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THE TRANSPOSITION OF LOCALLY COMPACT, CONNECTED TRANSLATION PLANES

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This note deals with the transposition of translation planes in the topological context. We show that a topological congruence C of the real vector space $\mathbb{R}^{2n}$ has the property that every hyperplane of $\mathbb{R}^{2n}$ contains a component of C. This makes it possible to define the transpose P^τ of the topological translation plane P associated with C; it is proved that the translation plane P^τ is topological also. The relationship between collineation groups and the relationship between coordinatizing quasifields of P and P^τ are also discussed.

§1.

Before we formulate our main results, we recall some facts and definitions.

The affine point set of a locally compact, connected, topological translation plane P can be identified topologically with a finite dimensional real vector space V (in its usual topology) in such a manner that the lines through the origin of V are linear subspaces ([3, p.12-13]). These lines form a congruence of V; and the translation plane P is the translation plane associated with this congruence in the sense of André [1]. Any congruence of a real finite dimensional vector space V which can be obtained in this manner will be called a *topological congruence* of V. It is known that the only possible dimensions of V are 2, 4, 8 and 16 ([3, p.47 Satz 1]).

By V^* we denote the dual space of V consisting of all linear forms $V \longrightarrow \mathbb{R}$; by the *polar* of a linear subspace C of V we mean as usual the annihilator $C^0 = \{\varphi \in V^* \,/\, \varphi(C) = 0\}$ of C in V^*.

Using results of Breuning [3] we shall prove

THEOREM. *Let* $\mathcal{C}$ *be a topological congruence of a real finite dimensional vector space* V. *Then*
(a) $\mathcal{C}$ *has the property that any hyperplane of* V *contains a component of* $\mathcal{C}$, *and*
(b) the set $\mathcal{C}^0 = \{C^0 \,/\, C \in \mathcal{C}\}$ *of the polars of the components of* $\mathcal{C}$, *which by (a) is a congruence of* V^*, *is a topological congruence.*

Remark: The topological hypothesis of our theorem cannot be omitted, since by [2, Teorema 2.2] there exist (necessarily nontopological) congruences of V which do not have the hyperplane property of part (a).

COROLLARY. *Let* P *be a locally compact, connected, topological translation plane, and let* $\mathcal{C}$ *be the associated topological congruence of the underlying real vector space* V. *Then the translation plane* P^τ *associated with the corresponding polar congruence* $\mathcal{C}^0$ *is also a locally compact, connected, topological translation plane.*

The plane P^τ is called the *transpose*[1)] of P.

Proof of the Theorem.
(a) Identify V with $\mathbb{R}^{2n}$ and give $\mathcal{C}$ the finest topology such that the map

(1) $\mathbb{R}^{2n} \setminus \{0\} \longrightarrow \mathcal{C}: x \longmapsto C_x$,

which sends each nonzero vector x to the unique component C_x of $\mathcal{C}$ containing x, is continuous. By Satz 2 of [3, p.14] $\mathcal{C}$ is homeomorphic to the n-sphere S^n.

1) The authors are indebted to Dr. Rainer Löwen for calling their attention to this elegant description of the transpose using polars.

Moreover, if we consider $\mathcal{C}$ as a subset of the Grassmann manifold $G_n(\mathbb{R}^{2n})$ of n-dimensional subspaces of $\mathbb{R}^{2n}$, then the topology of $\mathcal{C}$ is the topology induced from $G_n(\mathbb{R}^{2n})$ ([3, p.30 Satz 1 and p.17]).

Let H be an arbitrary but fixed hyperplane in $\mathbb{R}^{2n}$. Arguing indirectly, we assume that H does not contain a component of $\mathcal{C}$. Then obviously $n \geq 2$. Moreover, every component of $\mathcal{C}$ would intersect H in a linear subspace of dimension n-1. Thus

$$\mathcal{C}|H = \{C \cap H \;/\; C \in \mathcal{C}\}$$

would be a subset of the Grassmann manifold $G_{n-1}(H)$. By looking at the Grassmann manifolds as quotient spaces of Stiefel manifolds it is easy to see that the map

$$(2) \quad \mathcal{C} \longrightarrow G_{n-1}(H) : C \longmapsto C \cap H$$

would be continuous. Thus $\mathcal{C}|H$ as a subspace of $G_{n-1}(H)$ would be homeomorphic to S^n.

Consider the canonical (n-1)-dimensional vector bundle $\gamma^{n-1}(H)$ over $G_{n-1}(H)$ as defined for example in Milnor-Stasheff [6, p.59 ff.]. By restricting $\gamma^{n-1}(H)$ to the subspace $\mathcal{C}|H$ of $G_{n-1}(H)$ and by removing the zero vector from each fiber, we would obtain a locally trivial fiber bundle over the n-sphere $\mathcal{C}|H$ with fiber $\mathbb{R}^{n-1}\setminus\{0\}$, total space

$$E = \{(C \cap H, x) \in (\mathcal{C}|H) \times (H\setminus\{0\}) \;/\; x \in C\},$$

and bundle projection $E \longrightarrow \mathcal{C}|H$ given by the projection onto the first factor. The total space E would be homeomorphic to $H\setminus\{0\} = \mathbb{R}^{2n-1}\setminus\{0\}$ by the map $H\setminus\{0\} \longrightarrow E$: $x \longmapsto (C_x \cap H, x)$, which can be expressed using maps (1) and (2) to check continuity. We now note an inconsistency when we examine the following part of the exact homotopy sequence of this fiber bundle:

$$\longrightarrow \pi_{n-1}(S^n) \xrightarrow{\partial} \pi_{n-2}(\mathbb{R}^{n-1}\setminus\{0\}) \longrightarrow \pi_{n-2}(\mathbb{R}^{2n-1}\setminus\{0\}) \longrightarrow ;$$

since $n \geq 2$, the homotopy groups (or sets) $\pi_{n-1}(S^n)$ and $\pi_{n-2}(\mathbb{R}^{2n-1}\setminus\{0\})$ are zero, whereas $\pi_{n-2}(\mathbb{R}^{n-1}\setminus\{0\})$ is not zero. This contradiction proves part (a) of the theorem.

(b) By [6, p.57 Lemma 5.1] the map $C \longmapsto C^{o}$ from $\mathcal{C}$ into the Grassmann manifold $G_n(V^*)$ is continuous, since after appropriate identification of the vector spaces V and V^*, the polar C^{o} of C is nothing more than the orthogonal complement $C^{\perp}$. Therefore $\mathcal{C}^{o}$ is homeomorphic to S^n too.

Thus to complete the proof of part (b) it suffices to apply the following lemma, which is implicit in [3].

LEMMA. *Let* $\mathcal{C}$ *be a congruence of* $\mathbb{R}^{2n}$ *having linear subspaces as components. Then* $\mathcal{C}$ *is topological if and only if* $\mathcal{C}$ *is compact as a subset of* $G_n(\mathbb{R}^{2n})$.

The "only if" part of this lemma has already been used in part (a) of the proof; it follows immediately from [3, p.14 Satz 2, p.30 Satz 1 and p.17]. For the "if" part, we look at the canonical n-dimensional vector bundle $\gamma^n(\mathbb{R}^{2n})$ restricted to $\mathcal{C}$ having the total space

$$E = \{(C,x) \in \mathcal{C} \times \mathbb{R}^{2n} \ / \ x \in C\}$$

and the projection $E \longrightarrow \mathcal{C}: (C,x) \longmapsto C$. This vector bundle admits (in the terminology of [3, p.22]) the effective Gauss map $E \longrightarrow \mathbb{R}^{2n}: (C,x) \longmapsto x$. We can then apply [3, p.27 Satz, p.28 Korollar and p.34 Satz 1].

§2.

In this section we make a few remarks about the effects of transposition of a locally compact, connected translation plane as far as the collineation groups and coordinatizing quasifields are concerned.

Recall that a quasifield which coordinatizes a locally compact, connected translation plane $\mathcal{P}$ is a locally compact, connected, topological quasifield in the sense of Salzmann [7, §7]; conversely, any plane coordinatized by such a quasifield is a locally compact, connected translation plane. The kernel K of $\mathcal{P}$ is isomorphic as a topological field to $\mathbb{R}$, $\mathbb{C}$ or the quaternion field $\mathbb{H}$ ([3, p.8 Satz 3]).

We first claim that the transpose P^{τ} of P can be constructed equally well using K instead of $\mathbb{R}$ as the base field.

More generally let P be a (not necessarily topological) translation plane which is finite dimensional over its kernel K. By [1] the affine point set V of P may be considered as a one-sided vector space over K in such a manner that the components of the congruence C associated with P are K-linear subspaces. (In the case where P is locally compact and connected, the $\mathbb{R}$-linear structure which we have been considering up to now is, of course, simply given by restriction to the closed subfield $\mathbb{R}$ of K.) Given any subfield K' of K, we can regard V as a K'-vector space and form its dual

$$V^*_{K'} = \mathrm{Hom}_{K'}(V,K')$$

over K'. For $C \in C$ consider the K'-*polar*

$$C^o_{K'} = \{\psi \in V^*_{K'} \ / \ \psi(C) = 0\}$$

and the family

$$C^o_{K'} = \{C^o_{K'} \ / \ C \in C\}.$$

PROPOSITION 1. *Suppose* K' *is a subfield of* K *and* F *a central subfield of* K' *such that* V *has finite dimension over all these fields. Then there exists an* F*-linear isomorphism* $V^*_{K'} \longrightarrow V^*_F$ *which maps* $C^o_{K'}$ *onto* C^o_F *for every* $C \in C$. *In particular, if one of the families* C^o_F *and* $C^o_{K'}$ *is a congruence, the other is also.*

Proof: If $\delta: K' \longrightarrow F$ is any nonzero F-linear form, let

$$\Delta: V^*_{K'} \longrightarrow V^*_F: \psi \longmapsto \delta\circ\psi$$

be the induced map, which is F-linear, since F is central in K'. The map Δ is injective, and $\Delta(C^o_{K'}) \subseteq C^o_F$ for every $C \in C$. Since the F-dimensions of V^*_F and $V^*_{K'}$ are equal, and the F-dimensions of C^o_F and $C^o_{K'}$ are equal, Δ must be an isomorphism, and $\Delta(C^o_{K'}) = C^o_F$.

COROLLARY. *Let* P *be a locally compact, connected* *translation plane,* K *its kernel and* C *the congruence associated with* P. *Then* C_K^o *is a topological congruence, and the translation plane associated with* C_K^o *is topologically isomorphic to the transpose* P^τ *described in the Corollary on page 2.*

This follows from the Theorem on page 2 using Proposition 1 with $F = \mathbb{R}$ and $K' = K$.

Maduram [5] has described the relationships between some collineation groups and between coordinatizing quasifields of finite translation planes and their transposes. His general results as well as his proofs carry over to locally compact, connected translation planes. We give topological analogues of three of his propositions, which are of particular interest in this context, together with indications of proofs using our notation.

PROPOSITION 2 (cf. [5, Prop. 3]). *Let* P *be a locally compact, connected translation plane, let* Γ_0 *denote the isotropy subgroup of the affine collineation group of* P *with respect to the origin, and let* Σ_0 *denote the subgroup of continuous collineations in* Γ_0; *let* Γ_0^* *and* Σ_0^* *be the analogously defined groups of collineations of the transpose* P^τ. *Then* Γ_0 *and* Γ_0^* *are isomorphic--viewed as transformation groups of the congruences associated with* P *and* P^τ *respectively--via an isomorphism which maps* Σ_0 *onto* Σ_0^*. *The action of* Σ_0^* *on the translation group of* P^τ *is the contragredient representation of the* $\mathbb{R}$*-linear action of* Σ_0 *on the translation group of* P.

For the proof, recall that Γ_0 consists of all K-semilinear automorphisms of V which respect the congruence C ([1, p.178 Satz 19]). For $A \in \Gamma_0$ with $\alpha \in \mathrm{Aut}\, K$ its companion automorphism, define a K-semilinear transformation $A^\vee$ of V_K^* by

(3) $A^\vee(\psi) = \alpha \circ \psi \circ A^{-1} \qquad (\psi \in V_K^*)$.

A straightforward calculation shows that $A^{\vee}$ respects the congruence C_K^{o}, and thus by the corollary on page 5 may be considered as an element of Γ_0^*. One easily checks that $A \longmapsto A^{\vee}: \Gamma_0 \longrightarrow \Gamma_0^*$ is a group isomorphism; this and the bijection $\mathcal{C} \longrightarrow \mathcal{C}_K^{o}: C \longmapsto C_K^{o}$ together constitute a transformation group isomorphism

$$(4) \quad (\Gamma_0,\mathcal{C}) \xrightarrow{\cong} (\Gamma_0^*,\mathcal{C}_K^{o}): (A,C) \longmapsto (A^{\vee},C_K^{o}) .$$

Since the companion automorphism of $A^{\vee}$ is again α, the continuity of A, α and $A^{\vee}$ are equivalent.

To compare the actions of Σ_0 and Σ_0^*, take the $\mathbb{R}$-linear form $\delta: K \longrightarrow \mathbb{R}$ which associates to each $z \in K$ its real part $\delta(z) = \operatorname{Re} z$ (K is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), and identify V_K^* and $V_{\mathbb{R}}^*$ by the induced isomorphism Δ as defined in the proof of Proposition 1. We shall interprete formula (3) in $V_{\mathbb{R}}^*$: Making use of the fact that $\delta\circ\alpha = \delta$ when α is a continuous automorphism of K, we have

$$\Delta\circ A^{\vee}\circ\Delta^{-1}(\varphi) = \delta\circ\alpha\circ(\Delta^{-1}\varphi)\circ A^{-1} = \varphi\circ A^{-1} \quad (A\in\Sigma_0,\ \varphi\in V_{\mathbb{R}}^*),$$

which is the expression defining the contragredient representation.

PROPOSITION 3 (cf. [5, Prop. 4 and 5(3)]). *Let* Q *be a locally compact, connected quasifield (or a real finite dimensional division algebra), and let* P *be the translation plane coordinatized by* Q. *Then there is a locally compact, connected quasifield (respectively, a real division algebra)* Q', *which coordinatizes the transpose* P^{τ} *and thus has the same dimension as* Q, *such that the middle nucleus and the left nucleus of* Q' *are topologically isomorphic to the left nucleus and the middle nucleus of* Q *respectively.*

Remark: Note that Dembowski, André and Maduram consider quasifields with the distributive law $(a + b)c = ac + bc$, whereas the distributive law used by Breuning and Salzmann is $c(a + b) = ca + cb$. Proposition 3 is formulated according to the latter convention; in particular, the left nucleus here plays the role of the right nucleus in [4, p. 134] and [5].

For the proof of the proposition, let C_1 and C_2 denote the first and second coordinate axes in the coordinatization of P over Q, and call their respective points at infinity p_1 and p_2. Take Q' to be a locally compact, connected quasifield which coordinatizes P^τ with respect to the K-polars $(C_i)_K^o \in C_K^o$ $(i = 1,2)$ as first and second coordinate axes. To simplify notation we write these axes as C_i^o; denote their respective points at infinity by p_i^o $(i = 1,2)$.

By straightforward calculation we see that the transformation group isomorphism $(\Gamma_0, C) \cong (\Gamma_0^*, C_K^o)$ of (4) restricts to an isomorphism between the group $\Gamma(p_i, C_j)$ of central collineations of P with center p_i and axis C_j and the group $\Gamma^*(p_j^o, C_i^o)$ of central collineations of P^τ with center p_j^o and axis C_i^o $(i,j \in \{1,2\})$. These groups of central collineations are K-linear ([1, p. 180 Satz 22]); hence the groups carry the obvious topology, and the isomorphisms above are clearly continuous. The proof proceeds by using the well-known relationships of these groups to substructures of the coordinatizing quasifields (cf. [4, p.134, 3.1.30 ff.]): The left-middle interchange in the assertion regarding the nuclei of Q and Q' follows from the isomorphisms $\Gamma(p_1, C_2) \cong \Gamma^*(p_2^o, C_1^o)$ and $\Gamma(p_2, C_1) \cong \Gamma^*(p_1^o, C_2^o)$ and the relationships of these strain groups to nuclei. Moreover, Q' is a semifield when Q is, since the shears group $\Gamma(p_2, C_2)$ is transitive on $C \setminus \{C_2\}$ if and only if the corresponding group $\Gamma^*(p_2^o, C_2^o)$ is transitive on $C_K^o \setminus \{C_2^o\}$. Finally, a locally compact, connected semifield is a real finite dimensional (not necessarily associative) division algebra, because by the arguments of [3, p.8 Satz 3] it contains $\mathbb{R}$ in its center.

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