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The transition from rest to geostrophic balance of a dense volume of fluid

P. J. Montgomery & T. B. Moodie
Applied Mathematics Institute, Department of Mathematical Sciences, University of Alberta, Canada.

Abstract

The steady-state propagation of cold domes of dense fluid lying beneath a layer of less dense fluid and overlying a sloping bottom have been studied previously by Nof [1]. Such research is important in improving the understanding of heat and mass transport in the oceans, and much of the present-day understanding relies on the assumption that the flow is in geostrophic balance. Such an assumption is not valid either when inertial effects in the flow dominate, or when the flow is near the equator.

The research presented concerns the initial release problem of a finite volume of dense fluid underlying a deeper layer of less dense fluid. The flow is modelled by the shallow-water equations in three spatial dimensions which are expressed as a system of nonlinear hyperbolic conservation laws. Numerical solutions are achieved through the use of a finite-difference relaxation method developed previously by Jin and Xin [2], and modified for use in this specific situation by Montgomery [3]. Results are presented showing the resulting time-dependent flow for the initial release problem of a cylinder of dense fluid, with varying rotation rates and bottom slope to help investigate the times required for the inertial spreading to become primarily geostrophic.

1 Introduction

The motion of cold fronts in the atmosphere, fresh water rivers meeting a salt water ocean, and even snow powder avalanches may all be considered as the primarily horizontal motion of one fluid below

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a lighter fluid or above a denser fluid. These phenomena have been described under such general terms as density driven flow, or gravity currents, and an excellent review has been recently updated by Simpson [4]. In the situations where the earth's rotation is dynamically important, several interesting features exist which are not present for two-dimensional flow (see, for example, Griffiths [5]). The presence of the Coriolis Force permits steady-state solutions, or 'cold domes' to propagate along the bottom of the ocean, and exist in a balance between the gravitational downslope and Coriolis accelerations (Nof [1]).

In this paper, the shallow-water equations are used to model a cylindrical finite volume of dense fluid which evolves from rest within a semi-infinite layer of quiescent less dense fluid. The equations of motion are stated as a system of first-order partial differential equations representing the three variables of x and y momentum, as well as interface height. The rotation rate and bottom slope are varied, and a finite-difference numerical scheme is used to calculate solutions to the initial value problem.

2 The governing equations

The shallow-water equations for a single layer of fluid with density ρ_2, x velocity u, y velocity v, height h, and underlying a semi-infinite region of fluid with density $\rho_1 < \rho_2$, are given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g'\frac{\partial h}{\partial x} = fv - g'\frac{\partial h_b}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g'\frac{\partial h}{\partial y} = -fu - g'\frac{\partial h_b}{\partial y}, \tag{2}$$

and

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[(h - h_b)u] + \frac{\partial}{\partial y}[(h - h_b)v] = 0. \tag{3}$$

In eqns (1)-(3), the reduced gravity is defined as $g' = \frac{\rho_2 - \rho_1}{\rho_2}g$ for the gravitational acceleration $g = 9.8m/s^2$, and the bottom topography is given by the symbol $h_b(x, y)$ which is assumed to be a known function. The lower layer height h is measured above the bottom and is therefore always nonnegative.

A derivation of eqns (1)-(3) may be found in standard textbooks of fluid mechanics, in particular Pedlosky [6]. The Coriolis parameter, f represents the rotation of the earth, and is defined as $f = 2\Omega \sin\theta$, where the frequency of the earth's rotation is Ω, and the latitude of the motion occurs at an angle θ from the equator. Standard assumptions involved in deriving eqns (1)-(3) are that the fluids are incompressible, of constant density, Boussinesq, and inviscid while the flow is gentle enough so that entrainment and turbulence may be neglected. Of primary importance is the assumption that the flow maintains a low aspect ratio so that the accelerations in the horizontal direction are much greater than those in the vertical direction. This assumption allows the vertical and horizontal equations of motion to be decoupled.

To simplify eqns (1)-(3), they may be considered in nondimensional form through the introduction of a gravity current scaling and nondimensional variables, defined with a superscript tilde as

$$\begin{aligned}(x,y) = L(\tilde{x},\tilde{y}), \quad (h,h_b) = H(\tilde{h},\tilde{h}_b),\\ (u,v) = \sqrt{g'H}(\tilde{u},\tilde{v}), \quad t = \frac{L}{\sqrt{g'H}}\tilde{t}.\end{aligned} \tag{4}$$

In eqn (4), the characteristic horizontal and vertical length scales of the motion are given by L and H, while the velocity scaling is the standard gravity current one of $\sqrt{g'H}$.

Substitution of the variables from eqn (4) into eqns (1)-(3) give the three nondimensional equations

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{h}}{\partial \tilde{x}} = \varepsilon\tilde{v} - \frac{\partial \tilde{h}_b}{\partial \tilde{x}}, \tag{5}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{h}}{\partial \tilde{y}} = -\varepsilon\tilde{u} - \frac{\partial \tilde{h}_b}{\partial \tilde{y}}, \tag{6}$$

and

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}[(\tilde{h} - \tilde{h}_b)\tilde{u}] + \frac{\partial}{\partial \tilde{y}}[(\tilde{h} - \tilde{h}_b)\tilde{v}] = 0. \tag{7}$$

In eqns (5) and (6), the inverse of the Rossby Number is defined as

$$\varepsilon = \frac{fL}{\sqrt{g'H}}, \tag{8}$$

and is a dimensionless constant. It ranges from values of $\varepsilon = 0$ for purely inertial motion, to larger positive values as the effects of rotation become increasingly important.

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Eqns (5)-(7) may be put into conservative form using new variables defined as

$$\mu = \tilde{h}\tilde{u}, \quad \nu = \tilde{h}\tilde{v}, \quad \text{and} \quad \zeta = \tilde{h}. \tag{9}$$

The variables in eqn (9) represent the horizontal momenta, (μ, ν) and mass per unit height ζ in nondimensional form. Using these new variables, it can be shown (Montgomery [3]) that eqns (5)-(7) are equivalent to the system

$$\frac{\partial}{\partial \tilde{t}}\begin{bmatrix} \mu \\ \nu \\ \zeta \end{bmatrix} + \frac{\partial}{\partial \tilde{x}}\begin{bmatrix} \frac{\mu^2}{\zeta} + \frac{\zeta^2}{2} \\ \frac{\mu\nu}{\zeta} \\ \mu \end{bmatrix} + \frac{\partial}{\partial \tilde{y}}\begin{bmatrix} \frac{\mu\nu}{\zeta} \\ \frac{\nu^2}{\zeta} + \frac{\zeta^2}{2} \\ \nu \end{bmatrix} = \begin{bmatrix} \varepsilon\nu - \zeta\frac{\partial \tilde{h}_b}{\partial \tilde{x}} \\ -\varepsilon\mu - \zeta\frac{\partial \tilde{h}_b}{\partial \tilde{y}} \\ 0 \end{bmatrix}. \tag{10}$$

2.1 Discontinuous solutions to the initial value problem

Equations (5)-(7) have been investigated by Montgomery [3], and have been shown to be hyperbolic whenever the lower layer thickness, ζ, is positive. With such a result, the conservative form of the equations of motion, eqn (10), is initially well-posed for the initial value problem defined by eqn (10) and initial data of the form

$$\mu(\tilde{x}, \tilde{y}, 0) = \mu_0(\tilde{x}, \tilde{y}), \quad \nu(\tilde{x}, \tilde{y}, 0) = \nu_0(\tilde{x}, \tilde{y}), \quad \zeta(\tilde{x}, \tilde{y}, 0) = \zeta_0(\tilde{x}, \tilde{y}). \tag{11}$$

For hyperbolic initial value problems of the form (10)-(11), discontinuous initial data is permitted, and thus finite volumes of dense fluid may be considered which have vertical discontinuities in their original geometry. For example, a cylinder of radius 1 and horizontal upper surface may be considered, and defined as

$$\zeta_0(\tilde{x}, \tilde{y}) = \begin{cases} 1 - \tilde{h}_b(\tilde{x}, \tilde{y}), & \text{for} \quad \tilde{x}^2 + \tilde{y}^2 < 1, \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

with initial momenta given by

$$\mu_0(\tilde{x}, \tilde{y}) = 0, \quad \nu_0(\tilde{x}, \tilde{y}) = 0. \tag{13}$$

To choose the correct weak solution of eqn (10) which corresponds to the physical solution from eqns (5)-(7), various front conditions at discontinuities must be imposed, such as the Rankine-Hugoniot

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jump conditions. Although these are degenerate for the system (10), a expansion technique based on the methods of geometrical optics has been used to give a relation between the radial (u_r) and tangential (u_θ) velocities, and layer height $\tilde{h}$ when $\tilde{h}_b = 0$. The result may be found in Montgomery [3] and is quoted here as

$$u_r = \sqrt{2\tilde{h} - u_\theta(u_\theta + \varepsilon s)(8\ln s - 7)}, \tag{14}$$

for a discontinuity of height h and radial position $s(t)$. The relation (14) is an approximation for flow which is nearly axisymmetric.

3 Numerical results for the initial value problem

To obtain numerical solutions to the initial value problem of eqns (10)-(13), a finite-difference numerical method developed by Jin and Xin [2] is used. This is a relaxation scheme for a system of conservation laws in several space dimensions, which has been shown to be total variation diminishing in the scalar case. It is iterative, second-order, conservative, and is implemented with a slope limiter to capture discontinuous solutions to initial value problems without introducing nonphysical oscillations around a discontinuity.

This method has been used previously in one spatial variable for gravity currents in two spatial dimensions (Montgomery and Moodie [7]). Complete details may be found in Montgomery [3], and the method is used in the following two subsections. A spatial grid width of $\Delta x = 0.02$ was used throughout the calculations, with a stability condition of the Courant- Friedrichs-Levy Number equal to 1/2.

3.1 Initial value problem for $\tilde{h}_b = 0$

The first result, Figure 1, shows a representation of the three-dimensional geometry of a cold dome after the initial release of a cylinder of radius 1, height 1, and flat bottom topography. The initial dense volume expands radially outward along the bottom and beneath the less dense fluid. Soon after release, the initial discontinuity in height at the advancing front has been replaced by a smooth transition to zero height, as can be seen in Figure 1. This behaviour is contrary to the case for two-dimensional flow, where the front propagates from its initial discontinuity. The reason for this difference is the diverging nature

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of the flow as the dome expands radially, and the consequent rapid height decrease required for mass conservation.

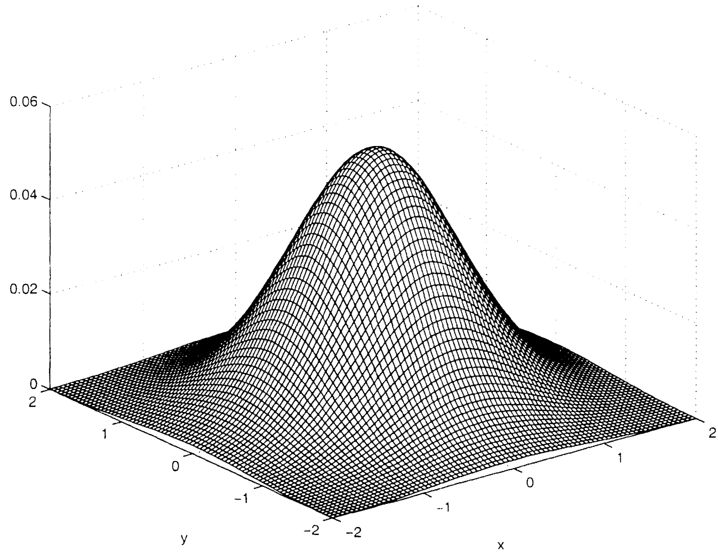

Figure 1: Lower layer height ζ at nondimensional time $\tilde{t} = 10$. Relevant parameters: $\varepsilon = 0$, $\zeta_0 = 1$, $\tilde{h}_b = 0$.

The shape of the cold dome in Figure 1 was found to be fairly generic and typical of the calculations conducted. The shape was not observed to change substantially for later times, except in the total height and radius. In addition, starting the numerical simulations with varying geometry such as a cube, resulted in similar profiles after the first several nondimensional time steps.

By tracking the position where the height ζ is larger than a specific value, chosen as 10^{-5}, the expanding front position of the cold dome may be charted when the flow possesses radial symmetry. The results for three different values of the Coriolis parameter are portrayed in Figure 2. There, it can be seen that the effect of rotation is to slow down the radial expansion of the rotating dome. For $\varepsilon = 2$, this expansion is actually checked, and the dome rotates in an almost solid-body fashion until lateral instabilities at the front (see Griffiths [5]) cause the numerical solution to break down. The small jumps in the curves are caused by computational constraints which necessitate larger grid widths when the front position reaches multiples of 10, and are purely a numerical artifact.

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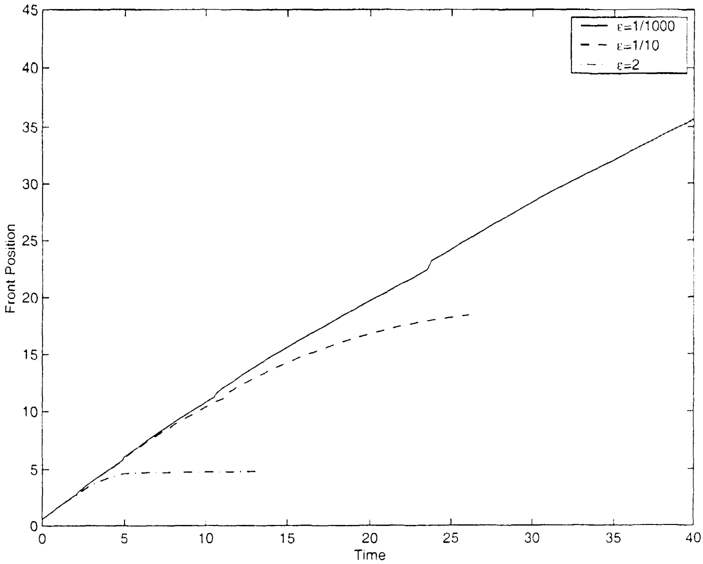

Figure 2: Front position versus time for increasing levels of rotation. Relevant parameters: $\zeta_0 = 1$, $\tilde{h}_b = 0$.

3.2 Initial value problem for $\tilde{h}_b = 0.1\tilde{x}$

Figures 1 and 2 show that over a horizontal bottom, the initial release of a dense cylinder causes a radially expanding flow which will spread outward as determined by the rotation rate. With sloping bottom topography, the initial release problem will cause the cold dome to move in the direction of decreasing bottom height, and a downslope acceleration is introduced.

The result of a constant bottom slope without rotation is shown in Figure 3, which is a contour plot showing the view from above. This diagram shows the height field, ζ, superimposed with a vector plot of the nondimensional velocity field, μ/ζ and ν/ζ. As expected, the expanding radial dome seen in Figure 1 is modified as it begins to drift downwards (to the left) under the influence of gravity. However, it is interesting to note that the radial symmetry is fairly well preserved in an reference frame which moves with the centre of the dome.

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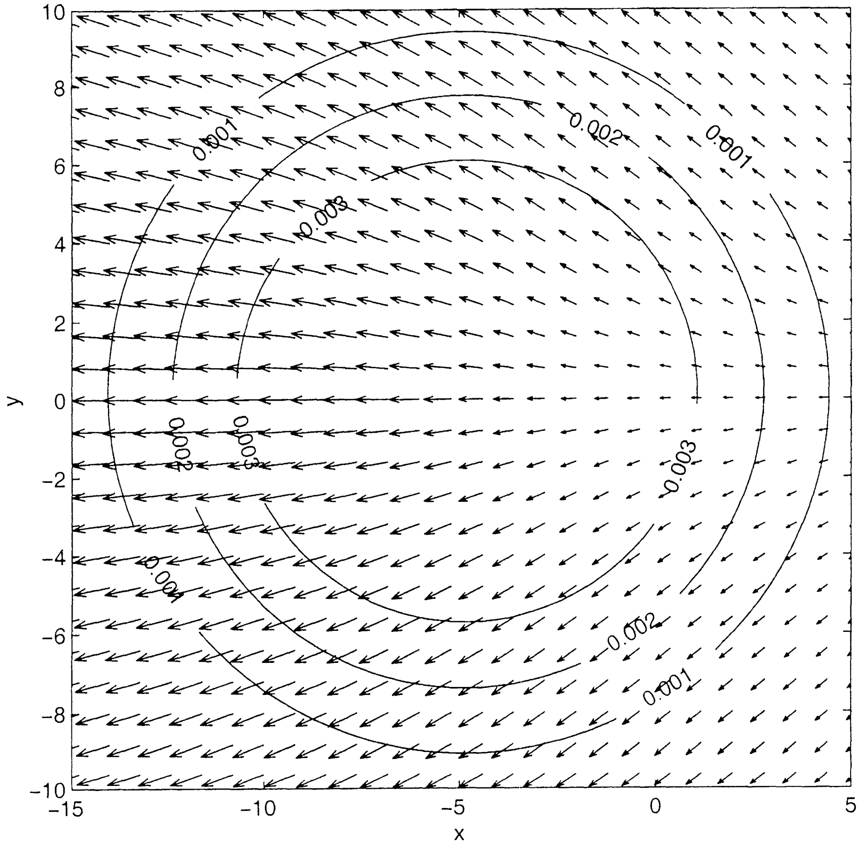

Figure 3: Height countours and velocity plot at time $\tilde{t} = 10$. Relevant parameters: $\zeta_0 = 1 + 0.1\tilde{x}$, $\tilde{h}_b = 0.1\tilde{x}$ $\varepsilon = 0$.

Once rotation is included in the calculations, an additional factor affects the flow of the dense fluid. A calculation which includes both sloping bottom topography and rotation is portrayed in Figure 4, which unfortunately looks asymmetrical due to the uneven axis ratio. It is observed from figure 4 that the cold dome begins to drift along the slope, and is deflected to the right so that it begins to move in a seemingly steady-state balance where the downslope acceleration is balanced by the Coriolis Force. Solutions for such cold domes have been given previously by Nof [1].

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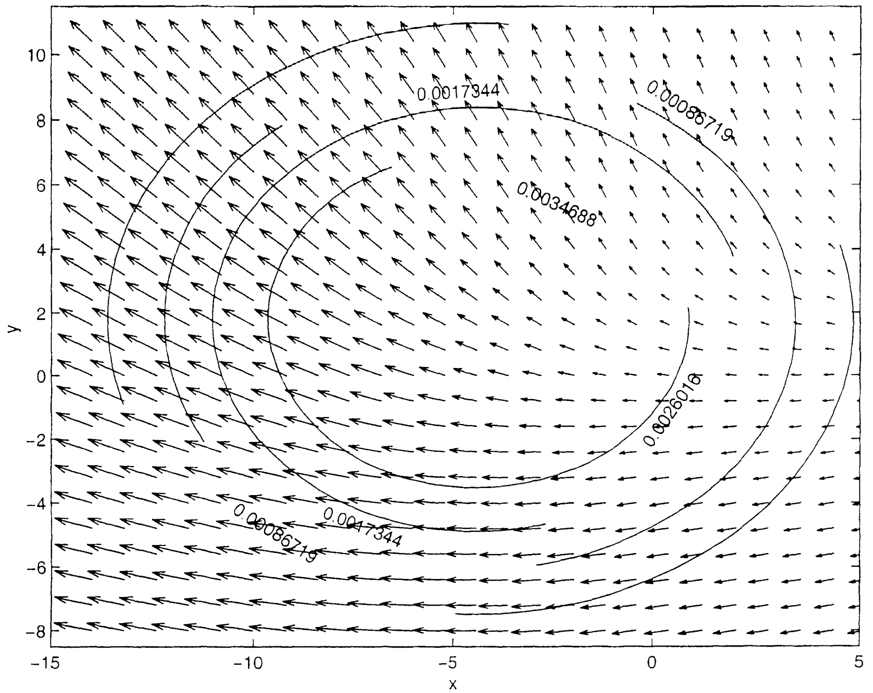

Figure 4: Height countours and velocity plot at time $\tilde{t} = 10$. Relevant parameters: $\zeta_0 = 1 + 0.1\tilde{x}$, $\tilde{h}_b = 0.1\tilde{x}$ $\varepsilon = 0.1$.

4 Concluding remarks

The initial release of a cylinder of dense fluid can be modelled effectively by the single-layer shallow-water equations. The effect of the Coriolis force is to deflect the developing cold dome to the right, which may be balanced by a downslope acceleration. Long time numerical calculations are difficult to obtain as lateral instabilities become apparent after the dome obtaines its steady state.

It was attempted to generalize thes previous methodology to two-layer flow in which the motion of the upper layer is not neglected. In such a case, there are six variables for the momenta and thicknesses of each layer. Unfortunately, only five equations were found which were in the flux-conserved form such as eqn (10), and the system could not be closed (Montgomery [3]). This contrasts with the single layer case, for which an infinite number of conservation laws may be calculated.

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Thus, the usefulness of the relaxation scheme may be limited to only the single layer, or axisymmetric case.

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