

The travelling salesman problem on randomly diluted lattices: Results for small-size systems

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Abstract. If one places N cities randomly on a lattice of size L , we find that $\bar{l}_E\sqrt{p}$ and $\bar{l}_M\sqrt{p}$ vary with the city concentration $p = N/L^2$, where \bar{l}_E is the average optimal travel distance per city in the Euclidean metric and \bar{l}_M is the same in the Manhattan metric. We have studied such optimum tours for visiting all the cities using a branch and bound algorithm, giving the exact optimized tours for small system sizes ($N \leq 100$) and near-optimal tours for bigger system sizes ($100 < N \leq 256$). Extrapolating the results for $N \rightarrow \infty$, we find that $\bar{l}_E\sqrt{p} = \bar{l}_M\sqrt{p} = 1$ for $p = 1$, and $\bar{l}_E\sqrt{p} = 0.73 \pm 0.01$ and $\bar{l}_M\sqrt{p} = 0.93 \pm 0.02$ with $\bar{l}_M/\bar{l}_E \simeq 4/\pi$ as $p \rightarrow 0$. Although the problem is trivial for $p = 1$, for $p \rightarrow 0$ it certainly reduces to the standard travelling salesman problem on continuum which is NP-hard. We did not observe any irregular behaviour at any intermediate point. The crossover from the triviality to the NP-hard problem presumably occurs at $p = 1$.

PACS. 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

1 Introduction

The travelling salesman problem (TSP) is a simple example of a multivariable combinatorial optimization problem and perhaps the most famous one. Given a certain set of cities and the intercity distance metric, a travelling salesman must find the shortest tour in which he visits all the cities and comes back to his starting point. It is a non-deterministic polynomial complete (NP-complete) problem [1–3]. In the standard formulation of TSP, we have N number of cities distributed randomly on a continuum plane and we determine the average optimal travel distance per city \bar{l}_E in the Euclidean metric (with $\Delta r_E = \sqrt{\Delta x^2 + \Delta y^2}$), or \bar{l}_M in the Manhattan metric (with $\Delta r_M = |\Delta x| + |\Delta y|$). Since the average distance per city scales (for fixed area) with the number of cities N as $1/\sqrt{N}$, we find that the normalized travel distance per city $\Omega_E = \bar{l}_E\sqrt{N}$ or $\Omega_M = \bar{l}_M\sqrt{N}$ become the optimized constants and their values depend on the method used to optimize the travel distance. Extending the analytic estimates of the average nearest neighbour distances, in particular within a strip and varying the width of the strip to extremize (single parameter optimization approximation), one gets $\frac{5}{8} < \Omega_E < 0.92$ [4] and $\frac{5}{2\pi} < \Omega_M < 1.17$ [5]. Careful (scaling, etc.) analysis of the numerical results obtained so far indicates that $\Omega_E \simeq 0.72$ [6].

Similar to many of the statistical physics problems re-defined on the lattices, *e.g.*, the statistics of self-avoiding

walks on lattices (for investigating the linear polymer conformational statistics), the TSP can also be defined on randomly dilute lattices. The (percolation) cluster statistics of such dilute lattices is now extensively studied [7]. The salesman's optimized path on a dilute lattice is necessarily a self-avoiding one; for optimized tour the salesman cannot afford to visit any city more than once and obviously it is one where the path is non-intersecting. The statistics of self-avoiding walks on dilute lattices has also been studied quite a bit (see *e.g.* [8]). However, this knowledge is not sufficient to understand the TSP on similar lattices. The TSP on dilute lattices is a very intriguing one, but has not been studied intensively so far.

The lattice version of the TSP was first studied by Chakrabarti [9]. In the lattice version of the TSP, the N cities are represented by randomly occupied lattice sites of a two-dimensional square lattice ($L \times L$), the fraction of sites occupied being p ($= N/L^2$, the lattice occupation concentration). One must then find the shortest tour in which the salesman visits each city only once and comes back to its starting point. The average optimal travel distance in the Euclidean metric \bar{l}_E , and in the Manhattan metric \bar{l}_M , are functions of the lattice occupation concentration p [10]. We intend to study here the variation of the normalised travel distance per city, $\Omega_E = \bar{l}_E\sqrt{p}$ and $\Omega_M = \bar{l}_M\sqrt{p}$, with the lattice concentration p for different system sizes. It is obvious that at $p = 1$, all the self-avoiding walks passing through all the occupied sites will satisfy the requirements of TSP and $\Omega_E = 1 = \Omega_M$ (the distance between the neighbouring

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cities is equal to the unit lattice constant and the path between neighbouring sites makes discrete angles of of $\pi/2$ or its multiples with the Cartesian axes). The problem becomes nontrivial as p decreases from unity: isolated occupied cities and branching configurations of occupied cities are found here with finite probabilities and self-avoiding walks through all the occupied cities, and only through the occupied cities, become impossible. As p decreases from unity, the discreteness of the distance of the path connecting the two cities and of the angle which the path makes with the Cartesian axes, tend to disappear. The problem reduces to the standard TSP on the continuum in the $p \rightarrow 0$ limit when all the continuous sets of distances and angles become possible. We study here the TSP on dilute lattice employing a computer algorithm which gives the exact optimized tours for small system sizes ($N \leq 100$) and near-optimal tours for bigger system sizes ($100 < N \leq 256$). Our study indeed indicates that Ω_E and Ω_M vary with p and $\Omega_E \simeq 0.73$ and $\Omega_M \simeq 0.93$ as $p \rightarrow 0$.

2 Computer simulation and results

We generate the randomly diluted lattice configurations following the standard Monte Carlo procedure for different system sizes. For each system size N , we vary the lattice size L so that the lattice concentration p varies. For each such lattice configuration, the optimum tour with open boundary conditions, is obtained with the help of the *GNU tsp_solve* [11] developed using a branch and bound algorithm (see Fig. 1). It claims to give exact results for $N \leq 100$ and near-optimal solutions for $100 < N \leq 256$. It may be noted that the program works essentially with the Euclidean distance. However there exists a geometric relationship between the Euclidean distance and the Manhattan distance. We may write $l_E = \sum_{i=1}^N r_i$, and $l_M = \sum_{i=1}^N r_i \alpha_i$, where r_i is the magnitude of the Euclidean path vector between two neighbouring cities and $r_i \alpha_i = r_i (|\sin \theta_i| + |\cos \theta_i|)$ is the sum of the components of the Euclidean path projected along the Cartesian axes. Naturally, $1 \leq \alpha_i \leq \sqrt{2}$. If l_E corresponds to the shortest Euclidean path, then $\sum_{i=1}^N r'_i > \sum_{i=1}^N r_i$, for any other path denoted by the primed set. If the optimized Euclidean path does not correspond to the optimized Manhattan path, then one will have $\sum_{i=1}^N r'_i \alpha'_i < \sum_{i=1}^N r_i \alpha_i$, where all the α_i and α'_i satisfy the previous bounds. Additionally, for random orientation of the Euclidean distance with respect to the Cartesian axes, $\langle \alpha_i \rangle = \langle \alpha'_i \rangle = (2/\pi) \int_0^{\pi/2} (\sin \theta + \cos \theta) d\theta = 4/\pi$. It seems, with all these constraints on α 's and α' 's, it would be impossible to satisfy the above inequalities on $\sum r_i$, and $\sum r_i \alpha_i$. In fact, we checked for a set of 50 random optimized Euclidean tours for small N (< 10), obtained using the algorithm, whether the optimized Manhattan tours correspond to different sequence (of visiting the cities), and did not find any. We believe that the optimized Euclidean tour necessarily corresponds to the optimized Manhattan tour. We then calculate l_E and l_M for each such optimized tour.

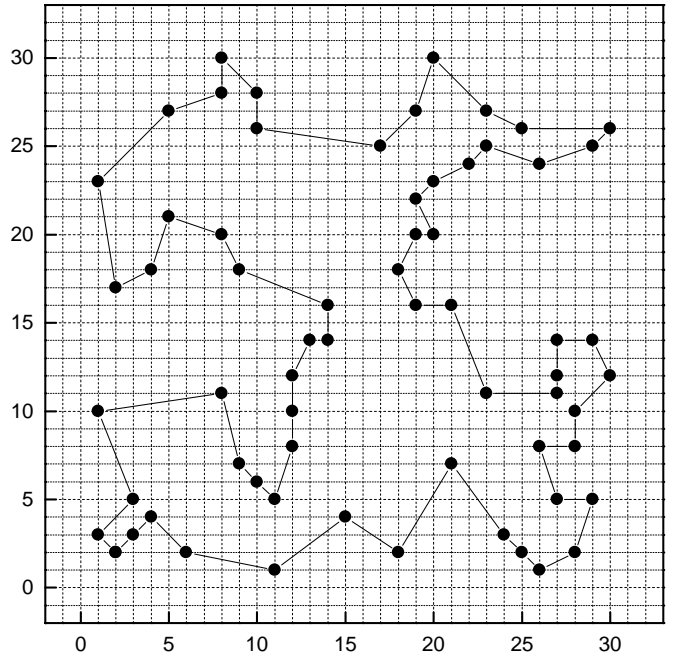


Fig. 1. A typical TSP for ($N =$) 64 cities on a dilute lattice of size $L = 30$. The cities are represented by black dots which are randomly occupied sites of the lattice with concentration $p = N/L^2 \simeq 0.07$. The optimized Euclidean path is indicated.

At each lattice concentration p , we take about 100 lattice configurations (about 150 configurations at some special points near $p \rightarrow 0$) and then obtain the averages \bar{l}_E and \bar{l}_M . We then determine $\Omega_E = \bar{l}_E \sqrt{p}$ and $\Omega_M = \bar{l}_M \sqrt{p}$ and study the variations of Ω_E and Ω_M , and of the ratio Ω_M/Ω_E with p . We find that Ω_E and Ω_M both have variations starting from the exact result of unity for $p = 1$ to the respective constants in the $p \rightarrow 0$ limit. In fact we noted that although Ω_M continuously decreases as $p \rightarrow 0$, it remains close to unity for all values of p . We studied the numerical results for $N = 64, 81, 100, 121, 144, 169, 196, 225$ and 256 . The results for $N = 64$ and 100 have been shown in Figures 2 and 3 respectively. We have studied the variations in the values of Ω_E and Ω_M against $1/N$ for $p \rightarrow 0$, to extrapolate its value in the $N \rightarrow \infty$ limit. It appears that for the large N limit (see Fig. 4), $\Omega_E(p \rightarrow 0)$ and $\Omega_M(p \rightarrow 0)$ eventually extrapolate to 0.73 ± 0.01 (as in continuum TSP) and to 0.93 ± 0.02 , respectively. This result for Ω_E (at $p \rightarrow 0$) compares very well with the previous estimates [6]. As p changes from 1 to 0, the ratio Ω_M/Ω_E changes continuously from 1 to about $1.27 (\simeq 4/\pi)$ (see Fig. 4), which is the average ratio of the Manhattan distance between two random points in a plane and the Euclidean distance between them [5, 10].

3 Conclusions

We note that the TSP on randomly diluted lattice is certainly a trivial problem when $p = 1$ (lattice limit) as it reduces to the one-dimensional TSP (the connections

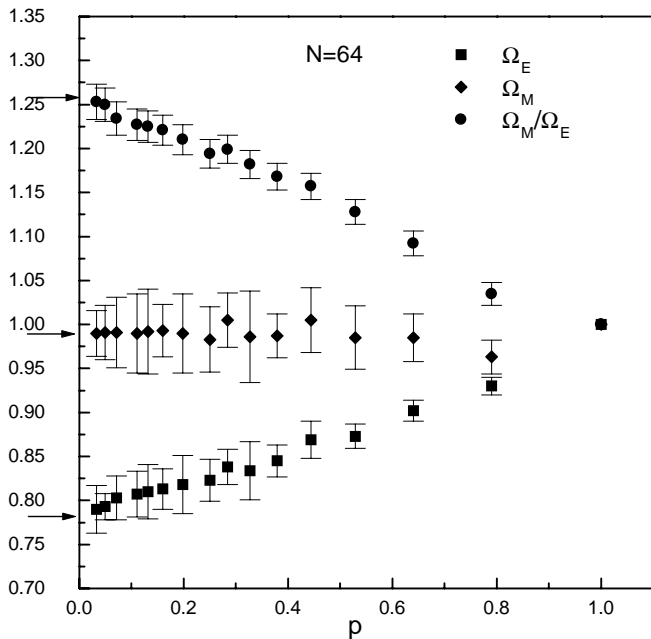


Fig. 2. Plot of Ω_E , Ω_M and Ω_M/Ω_E against p for $N = 64$ cities, obtained using the optimization programs (exact). The error bars are due to configurational fluctuations. The extrapolated values of Ω_E , Ω_M and Ω_M/Ω_E are indicated by horizontal arrows on the y -axis.

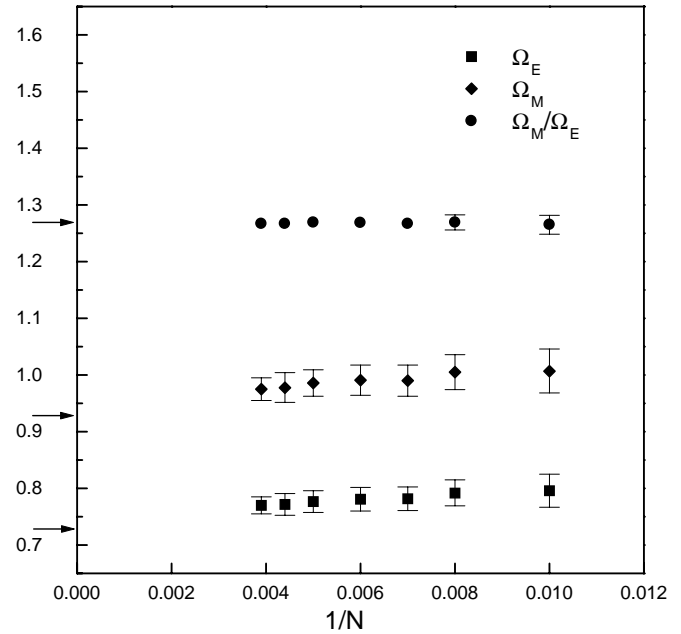


Fig. 4. Plots of Ω_E , Ω_M and of Ω_M/Ω_E in the $p \rightarrow 0$ limit, against $1/N$. The error bars are due to configurational fluctuations. The extrapolated value of Ω_E , Ω_M and Ω_M/Ω_E in this $p \rightarrow 0$ limit for $N \rightarrow \infty$ are indicated by horizontal arrows on the y -axis.

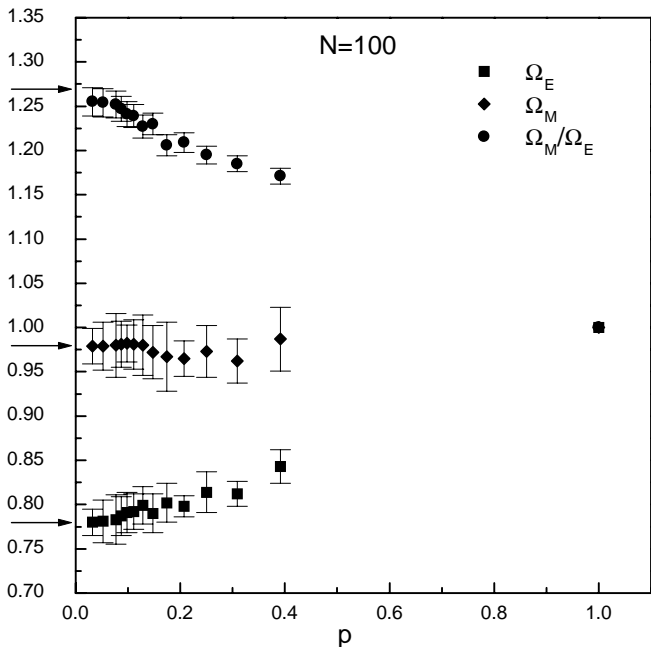


Fig. 3. Plot of Ω_E , Ω_M and Ω_M/Ω_E against p for $N = 100$ cities, obtained using the optimization programs (exact). The error bars are due to configurational fluctuations. The extrapolated values of Ω_E , Ω_M and Ω_M/Ω_E are indicated by horizontal arrows on the y -axis.

in the optimal tour are between the nearest neighbours along the lattice). Here $\Omega_E(p) = \Omega_M(p) = 1$. However, it is certainly NP-hard at the $p \rightarrow 0$ (continuum) limit, where $\Omega_E \simeq 0.73$ and $\Omega_M \simeq 0.93$ (extrapolated for large system sizes N). We note that Ω_M remains practically close to unity for all values of $p < 1$. Our numerical results also suggest that $\Omega_M/\Omega_E \simeq 4/\pi$ as $p \rightarrow 0$. It is clear that the problem crosses from triviality (for $p = 1$) to the NP-hard problem (for $p \rightarrow 0$) at a certain value of p . We did not find any irregularity in the variation of Ω at any p . A naive expectation might be that around the percolation point, beyond which the marginally connected lattice spanning path is snapped off [7], the Ω_E or Ω_M suffers some irregularity. The absence of any such irregularity can also be justified easily: the travelling salesman has to visit all the occupied lattice sites (cities), not necessarily those on the spanning cluster. Also, the TSP on dilute lattices has got to accommodate the same kind of frustration as the (compact) self-avoiding chains on dilute (percolating) lattices, although there the (collapsed) polymer is confined only to the spanning cluster. This indicates that the transition occurs either at $p = 1_-$ or at $p = 0_+$. From the consideration of frustration for the TSP even at $p = 1_-$, it is almost certain that the transition occurs at $p = 1$. However, this point requires further investigations.

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