THE TRANSVERSAL CREEPING VIBRATIONS OF A FRACTIONAL DERIVATIVE ORDER CONSTITUTIVE RELATION OF NONHOMOGENEOUS BEAM

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We considered the problem on transversal oscillations of two-layer straight bar, which is under the action of the lengthwise random forces. It is assumed that the layers of the bar were made of nonhomogenous continuously creeping material and the corresponding modulus of elasticity and creeping fractional order derivative of constitutive relation of each layer are continuous functions of the length coordinate and thickness coordinates. Partial fractional differential equation and particular solutions for the case of natural vibrations of the beam of creeping material of a fractional derivative order constitutive relation in the case of the influence of rotation inertia are derived. For the case of natural creeping vibrations, eigenfunction and time function, for different examples of boundary conditions, are determined. By using the derived partial fractional differential equation of the beam vibrations, the almost sure stochastic stability of the beam dynamic shapes, corresponding to the *n*th shape of the beam elastic form, forced by a bounded axially noise excitation, is investigated. By the use of S. T. Ariaratnam's idea, as well as of the averaging method, the top Lyapunov exponent is evaluated asymptotically when the intensity of excitation process is small.

1. Introduction

The use of composite beams is now a real trend in many engineering applications. This trend calls for the development of efficient tools, suitable for the analysis of beams exhibiting three-dimensional effects, for which the classical beam theory assumptions are no more valid.

Transversal vibration beam problem is classical, but in current university books on vibrations we can find only Euler-Bernoulli's classical partial differential equation (see [24, 31]) for describing transversal beam vibrations. In some monographs [25–27], we can find a nonlinear partial differential equation for describing transversal vibrations of

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the beam with nonlinear constitutive stress-strain relation of the beam ideal elastic material. Recently new models of the constitutive stress-strain relations of the rheological new beam materials [10] were found in books [5] and as applications in journal papers [2, 4, 19–22]. In the university book by Rašković [31], extended partial differential equation of the transversal ideally elastic beam vibration is presented with members by which influences of the inertia rotation of the beam cross-section and shear of the cross-section by transversal forces are presented.

In the paper by Tabaddor [36], compare the experimentally and theoretically obtained single-mode responses of a cantilever beam. The analytical portion involves solving an integro-differential equation via the method of multiple scales. For the single-mode response, a large discrepancy is found between theory and experiment for an assumed ideal clamp model.

Bypassing the complexity of a full three-dimensional elasticity analysis, Crespo da Silva derived nonlinear equations governing the dynamics of 3D motions of beams.

The purpose of the paper by Fatmi and Zenzri [6] was to simplify the numerical implementation of the exact elastic beam theory in order to allow an inexpensive and large use of it. A finite element method is proposed for the computation of the beam operators involved in this theory. The discretization is reduced since only one element is required in the longitudinal direction of the beam. The proposed method is applied to homogeneous and composite beams made of isotropic materials and to symmetric and antisymmetric laminated beams made of transversely isotropic materials. Structural beam rigidities, elastic couplings, warpings, and three-dimensional stresses are provided and compared to available results.

The integral theory of analytical dynamics of discrete hereditary systems is presented in the monograph by Goroshko and Hedrih [10] and their applications are published in the following papers by Hedrih (Stevanović) [12, 13, 16–20].

In the paper by Machado [37] we learn that the papers by Gemant [8] and Oldham [28], among other cited papers, contain the basic aspects of the fractional calculus theory and the study of its properties can be addressed in these references, while research results can be found in papers by Osler [29], Ross [32], Campos [3], Samko [33], and others. We must also refer to Gorenflo and Mainardi's [9].

In [9, 34] fractional calculus is mathematically based on corresponding integral and fractional order differential equations and in [5] fractional calculus is coupled with constitutive relation of real creeping material. In [2, 4] the authors presented new results of the *stability and creeping* and dynamical stability of viscoelastic column with fractional derivative constitutive relation of rod material. Papers by Hedrih (Stevanović) [19, 20] are in relation to the transversal vibrations of the beam of the hereditary material and the stochastic *stability of the beam dynamic shapes, corresponding to the nth shape of the beam elastic form.* Also, in [19] the transversal vibrations of the beam of the new models of the constitutive stress-strain relations material in the form of *a fractional derivative order constitutive relation beam are* studied, and as well, the stochastic *stability of the beam dynamic shapes, corresponding to the nth shape of the beam elastic form, is examined by using ideas of S. T. Ariaratnam* [1]. By Isayev and Mamedov [23] some results on dynamic stability of nonhomogenous bars are presented.

Foster and Berdichevsky [7] apply the quantitative method to estimate the violation of Saint-Venant's principle in the problem of flexural vibration of a two-dimensional strip. A probabilistic approach is used to determine the relative magnitude of the penetrating stress state and the results of computations are presented as a function of frequency. The results are not dependent on material properties except for the Poisson ratio. The major conclusion of these papers is that over a wide range of frequencies, the maximum propagating stress is always small compared with the maximum applied stress; hence, Saint-Venant's principle may be said to apply to this problem. An interesting outcome of the study is that the accuracy of engineering theories for flexural vibrations is much higher than for longitudinal vibrations.

In [1] the stochastic stability of viscoelastic systems under bounded noise excitation by S. T. Ariaratnam is investigated and some new interesting results for applications are found. The paper by Parks and Pritchard [30] is a contribution on the construction and the use of the Lyapunov functionals. The monograph by Stratonovich [35] is the monograph with topics in the theory of random noise used in [1] and in this paper. Asymptotic method of averaging applied to the nonstationary nonlinear processes and to the nonlinear vibrations of deformable bodies is the topic of the three monographs [25–27]. Krilov-Bogolyubov-Mitropol'skiĭ's method is presented in the book by Hedrih (Stevanović) [14, 15]. This paper contains new results on transversal vibrations on nonhomogeneous beams based on the contents of the cited references and books.

2. Model of creeping rheological body

For modeling processes of solidification and relaxation, models of Kelvin's viscous-elastic material and Maxwell's ideal-elastic-viscous fluid are being used. In their paper, Goroshko and Puchko [11] have used model of standard hereditary body to modeling dynamics of mechanical systems with rheological links. Studying elements of mechanics of hereditary systems in their monograph, G. N. Savin and Yu. Ya. Ruschisky gave survey of both structure and analysis of the rheological models of simple and complex laws for linear deformable hereditary-elastic media, as well as theory of growing old of hereditary-elastic systems.

Recently, there is a noticeable interest in using fractional derivatives to describe creep behavior of material. In solid mechanics particularly for describing problems related to material creep behavior including viscoelastic and viscoplastic effects, fractional derivatives have a longer history (see [5, 9]). Mathematical basis of the fractional derivative and short complete of fractional calculus are presented in the monograph paper by Gorenflo and Mainardi [9].

The paper by Dli et al. [4] contains the consideration of dynamical stability of viscoelastic column with fractional derivative constitutive relation. The paper by Bačlić and Atanacković [2] considered stability and creep of a fractional derivative order viscoelastic rod.

We introduce that material of the one layer beam is a creeping material. Parameters of the beam creep material are the following: α is proper material constant of the characteristic creep law of material, E_0 and E_α are modulus of elasticity and creeping properties of material.

By using stress-strain relation from the cited references, a single-axis stress state of the creep hereditary-type material is described by fractional order time derivative differential relation in the form of three-parameter model. For line element of beam creep material, constitutive stress-strain state relation is expressed by fractional derivative constitutive relation in the following form:

$$\sigma_z(z,y,t) = y\left\{E_0\frac{\partial\varphi(z,t)}{\partial z} + E_\alpha D_t^\alpha\left[\frac{\partial\varphi(z,t)}{\partial z}\right]\right\}, \tag{A}$$

where $D_t^\alpha[\cdot]$ is notation of the fractional derivative operator defined by the following expression:

$$D_t^\alpha\left[y\frac{\partial\varphi(z,t)}{\partial z}\right] = \frac{y}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{[\partial\varphi(z,\tau)/\partial z]}{(t-\tau)^\alpha}d\tau = yD_t^\alpha\left[\frac{\partial\varphi(z,t)}{\partial z}\right], \tag{B}$$

where α is ratio number from interval $0<\alpha<1$; $\sigma_z(z,y,t)$ is normal stress in the point of cross-section of the line element, at distance z from the left beam end and at point with distance y from neutral axis—bending beam axis; $\varphi(z,t)$ is turn angle of the beam cross-section for pure bending; and $\varepsilon_z(z,y,t) = y(\partial\varphi(z,t)/\partial z)$ is dilatation of the line element.

3. Partial fractional differential equation

The formulation of the problem of stochastic stability of nonhomogenous creeping bars of a fractional order derivative constitutive relation of material is assumed to be a continuous function of the length coordinate. Let us consider the problem on transversal oscillations of two-layer straight bar, which is under the action of the lengthwise random forces. The excitation process is a bounded noise excitation.

It is assumed that the layers of the bar were made of continuously creeping nonhomogenous material and the corresponding modulus of elasticity and creeping fractional order derivative constitutive relation of each layer are continuous functions of the length coordinate and thickness coordinates and are changed under the following laws (see Figure 3.1):

$$\begin{aligned}
E_e^{(1)}(z,y) &= E_0^{(1)}f_e^{(1)}(z)f_e^{(11)}(y),\\
E_e^{(2)}(z,y) &= E_0^{(2)}f_e^{(2)}(z)f_e^{(22)}(y),\\
E_\alpha^{(1)}(z,y) &= E_{0\alpha}^{(1)}f_\alpha^{(1)}(z)f_\alpha^{(11)}(y),\\
E_\alpha^{(2)}(z,y) &= E_{0\alpha}^{(2)}f_\alpha^{(2)}(z)f_\alpha^{(22)}(y),\\
0\le\alpha\le 1,\qquad 0\le z&\le \ell,\qquad -h_1\le y\le h_2.
\end{aligned} \tag{3.1}$$

In this case connection between increments of stresses and deformations in each layer is represented in view:

$$\begin{aligned}
\Delta\sigma_z^{(1)} &= E_e^{(1)}\Delta\varepsilon_z^{(1)} + E_\alpha^{(1)}D_t^\alpha[\Delta\varepsilon_z^{(1)}],\quad -h_1\le y\le 0,\\
\Delta\sigma_z^{(2)} &= E_e^{(2)}\Delta\varepsilon_z^{(2)} + E_\alpha^{(2)}D_t^\alpha[\Delta\varepsilon_z^{(2)}],\quad 0\le y\le h_2,
\end{aligned} \tag{3.2}$$

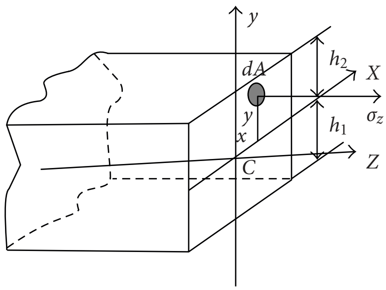

Figure 3.1

where

$$D_t^{\alpha}[\varepsilon_z(t)] = \frac{d^{\alpha}\varepsilon_z(t)}{dt^{\alpha}} = \varepsilon_z^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varepsilon_z(\tau)}{(t-\tau)^{\alpha}}d\tau. \tag{3.3}$$

Here h_1 and h_2 are thicknesses of the corresponding layers.

Dilatations are

$$\varepsilon_z = y\frac{\partial\varphi(z,t)}{\partial z}, \qquad \Delta\varepsilon_z = y\frac{\partial\Delta\varphi(z,t)}{\partial z}, \tag{3.4}$$

where $\varphi(z,t)$ is angle of pure bending. The normal stress of pure bending is

$$\begin{aligned} d\sigma_z^{(1)} &= E_0^{(1)} f_e^{(1)}(z) f_e^{(11)}(y)\,dy\frac{\partial\varphi(z,t)}{\partial z} + E_{0\alpha}^{(1)} f_\alpha^{(1)}(z) f_\alpha^{(11)}(y) D_t^{\alpha}\left[dy\frac{\partial\varphi(z,t)}{\partial z}\right], \quad -h_1 \le y \le 0,\\ d\sigma_z^{(2)} &= E_0^{(2)} f_e^{(2)}(z) f_e^{(22)}(y)\,dy\frac{\partial\varphi(z,t)}{\partial z} + E_{0\alpha}^{(2)} f_\alpha^{(2)}(z) f_\alpha^{(22)}(y) D_t^{\alpha}\left[dy\frac{\partial\varphi(z,t)}{\partial z}\right], \quad 0 \le y \le h_2. \end{aligned} \tag{3.5}$$

From the equilibrium conditions we can write

$$\sum_{i=1}^{N}\vec{F}_i = 0, \qquad \sum_{i=1}^{N}\vec{M}_0^{\vec{F}_i} = \vec{M}_{fx} = \vec{M}_{fx}^{\vec{F}_i} \tag{3.6}$$

or

$$\begin{gathered} \iint_{A''}\sigma_z^{(1)}dx\,dy + \iint_{A''}\sigma_z^{(2)}dx\,dy = 0, \qquad \iint_{A''}\sigma_z^{(1)}x\,dx\,dy + \iint_{A''}\sigma_z^{(2)}x\,dx\,dy \cong 0,\\ \iint_{A''}\sigma_z^{(1)}y\,dx\,dy + \iint_{A''}\sigma_z^{(2)}y\,dx\,dy = M_{fx} \end{gathered} \tag{3.7}$$

or

$$\begin{aligned} &\int_{-b/2}^{b/2}\int_{-h_1}^{0}\left\{E_0^{(1)} f_e^{(1)}(z) f_e^{(11)}(y)\,y\frac{\partial\varphi(z,t)}{\partial z} + E_{0\alpha}^{(1)} f_\alpha^{(1)}(z) f_\alpha^{(11)}(y) D_t^{\alpha}\left[y\frac{\partial\varphi(z,t)}{\partial z}\right]\right\}dx\,dy\\ &\quad -\int_{-b/2}^{b/2}\int_{0}^{h_2}\left\{E_0^{(2)} f_e^{(2)}(z) f_e^{(22)}(y)\,y\frac{\partial\varphi(z,t)}{\partial z}\right.\\ &\qquad\qquad \left. + E_{0\alpha}^{(2)} f_\alpha^{(2)}(z) f_\alpha^{(22)}(y) D_t^{\alpha}\left[y\frac{\partial\varphi(z,t)}{\partial z}\right]\right\}dx\,dy = 0, \end{aligned}$$

$$\int_{-b/2}^{b/2}\int_{-h_1}^{0}\left\{E_0^{(1)}f_e^{(1)}(z)f_e^{(11)}(y)y^2\frac{\partial\varphi(z,t)}{\partial z}+E_{0\alpha}^{(1)}f_\alpha^{(1)}(z)f_\alpha^{(11)}(y)D_t^\alpha\left[y^2\frac{\partial\varphi(z,t)}{\partial z}\right]\right\}dx\,dy$$
$$-\int_{-b/2}^{b/2}\int_{0}^{h_2}\left\{E_0^{(2)}f_e^{(2)}(z)f_e^{(22)}(y)y^2\frac{\partial\varphi(z,t)}{\partial z}\right.$$
$$\left.+E_{0\alpha}^{(2)}f_\alpha^{(2)}(z)f_\alpha^{(22)}(y)D_t^\alpha\left[y^2\frac{\partial\varphi(z,t)}{\partial z}\right]\right\}dx\,dy=M_{fx}. \tag{3.8}$$

If we introduce the following notations:

$$\begin{aligned}
a_e^{(1)(1)}&=\int_{-h_1}^{0}f_e^{(11)}(y)y\,dy, & a_e^{(2)(1)}&=\int_{0}^{h_2}f_e^{(22)}(y)y\,dy,\\
a_\alpha^{(1)(1)}&=\int_{-h_1}^{0}f_\alpha^{(11)}(y)y\,dy, & a_\alpha^{(2)(1)}&=\int_{0}^{h_2}f_\alpha^{(22)}(y)y\,dy,\\
a_e^{(1)(2)}&=\int_{-h_1}^{0}f_e^{(11)}(y)y^2\,dy, & a_e^{(2)(2)}&=\int_{0}^{h_2}f_e^{(22)}(y)y^2\,dy,\\
a_\alpha^{(1)(2)}&=\int_{-h_1}^{0}f_\alpha^{(11)}(y)y^2\,dy, & a_\alpha^{(2)(2)}&=\int_{0}^{h_2}f_\alpha^{(22)}(y)y^2\,dy,
\end{aligned} \tag{3.9}$$

or set the following form:

$$a_\alpha^{(1)(n)}=\int_{-h_1}^{0}f_\alpha^{(11)}(y)y^n\,dy,\quad a_\alpha^{(2)(n)}=\int_{0}^{h_2}f_\alpha^{(22)}(y)y^n\,dy,\quad n=0,1,2, \tag{3.10}$$

we can write the previous equilibrium conditions in the following relations:

$$\begin{aligned}
a_e^{(1)(1)}E_0^{(1)}f_e^{(1)}(z)-a_e^{(2)(1)}E_0^{(2)}f_e^{(2)}(z)=0&\Longrightarrow f_e^{(2)}(z)=f_e^{(1)}(z)\frac{E_0^{(1)}}{E_0^{(2)}}\frac{a_e^{(1)(1)}}{a_e^{(2)(1)}},\\
a_\alpha^{(1)(1)}E_{0\alpha}^{(1)}f_\alpha^{(1)}(z)-a_\alpha^{(2)(1)}E_{0\alpha}^{(2)}f_\alpha^{(2)}(z)=0&\Longrightarrow f_\alpha^{(2)}(z)=f_\alpha^{(1)}(z)\frac{E_{0\alpha}^{(1)}}{E_{0\alpha}^{(2)}}\frac{a_\alpha^{(1)(1)}}{a_\alpha^{(2)(1)}}
\end{aligned} \tag{3.11}$$

and write the following expression for bending moment:

$$\begin{aligned}
M_{fx}(z,t)&=b\frac{\partial\varphi(z,t)}{\partial z}\left\{E_0^{(1)}a_e^{(1)(2)}f_e^{(1)}(z)+E_0^{(2)}a_e^{(2)(2)}f_e^{(2)}(z)\right\}\\
&\quad+bD_t^\alpha\left[\frac{\partial\varphi(z,t)}{\partial z}\right]\left\{E_{0\alpha}^{(1)}a_\alpha^{(1)(2)}f_\alpha^{(1)}(z)+E_{0\alpha}^{(2)}a_\alpha^{(2)(2)}f_\alpha^{(2)}(z)\right\},
\end{aligned} \tag{3.12}$$

also with respect to the relations (3.10) we can write (3.12) in the following form:

$$
\begin{aligned}
M_{fx}(z,t) = {} & E_0^{(1)} b \frac{\partial \varphi(z,t)}{\partial z} f_e^{(1)}(z) \left\{ a_e^{(1)(2)} + a_e^{(2)(2)} \frac{a_e^{(1)(1)}}{a_e^{(2)(1)}} \right\} \\
& + E_{0\alpha}^{(1)} b f_\alpha^{(1)}(z) D_t^\alpha \left[\frac{\partial \varphi(z,t)}{\partial z} \right] \left\{ a_\alpha^{(1)(2)} + a_\alpha^{(2)(2)} \frac{a_\alpha^{(1)(1)}}{a_\alpha^{(2)(1)}} \right\}.
\end{aligned}
\tag{3.13}
$$

We take into account the rotatory inertia of cross-section and we can write the following equations of bar dynamics:

$$
\begin{aligned}
dJ_x \frac{\partial^2 \varphi(z,t)}{\partial t^2} &= -dM_f(z,t) + F_T(z,t)dz + F_N(\Xi,z,t)dv(z,t), \\
dm \frac{\partial^2 v(z,t)}{\partial t^2} &= dF_T(z,t).
\end{aligned}
\tag{3.14}
$$

If we introduce

$$
dm = \rho_1 A_1 + \rho_2 A_2, \qquad d\mathbf{J}_x = \left[\rho_1 \mathbf{I}_{\mathbf{x}}^{(\mathbf{1})} + \rho_2 \mathbf{I}_{\mathbf{x}}^{(2)} \right] dz,
\tag{3.15}
$$

we can write

$$
\begin{aligned}
& [\rho_1 \mathbf{I}_x^{(1)} + \rho_2 \mathbf{I}_x^{(2)}] \frac{\partial^2 \varphi(z,t)}{\partial t^2} \\
& \quad = E_0^{(1)} b \left[a_e^{(1)(2)} + a_e^{(2)(2)} \frac{a_e^{(1)(1)}}{a_e^{(2)(1)}} \right] \frac{\partial}{\partial z} \left[\frac{\partial \varphi(z,t)}{\partial z} f_e^{(1)}(z) \right] \\
& \quad\quad + E_{0\alpha}^{(1)} b \left[a_\alpha^{(1)(2)} + a_\alpha^{(2)(2)} \frac{a_\alpha^{(1)(1)}}{a_\alpha^{(2)(1)}} \right] \frac{\partial}{\partial z} \left\{ f_\alpha^{(1)}(z) D_t^\alpha \left[\frac{\partial \varphi(z,t)}{\partial z} \right] \right\} + \mathbf{F}_T + F_N \frac{\partial v(z,t)}{\partial z}, \\
& (\rho_1 A_1 + \rho_2 A_2) \frac{\partial^2 v(z,t)}{\partial t^2} = \frac{\partial F_T(z,t)}{\partial z}.
\end{aligned}
\tag{3.16}
$$

After applying derivative with respect to time, we can write

$$
\frac{\partial \varphi(z,t)}{\partial z} = \frac{\partial^2 v(z,t)}{\partial z^2}, \qquad \frac{\partial^3 \varphi(z,t)}{\partial z \partial t^2} = \frac{\partial^4 v(z,t)}{\partial z^2 \partial t^2},
\tag{3.17}
$$

$$
\begin{aligned}
& [\rho_1 \mathbf{I}_x^{(1)} + \rho_2 \mathbf{I}_x^{(2)}] \frac{\partial^3 \varphi(z,t)}{\partial t^2 \partial z} \\
& \quad = E_0^{(1)} b \left[a_e^{(1)(2)} + a_e^{(2)(2)} \frac{a_e^{(1)(1)}}{a_e^{(2)(1)}} \right] \frac{\partial^2}{\partial z^2} \left[\frac{\partial \varphi(z,t)}{\partial z} f_e^{(1)}(z) \right] \\
& \quad\quad + E_{0\alpha}^{(1)} b \left[a_\alpha^{(1)(2)} + a_\alpha^{(2)(2)} \frac{a_\alpha^{(1)(1)}}{a_\alpha^{(2)(1)}} \right] \frac{\partial^2}{\partial z^2} \left\{ f_\alpha^{(1)}(z) D_t^\alpha \left[\frac{\partial \varphi(z,t)}{\partial z} \right] \right\} \\
& \quad\quad + \frac{\partial \mathbf{F}_T}{\partial z} + \frac{\partial}{\partial z} \left[F_N \frac{\partial v(z,t)}{\partial z} \right].
\end{aligned}
\tag{3.18}
$$

By introducing derivatives (3.17) into (3.18) we obtain the following partial fractional differential equation:

$$\begin{aligned}
&[\rho_1\mathbf{I}_x^{(1)}+\rho_2\mathbf{I}_x^{(2)}]\frac{\partial^4 v(z,t)}{\partial t^2\partial z^2}\\
&\quad=E_0^{(1)}b\left[a_e^{(1)(2)}+a_e^{(2)(2)}\frac{a_e^{(1)(1)}}{a_e^{(2)(1)}}\right]\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2}f_e^{(1)}(z)\right]\\
&\qquad+E_{0\alpha}^{(1)}b\left[a_\alpha^{(1)(2)}+a_\alpha^{(2)(2)}\frac{a_\alpha^{(1)(1)}}{a_\alpha^{(2)(1)}}\right]\frac{\partial^2}{\partial z^2}\left\{f_\alpha^{(1)}(z)D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\}\\
&\qquad+(\rho_1A_1+\rho_2A_2)\frac{\partial^2 v(z,t)}{\partial t^2}+\frac{\partial}{\partial z}\left[F_N\frac{\partial v(z,t)}{\partial z}\right],\\
&\frac{\partial^2 v(z,t)}{\partial t^2}+\frac{E_0^{(1)}b[a_e^{(1)(2)}+a_e^{(2)(2)}(a_e^{(1)(1)}/a_e^{(2)(1)})]}{(\rho_1A_1+\rho_2A_2)}\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2}f_e^{(1)}(z)\right]\\
&\qquad+\frac{1}{(\rho_1A_1+\rho_2A_2)}\frac{\partial}{\partial z}\left[F_N\frac{\partial v(z,t)}{\partial z}\right]+\frac{E_{0\alpha}^{(1)}b[a_\alpha^{(1)(2)}+a_\alpha^{(2)(2)}(a_\alpha^{(1)(1)}/a_\alpha^{(2)(1)})]}{(\rho_1A_1+\rho_2A_2)}\\
&\qquad\times\frac{\partial^2}{\partial z^2}\left\{f_\alpha^{(1)}(z)D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\}+\frac{[\rho_1\mathbf{I}_x^{(1)}+\rho_2\mathbf{I}_x^{(2)}]}{(\rho_1A_1+\rho_2A_2)}\frac{\partial^4 v(z,t)}{\partial t^2\partial z^2}=0.
\end{aligned}\tag{3.19}$$

By introducing the following notations:

$$\begin{gathered}
\tilde{c}_{0x}^2=\frac{\tilde{E}_0^{(1)}}{\rho}\tilde{i}_{xe}^2=\frac{E_0^{(1)}b[a_e^{(1)(2)}+a_e^{(2)(2)}(a_e^{(1)(1)}/a_e^{(2)(1)})]}{(\rho_1A_1+\rho_2A_2)},\\
\tilde{c}_{0x\alpha}^2=\frac{\tilde{E}_{0\alpha}^{(1)}}{\rho}\tilde{i}_{x\alpha}^2=\frac{E_{0\alpha}^{(1)}b[a_\alpha^{(1)(2)}+a_\alpha^{(2)(2)}(a_\alpha^{(1)(1)}/a_\alpha^{(2)(1)})]}{(\rho_1A_1+\rho_2A_2)},\\
\tilde{i}_x^2=\frac{[\rho_1\mathbf{I}_x^{(1)}+\rho_2\mathbf{I}_x^{(2)}]}{(\rho_1A_1+\rho_2A_2)},\qquad \hat{i}_x^2=\frac{[\rho_1\mathbf{I}_x^{(1)}+\rho_2\mathbf{I}_x^{(2)}]}{A\rho},\\
\tilde{i}_{xe}^2=\frac{b[a_e^{(1)(2)}+a_e^{(2)(2)}(a_e^{(1)(1)}/a_e^{(2)(1)})]}{A},\qquad \tilde{i}_{x\alpha}^2=\frac{b[a_\alpha^{(1)(2)}+a_\alpha^{(2)(2)}(a_\alpha^{(1)(1)}/a_\alpha^{(2)(1)})]}{A},
\end{gathered}\tag{3.20}$$

we obtain the following partial fractional differential equation of transversal vibrations of creeping of two-layer straight bar, which is under the action of the lengthwise random forces:

$$\begin{aligned}
&\frac{\partial^2 v(z,t)}{\partial t^2}+\tilde{c}_{0x}^2\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2}f_e^{(1)}(z)\right]+\frac{1}{(\rho_1A_1+\rho_2A_2)}\frac{\partial}{\partial z}\left[F_N\frac{\partial v(z,t)}{\partial z}\right]\\
&\qquad+\tilde{c}_{0x\alpha}^2\frac{\partial^2}{\partial z^2}\left\{f_\alpha^{(1)}(z)D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\}-\tilde{i}_x^2\frac{\partial^4 v(z,t)}{\partial t^2\partial z^2}=0.
\end{aligned}\tag{3.21}$$

We study the following special case: from (3.21), we exclude members which contain axial forces, or we suppose that axial forces are equal to zero, and we solve the following equation:

$$\frac{\partial^2 v(z,t)}{\partial t^2}+\tilde{c}_{0x}^2\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2}f_e^{(1)}(z)\right]+\tilde{c}_{0x\alpha}^2\frac{\partial^2}{\partial z^2}\left\{f_\alpha^{(1)}(z)D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\}-\tilde{i}_x^2\frac{\partial^4 v(z,t)}{\partial t^2\partial z^2}=0 \tag{3.22}$$

when

$$f_e^{(1)}(z)=f_\alpha^{(1)}(z)=f(z) \tag{3.23}$$

and we can write

$$\frac{\partial^2 v(z,t)}{\partial t^2}+\tilde{c}_{0x}^2\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2}f(z)\right]+\tilde{c}_{0x\alpha}^2\frac{\partial^2}{\partial z^2}\left\{f(z)D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\}-\tilde{i}_x^2\frac{\partial^4 v(z,t)}{\partial t^2\partial z^2}=0. \tag{3.24}$$

4. Solution of the partial fractional differential equation of the beam transversal vibrations with creep material properties

By using Bernoulli's method for obtaining solution and for solution of the partial fractional differential equation (3.24), we can write a product of the two functions depending on separate coordinate z and time t in the following form:

$$v(z,t)=Z(z)T(t). \tag{4.1}$$

By introducing this solution into (3.24) we obtain

$$Z(z)\ddot{T}(t)+\tilde{c}_{0x}^2T(t)\frac{d^2}{dz^2}[Z''(z)f(z)]+\tilde{c}_{0x\alpha}^2\frac{d^2}{dz^2}\{Z''(z)f(z)\}D_t^\alpha[T(t)]-\tilde{i}_x^2Z''(z)\ddot{T}(t)=0 \tag{4.2}$$

or

$$Z(z)+\frac{d^2}{dz^2}[Z''(z)f(z)]\left\{\tilde{c}_{0x}^2\frac{T(t)}{\ddot{T}(t)}+\tilde{c}_{0x\alpha}^2\frac{1}{\ddot{T}(t)}D_t^\alpha[T(t)]\right\}-\tilde{i}_x^2Z''(z)=0 \tag{4.3}$$

or we obtain two equations

$$\begin{gathered}\tilde{c}_{0x}^2\frac{T(t)}{\ddot{T}(t)}+\tilde{c}_{0x\alpha}^2\frac{1}{\ddot{T}(t)}D_t^\alpha[T(t)]=-\frac{1}{k^4},\\ Z(z)-\frac{d^2}{dz^2}[Z''(z)f(z)]\frac{1}{k^4}-\tilde{i}_x^2Z''(z)=0\end{gathered} \tag{4.4}$$

or

$$\ddot{T}(t)+\tilde{\omega}_{\alpha x}^2D_t^\alpha[(t)]+\tilde{\omega}_{0x}^2T(t)=0, \tag{4.5}$$

$$\frac{d^2}{dz^2}[Z''(z)f(z)]+\tilde{i}_x^2k^4Z''(z)-k^4Z(z)=0, \tag{4.6}$$

where

$$\tilde{\omega}_{0x}^2 = k^4\tilde{c}_{0x}^2 = k^4\frac{E_0^{(1)}b\big[a_e^{(1)(2)} + a_e^{(2)(2)}\big(a_e^{(1)(1)}/a_e^{(2)(1)}\big)\big]}{(\rho_1A_1+\rho_2A_2)},$$
$$\tilde{\omega}_{\alpha x}^2 = k^4\tilde{c}_{0x\alpha}^2 = k^4\frac{E_{0\alpha}^{(1)}b\big[a_\alpha^{(1)(2)} + a_\alpha^{(2)(2)}\big(a_\alpha^{(1)(1)}/a_\alpha^{(2)(1)}\big)\big]}{(\rho_1A_1+\rho_2A_2)}, \tag{4.7}$$
$$\tilde{i}_x^2 = \frac{\big[\rho_1\mathbf{I}_x^{(1)} + \rho_2\mathbf{I}_x^{(2)}\big]}{(\rho_1A_1+\rho_2A_2)}.$$

We can obtain the solution of the fractional differential equation of system (4.5) by using the Laplace transform, and by having in mind that, in initial moment, $d^{\alpha-1}T(t)/dt^{\alpha-1}|_{t=0} = 0$. Solutions for special cases when $\alpha = 0$ and $\alpha = 1$, and for beam kinetic parameters: $\omega_0 > (1/2)\omega_1^2$ for soft creep and $\omega_0 < (1/2)\omega_1^2$ for strong creep, are solutions of classical ordinary differential equation. It is the same for $\alpha = 1$ and $\omega_{0x} = (1/2)\omega_{1x}^2$.

For the general case when $\omega_{0x}^2 \neq 0$, the Laplace transform of the solution $L\{T(t)\}$ of the fractional differential equation of system (4.5) can be developed, in two steps, into series with respect to binoms $(p^\alpha + \omega_{0x}^2/\omega_{\alpha x}^2)$, and with respect to p^α. Then we obtain the following expression:

$$L\{T(t)\} = \left(T_0 + \frac{\dot{T}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{\alpha x}^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{p^{\alpha j}\omega_{\alpha x}^{2(j-k)}}{\omega_{ox}^{2j}}. \tag{4.8}$$

By using inverse of the Laplace transform of the solution $L\{T(t)\}$, for general case, when beam material parameter is from interval $0 \le \alpha \le 1$, for solution of the fractional-differential equation of system (4.5), we obtain the following expression in the form of potential series of the time t:

$$\begin{aligned} T(t) &= L^{-1}\{T(t)\} \\ &= \sum_{k=0}^{\infty}(-1)^k\omega_{\alpha x}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{\alpha x}^{2j}t^{-\alpha j}}{\omega_{ox}^{2j}}\left[\frac{T_0}{\Gamma(2k+1-\alpha j)} + \frac{\dot{T}_0 t}{\Gamma(2k+2-\alpha j)}\right]. \end{aligned} \tag{4.9}$$

And in that case there are special cases when $\omega_{0x}^2 = 0$ for $\alpha = 0$ and for $\alpha = 1$.

In Figure 4.1 numerical simulations and graphical presentation of the solution of the fractional differential equation of system (4.5) are presented. Time functions $T(t,\alpha)$ surfaces for the different beam transversal vibrations kinetic and creep material parameters in the space $(T(t,\alpha), t, \alpha)$ for interval $0 \le \alpha \le 1$ are visible in (a) for $(\omega_{\alpha x}/\omega_{0x}) = 1$, (b) for $(\omega_{\alpha x}/\omega_{0x}) = 1/4$, (c) for $(\omega_{\alpha x}/\omega_{0x}) = 1/3$, and (d) for $(\omega_{\alpha x}/\omega_{0x}) = 3$.

In Figure 4.2 the time functions $T(t,\alpha)$ surfaces and curves families for the different beam transversal vibrations kinetic and discrete values of the creeping material parameters $0 \le \alpha \le 1$ are presented in (a) and (c) for $(\omega_{\alpha x}/\omega_{0x}) = 1$, (b) and (d) for $(\omega_{\alpha x}/\omega_{0x}) = 1/4$, (e) for $(\omega_{\alpha x}/\omega_{0x}) = 1/3$, and (f) for $(\omega_{\alpha x}/\omega_{0x}) = 3$.

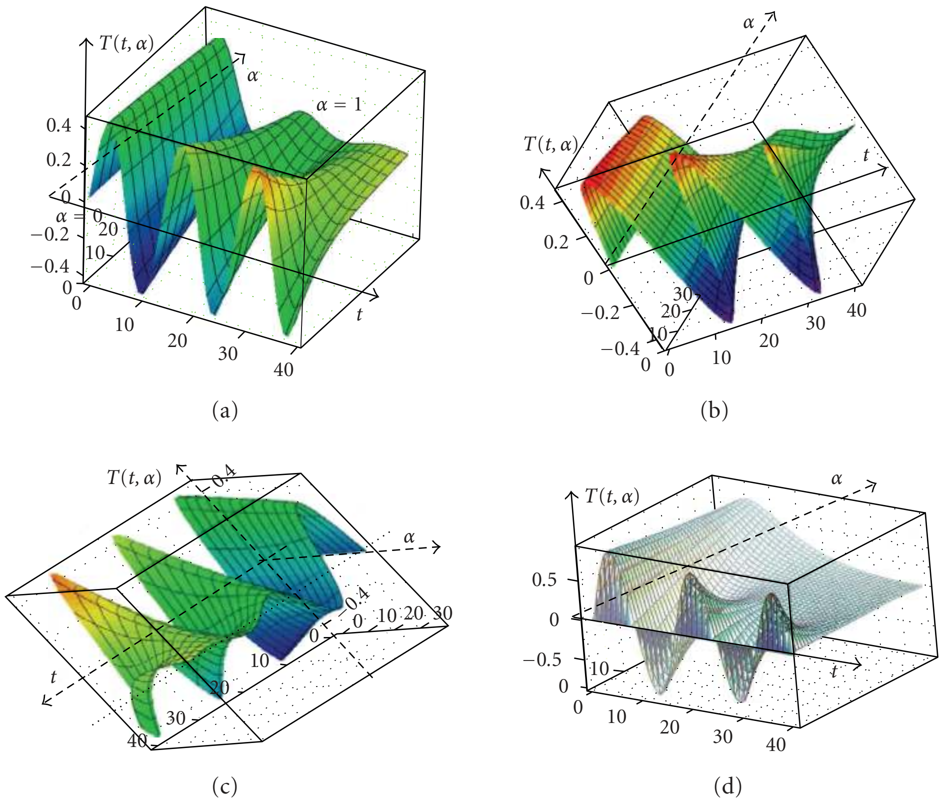

Figure 4.1. Numerical simulations and graphical presentation of the results. Time functions $T(t,\alpha)$ surfaces for the different beam transversal vibrations kinetic and creep material parameters: (a) $(\omega_{\alpha x}/\omega_{0x}) = 1$, (b) $(\omega_{\alpha x}/\omega_{0x}) = 1/4$, (c) $(\omega_{\alpha x}/\omega_{0x}) = 1/3$, and (d) $(\omega_{\alpha x}/\omega_{0x}) = 3$.

5. The S. T. Ariaratnam idea applied to the stochastic stability of the creep beam transversal vibrations dynamic shapes under axial bounded noise excitation

In the case

$$f(z) = 1 + m\frac{z}{\ell} \tag{5.1}$$

equation (3.21) transforms to the form

$$\begin{aligned}\frac{\partial^2 v(z,t)}{\partial t^2} &+ \tilde{c}_{0x}^2 \frac{\partial^2}{\partial z^2}\left[\frac{\partial^2 v(z,t)}{\partial z^2} f_e^{(1)}(z)\right] + \frac{1}{(\rho_1 A_1 + \rho_2 A_2)}\frac{\partial}{\partial z}\left[F_N \frac{\partial v(z,t)}{\partial z}\right] \\ &+ \tilde{c}_{0x\alpha}^2 \frac{\partial^2}{\partial z^2}\left\{f_\alpha^{(1)}(z) D_t^\alpha\left[\frac{\partial^2 v(z,t)}{\partial z^2}\right]\right\} - \tilde{i}_x^2 \frac{\partial^4 v(z,t)}{\partial t^2 \partial z^2} = 0\end{aligned} \tag{5.2}$$

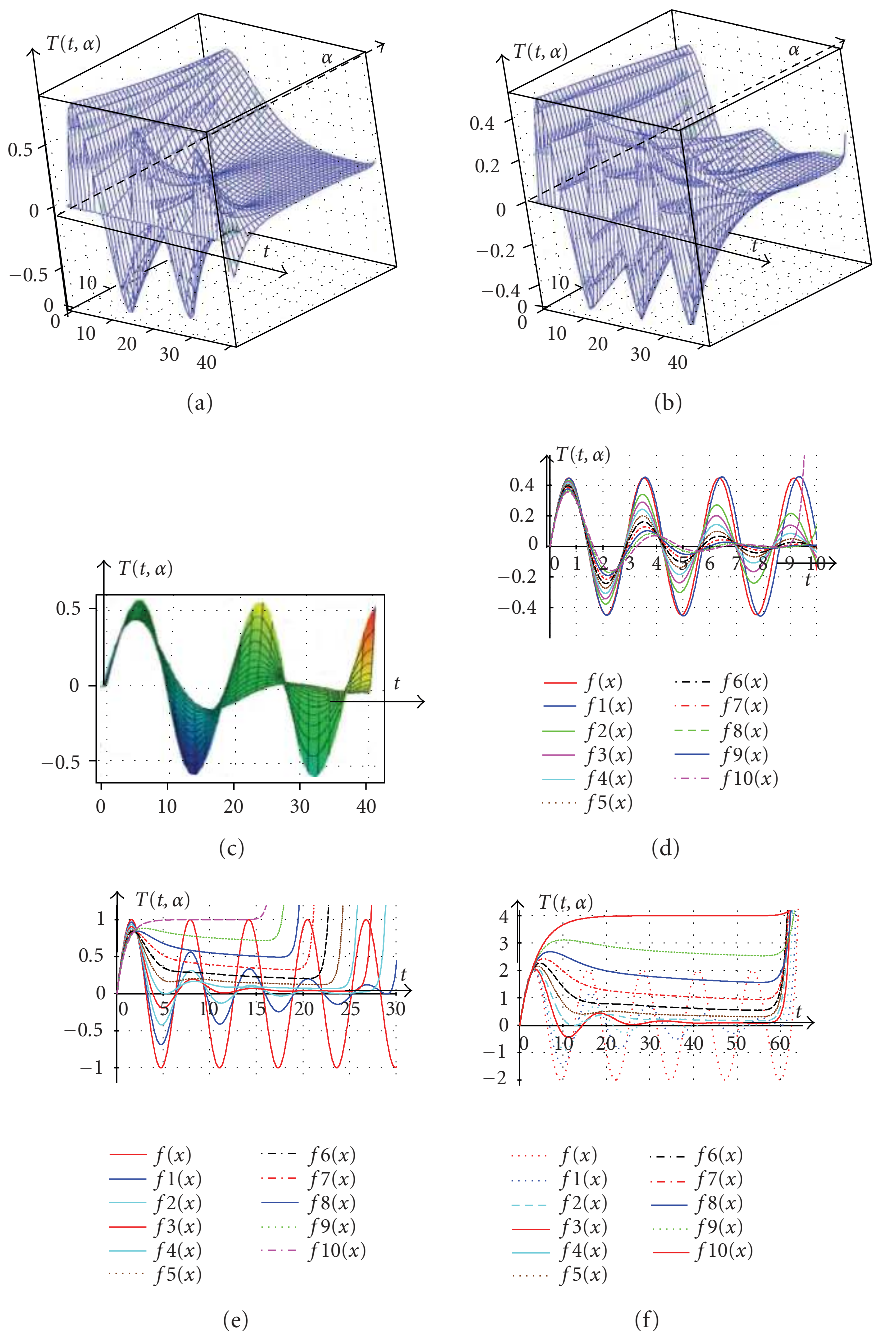

Figure 4.2. Numerical simulations and graphical presentation of the results. Time functions $T(t,\alpha)$ surfaces and curves families for the different beam transversal vibrations kinetic and discrete values of the creeping material parameters $0 \le \alpha \le 1$: (a) and (c) $(\omega_{\alpha x}/\omega_{0x}) = 1$, (b) and (d) $(\omega_{\alpha x}/\omega_{0x}) = 1/4$, (e) $(\omega_{\alpha x}/\omega_{0x}) = 1/3$, and (f) $(\omega_{\alpha x}/\omega_{0x}) = 3$.

and (4.6) transforms to the form

$$\frac{d^2}{dz^2}\left[Z''(z)\left(1+m\frac{z}{\ell}\right)\right]+\tilde{i}_x^2k^4Z''(z)-k^4Z(z)=0. \tag{5.3}$$

In the case of hinge fixing of the ends of the bar, solution of (5.1) into first unperturbed form is found in a view

$$v(z,t)=T(t)\sin\frac{n\pi z}{\ell}. \tag{5.4}$$

By introducing (5.4) into (5.2) and applying the Bubnov-Galerkin method, we obtain the following equations:

$$\begin{aligned}
&\ddot{T}(t)\sin\frac{n\pi z}{\ell}+\tilde{c}_{0x}^2T(t)\frac{d^2}{dz^2}\left\{-\left(\frac{n\pi}{\ell}\right)^2\left(1+m\frac{z}{\ell}\right)\sin\frac{n\pi z}{\ell}\right\}\\
&\quad-\frac{F_N(t)}{(\rho_1A_1+\rho_2A_2)}\left(\frac{n\pi}{\ell}\right)^2T(t)\sin\frac{n\pi z}{\ell}+\tilde{c}_{0x\alpha}^2\frac{d^2}{dz^2}\left\{-\left(\frac{n\pi}{\ell}\right)^2\left(1+m\frac{z}{\ell}\right)\sin\frac{n\pi z}{\ell}\right\}D_t^\alpha[T(t)]\\
&\quad+\tilde{i}_x^2\left(\frac{n\pi}{\ell}\right)^2\ddot{T}(t)\sin\frac{n\pi z}{\ell}=0,\\
&\ddot{T}(t)+\frac{\tilde{c}_{0x}^2(n\pi/\ell)^4[1+m/2]}{[1+\tilde{i}_x^2(n\pi/\ell)^2]}\left\{1-\frac{\tilde{F}_N(t)(n\pi/\ell)^2}{\tilde{c}_{0x}^2(n\pi/\ell)^4[1+m/2]}\right\}T(t)\\
&\quad+\frac{\tilde{c}_{0x\alpha}^2(n\pi/\ell)^4[1+m/2]}{[1+\tilde{i}_x^2(n\pi/\ell)^2]}D_t^\alpha[T(t)]=0.
\end{aligned} \tag{5.5}$$

We pointed out the following notations:

$$\begin{aligned}
\tilde{\omega}_{0xn}^2&=\frac{\tilde{c}_{0x}^2(n\pi/\ell)^4[1+m/2]}{[1+\tilde{i}_x^2(n\pi/\ell)^2]},\\
h_{xn}\xi(t)&=\frac{\tilde{F}_N(t)}{\tilde{c}_{0x}^2(n\pi/\ell)^2[1+m/2]},\\
\tilde{\omega}_{0x\alpha n}^2&=\frac{\tilde{c}_{0x\alpha}^2(n\pi/\ell)^4[1+m/2]}{[1+\tilde{i}_x^2(n\pi/\ell)^2]}
\end{aligned} \tag{5.6}$$

and we obtain the following fractional differential equation with respect to the time function:

$$\ddot{T}(t)+\tilde{\omega}_{0xn}^2\{1-h_{xn}\xi(t)\}T(t)+\tilde{\omega}_{0x\alpha n}^2D_t^\alpha[T(t)]=0. \tag{5.7}$$

To solve the previous equation we can apply Ariaratnam's idea [1]. The random bounded noise axial excitation $\xi(t)$ is taken in the following form:

$$F(t)=F_0\xi(t)=F_0\sin[\Omega t+\sigma B(t)+\gamma], \tag{5.8}$$

where $B(t)$ is the standard Wiener process, and γ is a random uniformly distributed variable in interval $[0,2\pi]$, then $\xi(t)$ is a stationary process having autocorrelation function and spectral density function:

$$R(\tau,\Omega)=\frac{1}{2}h_{on}^2e^{-\sigma^2\tau/2}\cos\Omega\tau,$$
$$S(\omega,\Omega)=\int_{-\infty}^{+\infty}R(\tau,\Omega)e^{i\omega\tau}d\tau=\frac{1}{2}h_{0n}\sigma^2\frac{\omega^2+\Omega^2+\sigma^2/4}{\left[(\omega^2-\Omega^2-\sigma^2/4)^2+\sigma^2\omega^2\right]}. \tag{5.9}$$

Stochastic process $|\xi(t)|\leq 1$ is bounded for all values of time t.

The next idea of Ariaratnam is to apply the averaging method, and for that reason we must introduce the amplitude $a_n(t)$ and the phase $\Phi_n(t)$, which are time unknown functions, by means of the transformation relation of $T_n(t)$:

$$T_n(t)=a_n(t)\cos\Phi_n(t),\qquad \dot{T}_n(t)=-a_n(t)\omega_{0n}\sin\Phi_n(t). \tag{5.10}$$

Substituting these relations in (5.7) and using (5.8) as well as $\Phi_n(t)=(\Omega/2)t+\tilde{\phi}_n(t)$ and $\Delta_n=\omega_{0xn}-\Omega/2$, we can write the following system fractional differential equation with respect to the amplitude $a_n(t)$ and the phase $\Phi_n(t)$, exactly equivalent to (5.7):

$$\dot{a}_n(t)=-\frac{1}{2}\omega_{0xn}a_n(t)h_{0n}\sin(\Omega t+\psi)\sin\left(\Omega t+2\tilde{\phi}_n\right)+\frac{\omega_{\alpha xn}^2}{\omega_{0xn}}\sin\Phi_n(t)D_t^{\alpha}\left[a_n(t)\cos\Phi_n(t)\right], \tag{5.11}$$

$$\begin{aligned}\dot{\tilde{\phi}}_n(t)&=\Delta_n(t)-\frac{1}{2}\omega_{0xn}h_{0n}\sin(\Omega t+\psi)\left[1+\cos\left(\Omega t+2\tilde{\phi}_n\right)\right]\\&\quad+\frac{\omega_{\alpha xn}^2}{a_n(t)\omega_{oxn}}\cos\Phi_n(t)D_t^{\alpha}\left[a_n(t)\cos\Phi_n(t)\right],\qquad \dot{\psi}(t)=\sigma\dot{B}(t),\end{aligned} \tag{5.12}$$

where

$$D_t^{\alpha}\left[a_n(t)\cos\Phi_n(t)\right]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{a_n(\tau)\cos\Phi_n(\tau)}{(t-\tau)^{\alpha}}d\tau. \tag{5.13}$$

By applying the averaging method, we assume that excitation and beam kinetic parameters values h_{0n}, $\omega_{\alpha xn}^2$, $\Delta_n(t)$ are small depending on small parameter ε (see [12, 16]) as $\beta_n=O(\varepsilon)$, $\Delta_n=O(\varepsilon)$ and $\mu=O(\varepsilon)$, where $0<\varepsilon\leq 1$. The assumption or the condition $\Delta_n=O(\varepsilon)$ shows that frequencies of external random bounded excitation Ω are in the vicinity of the frequency $2\omega_{0xn}$ of fundamental parametric resonance in the nth form of perturbed parametric resonance state.

We introduce the following notations:

$$\int_0^{+\infty}R(\tau)e^{i\omega\tau}d\tau=H_c(\omega)+iH_s(\omega),\quad \text{where the kernel is in the form } R(\tau)=\tau^{-\alpha}. \tag{5.14}$$

Having in consideration that (see [5])

$$D_t^{\alpha}[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}d\tau = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau + f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

$$D_t^{\alpha}[a_n(t)\cos\Phi_n(t)] = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{-a_n(\tau)\omega_{0xn}\sin\Phi_n(\tau)}{(t-\tau)^{\alpha}}d\tau + a_n(0^+)\cos\Phi_n(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{5.15}$$

and after averaging the right-hand side of (5.12) with respect to total phase Φ_n and taking that $\Omega t = 2\Phi_n - 2\widetilde{\phi}_n$, we obtain the averaged equations:

$$\begin{gathered}
\dot{a}_n(t) = -\frac{1}{4}\omega_{0xn}a_n(t)h_{0n}\cos(\psi - 2\widetilde{\phi}_n) - \frac{\omega_{\alpha xn}^2}{2\Gamma(1-\alpha)}a_n(t)H_{en}\left(\frac{\Omega}{2}\right),\\
\dot{\widetilde{\phi}}_n(t) = \Delta_n(t) - \frac{1}{4}\omega_{0xn}h_{on}\sin(\psi - 2\widetilde{\phi}_n) + \frac{\omega_{\alpha xn}^2}{2\Gamma(1-\alpha)}H_{en}\left(\frac{\Omega}{2}\right),\\
\dot{\psi}(t) = \sigma\dot{B}(t),
\end{gathered} \tag{5.16}$$

where $\lim_{T\to\infty}(1/T)\int_0^T e^{i\Phi_n(t)}t^{-\alpha}\,dt = 0$.

By introducing the change of the variables by the relations $\rho_n(t) = \ln a_n(t)$ and $\theta_n = \widetilde{\phi}_n - \psi/2$ into the previous pair of the stochastic differential equations, we obtain the system

$$\begin{aligned}
d\rho_n(t) &= \left[-\frac{1}{4}\omega_{0xn}h_{0n}\cos 2\theta_n - \frac{\omega_{\alpha xn}^2}{2\Gamma(1-\alpha)}(t)H_{en}\left(\frac{\Omega}{2}\right)\right]dt,\\
d\theta_n(t) &= \left[\Delta_n(t) + \frac{1}{4}\omega_{0xn}h_{0n}\sin 2\theta_n + \frac{\omega_{\alpha xn}^2}{2\Gamma(1-\alpha)}H_{sn}\left(\frac{\Omega}{2}\right)\right]dt - \frac{1}{2}\sigma dB(t).
\end{aligned} \tag{5.17}$$

6. The Lyapunov exponent and stochastic stability

The Lyapunov exponent (see [1]) of the creeping beam stochastic transversal vibrations in the nth form of perturbed parametric resonance state given by the averaged stochastic equations system (5.17) may be defined by the following expression:

$$\lambda_n = \lim_{t\to\infty}\frac{1}{2t}\ln\left\{[T_n(t)]^2 + \frac{1}{\omega_{on}^2}[\dot{T}_n(t)]^2\right\} \Longrightarrow \lambda_n = \lim_{t\to\infty}\frac{1}{t}\ln[a_n(t)] = \lim_{t\to\infty}\frac{1}{t}\rho_n(t). \tag{6.1}$$

Now, the Lyapunov exponent is a measure of the average exponential growth of the amplitude process $a_n(t)$ of the creep beam transversal vibrations in the nth form of perturbed parametric resonance process. λ_n is a deterministic number with probability one (w.p.1) for the system given by (5.17). Solutions of the averaged differential equations depending on initial values $T_n(t_0)$ and $\dot{T}_n(t_0)$, in general, are two values of the Lyapunov exponent λ_n in the corresponding nth form of perturbed parametric resonance process. If both Lyapunov exponents are negative, the trivial solution in the corresponding nth form of perturbed parametric resonance process are stable processes.

In order to calculate λ_n, we must integrate both sides of (5.17) and we obtain the following expression:

$$\lambda_n = \lim_{t\to\infty} \frac{1}{t}\rho(t) = -\frac{1}{4}\omega_{0xn}h_{0n}E[\cos 2\theta_n] - \frac{1}{2}\frac{\omega^2_{\alpha xn}}{\Gamma(1-\alpha)}H_{en}\left(\frac{\Omega}{2}\right). \tag{6.2}$$

By using corresponding results obtained by Stratonovich [35] and Ariaratnam [1] for the Lyapunov exponent we obtain the following asymptotic result:

$$\lambda_n = -\frac{1}{4}\omega_{oxn}h_{on}\mathbf{F}\left(\frac{h_{on}\omega_{oxn}}{\sigma^2}, \frac{4\Delta_n}{\sigma^2}\right) - \frac{1}{2}\frac{\omega^2_{\alpha xn}}{\Gamma(1-\alpha)}H_{en}\left(\frac{\Omega}{2}\right). \tag{6.3}$$

In the previous expressions and calculations we used invariant (stationary) probability density function satisfying the periodicity condition when

$$\Delta_{0n}(t) = \omega_{on} - \frac{\Omega}{2} + \frac{1}{2}\frac{\omega^2_{\alpha xn}}{\Gamma(1-\alpha)}H_{sn}\left(\frac{\Omega}{2}\right) = 0. \tag{6.4}$$

For the Lyapunov exponent we obtain

$$\lambda_n = -\frac{1}{4}h_{0n}\omega_{0xn}\frac{I_1(h_{0n}\omega_{0xn}/\sigma^2)}{I_0(h_{0n}\omega_{0xn}/\sigma^2)} - \frac{\omega^2_{\alpha xnn}}{2\Gamma(1-\alpha)}H_{cn}\left(\frac{\Omega}{2}\right), \tag{6.5}$$

where I_0, I_1 are Bessel functions of real argument, and $\mathbf{F}(\nu \cdot q)$ is a function of Bessel functions of imaginary argument.

7. Concluding remarks

From the obtained analytical and numerical results for natural transversal creeping vibrations of a fractional order derivative hereditary rod with two layers, it can be seen that fractional order derivative hereditary properties are convenient for changing time function depending on material creep parameters, and that fundamental eigenfunction depending on space coordinate is dependent only on boundary conditions and geometrical properties of layers.

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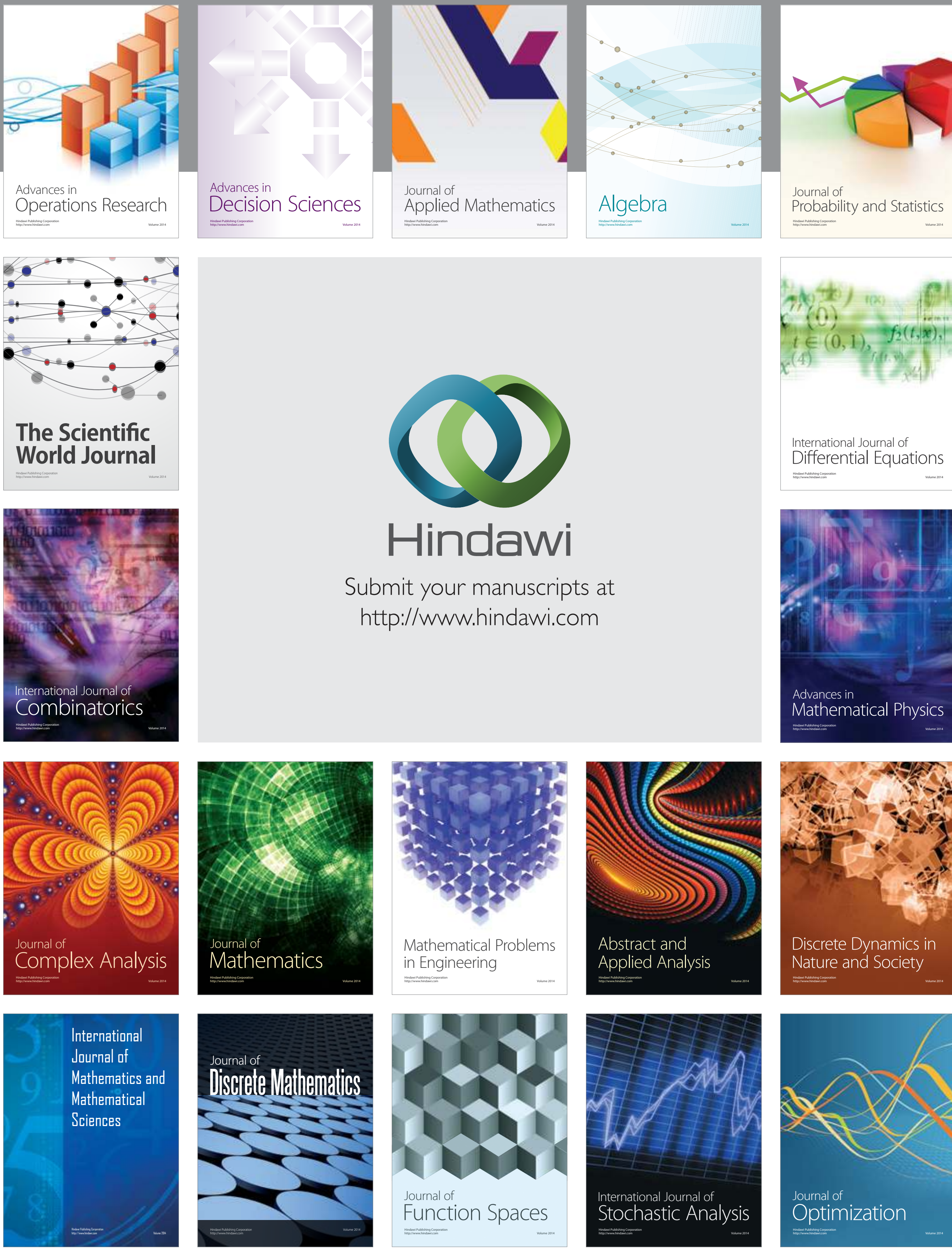
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