The Transport of Vorticity and Heat through Fluids in Turbulent Motion.

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In Reynolds' well-known theory of turbulent flow the effect of turbulence on the mean flow of a fluid is conceived as the same as that of a system of stresses which, like those due to viscosity, may have tangential as well as normal components across any plane element. Taking the case of laminar mean flow, that is when the mean flow is, say, horizontal and constant in direction and magnitude at any given height, the components of stress over a horizontal plane at height z are F_x and F_y where $F_x = -\rho\overline{uw}$, $F_y = -\rho\overline{vw}$, and u, v, w are the components of turbulent velocity parallel to two horizontal axes x and y and the vertical axis z. The bar denotes that mean values have been taken over a large horizontal area and ρ is the density of the fluid. The stress F_x is therefore due to the existence of a correlation between u and w. In the extension of Reynolds' theory due to Prandtl this correlation depends on the rate of change in mean velocity. In its most simplified form the theory may be expressed as follows. A portion of fluid possessing the mean velocity of a level z_0 may be conceived to move upwards to a layer of height $z_0 + l$ preserving the mean velocity U_0 of the layer from which it originated. At this height it is conceived to mix with its surroundings. If l is small the mean velocity of this layer is $U_0 + l\dfrac{dU}{dz}$, U being the mean velocity at height z, so that $u = -l\dfrac{dU}{dz}$, and hence

$$F_x = \rho\overline{wl}\,\frac{dU}{dz}. \tag{1}$$

The quantity $\rho\overline{wl}$ is therefore of the same dimensions as viscosity and in Prandtl's theory it is treated as though it were in fact a coefficient of viscosity, though not necessarily as one which has the same value at all points in the field.

In deriving the expression (1) it is assumed that the pressure gradients on the fluid which accompany the eddying motion have no effect on the final result, so that each particle continues moving with the horizontal momentum of the layer from which it originated till at some stage it mixes with the fluid at the level to which it penetrates. Without knowing how $\overline{wl}$ depends on the boundary conditions and on the law of variation of U with z it is not possible to apply any direct test to the theory by analysing observations of mean velocity in two dimensions. All that can be done is to make plausible assumptions about $\overline{wl}$ and thus calculate a corresponding distribution of mean velocity which can be compared with the observed distribution; or, alternatively, to calculate values of $\overline{wl}$ corresponding with an observed distribution of mean velocity.

Some years ago, at a time when I was ignorant of the work of Prandtl, I put forward a somewhat similar theory,* but it differed in one very significant feature. I supposed that each particle of fluid retained the vorticity, but not the momentum, of the layer from which it started. Otherwise the theories are identical, the mixture length l serving the same purpose in both theories. My object in concentrating attention on the transference of vorticity rather than momentum was that if the motion is limited to two dimensions the local differences in pressure do not affect the vorticity of an element, whereas Prandtl has to neglect them or to assume arbitrarily that they do not affect the mean transfer of momentum even though they certainly affect the momentum of individual elements of fluid. To illustrate the difference between the two theories we may consider the case in which the mean flow is parallel to the axis of x and the whole motion is limited to two dimensions, x horizontal, and z vertical. According to Prandtl's theory the tangential stress F is $\rho\overline{lw}\,\frac{d\mathrm{U}}{dz}$ and the rate at which momentum is communicated to unit area of a layer of thickness δz is

$$\rho\,\frac{d}{dz}\left[\overline{lw}\,\frac{d\mathrm{U}}{dz}\right]\delta z,$$

so that if the flow is that due to a uniform pressure gradient $\overline{dp}/dx$ the equation for U is

$$\frac{d}{dz}\left\{\overline{lw}\,\frac{d\mathrm{U}}{dz}\right\} = \frac{1}{\rho}\,\frac{\overline{dp}}{dx}, \tag{2}$$

and if the motion is the shearing motion which would occur in the space between two horizontal planes in relative movement

$$\overline{lw}\,\frac{d\mathrm{U}}{dz} = \text{constant}. \tag{3}$$

* 'Phil. Trans.,' A, vol. 215, p. 1 (1915).

In order to express the idea that vorticity rather than momentum is conveyed from one level to another by means of eddies, the equation of motion, neglecting viscosity, may be written

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z},$$

and writing $\eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$ for the vorticity, (3) becomes

$$-\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{1}{2}u^2 + \frac{1}{2}v^2\right) = \frac{\partial u}{\partial t} + 2w\eta.$$

Taking mean values, and supposing that the eddying motion is on the average uniform in the direction of x, this becomes

$$-\frac{1}{\rho}\frac{d\overline{p}}{dx} = 2\overline{w\eta}. \tag{4}$$

Now make the hypothesis, which is known to be true in the case of a non-viscous fluid moving in two dimensions, that the vorticity is conveyed without change by eddies just as Reynolds and Prandtl assume that horizontal momentum is carried. In this case the correlation between η and w might arise in exactly the same manner as the correlation between u and w in Prandtl's theory, a portion of fluid which has come through a vertical distance l since it possessed the mean vorticity of the layer from which it originated has a vorticity greater than that of the layer to which it has penetrated by an amount $-l\frac{d}{dz}\left(\frac{1}{2}\frac{dU}{dz}\right)$ for the vorticity of the mean motion at any layer is $\frac{1}{2}\frac{dU}{dz}$.

Hence

$$2w\eta = -\overline{wl}\frac{d^2U}{dz^2}$$

and (5) becomes

$$\frac{1}{\rho}\frac{d\overline{p}}{dx} = \overline{wl}\frac{d^2U}{dz^2}. \tag{5}$$

Comparing (5) with (2) it will be seen that the two formulæ appear at a first glance to be very similar. In the particular case when $\overline{wl}$ does not vary with z they are identical.

In considering how it would be possible to devise experimental tests to find out which, if either, of the two theories is correct, it will be seen that measurements of the distribution of velocity alone are not in general capable of distinguishing between them. If the mean velocity and pressure are measured at every point in a field of turbulent flow it is possible to deduce M, the rate at which momentum is communicated to unit volume. According to Prandtl's theory

$$M = \rho \frac{d}{dz}\left(\overline{wl}\,\frac{dU}{dz}\right),$$

while according to the vorticity transport theory in two dimensions

$$M = \rho\,\overline{wl}\,\frac{d^2U}{dz^2},$$

so that each theory enables $\overline{wl}$ to be determined from the observations as a function of z, but they would in general be different functions of z in the two cases.

On the other hand, if at the same time experiments on the distribution of temperature were carried out, another estimate of the value of $\overline{wl}$ would be found, for if Q be the rate at which heat is transferred across unit area perpendicular to the axis of z, and σ is the specific heat

$$Q = \rho\sigma\overline{wl}\,\frac{d\theta}{dz},$$

where θ is the mean temperature at any point, so that measurements of Q and $d\theta/dz$ would enable us to obtain an independent value of $\overline{wl}$. This could be compared with those deduced from the distribution of mean velocity by applying the two theories.

In the simple form presented above, the vorticity transport theory deals only with cases where the turbulent velocities as well as the mean velocity are confined to two dimensions. Some recent observations by Fage and Townend* have shown that near a solid surface the component of turbulent velocity parallel to the surface but perpendicular to the direction of mean flow is considerably greater than either of the other two components, so that the flow is certainly not two-dimensional. Experiments on the distribution of temperature near a heated surface past which a turbulent stream is flowing are therefore not suitable for our purpose. On the other hand, it seems possible that the turbulence which occurs in the wake behind a cylindrical obstacle with its axis perpendicular to the direction of the wind may be largely two-dimensional. This might be anticipated on theoretical grounds because the distribution of mean velocity in the wake behind an obstacle is of a type which for a non-viscous fluid is unstable for two-dimensional disturbances.

* 'Proc. Roy. Soc.,' A, vol. 135, p. 656 (1932).

There is little experimental evidence as to the character of the turbulent motion in the wake behind an obstacle. The hypothetical Kármán street of vortices is, of course, a two-dimensional motion and experiments which confirm the existence of such a system may therefore be regarded as being in favour of a tendency to a two-dimensional character in the eddying flow in the wake. For this reason it seemed desirable to examine both theoretically and experimentally the connection between the distributions of temperature and velocity in the wake behind a heated obstacle placed in a stream of wind.

Diffusion of Momentum in the Wake behind a Cylindrical Obstacle.

The distribution of velocity in the wake behind a cylindrical obstacle has recently been examined theoretically and experimentally by H. Schlichting.*

It is well known that the width of the wake behind an obstacle placed in a steady stream increases as the distance from the obstacle increases, while at the same time the difference between the velocity in the wake and that in the stream outside it decreases. Schlichting found that at some distance behind a cylindrical obstacle (more than 30 diameters) the expanding wake settles down to a steady regime in which the velocity at its centre is proportional to $x^{-\frac{1}{2}}$, while the width is proportional to $x^{\frac{1}{2}}$, x being the distance down stream from the obstacle.

If U_0 is the velocity of the stream in the absence of the obstacle and $U_0 - u$, v are the components of velocity in the wake, Schlichting's experimental results show that u can be expressed in the form

$$u/U_0 = x^{-\frac{1}{2}} f(\eta), \tag{6}$$

where $\eta = yx^{-\frac{1}{2}}$ and y is the distance of any point from the centre line of the wake. The corresponding expression for v which satisfies the equation of continuity, $\dfrac{\partial}{\partial x}(U - u) + \dfrac{\partial v}{\partial y} = 0$, is

$$v/U_0 = -\tfrac{1}{2} x^{-1} \eta f(\eta). \tag{7}$$

If the wake is assumed to be narrow so that in it y is small compared with x, and if u is small compared with U_0, the dynamical equation of motion representing Prandtl's theory of momentum transport is

$$-U_0 \frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}\left(\kappa \frac{\partial u}{\partial y}\right), \tag{8}$$

* H. Schlichting, "Ueber das ebene Windschatten problem," 'Ingenieur Archiv.,' (1930). See also "Turbulente Stromungen," W. Tolmien, 'Handbd. Exp. Physik.,' vol. 4, p. 323, Leipzig, where a figure showing comparison of Schlichting's theory with observation is reproduced.

where κ is written for the coefficient of diffusion by turbulence, *i.e.*, $\overline{lw}\dfrac{\partial u}{\partial y}$ according to equation (2). If it is assumed that the turbulent as well as the mean motion is confined to two dimensions, the dynamical equation of the vorticity transport theory is

$$-\,U_0\frac{\partial u}{\partial x} = -\,\kappa\,\frac{\partial^2 u}{\partial y^2}\,. \tag{9}$$

Expressing these equations in terms of η

$$\frac{1}{U_0}\left[\frac{\partial u}{\partial x}\right]_{y\ \text{constant}} = -\,\tfrac{1}{2}x^{-3/2}f(\eta) - \tfrac{1}{2}x^{-3/2}\,\eta\,f'(\eta) = -\,\tfrac{1}{2}x^{-3/2}\frac{d}{d\eta}\{\eta\,f(\eta)\}, \tag{10}$$

and since the wake is assumed to be narrow

$$\frac{\partial u}{\partial y} = x^{-\frac{1}{2}}\frac{du}{d\eta} = U_0\,x^{-1}f'(\eta).$$

Hence (8) becomes

$$\tfrac{1}{2}\frac{d}{d\eta}(\eta f\eta) = -\,\frac{d}{d\eta}\{\kappa f'(\eta)\},$$

This may be integrated, the constant of integration being omitted because $f'(\eta) = 0$ when $\eta = 0$ so that

$$\tfrac{1}{2}\eta\,f(\eta) = -\,\kappa f'(\eta). \tag{11}$$

Similarly (9) becomes

$$\tfrac{1}{2}\frac{d}{d\eta}\{\eta\,f(\eta)\} = -\,\kappa\,\frac{d^2 f(\eta)}{d\eta^2}\,. \tag{12}$$

It will be seen that (11) and (12) contain η only so that Schlichting's experimental result mentioned above implies that κ is a function of η only.

In Schlichting's experiments u was found from his measurements with a Pitot tube in the wake, so that $f(\eta)$ was determined. It will be seen that according to Prandtl's theory

$$-\,\kappa = \frac{\eta\,f(\eta)}{2f'(\eta)}\,, \tag{13}$$

and according to the vorticity transport theory of turbulence

$$-\,\kappa = \frac{f(\eta) + \eta\,f'(\eta)}{2f''(\eta)}\,. \tag{14}$$

In either case κ can be found as a function of η from the observed distribution of velocity in the wake, but it is not possible from these measurements alone to distinguish which theory, if either, is correct.

Heat Transport.

Suppose now that Schlichting's obstacle had been heated. The heat would be spread out in the wake and the equation of heat transport is

$$U_0 \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial y}\left(\kappa \frac{\partial \theta}{\partial y}\right), \tag{15}$$

where θ is the difference in temperature between any point in the wake and that in the main stream. This equation is identical with (8) except that θ replaces u so that θ must be of the form

$$\theta = x^{-\frac{1}{2}} \phi(\eta), \tag{16}$$

and substituting this expression in (15) it will be seen that

$$\kappa = -\frac{\eta\phi(\eta)}{2\phi'(\eta)}. \tag{17}$$

We are now in a position to compare the distribution of temperature which might be expected according to Prandtl's theory and according to the vorticity transport theory. According to Prandtl's theory of transport of momentum

$$\frac{\phi'(\eta)}{\phi(\eta)} = -\frac{\eta}{2\kappa} = \frac{f'(\eta)}{f(\eta)},$$

so that

$$\phi(\eta)/f(\eta) = \text{constant}, \tag{18}$$

i.e., the distribution of velocity and temperature across the wake should be identical, as is obvious *a priori*.

According to the vorticity transport theory

$$\frac{\phi'(\eta)}{\phi(\eta)} = -\frac{\eta}{2\kappa} = \frac{\eta f''(\eta)}{f(\eta) + \eta f'(\eta)},$$

so that

$$\log\{\phi(\eta)\} = \int \frac{\eta f''(\eta)\, d\eta}{f(\eta) + \eta f'(\eta)} + \text{constant}. \tag{19}$$

From the measured distribution of velocity in the wake the distribution of temperature can therefore be predicted, but in this case it is not identical with the distribution of velocity.

Distribution of Velocity in the Wake.

In order to predict the distribution of velocity in the wake behind an obstacle Schlichting used Prandtl's hypothesis that

$$\kappa = Al^2 \left|\frac{du}{dy}\right|, \tag{20}$$

where A is a constant and l is a "mixture length," and the straight bars surrounding $\frac{du}{dy}$ indicate that the absolute value is taken. As a further special hypothesis he assumed that l is constant over any given section of the wake and that it is proportional to $x^{\frac{1}{2}}$ so that l is always a given fraction of the width of the wake. Taking $l = ax^{\frac{1}{2}}$ it will be seen that Schlichting's assumed value of κ is

$$\kappa = -\,\mathrm{A}a^2 f'(\eta). \tag{21}$$

κ is thus a function of η only so that it satisfies the condition which is necessary in order that the breadth of the wake may be proportional to $x^{\frac{1}{2}}$ and thus be in accordance with Schlichting's observations. Making this assumption, the equation (11) becomes

$$\eta f(\eta) = 2\mathrm{A}a^2 [f'(\eta)]^2. \tag{22}$$

The integral of (22) is

$$[f(\eta)]^{\frac{1}{2}} = (18a^2\mathrm{A})^{-\frac{1}{2}} \eta^{\frac{3}{2}} + \text{constant}. \tag{23}$$

If η_0 is the value of η at the edge of the wake, which in this theory is entirely enclosed within the parabola $\eta = \eta_0$ then $f(\eta) = 0$ at $\eta = \eta_0$. If ξ written for η/η_0 (23) becomes

$$f(\eta) = (18\mathrm{A}a^2)^{-1} \eta_0^{\,3} (1 - \xi^{3/2})^2. \tag{24}$$

If u_0 is value of u at the centre of the wake in any given section at distance x from the obstacle, (24) may be written

$$u/u_0 = (1 - \xi^{3/2})^2. \tag{25}$$

Schlichting's observations provide a remarkable confirmation of the accuracy of this formula, and experiments recently made by Messrs. Fage and Falkner at my request in the National Physical Laboratory also confirm its substantial accuracy in the two cases which they examined.* The comparison between the observed values of u/u_0 and ξ and Schlichting's relationship (25) is shown in figs. 3 and 4.† It will be seen that the agreement is good. At first sight this might be taken as indicating that Prandtl's theory of momentum transport is correct, and that consequently the theory of vorticity transport is incorrect; such an inference, however, is not justifiable, for the vorticity transport theory predicts exactly the same distribution if the same assumption is made regarding

* Observations made in a tunnel specially designed to be free of turbulence gave rather different results, but it seems probable that the final steady regime had not been attained. The matter is being investigated further.

† *See* appendix, p. 702.

κ. To prove this, substitute $\kappa = -\mathrm{A}a^2 f'(\eta)$ in (12). The differential equation for $f(\eta)$ is then

$$\mathrm{A}a^2 f'(\eta) = \frac{f(\eta) + \eta f'(\eta)}{2f''(\eta)}. \tag{26}$$

The integral of (26) is

$$\eta f(\eta) = \mathrm{A}a^2 [f'(\eta)]^2. \tag{27}$$

It will be seen that (27) is the same as (22) except that the constant $2\mathrm{A}a^2$ in (22) has become $\mathrm{A}a^2$ in (27). The integral of (27) is

$$f(\eta) = (9\mathrm{A}a^2)^{-1} \eta_0^3 (1 - \xi^{3/2})^2, \tag{28}$$

and if u_0 is the value of u at the centre of the wake, (28) becomes

$$u/u_0 = (1 - \xi^{3/2})^2, \tag{29}$$

which is identical with (25). It will be seen, therefore, that with the particular assumption that κ is proportional to $f'(\eta)$ (or for constant x, to du/dy) both theories lead to the same predicted distribution of velocity in the wake and this distribution is very closely verified by experiment.

Distribution of Temperature in the Wake when $u/u_0 = (1 - \xi^{3/2})^2$.

According to the momentum transport theory of turbulence the distribution of temperature and velocity are identical, so that if θ_0 is the value of θ at the centre of the wake

$$\theta/\theta_0 = (1 - \xi^{3/2})^2. \tag{30}$$

The distribution of temperature which would be expected according to the vorticity transport theory is found by putting

$$f(\eta) = \eta_0^3 (9\mathrm{A}a^2)^{-1} \{1 - (\eta/\eta_0)^{3/2}\}^2$$

in (19). Using this expression for $f(\eta)$ and putting $\xi = \eta/\eta_0$

$$\int \frac{\eta f''(\eta)\, d\eta}{f(\eta) + \eta f'(\eta)}$$

becomes

$$\int \frac{-\frac{3}{2}\xi^{1/2} + 6\xi^2}{1 - 5\xi^{3/2} + 4\xi^3}\, d\xi$$

the integral of which is

$$\log(1 - \xi^{3/2}) + \text{constant}. \tag{31}$$

Hence from (19)

$$\log \phi(\eta) = \log(1 - \xi^{3/2}) + \text{constant},$$

or

$$\theta/\theta_0 = 1 - \xi^{3/2}. \tag{32}$$

At first sight it seems a paradox that the two theories should lead to identical distributions for velocity but different distributions for temperature, though the equations governing the velocity distribution are different in the two theories while the same equation for heat flow is used in each case. The explanation is that the parameter Aa^2 necessary to produce a *given* distribution of velocity in the wake, according to the vorticity transport theory, is twice as great as the parameter necessary to produce the *same* distribution according to the momentum transport theory. This parameter Aa^2 determines the thermal conductivity due to turbulence, so that for a given distribution of velocity the distribution of temperature in the wake of a heated obstacle is determined by a thermal conductivity which the vorticity transport theory predicts to be twice as great as that required by the momentum transport theory. It may be noted that the effect of the local pressure gradients which were neglected in Prandtl's theory is in this case to *increase* the rate of diffusion of heat compared with the rate of diffusion of momentum. Previous workers who have considered the effect in a qualitative way have predicted a *decrease*.*

Comparison with Experiment.

With a view to testing which (if either) of these two theories is correct, measurements of the distribution of velocity and temperature in the wake of a heated cylindrical obstacle were made at the National Physical Laboratory by Messrs. Fage and Falkner. These experiments are described in their note at the end of this paper. The velocity distribution was measured by means of a Pitot tube traversed across a section situated some 25 to 40 diameters of the obstacle down stream. From these measurements the width ($2b$) of the wake and the position of its centre were determined. The diagram, fig. 1, shows the scheme, the shaded portion representing the observed distribution of velocity. The non-dimensional variable $\xi = y/b$ measuring the distance from the centre of the wake was next calculated for each position where measurements were made.

The differences θ in temperature, and u in velocity between the heated wake and the main air stream were measured, and these were reduced to non-dimensional form by dividing by θ_0, u_0 the values of θ and u at the centre of the wake.

* *Cf.* V. W. Ekman, "Meeresstromungen," 'Handbd. Phys. and Tech. Mech.' (Auerbach and Hort), p. 203.

The results of these observations are shown in figs. 3 and 4 where the values of u/u_0 and θ/θ_0 are plotted against ξ. Since the wake is symmetrical only half of it is shown, positive and negative values of ξ being represented on the same side of the axis. In each case the curves whose ordinates are $(1 - \xi^{3/2})^2$ and $1 - \xi^{3/2}$ are shown by broken lines.

In fig. 3, which represents the results of experiments with a $\frac{5}{8}$-inch circular cylinder in a 3-feet open wind tunnel with parallel walls, it will be seen that the points representing u/u_0 fall very close to the curve $(1 - \xi^{3/2})^2$. This is the relationship which would be expected both on the vorticity transport and on the momentum transport theories of turbulence when Prandtl's special assumption is made for κ. The points representing θ/θ_0, on the other hand, are scattered round the curve $1 - \xi^{3/2}$ as predicted by vorticity transport theory and are very far indeed from the curve $(1 - \xi^{3/2})^2$ which is predicted by the momentum transport theory.

The scatter of the points representing θ/θ_0 in this experiment is rather large, and it was suspected that this might be due to casual variations in temperature of the building in which the open tunnel was situated; accordingly experiments were made in a 1-foot open jet tunnel of the return flow type. Experiments with an $\frac{1}{2}$-inch cylinder in this tunnel showed that much greater steadiness of temperature could be obtained, but the smallness of the diameter of the jet (1 foot) made the flow behind an $\frac{1}{2}$-inch cyclinder lose its two-dimensional character at a distance of 30 diameters down stream.*

To regain the two-dimensional character of the wake it was necessary to decrease its width, and for this purpose the $\frac{1}{2}$-inch circular cylinder was replaced by a lenticular cylinder 0·53 inch thick by 2·6 inches wide. With this obstacle the results shown in fig. 4 were obtained. It will be seen that the scatter of points representing θ/θ_0 has disappeared. The points representing u/u_0 are now not so close to Schlichting's curve $(1 - \xi^{3/2})^2$ as before; there is a systematic variation which may be due to the fact that the width of the wake is only five times that of the obstacle so that the flow may not have settled down to its permanent regime. The same kind of systematic variation of the points representing θ/θ_0 from the curve $1 - \xi^{3/2}$ will be seen, but in spite of this variation the confirmation of the vorticity transport theory of turbulence

* That the mean flow in the 3-feet tunnel was two-dimensional in the central part of the tunnel is shown by the fact that there is no systematic difference between the points marked ⊗ which were taken along a line distant 3 inches from the centre of the tunnel and those marked ⊙ which were taken along a line passing through the centre of the tunnel.

is strikingly verified for, even at its greatest, this variation is small compared with the difference between $1 - \xi^{3/2}$ and $(1 - \xi^{3/2})^2$.

Proof that "Momentum Transport" Theory is untrue when Motion is confined to Two Dimensions.

Though Prandtl's theory is usually expressed in a mathematical form into which only two dimensions enter, it is possible to give a mathematical proof that it would be untrue if the turbulence as well as the mean motion were limited to two dimensions. If ψ is the stream function at any instant of any two-dimensional motion of a viscous incompressible fluid, then the whole system may be rotated with uniform angular velocity Ω about an axis perpendicular to the plane of motion, and a motion relative to the rotating axes identical in every respect with the original motion is possible. If p is the pressure corresponding with the original motion, the pressure when the whole system is rotated is $p + 2\rho\,\Omega\psi + \frac{1}{2}\rho\,\Omega^2 r^2$, where r is the distance from the centre of rotation. The stresses due to viscosity are unaltered by the rotation as also are the stresses due to turbulence.

Prandtl's expression for the tangential stress due to turbulence in the case where the mean flow is in circles is

$$\mathrm{F} = \rho\overline{wl}\left[\frac{1}{r}\frac{d}{dr}(\mathrm{V}r)\right],$$

where V is the tangential velocity. A rotation of the whole system about the centre merely adds to V an amount Ωr without altering $\overline{wl}$. Prandtl's expression therefore involves an increase in F of amount $2\rho\overline{wl}\,\Omega$ due to the rotation.

As we have seen, the assumption that the whole fluid motion is limited to two dimensions necessarily implies that a rotation of the whole system makes no difference to anything but the pressure, so that the tangential stresses are unaltered by rotation. This conclusion would also follow from the vorticity transport theory because the addition of a constant vorticity to all parts of the field leaves the transport of vorticity unchanged.

One must therefore conclude that the reason why Prandtl's theory does not apply to two-dimensional flow is that he neglects the effect of the local pressure distribution in a turbulent system in altering the momentum of the portions of fluid which act as transporters of momentum from one layer to the next.

This source of error must also exist in all cases of turbulent flow except those in which the turbulence is confined to planes perpendicular to the mean flow.*

* *i.e.*, cases where lines of particles parallel to the direction of mean flow remain parallel to that direction.

In the case of flow between concentric circular cylinders for instance, Prandtl's theory might be expected to apply if the turbulence consisted entirely of the ring-shaped vortices symmetrical about the common axis, which do occur under certain conditions of rotation of the inner and outer cylinders.

Extension of "Vorticity Transport" Theory to Three Dimensions.

The equations of steady mean motion of a non-viscous fluid may be expressed in the form

$$X - \frac{1}{\rho}\frac{\partial P}{\partial x} = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} + \tfrac{1}{2}\frac{\partial}{\partial x}\overline{q^2} + 2\,(\overline{w\eta' - v\zeta'}) \qquad (33)$$

with two similar equations. In these equations X, Y, Z are the components of external force acting on unit mass of the fluid, U, V, W are the components of mean velocity and P is the mean pressure. The velocity at any point is $U + u$, $V + v$, $W + w$, $q^2 = u^2 + v^2 + w^2$, and ξ', η', w' are the components of vorticity of the turbulent motion so that

$$\xi' = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),$$

etc., and since $\overline{u} = \overline{v} = \overline{w} = 0$, $\xi' = \eta' = \zeta' = 0$.

Equations (33) are the three-dimensional analogy of (4). It is not possible, however, to proceed quite in the manner previously adopted and thus deduce a three-dimensional version of (5), because the vorticity of an element only remains unchanged when the motion is limited to two dimensions. Using the equations of vorticity in the Lagrangian form we may express the components of vorticity at any point in terms of its vorticity components at some previous time in the form*

$$\left.\begin{aligned} \xi + \xi' &= \xi_0\frac{\partial x}{\partial a} + \eta_0\frac{\partial x}{\partial b} + \zeta_0\frac{\partial x}{\partial c} \\ \eta + \eta' &= \xi_0\frac{\partial y}{\partial a} + \eta_0\frac{\partial y}{\partial b} + \zeta_0\frac{\partial y}{\partial c} \\ \zeta + \zeta' &= \xi_0\frac{\partial z}{\partial a} + \eta_0\frac{\partial z}{\partial b} + \zeta_0\frac{\partial z}{\partial c} \end{aligned}\right\}, \qquad (34)$$

where ξ, η, ζ are the components of vorticity of the mean motion, (a, b, c) are the co-ordinates of the particle of fluid at time t_0 which at time t occupies the position (x, y, z). ξ_0, η_0, ζ_0 are the values of ξ, η, ζ at the point (a, b, c) at

* See Lamb, "Hydrodynamics," 4th ed., p. 197.

time t_0. The assumption previously made in the case of two-dimensional motion, namely that inherent in the idea of a "mixture length," may now be made. This is that the portion of fluid which at time t occupies the position x, y, z, had at some previous time t_0 the same components of vorticity as its surroundings, namely the mean vorticity at the point a, b, c. Under these circumstances if the mixture length be supposed small and the mean motion steady, we may expand ξ_0 in terms of ξ using only the first order terms of the Taylor series so that

$$\xi_0 = \xi - (x-a)\frac{\partial \xi}{\partial x} - (y-b)\frac{\partial \xi}{\partial y} - (z-c)\frac{\partial \xi}{\partial z} \tag{35}$$

with similar expressions for η_0, ζ_0.

Inserting these expressions in (34) $\overline{w\eta' - v\zeta'}$ therefore becomes

$$\begin{aligned}
&\xi\left(\overline{w\frac{\partial y}{\partial a} - v\frac{\partial z}{\partial a}}\right) + \eta\left(\overline{w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial b}}\right) + \zeta\left(\overline{w\frac{\partial y}{\partial c} - v\frac{\partial z}{\partial c}}\right) \\
&\quad + \overline{(x-a)\left(v\frac{\partial z}{\partial a} - w\frac{\partial y}{\partial a}\right)}\frac{\partial \xi}{\partial x} + \overline{(y-b)\left(v\frac{\partial z}{\partial a} - w\frac{\partial y}{\partial a}\right)}\frac{\partial \xi}{\partial y} + \overline{(z-c)\left(v\frac{\partial z}{\partial a} - w\frac{\partial y}{\partial a}\right)}\frac{\partial \xi}{\partial z} \\
&\quad + \overline{(x-a)\left(v\frac{\partial z}{\partial b} - w\frac{\partial y}{\partial b}\right)}\frac{d\eta}{\partial x} + \overline{(y-b)\left(v\frac{\partial z}{\partial b} - w\frac{\partial y}{\partial b}\right)}\frac{\partial \eta}{\partial x} + \overline{(z-c)\left(v\frac{\partial z}{\partial b} - w\frac{\partial y}{\partial b}\right)}\frac{\partial \eta}{\partial z} \\
&\quad + \overline{(x-a)\left(v\frac{\partial z}{\partial c} - w\frac{\partial y}{\partial c}\right)}\frac{\partial \zeta}{\partial x} + \overline{(y-b)\left(v\frac{\partial z}{\partial c} - w\frac{\partial y}{\partial c}\right)}\frac{\partial \zeta}{\partial y} + \overline{(z-c)\left(v\frac{\partial z}{\partial c} - w\frac{\partial y}{\partial c}\right)}\frac{\partial \zeta}{\partial z}.
\end{aligned} \tag{36}$$

This expression, together with the two similar ones obtained by permuting cyclically xyz, abc, $\xi\eta\zeta$, represents the effect of turbulent motion on the mean motion according to the vorticity transport theory. In general it is so complicated that it is of little practical use, but in certain special cases considerable simplifications may occur.

Case of Laminar Mean Flow.

Let us now take the case previously discussed for which the mean velocity U is parallel to the axis of x and is a function of z only. In that case $\xi = \zeta = 0$ and $\eta = \frac{1}{2}\frac{dU}{dz}$, and since $\overline{q^2}$ may be assumed a function of z only, equation (33) becomes

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{dU}{dz}\left(\overline{w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial c}}\right) - \frac{d^2U}{dz^2}\overline{(z-c)\left(w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial c}\right)}. \tag{37}$$

It will now be shown that the equation (37) includes Prandtl's momentum transfer equation and my simple vorticity transfer equation for two-dimensional turbulent motion as special cases.

Case A. Two-dimensional Turbulent Motion Parallel to the Plane (xz).—In this case $v = 0$, $\partial y/\partial b = 1$, so that (37) becomes

$$-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x} = -\frac{d^2\mathrm{U}}{dz^2}\,\overline{(z-c)\,(w)}. \tag{38}$$

In this equation $z - c$ represents the same physical quantity as l in equation (5) so that (38) is identical with (5).

Case B. Turbulence confined to the Plane (yz).—This is the case to which Prandtl's momentum transfer theory must apply because there can be no pressure gradients to alter the momentum of lines of particles parallel to the axis of x.

Since y, z are functions of b, c, t, we can also consider b, c as functions of y, z, t. Remembering that the equation of continuity for an incompressible fluid in Lagrangian co-ordinates is

$$\frac{\partial y}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial y}{\partial c}\frac{\partial z}{\partial b} = 1. \tag{39}$$

It will be found that the equations for transformation are

$$\frac{\partial b}{\partial y} = \frac{\partial z}{\partial c}, \quad \frac{\partial c}{\partial y} = -\frac{\partial z}{\partial b}, \quad \frac{\partial c}{\partial z} = \frac{\partial y}{\partial b}, \quad \frac{\partial b}{\partial z} = -\frac{\partial y}{\partial c}. \tag{40}$$

Hence

$$\overline{w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial b}} = \overline{w\frac{\partial c}{\partial z} + v\frac{\partial c}{\partial y}}.$$

Taking the average value of $\overline{w\dfrac{\partial c}{\partial z} + v\dfrac{\partial c}{\partial y}}$ over a great breadth L in the direction of the axis of y, then since

$$\frac{1}{\mathrm{L}}\int_0^{\mathrm{L}} w\,dy \to 0, \qquad \overline{w\frac{\partial c}{\partial z} + v\frac{\partial c}{\partial y}} = -\frac{1}{\mathrm{L}}\int_0^{\mathrm{L}}\left\{w\frac{\partial (z-c)}{\partial z} + v\frac{\partial (z-c)}{\partial y}\right\}dy,$$

$$-\frac{1}{\mathrm{L}}\int_0^{\mathrm{L}} v\frac{\partial (z-c)}{\partial y}\,dy = -\frac{1}{\mathrm{L}}\Big[v\,(z-c)\Big]_0^{\mathrm{L}} + \frac{1}{\mathrm{L}}\int_0^{\mathrm{L}}(z-c)\frac{\partial v}{\partial y}\,dy.$$

Since $v\,(z-c)$ does not increase as L increases

$$\frac{1}{\mathrm{L}}\Big[v\,(z-c)\Big]_0^{\mathrm{L}} \to 0,$$

and since

$$\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z},$$

$$\int_0^L (z-c)\frac{\partial v}{\partial y}\,dy = -\int_0^L (z-c)\frac{\partial w}{\partial z}\,dy,$$

hence

$$\overline{w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial c}} = -\frac{1}{L}\int_0^L \left\{w\frac{\partial}{\partial z}(z-c) + (z-c)\frac{\partial w}{\partial z}\right\}dy$$

$$= -\frac{\partial}{\partial z}\overline{w(z-c)}. \qquad (41)$$

Transforming the coefficient of d^2U/dz^2 in (37) in the same way it is found that

$$-\overline{(z-c)\left(w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial c}\right)} = -\overline{(z-c)\left(w\frac{\partial c}{\partial z} + v\frac{\partial c}{\partial y}\right)},$$

and

$$-\overline{(z-c)\,w\frac{\partial c}{\partial z}} = \overline{(z-c)\,w\frac{\partial}{\partial z}(z-c)} - \overline{w(z-c)} = \overline{\tfrac{1}{2}w\frac{\partial}{\partial z}(z-c)^2} - \overline{w(z-c)}$$

$$-\overline{(z-c)\,v\frac{\partial c}{\partial y}} = \overline{(z-c)\,v\frac{\partial}{\partial y}(z-c)} = \overline{\tfrac{1}{2}v\frac{\partial}{\partial y}(z-c)^2}$$

$$= \mathrm{Lt}_{L\to\infty}\left[\frac{1}{2L}\left[v(z-c)^2\right]\right]_0^L - \frac{1}{2L}\int_0^L (z-c)^2\frac{\partial v}{\partial y}\,dy$$

$$= \mathrm{Lt}_{L\to\infty}\frac{1}{2L}\int_0^L (z-c)^2\frac{\partial w}{\partial z}\,dy = \tfrac{1}{2}\overline{(z-c)^2\frac{\partial w}{\partial z}},$$

Hence

$$-\overline{(z-c)\left(w\frac{\partial y}{\partial b} - v\frac{\partial z}{\partial c}\right)} = -\overline{w(z-c)} + \tfrac{1}{2}\frac{d}{dz}\overline{w(z-c)^2}. \qquad (42)$$

Now in deriving (36) and (37) it was assumed that the mixture lengths $z-c$ are small so that terms in $(z-c)^2$ can be neglected compared with terms containing $(z-c)$. To this order of approximation therefore the equation of motion (37) becomes

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{dU}{dz}\left[\frac{\partial}{\partial z}\overline{w(z-c)}\right] - \frac{d^2U}{dz^2}\overline{[w(z-c)]},$$

so that

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{d}{\partial z}\left[\overline{w(z-c)}\,\frac{dU}{dz}\right]$$

or in the notation of (2),

$$= \frac{d}{dz}\left(\overline{wl}\frac{dU}{dz}\right). \qquad (43)$$

This is identical with Prandtl's momentum transfer equation.

Concluding Remarks.

The only other case besides that discussed in this paper in which a complete set of measurements of the distribution of temperature and velocity in a turbulent fluid seems to have been made is that of the air near a heated flat plate in a wind stream. The very careful experiments of F. Elias* have shown that the distributions of velocity and temperature are then very nearly identical, as would be expected according to Reynolds' and Prandtl's theory of momentum transfer. We have seen that there is one type of turbulence for which the momentum transport theory is identical with the vorticity transport theory, namely when the turbulence is confined to the plane perpendicular to the direction of the mean flow (see (43) above). In a recent note† I have shown that the observations carried out a short while ago by Fage and Townend on the maximum values of the components of turbulent motion in a pipe suggest that the motion near the surface of the pipe is approximately of this type. More observations are required before it can be known definitely whether this is the true cause of the agreement between the observed distributions of temperature and velocity near a heated plate.

The theory of vorticity transport was developed in the essay for which the Adams Prize was awarded in 1915, but the experimental confirmation afforded by the experiments here described results from a study of Schlichting's paper.

Summary.

The theory that the dynamics of turbulent motion should be regarded as an effect of diffusion of vorticity rather than as a diffusion of momentum was put forward by the present writer in 1915, and the particular case when the whole motion is limited to two dimensions was then discussed, though so briefly that it appears to have escaped notice. The analysis is now extended to three-dimensional motion and it is shown that the "momentum transport" theory of Reynolds and Prandtl agrees with the "vorticity transport" theory in one case only, namely when the turbulent motion is of a two-dimensional type, being confined to the plane perpendicular to the mean motion.

When the turbulent motion as well as the mean motion is confined to two dimensions the vorticity transport theory yields results which are quite different from those predicted by the momentum transport theory.

* "Die Warmenbegang einen gleitzten Platte an Stromende Luft," 'Abh. Aerodynamischen Institut Aachen,' vol. 9, p. 10 (1930).

† 'Proc. Roy. Soc.,' A, vol. 135, p. 678 (1932).

A searching test of the comparative merits of the two theories is provided by comparing the distribution of temperature and velocity in the wake behind a heated obstacle. According to the momentum transport theory they should be identical, at any rate at some distance down stream, while according to the vorticity transport theory they should be related to one another by an equation which is given. Measurements made at the National Physical Laboratory show a large difference between the distributions of temperature and velocity and confirm the accuracy of the theoretical distributions given by the vorticity transport theory for the case of two-dimensional motion when the turbulent motion is confined to the plane of the mean motion.

APPENDIX.

Note on Experiments on the Temperature and Velocity in the Wake of a Heated Cylindrical Obstacle.

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Description of Obstacles.

For the experiments, the results of which are shown in fig. 3, a solid carbon cylindrical rod of diameter ⅝-inch was mounted in a 3-feet wind tunnel of the N.P.L. type. The length of the rod was 3 feet. The rod was directly heated by passing through it a current of about 70 amps.

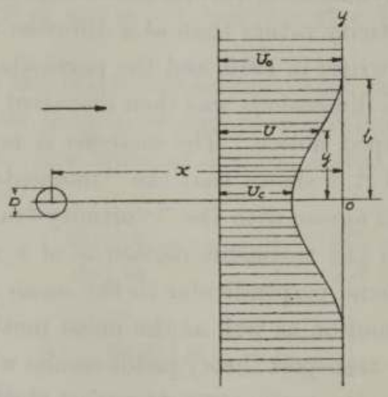

Fig. 1.

The experiments, results of which are given in fig. 4, were made with a thin-walled copper cylinder having a lenticular section (2·60 inches by 0·53 inch)