be a generalized G-cohomology theory. Let $\alpha = V - W \in RO(G)$ and $\beta = V' - W' \in RO(G)$. Let $x \in h_G^\alpha(Y)$ and $u \in \pi_G^\beta(X)$, and let

$$f\colon (U' \oplus W')^c \wedge X_+) \longrightarrow (U' \oplus V')^c$$

be a representative of u. Put

$$u \otimes x = (\sigma^{\alpha+\beta, U' \oplus V'})^{-1} (f \wedge id_Y)^* \sigma^{\alpha, U' \otimes V'}(x).$$

Then we obtain a welldefined bilinear pairing

$$\otimes\colon h_G^\alpha(Y_G^\beta) \otimes \pi(X) \longrightarrow h_G^{\alpha+\beta}(X \times Y).$$

If $X = Y$, then by use of the diagonal map $X \to X \times X$, we obtain a homomorphism

$$h_G^\alpha(X) \otimes \pi_G^\beta(X) \longrightarrow h_G^{\alpha+\beta}(X),$$

and we can check easily the following

Proposition 3.1. *Any generalized G-cohomology theory $h_G^*(X)$ has a natural $\pi_G^*(X)$-module structure.*

Let $\pi\colon X \to \text{pt.}$ be the unique map. Then *via* the ring homomorphism $\pi^*\colon \pi_G^*(\text{pt.}) \to \pi_G^*(X)$, we may consider $h_G^*(X)$ as a $\pi_G^*(\text{pt.})$-module.

Ex. 2. (*K-cohomology*). Let $\alpha = V - W \in RO(G)$ as before. Put

$$\tilde{K}_G^\alpha(X) = \tilde{K}_G((V \oplus W)^c \wedge X)$$

where $\tilde{K}_G$ denotes the reduced equivariant K-group of Atiyah-Segal (see [13]). By the Bott periodicity $\beta\colon \tilde{K}_G(X) \cong \tilde{K}_G(X \wedge (V \otimes \mathbf{C})^c)$, one can define the suspension isomorphism

$$\sigma\colon \tilde{K}_G^\alpha(X) \longrightarrow \tilde{K}_G^{\alpha+U}(U^c \wedge X).$$

by

$$\beta: \tilde{K}_G((V\oplus W)^c \wedge X) \longrightarrow \tilde{K}_G((V\oplus W)^c \wedge X \wedge (U\otimes \mathbf{C})^c)$$
$$\cong \tilde{K}_G((V\oplus W\oplus 2U)^c \wedge X).$$

As non equivariant K-theory, one can easily see that K_G^* is a generalized G-cohomology theory.

§4. The transfer homomorphism

Let h_G^* be a generalized G-cohomology theory. Let $\xi=(p: E\to X)$ be an admissible G-bundle with fibre F.

Definition 4.1. The transfer homomorphism for ξ

$$p_!: h_G^*(E) \longrightarrow h_G^*(X)$$

is defined by

$$p_!(x)=(\sigma^{*,V})^{-1}t^*\sigma^{*,V}(x)$$

where $t: (X\times V)^c = X_+ \wedge V^c \to (E\times V)^c = E_+ \wedge V^c$ is a trace of the bundle ξ.

This definition is well-defined, because of

Lemma 4.2. *The suspension image of t in $\{X_+, E_+\}^G = \varinjlim_V [X_+ \wedge V^c, E_+ \wedge V^c]^G$ is uniquely determined by $\xi=(p: E\to X)$.*

Proof. The definition of t depends on a choice of an embedding $i: F\to W$ and a choice of $\eta^\perp$. It is known [18] that if W is large enough (contains each irreducible representation enough many times), then any embedding $F\to W$ are G-isotopic each other. On the other hand, for a given embedding $F\to W$, the stable class of $\eta^\perp$ is unique. Then one can easily verify that the equivariant stable class of t is independent on choices above. q.e.d.

If $G=e$, one can see easily that our definition of the transfer coincides with that of Becker and Gottlieb [4].

Proposition 4.3. *Let $\xi=(p: E\to X)$ be an admissible G-bundle and let h_G^* and k_G^* be generalized G-cohomology theories. Then we have the following.*

i) *The transfer $p_!: h_G^*(E)\to h_G^*(X)$ is a $\pi_G^*(\mathrm{pt.})$ module homomorphism*

ii) *If $\varphi: h_G^*\to k_G^*$ is a stable natural transformation then the following diagram* is commutative

$$\begin{array}{ccc} h_G^*(E) & \xrightarrow{\varphi} & k_G^*(E) \\ \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_!} \\ h_G^*(X) & \xrightarrow[\varphi]{} & k_G^*(X). \end{array}$$

Proof. Let $x\in h_G^*(E)$ and $u\in \pi_G^*(\mathrm{pt.})$, and let $f: W^c\to U^c$ be a representative of u. Let $t: X_+ \wedge V^c \to E_+ \wedge V^c$ be a trace of ξ. Note that

$$t\wedge id_W\colon X_+\wedge V^c\wedge W^c \longrightarrow E_+\wedge V^c\wedge W^c$$

is also a trace of ξ. Then

$$\begin{aligned} p_!(ux) &= (\sigma^{*,V\oplus W})^{-1}(t\wedge id_W)^*(id_{E_+\wedge V^c}\wedge f)^*\sigma^{*,V\oplus U}(x) \\ &= (\sigma^{*,V\oplus W})^{-1}(id_{X_+\wedge V^c}\wedge f)^*(t\wedge id_U)^*\sigma^{*,V\oplus U}(x) \\ &= up_!(x)\,. \end{aligned}$$

This proves i), and ii) is clear from the definition. q.e.d.

Now we consider the naturality of the transfer. Let $\tilde{\xi}=(\tilde{E}, \tilde{p}, X)$ be a principal (Γ, α, G)-bundle and let $f\colon Y\to X$ be a G-map. Then the induced principal Γ-bundle $f^*\tilde{\xi}$ clearly has a principal (Γ, α, G)-bundle structure induced from $\tilde{E}$. Let $\xi=(p\colon E\to X)$ be the (Γ, α, G)-bundle with fibre F associated with $\tilde{\xi}$, and let $f^*\xi=(p'\colon f^*E\to Y)$ be associated with f^*. Then we obtain a pull-back diagram of (Γ, α, G)-bundles

$$\begin{array}{ccc} f^*E & \xrightarrow{\ \bar{f}\ } & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\ f\ ]{} & X \end{array}$$

Proposition 4.4. *Let ξ be an admissible G-bundle. Given a pull-back diagram as above, we have the following commutative diagram*

$$\begin{array}{ccc} h_G^*(E) & \xrightarrow{\ \bar{f}^*\ } & h_G^*(f^*E) \\ {\scriptstyle p_!}\downarrow & & \downarrow{\scriptstyle p'_!} \\ h_G^*(X) & \xrightarrow[\ f^*\ ]{} & h_G^*(Y) \end{array}$$

Proof. Let $i\colon F\to W$ be a $\Gamma\times_\alpha G$-embedding in a Euclidian $\Gamma\times_\alpha G$-space W and let $\eta=\tilde{E}\times_\Gamma W$ be the G-vector bundle over X. Note that

$$\eta'=f^*\tilde{E}\times_\Gamma W\cong f^*(\eta)$$

where $f^*(\eta)$ is the induced G-vector bundle over Y. Hence for the construction of a trace of f^*E, one may choose

$$(\eta')^\perp=(f^*\tilde{E}\times_\Gamma W)^\perp\cong f^*(\eta^\perp).$$

Then by an easy diagram chasing, we have the commutative diagram

$$\begin{array}{ccc} f^*E_+\wedge V^c & \xrightarrow{\ \bar{f}\wedge id\ } & E_+\wedge V^c \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle t} \\ Y_+\wedge V^c & \xrightarrow[\ f\wedge id\ ]{} & X_+\wedge V^c \end{array}$$

This shows the proposition. q.e.d.

Now let $\xi=(p\colon E\to X)$ be an admissible G-bundle and let $t\colon X_+\wedge V^c\to E_+\wedge V^c$

be a trace. The stable class of the composition map

$$X_+ \wedge V^c \xrightarrow{t} E_+ \wedge V^c \xrightarrow{\pi} V^c$$

will be denoted by $w(\xi) \in \pi_G^0(X)$, where π denotes the canonical projection. If $\xi = (p: M \to \text{point})$, then $w(\xi) \in \pi_G^0(\text{pt.})$ is denoted by $w(M)$. The class $w(\xi)$ is natural, i.e., for a G-map $f: Y \to X$ and for an admissible G-bundle $\xi = (p: E \to X)$, we have

$$f^* w(\xi) = w(f^* \xi)$$

by Proposition 4.4.

Lemma 4.5. *$w(\xi) = p_! p^*(1)$, where $1 \in \pi_G^0(X)$ denotes the unit.*

This is clear by definition.

Theorem 4.6. *Let $\xi = (p: E \to X)$ be an admissible G-bundle. Let h_G^* be a generalized G-cohomology theory and let $x \in h_G^*(X)$. Then we have*

$$p_! p^*(x) = w(\xi) x.$$

Proof. Let $d: X \to X \times X$ be the diagonal map and let $\Delta = p \times id: E \to X \times E$. Note that

$$id \times p: X \times E \longrightarrow X \times X$$

is an admissible G-bundle and we have the following pull-back diagram

$$\begin{array}{ccc} E & \xrightarrow{\Delta} & X \times E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle id \times p} \\ X & \xrightarrow[d]{} & X \times X \end{array}$$

From the definition, we see that a trace of $id \times p: X \times E \to X \times X$ is given by

$$t' = id_X \wedge t: (X \times X)_+ \wedge V^c \longrightarrow (X \times E)_+ \wedge V^c$$

where $t: X_+ \wedge V^c \to E_+ \wedge V^c$ is a trace of $\xi = (p: E \to X)$. Then for $x \in h_G^*(X)$ and $y \in \pi_G^*(E)$, we have

$$(id \times p)_!(y \otimes x) = p_!(y) x.$$

Now by Proposition 4.4 and by the naturality of π_G^* action on h_G^*, we have a commutative diagram

$$\begin{array}{ccccc} h_G^*(E) & = & h_G^*(E) & \xrightarrow{p_!} & h_G^*(X) \\ {\scriptstyle d^*}\uparrow & & \uparrow{\scriptstyle \Delta^*} & & \uparrow{\scriptstyle d^*} \\ h_G^*(E \times E) & \xrightarrow[(p \times id)^*]{} & h_G^*(X \times E) & \xrightarrow[(id \times p)_!]{} & h_G^*(X \times X) \\ \uparrow & & \uparrow & & \uparrow \\ h_G^*(E) \otimes \pi_G^*(E) & \xrightarrow[p^* \otimes id]{} & h_G^*(X) \otimes \pi_G^*(E) & \xrightarrow[id \otimes p_!]{} & h_G^*(X) \otimes \pi_G^*(X) \end{array}$$

Then by a simple diagram chasing starting from $x\otimes 1\in h_G^*(X)\otimes\pi_G^*(E)$, the theorem is proved. q.e.d.

As an application of the theorem, we consider an admissible G-bundle with a base space with trivial G-action. Let $\xi=(p\colon E\to X)$ be such a bundle. Let $x_0\in X$ and $F=p^{-1}(x_0)$. Then the inclusion $i\colon x_0\to X$ is a G-map and F is a G-manifold. Moreover we see that

$$i^*w(\xi)=w(F)\in\pi_G^0(x_0).$$

For a π_G^*(pt.)-module M and for $\chi\in\pi_G^*$(pt.), let $M[\chi^{-1}]$ denotes the localization of M by the multiplicative set $\{\chi^n\}_{n=1,2,\dots}$. Then we have

Theorem 4.7. *Let $\xi=(p\colon E\to X)$ be an admissible G-bundle with fibre F. Suppose that X is a connected finite CW-complex with trivial G-action. Then the composition of $\pi_G^*(pt.)[w(F)^{-1}]$-module homomorphisms*

$$p_!p^*\colon h_G^*(X)[w(F)^{-1}]\longrightarrow h_G^*(X)[w(F)^{-1}]$$

is an isomorphism.

Proof. Let x_0 be a vertex of X. Let $\pi\colon X\to x_0$ be the unique map and let $i\colon X^{(0)}\to X$ be the inclusion of 0-skeleton. Since X is connected and trivial as G-space, we see that

$$i^*(w(\xi)-\pi^*w(F))=0.$$

Hence we can write $w(\xi)=\pi^*w(F)+z$, $z\in\ker i^*$. Therefore in $\pi_G^*(X)[w(F)^{-1}]$, we can write

$$w(\xi)=\pi^*w(F)(1+z'),$$

where $z'\in\ker[i^*\colon\pi_G^*(X)[w(F)^{-1}]\to\pi_G^*(pt.)[w(F)^{-1}]]$. Since X is a finite CW complex, the element $1+z'$ is invertible as usual. Hence the multiplication with $w(\xi)$ is an isomorphism in $h_G^*(X)[w(F)^{-1}]$, and the theorem follows from Theorem 4.6. q.e.d.

Remark. If G is a finite group, Segal [14] has shown that $\pi_G^0(pt.)$ is isomorphic to the Burnside ring $A(G)$. For a compact Lie group G, the structure of $\pi_G^0(pt.)$ is determined by Rubinsztein [11].

§5. The transfer in K_G-theory

Let $K_G(X)=K_G^0(X)$ be the equivariant K-group. Recall that the suspension isomorphism in $\tilde{K}_G^*$-theory $\sigma\colon\tilde{K}_G^0(E_+)\to\tilde{K}_G^V(E_+\wedge V^c)$ is given by the Bott periodicity, or in other word the Thom isomorphism (see e.g. [3])

$$\Phi\colon K_G(E)\xrightarrow{\cong}K_G(E\times V_{\mathbf{C}})=K_G(E\times(2V)).$$

where $V_{\mathbf{C}}$ denotes the complexification of V.

Let $\xi=(p\colon E\to X)$ be an admissible G-bundle. Then in the construction of a trace $t\colon X_+\wedge V^c\to E_+\wedge V^c$, one may suppose that V is a complex G-vector space. Then it is easy to see that the transfer for ξ is given by the composition

$$K_G(E)\xrightarrow[\Phi]{} K_G(E\times V)\xrightarrow[t_*]{} K_G(X\times V)\xrightarrow[\Phi^{-1}]{} K_G(X).$$

Now let M be a closed G-manifold and let τM be the tangent bundle of M. Let

$$t\text{-ind}\colon K_G(\tau M)\longrightarrow K_G(pt.)\cong R(G)$$

be the topological index [3], where $R(G)$ denotes the complex representation ring of G. Let

$$\pi\colon \tau M\longrightarrow M$$

be the bundle projection and let $\lambda^*(\tau M)$ denote the exterior algebra of the vector bundle τM. Then $\pi^*(\lambda^*(\tau M))$ is a complex of real vector bundles over τM exact outside the 0-section. Hence its complexification defines an element of $K_G(\tau M)$, so put

$$u(M)=\pi^*(\lambda^*(\tau M))\otimes\mathbf{C}\in K_G(\tau M).$$

Then by the multiplication with $u(M)$, we obtain a homomorphism

$$\psi\colon K_G(M)\longrightarrow K_G(\tau M).$$

Now we have

Theorem 5.1. *Let M be a closed G-manifold and let $\xi=(p\colon M\to point)$ be the admissible G-bundle. Then*

$$p_!=t\text{-ind}\,\psi\colon K_G(M)\longrightarrow K_G(pt.).$$

Proof. Let $i\colon M\to V$ be a G-embedding of M into a real G-vector space. For a real vector bundle ξ, its complexification is denoted by $\xi_{\mathbf{C}}$. Let $\pi_{\mathbf{C}}\colon \tau_{\mathbf{C}}M\to M$ be the projection and let

$$k\colon \tau M\longrightarrow \tau M_{\mathbf{C}}$$

be the inclusion onto the real part of $\tau M_{\mathbf{C}}$. Denote by νM the normal bundle of $M\subset V$. Then $\tau M_{\mathbf{C}}\oplus\nu M_{\mathbf{C}}\cong M\times V_{\mathbf{C}}$. Clearly $\pi_{\mathbf{C}}k=\pi\colon \tau M\to M$ and we have a pull-back diagram of G-vector bundles

$$\begin{array}{ccccc}
\pi^*(\nu M_{\mathbf{C}}) & \longrightarrow & M\times V_{\mathbf{C}} & \longrightarrow & \nu M_{\mathbf{C}}\\
\downarrow & & \downarrow & & \downarrow\\
\tau M & \xrightarrow[k]{} & \tau M_{\mathbf{C}} & \xrightarrow[\pi_{\mathbf{C}}]{} & M
\end{array}$$

As shown in [3], $N=\pi^*(\nu M_{\mathbf{C}})$ may be considered as the normal bundle of τM in $\tau V\cong V_{\mathbf{C}}$. We embed M in τM by the 0-section. Consider the embedding

$$i' \colon M \subset \tau M \subset \tau V = V_{\mathbf{C}}.$$

The total space of the normal bundle of M in $V_{\mathbf{C}}$ is clearly N.

Now define the transfer $p_! \colon K_G(M) \to K_G(pt.)$ using the embedding i'. Then $p_!$ is given by the composite

$$K_G(M) \xrightarrow{\Phi} K_G(M \times V_{\mathbf{C}}) \xrightarrow{l^*} K_G(N) \xrightarrow{j_*} K_G(V_{\mathbf{C}}) \xrightarrow{\Phi^{-1}} K_G(pt.)$$

where $l \colon N = \nu(M, V_{\mathbf{C}}) \to \nu(M, V_{\mathbf{C}}) \oplus \tau M \cong M \times V_{\mathbf{C}}$ and $j \colon N \to V_{\mathbf{C}}$ are natural inclusions, and j_* is the homomorphism induced from the map $V_{\mathbf{C}}^c \to V_{\mathbf{C}}^c/(V_{\mathbf{C}} - \operatorname{Im} j)^c \cong N^c$.

Let $\lambda \in K_G(\tau M_{\mathbf{C}})$ be the canonical Thom class of the bundle $\tau M_{\mathbf{C}}$. Then clearly we see that

$$k^* \lambda = u(M) \in K_G(\tau M).$$

Then by the naturality of the Thom homomorphism in the pull back diagram above, we have

$$p_!(x) = \Phi^{-1} J_* l^* \Phi(x) = \Phi^{-1} J_* \Phi \psi(x) = t\text{-ind}\, \psi(x)$$

where Φ denotes the Thom isomorphism for appropreate bundles. q.e.d.

As a corollary, we shall show that the induction homomorphism of representations of compact Lie groups defined by Segal [12] can be also defined by the transfer.

Let H be a closed subgroup of a compact Lie group G. The homogeneous space G/H has a usual left G-action. Recall that $K_G(G/H) \cong R(H)$. Then the transfer for ($p \colon G/H \to$ point) gives a homomorphism

$$p_! \colon R(H) \longrightarrow R(G).$$

Now we recall [12] the definition of the induced representation

$$i_* \colon R(H) \longrightarrow R(G).$$

Let M be a complex representation of H. Let ξ_M denote the G-vector bundle

$$G \times_H M \longrightarrow G/H.$$

By this correspondence, we see that $R(H) \cong K_G(G/H)$. Let $T^* \to G/H$ be the cotangent bundle of G/H. For a complex G-vector bundle ξ, $D(\xi)$ denotes the G-space of smooth sections. Then by use of a linear connection

$$\nabla_M \colon D(\xi_M) \longrightarrow D(\xi_M \otimes T_{\mathbf{C}}^*),$$

Segal defined an elliptic operator

$$\nabla_M + \nabla_M^* \colon \coprod_{i:\text{even}} D(\xi_M \otimes \lambda^i T_{\mathbf{C}}^*) \longrightarrow \coprod_{i:\text{odd}} D(\xi_M \otimes \lambda^i T_{\mathbf{C}}^*)$$

where λ^i denotes the exterior power and ∇_M^* is the adjoint of ∇_M. Then the analytic index

$$a\text{-ind}(\nabla_M+\nabla_M^*)\in R(G)$$

and by linearity, this defines a homomorphism

$$i_*: R(H) \longrightarrow R(G).$$

Now consider the symbol $\sigma(\nabla_M+\nabla_M^*)\in K_G(T^*)$. Since

$$\nabla_M: D(\xi_M\otimes\lambda^i T_{\mathbf{C}}^*) \longrightarrow D(\xi_M\otimes\lambda^{i+1}T_{\mathbf{C}}^*)$$

is given by the covariant exterior derivative, for $v\in T^*$ we see that

$$\sigma(\Delta_M)_v: \pi^*(\xi_M\otimes\lambda^i T_{\mathbf{C}}^*)_v \longrightarrow \pi^*(\xi_M\otimes\lambda^{i+1}T_{\mathbf{C}}^*)_v$$

is given by the product with v. Thus

$$\sigma(\nabla_M+\nabla_M^*)=\xi_M\otimes(\sum(-1)^i\pi^*\lambda^i T_{\mathbf{C}}^*)=\xi_M\otimes u(G/H).$$

where we have identified T^* with $\tau(G/H)$ by use of a G-invariant metric. Now by the index theorem [3]

$$a-\text{ind}(\nabla_M+\nabla_M^*)=t-\text{ind}(\sigma(\nabla_M+\nabla_M^*))$$

and by Theorem 5.1, we obtain

Theorem 5.2. $p_!=i_*: R(H)\to R(G)$.

Remark. If G/H is not merely a G-manifold but has another structure, there may exist a finer induction homomorphism. For example, if G/H is a complex manifold (e.g. $U(n)/T^n$), then by using the Thom class $\lambda\in K_G(\tau(G/H))$ instead of $u(G/H)$, we obtain another homomorphism $R(H)\to R(G)$.

Now we recall that $K_G(\text{pt.})=K_G^0(\text{pt.})$ is a $\pi_G^0(\text{pt.})$-module. We define the degree homomorphism

$$d: \pi_G^0(\text{pt.}) \longrightarrow K_G^0(\text{pt.})=R(G)$$

by $d(u)=u\cdot 1$ where $1\in K_G(\text{pt.})$ denotes the unit.

Let M be a compact G-manifold, then the equivariant Euler characteristic is defined by

$$\chi_G(M)=\sum(-1)^i H^i(M: \mathbf{C})\in R(G).$$

We have defined $w(M)\in\pi_G^0(\text{pt.})$ in §4. Then we have

Theorem 5.3. *Let M be a closed G-manifold. Then*

$$d(w(M))=\chi_G(M).$$

Proof. Let Ω^* be the complex valued de Rham complex of M. Then $\chi_G(M)$ is given by the Euler characteristic of Ω^*, $\chi(\Omega^*)$, and by the index theorem we see

$$\chi(\Omega^*)=t-\text{ind}(\sigma(\Omega^*)).$$

On the other hand, the symbol $\sigma(\Omega^*)$ is the complex of the exterior algebra of $T^*M_{\mathbf{C}}$. Hence

$$\sigma(\Omega^*)=u(M)\in K_G(\tau M)$$

and by Theorem 5.1 and by the definition of w, we see

$$\chi(\Omega^*)=t\text{-ind}\,\psi(1)=p_!(1)=w(M)\cdot 1=d(w(M)).$$

q.e.d.

Remark. If G is a finite group, then one can identify $\pi_G^0(\text{pt.})$ with the Burnside ring of G (Segal [14]). Let l: $A(G)\to R(G)$ be defined by $l(S)=G$-vector space generated by S for a finite G set S. Then one can prove that $l=d$: $\pi_G^0(\text{pt.})\to R(G)$.

Finally we relate the equivariant transfer with the non equivariant one. Let E be a compact free G-space and let H be a closed subgroup of G. Let

$$p_!: K(E/H)\longrightarrow K(E/G)$$

be the transfer in K-theory for the fibre bundle

$$G/H\longrightarrow E/H\xrightarrow{\ p\ }E/G.$$

For a G-vector space M, correspond the vector bundle $E\times_G M\to E/G$. Such a homomorphism is denoted by

$$\alpha: R(G)\longrightarrow K(E/G).$$

We denote here the transfer $R(H)\to R(G)$ for $\xi=(G/H\to\text{point})$ by τ. Ten we have

Proposition 5.4. *The following diagram is commutative*

$$\begin{array}{ccc} R(H) & \xrightarrow{\ \alpha\ } & K(E/H) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle p_!} \\ R(G) & \xrightarrow[\ \alpha\ ]{} & K(E/G) \end{array}$$

Proof. We can identify $R(H)=K_G(G/H)$, $K(E/H)=K_G(G/H\times E)$ and $K(E/G)=K_G(E)$. Then the homomorphisms

$$\alpha: R(H)\longrightarrow K(E/H)$$

$$\alpha: R(G)\longrightarrow K(E/G)$$

may be defined by the projections $G/H\times E\to G/H$ and $E\to$point, respectively. Consider the pull-back diagram

$$\begin{array}{ccc} G/H\times E & \longrightarrow & G/H \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ E & \longrightarrow & pt. \end{array}$$

where all maps are appropriate projections. Then by Proposition 4.4, we have a commutative diagram

$$\begin{array}{ccc} K_G(G/H) & \xrightarrow{\alpha} & K_G(G/H\times E) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\pi_!} \\ K_G(pt.) & \xrightarrow[\alpha]{} & K_G(E). \end{array}$$

Note that if ξ is a complex G-vector bundle over E, then there exists $\xi^\perp$ such that $\xi\oplus\xi^\perp\cong E\times\mathbf{C}^N$ where $\mathbf{C}^N$ is a trivial G-space. For using the isomorphism $K_G(E)\cong K(E/G)$, we can choose a complementary bundle of ξ in $K(E/G)$. Noting this observation, we consider the transfer for $\pi: G/H\times E\to E$. Let $G/H\subset V$ be a G-embedding into a complex G-vector space V. Let $\eta=(E\times V\to E)$ and choose $\eta^\perp$ such that $\eta\oplus\eta^\perp\cong E\times\mathbf{C}^N$. Then a trace of π is induced from

$$E\times\mathbf{C}^N\cong\eta\oplus\eta^\perp\supset(E\times\nu(G/H))\oplus\eta^\perp\subset p^*(\eta\oplus\eta^\perp)\cong E\times G/H\times\mathbf{C}^N.$$

Taking the G quotient spaces above, we obtain

$$E/G\times\mathbf{C}^N\supset(E\times_G\nu(G/H))\oplus\eta^\perp/G\subset(E\times_G G/H)\times\mathbf{C}^N=E/H\times\mathbf{C}^N,$$

and the induced map of one point compactification

$$(E/G\times\mathbf{C}^N)^c\longrightarrow(E/H\times\mathbf{C}^N)^c$$

may be considered as a trace of the fibre bundle $G/H\to E/H\to E/G$.

Note that the isomorphism $K_G(E)\cong K(E/G)$ holds when E is a locally compact free G-space and the diagram

$$\begin{array}{ccc} K_G(E) & \xrightarrow{\Phi} & K_G(E\times\mathbf{C}^N) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ K(E/G) & \xrightarrow[\Phi]{} & K(E/G\times\mathbf{C}^N) \end{array}$$

is commutative where Φ denotes the Thom isomorphism. Then we have a commutative diagram

$$\begin{array}{ccc} K_G(G/H\times E) & \xrightarrow{\cong} & K(E/H) \\ {\scriptstyle\pi_!}\downarrow & & \downarrow{\scriptstyle p_!} \\ K_G(E) & \xrightarrow[\cong]{} & K(E/G) \end{array}$$

This completes the proof. q.e.d.

§6. The Adams conjecture

In this section we prove the Adams conjecture for complex vector bundles. Our method is reducing the vector bundle problem to representation theory by using the transfer. So if we know a similar result on $RO(G)$ as mentioned in introduction, then our method can apply immediately to real vector bundles.

We formulate the Adams conjecture as follows.

Let F_n denote the monoid of proper homotopy equivalences of R^n. Let BF_n be the classifying space of F_n, and let $BF=\lim BF_n$. For a finite CW-complex X, put

$$Sph(X)=[X_+, BF\times Z].$$

According to Stasheff [16], the homotopy set $[X_+, BF]$ is isomorphic to the group of stable fibre homotopy equivalence classes of spherical fibre spaces. Let

$$J\colon K(X)\longrightarrow Sph(X)$$

be the J-homomorphism defined by

$$J(\xi)=([\xi], \dim\xi)$$

for a complex vector bundle ξ where $[\xi]$ denotes the class of the associated sphere bundle.

Let p be a prime number. For an abelian group A, $A\otimes Z\left[\frac{1}{p}\right]$ is denoted by $A\left[\frac{1}{p}\right]$.
Then

Theorem 6.1. (Quillen [10], Sallivan [17], Friedlander [7], Becker-Gottlieb [4]). *Let X be a finite CW-complex and let ψ^p be the Adams operation. Then*

$$J(\psi^p-1)=0\colon K(X)\left[\frac{1}{p}\right]\longrightarrow \mathrm{Sph}(X)\left[\frac{1}{p}\right].$$

Adams [1] has proved this for line bundles. So we shall prove the theorem by saying that we can reduce the problem to line bundles. In the proof, the following facts are crucial.

i) Segal [15] has shown that the monoid $\coprod BF_n$ is a Γ-space and hence its group completion $BF\times \mathbf{Z}$ is an infinite loop space. Furthermore the natural map

$$\coprod BU_n\longrightarrow \coprod BF_n$$

is a morphism of Γ-spaces. Thus we obtain an infinite loop map

$$j\colon BU\times \mathbf{Z}\longrightarrow BF\times \mathbf{Z}$$

which induces the J-homomorphism $j^*=J\colon K(X)\to Sph(X)$. From this we see that $Sph(X)$ is a 0-th group of a generalized cohomology theory and J is a stable natural transformation. Then one can think of the transfer in $Sph(X)$, and we see that the J-homomorphism commutes with the transfers by Proposition 4.3.

ii) The second fact is that the Adams operation ψ^p is a stable operation on $K(X)\left[\frac{1}{p}\right]$. This is well-known. Therefore ψ^p also commutes with transfers in $K(X)\left[\frac{1}{p}\right]$

iii) Finally if G is a compact Lie group, Segal [12] has shown that any complex representation of G is a linear combination of monomial representations, i.e., induced from one dimensional representations of appropriate subgroups.

Now we prove the theorem. Let H be a closed subgroup of G and let E be a compact free G-space. Then by the fact i) and by the localized version of Proposition 5.4, we have a commutative diagram

$$\begin{array}{ccccc} R(H)\left[\frac{1}{p}\right] & \xrightarrow{\alpha} & K(E/H)\left[\frac{1}{p}\right] & \xrightarrow{J} & \mathrm{Sph}(E/H)\left[\frac{1}{p}\right] \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle p_!} & & \downarrow{\scriptstyle p_!} \\ R(G)\left[\frac{1}{p}\right] & \xrightarrow{\alpha} & K(E/G)\left[\frac{1}{p}\right] & \xrightarrow{J} & \mathrm{Sph}(E/G)\left[\frac{1}{p}\right] \end{array}$$

where $p_!$ is the transfer for the bundle $G/H \to E/H \to E/G$.

From the fact ii), we have $\psi^p p_! = p_! \psi^p$, and clearly $\alpha\psi^p = \psi^p\alpha$. Hence we see that

$$(\alpha\tau)\psi^p = \psi^p(\alpha\tau).$$

Now let ξ be an n-dim. complex vector bundle over X. Let $E \to X$ be the associated principal $U(n)$-bundle. Let $\iota_n \in R(U(n))$ be the identity representation, then clearly $\xi = \alpha(\iota_n)$. We apply the fact iii) to $\iota_n \in R(U(n))$. Then

$$\iota_n = \sum_H i_{H*}(\lambda_H)$$

for some one dimensional representations λ_H of subgroups H of $U(n)$. Here we can identify i_{H*} with the transfer τ by Theorem 5.2. Then we have

$$\begin{aligned} J(\psi^p - 1)(\xi) &= J(\psi^p - 1)(\iota_n) \\ &= \sum_H J(\psi^p - 1)\alpha\tau(\lambda_H) \\ &= \sum_H J\alpha\tau(\psi^p - 1)(\lambda_H) \\ &= \sum_H p_! J\alpha(\psi^p - 1)(\lambda_H) \\ &= \sum_H p_! J(\psi^p - 1)(\alpha\lambda_H) = 0. \end{aligned}$$

This completes the proof.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] J. F. Adams: On the group $J(X)-I$, Topology **2** (1963), 181–195.

[2] M. F. Atiyah and G. B. Segal: Equivariant K-theory and completions, J. Defferential Geometry **3** (1969), 1–18.

[3] M. F. Atiyah and I. M. Singer: The index of elliptic operators; I, Ann. of Math. **87** (1968), 484–530.

[4] J. C. Becker and D. H. Gottlieb: The transfer map and fibre bundles, Topology **14** (1975), 1–12.

[5] G. Bredon: Introduction to compact transformation groups, Academic Press 1972.

[6] T. tom Dieck: Faserbundel mit Gruppen operation, Arch. Math. **20** (1969), 136–143.

[7] E. Friedlander: Fibrations in etale homotopy theory, Publ. Math. I.H.E.S. **42** (1972).

[8] D. S. Kahn and S. B. Priddy: Applications of transfer to stable homotopy theory, Bull. A.M.S. **78** (1972), 981–987.

[9] C. Kosniowski: Equivariant cohomology and stable cohomotopy, Math. Ann. **210** (1974), 83–104.

[10] D. Quillen: The Adams conjecture, Topology **10** (1971), 67–80.

[11] R. L. Rubinsztein: Thesis (Polish Acad. Science), 1973.

[12] G. B. Segal: The representation ring of a compact Lie group, Publ. Math. I.H.E.S. **34** (1968), 113–128.

[13] G. B. Segal: Equivariant *K*-theory, Publ. Math. I.H.E.S. **34** (1968), 129–151.

[14] G. B. Segal: Equivariant stable homotopy theory, Actes. Congress. Intern. Math. **2** (1970), 59–63.

[15] G. B. Segal: Categories and cohomology theories, Topology **13** (1974), 293–312.

[16] D. Stasheff: A classification theorem for fibre spaces, Topology **2** (1963), 239–246.

[17] D. Sullivan: Geometric topology, Part I, Localization, Periodicity and Galoi's Symmetry, Mimeographed, M.I.T. (1970).

[18] A. G. Wasserman: Equivariant differential topology, Topology **8** (1969), 127–151.