

Marsha F. Foregger; Thomas H. Foregger
The tree-covering number of a graph

Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 4, 633–639

Persistent URL: <http://dml.cz/dmlcz/101711>

Terms of use:

© Institute of Mathematics AS CR, 1980

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

THE TREE-COVERING NUMBER OF A GRAPH

MARSHA F. FOREGGER and THOMAS H. FOREGGER, Murray Hill

(Received January 18, 1979)

INTRODUCTION

The *point-arboricity* $\varrho(G)$ of a graph G is the minimum number of subsets in any partition of the point set of G such that each subset induces an acyclic subgraph. We let $\varrho'(G)$ denote the minimum number of subsets in any partition of the point set of G such that each subset induces a *connected* acyclic graph. Since a connected acyclic graph is a tree, we refer to $\varrho'(G)$ as the *tree-covering number* for G .

If V_1, \dots, V_n is a partition of the point set of G such that the subgraph $\langle V_i \rangle$ of G induced by V_i is connected and acyclic, $i = 1, \dots, n$, we call V_1, \dots, V_n a *connected acyclic partition* for G (or simply a partition for G when there is no chance for confusion). Any connected acyclic partition for G is an acyclic partition, so $\varrho(G) \leq \varrho'(G)$ for any graph G .

If H is a subgraph of G , it is always true that $\varrho(H) \leq \varrho(G)$. This property does not hold for ϱ' , as the graphs in Fig. 1 indicate. Consequently many theorems about point-arboricity whose proofs rely on induction on the number of points or edges in the graph do not easily carry over to results about the tree-covering number.

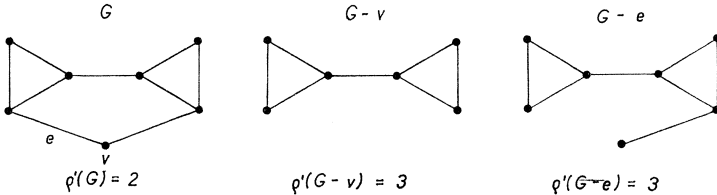


Fig. 1.

All definitions not given here may be found in [1].

SOME BOUNDS FOR $\varrho'(G)$.

If V_1, \dots, V_n is a partition of the points of G such that each V_i contains at most two points, then each $\langle V_i \rangle$ is acyclic. Thus $\varrho(G) \leq \lceil n/2 \rceil$ where G has order n and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . However, it is not always

possible to partition the points of G into $\lceil n/2 \rceil$ sets of size at most two such that each set induces a connected acyclic subgraph. For example, if G is the graph of order 10 formed by three K_4 's which share a common point, then it is not possible to find five point-disjoint edges in G .

Nevertheless, the same upper bound of $\lceil n/2 \rceil$ also holds for $g'(G)$ if G is connected. To prove this we need the following lemma about blocks.

Lemma 1. *Let G be a block with n points.*

- a) *If $n \geq 3$ there exists an edge v_1v_2 in G such that $G(v) = G - v_1 - v_2$ is connected.*
- b) *If $n \geq 4$ there exist disjoint edges u_1u_2 and v_1v_2 in G such that $G(u) = G - u_1 - u_2$ and $G(v) = G - v_1 - v_2$ are connected.*

The proof of this lemma uses the following theorem by ROBERTS [3].

Theorem (Roberts). *If G is a 2-connected graph which is not a single point, then we can write G as the union of subgraphs $G = P_0 \cup P_1 \cup \dots \cup P_k$ where P_0 is an edge graph and for $i = 1, \dots, k$, P_i is a simple path which avoids $B_{i-1} = P_0 \cup P_1 \cup \dots \cup P_{i-1}$ except for its distinct endpoints which are contained in $V(B_{i-1})$.*

Roberts's theorem is proved by induction on the number of edges in G and relies on a theorem by WHITNEY.

Theorem (Whitney). *Let G be a 2-connected graph. Suppose K is a 2-connected proper subgraph of G containing at least one edge. Then we can write G as $H \cup L$, where H is a 2-connected proper subgraph of G containing K , and L is a simple path in G that avoids H except for its distinct endpoints which are in $V(H)$.*

For a proof of Whitney's theorem see [4] p. 19.

Proof of Lemma 1. a) For $n \geq 3$ the concepts of block and 2-connected graph are identical so by Roberts's theorem we can write $G = P_0 \cup \dots \cup P_k$ where P_0 is an edge and P_i is a path avoiding $B_{i-1} = P_0 \cup \dots \cup P_{i-1}$ except for its distinct endpoints which lie in B_{i-1} , $i = 1, \dots, k$. It is easy to see that each B_i is a block. Since P_1 has length at least 2 there exists a largest $j \geq 1$ such that P_j has length at least 2. Then for $i > j$, P_i is an edge between two points of B_{j-1} . Let $V(P_j) = \{v_1, v_2, \dots, v_m\}$ where $v_1, v_m \in V(B_{j-1})$, $v_i \notin V(B_{j-1})$ for $2 \leq i \leq m-1$, and $m \geq 3$. Let $G(v) = G - v_1 - v_2$. Since B_{j-1} is a block, $B_{j-1} - v_1$ is connected. $B_j - v_1 - v_2$ consists of $B_{j-1} - v_1$ plus a path joined to $B_{j-1} - v_1$ at v_m , so it is also connected. Since $G(v)$ is just $B_j - v_1 - v_2$ plus additional edges, $G(v)$ is also connected.

b) The proof will be by induction on n . If $n = 4$, G is K_4 , K_4 minus an edge, or C_4 and the result holds by inspection. Now assume that if H is any block on m points with $4 \leq m < n$, there exist two disjoint edges such that removing either one from H leaves a connected graph. Let G be a block of order $n \geq 5$. By part a)

there exists an edge v_1v_2 such that $G(v)$ is connected. Moreover if the path P_j used in the proof of part a) has length at least 3 we can let $u_1 = v_m, u_2 = v_{m-1}$. By the same reasoning as above $G(u)$ is connected and we have the desired result. Thus we may assume that P_j has length 2. Then G is $B_{j-1} + v_1v_2 + v_2v_3$ plus possibly other edges, so we may regard G as a block H together with the point v_2 joined to H at v_1, v_3 , and possibly other points.

Since H has order $n - 1$ the induction hypothesis implies the existence of disjoint edges x_1x_2 and y_1y_2 such that $H(x) = H - x_1 - x_2$ and $H(y) = H - y_1 - y_2$ are connected. If $\{x_1, x_2\} = \{v_1, v_3\}$ the edges v_1v_2 and y_1y_2 in G have the desired properties. If $x_1 = v_1$ but $x_2 \neq v_3$ use the edges x_1x_2 and v_2v_3 . Similar arguments apply when $\{y_1, y_2\} = \{v_1, v_3\}$ and $y_1 = v_1$ but $y_2 \neq v_3$. Thus we may suppose x_1, x_2, y_1, y_2, v_1 , and v_3 are all distinct, and the edges x_1x_2 and y_1y_2 have the desired property. This completes the proof.

Theorem 2. *If G is a connected graph of order n then $q'(G) \leq \lceil n/2 \rceil$.*

Proof. We use induction on n . If n is 2 or 3 the result is easily checked. Suppose that if H is any connected graph of order $m < n$ then $q'(H) \leq \lceil m/2 \rceil$, and let G be a connected graph of order $n \geq 4$. If G is a block then Lemma 1a implies the existence of an edge v_1v_2 such that $G(v) = G - v_1 - v_2$ is connected. If V_1, \dots, V_k is a connected acyclic partition for $G(v)$, $\{v_1, v_2\}, V_1, \dots, V_k$ is a partition for G . Since the induction hypothesis applies to $G(v)$ we have

$$q'(G) \leq 1 + q'(G(v)) \leq 1 + \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

Now suppose G is not a block. Then there exists a cut-point v of G such that all blocks of G containing v with at most one exception are endblocks (see [1], p. 29). Let B_1, \dots, B_k be the endblocks of G containing v , with $k \geq 1$. If $|V(B_i)| \geq 4$ for some i then Lemma 1b implies the existence of disjoint edges u_1u_2 and v_1v_2 in B_i such that removing either one from B_i leaves a connected subgraph of B_i . Without loss of generality u_1u_2 does not contain the point v , so $G(u) = G - u_1 - u_2$ is connected. Applying the induction hypothesis to $G(u)$ we again have

$$q'(G) \leq 1 + q'(G(u)) \leq 1 + \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

If $|V(B_i)| = 3$ for some i then $B_i = K_3$ and there exists an edge u_1u_2 in $B_i - v$ whose removal leaves a connected subgraph of B_i , namely $\langle v \rangle$. Then $G(u)$ is again connected and use of the induction hypothesis on $G(u)$ gives the desired bound.

Thus we may assume $|V(B_i)| = 2$ for $i = 1, \dots, k$ so that $\langle \bigcup_{i=1}^k V(B_i) \rangle$ is acyclic. Let $H = G - \bigcup_{i=1}^k V(B_i) + v$. Then $H - v$ is connected of order $n - k - 1$ so by induction

$$\varrho'(G) \leq 1 + \varrho'(H - v) \leq 1 + \left\lceil \frac{n - k - 1}{2} \right\rceil = \left\lceil \frac{n - k + 1}{2} \right\rceil \leq \left\lceil \frac{n}{2} \right\rceil$$

since $k \geq 1$. This completes the proof.

The graphs for which the inequality of this theorem is an equality are characterized in the final section of this paper.

Corollary 3. *If G is a graph on n points with k components then*

$$\varrho'(G) \leq \left\lceil \frac{n + k - 1}{2} \right\rceil.$$

Proof. Let C_1, \dots, C_k be the components of G and let $n_i = |V(C_i)|$, $i = 1, \dots, k$. Then for each i , $\varrho'(C_i) \leq \lceil n_i/2 \rceil$ by Theorem 2, so

$$\varrho'(G) = \sum_{i=1}^k \varrho'(C_i) \leq \sum_{i=1}^k \left\lceil \frac{n_i}{2} \right\rceil \leq \sum_{i=1}^k \frac{n_i + 1}{2} = \left\lceil \frac{n + k}{2} \right\rceil.$$

Since $\varrho'(G)$ is an integer,

$$\varrho'(G) \leq \left\lceil \frac{n + k - 1}{2} \right\rceil.$$

If G is a connected graph of order n , then $\varrho(G)$ and $\varrho'(G)$ are both bounded by $\lceil n/2 \rceil$. However, the ratio $\varrho'(G)/\varrho(G)$ can be arbitrarily large. To see this consider the graph G consisting of k copies of K_m all sharing a common point v , where $m \geq 4$ is even. Let V_i consist of a pair of points from each $K_m - v$ for $i = 1, \dots, (m - 2)/2$. The remaining points induce an acyclic graph, so $\varrho(G) = (m - 2)/2 + 1 = m/2$. It is not hard to see that

$$\varrho'(G) = k \left(\frac{m - 2}{2} \right) + 1 = \frac{k(m - 2) + 2}{2}.$$

Thus

$$\frac{\varrho'(G)}{\varrho(G)} = \frac{k(m - 2) + 2}{m}$$

which becomes arbitrarily large as k increases.

Given any graph G , its complement \bar{G} has $V(\bar{G}) = V(G)$ and $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. NORDHAUS and GADDUM [2] found bounds on the chromatic number χ of a graph and its complement:

Theorem (Nordhaus and Gaddum). *Let G be a graph of order n . Then*

$$2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n + 1, \quad n \leq \chi(G)\chi(\bar{G}) \leq (n + 1)^2/4.$$

We prove a similar theorem for ϱ' .

Theorem 4. Let G be a graph of order n . Then

$$\sqrt{(n)} \leq \varrho'(G) + \varrho'(\bar{G}) \leq \lceil 3n/2 \rceil, \quad n/4 \leq \varrho'(G) \varrho'(\bar{G}) \leq n(n+1)/2.$$

Proof. Since $\varrho'(H) \geq \varrho(H) \geq \chi(H)/2$ for any graph H , the lower bounds in each case follow from the theorem of Nordhaus and Gaddum. Observe that for any graph G , at least one of G and \bar{G} must be connected, so without loss assume G is connected. Then $\varrho'(G) \leq \lceil n/2 \rceil$ and clearly $\varrho'(\bar{G}) \leq n$, so

$$\begin{aligned} \varrho'(G) + \varrho'(\bar{G}) &\leq n + \lceil n/2 \rceil = \lceil 3n/2 \rceil, \\ \varrho'(G) \cdot \varrho'(\bar{G}) &\leq n \lceil n/2 \rceil \leq n(n+1)/2. \end{aligned}$$

The upper bounds in Theorem 4 are best possible, since equality holds when $G = K_n$ with n odd. If G is a path v_1, v_2, \dots, v_{4k} , then any 5 points of \bar{G} induce a cyclic subgraph. Since the sets $\{v_1, \dots, v_4\}, \{v_5, \dots, v_8\}, \dots, \{v_{4k-3}, \dots, v_{4k}\}$ form a connected acyclic partition for \bar{G} , we have $\varrho'(\bar{G}) = k$. Thus $\varrho'(G) \varrho'(\bar{G}) = n/4$ and the lower bound in the second inequality is best possible.

The lower bound in the first inequality is also best possible as the following example shows. Take $n = 16k^2$ and let G' be the complete $2k$ -partite graph $K(8k, \dots, 8k)$. Let $U_j, j = 1, 2, \dots, 2k$ be the basic partitioning sets for G' , so that each $\langle U_j \rangle$ consists of $8k$ isolated points. Let $U_j = \{v_{1,j}, v_{2,j}, \dots, v_{8k,j}\}$. Form G by adding to G' the edges $v_{i,j}v_{i+1,j}, 1 \leq i < 8k, 1 \leq j \leq 2k$ (so that each U_j induces

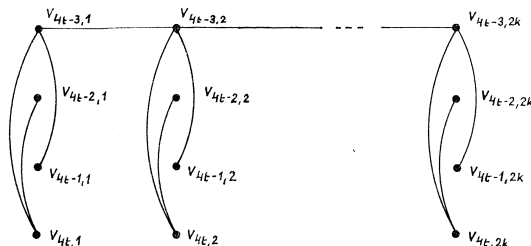


Fig. 2. Subgraph of \bar{G} induced by V_t .

a path in G) and deleting from G' the edges $v_{i,j}v_{i,j+1}, 1 \leq j < 2k, 1 \leq i < 8k, i \equiv 1 \pmod{4}$. The sets U_1, \dots, U_{2k} form a connected acyclic partition for G , so $\varrho'(G) \leq 2k$. Let $V_t = \{v_{i,j} \mid 4t-3 \leq i \leq 4t, 1 \leq j \leq 2k\}$ for $t = 1, \dots, 2k$. Then the subgraph of \bar{G} induced by each V_t has the form indicated in Fig. 2, so the sets V_1, \dots, V_{2k} form a partition form \bar{G} . Thus $\varrho'(\bar{G}) \leq 2k$ so that $\varrho'(G) + \varrho'(\bar{G}) \leq 4k = \sqrt{n}$.

THE CASE OF EQUALITY IN THEOREM 2

In this section we characterize those connected graphs for which the tree-covering number is as large as possible. Before stating the theorem, we present some notation.

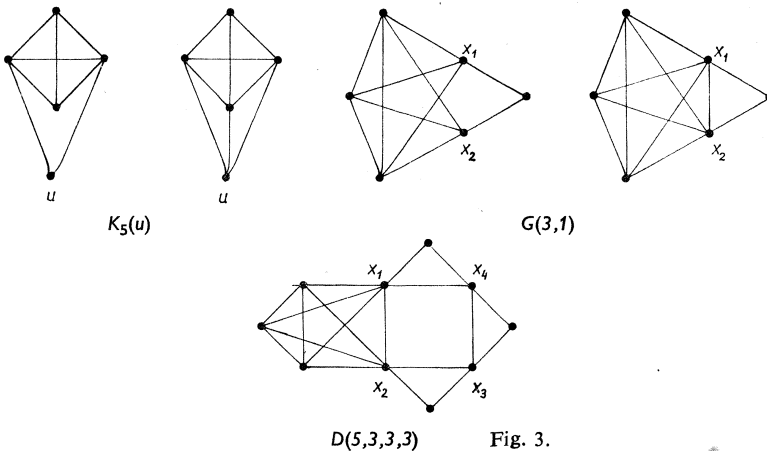
We use $K_n(u)$ to denote any graph obtained from K_n by the deletion of at most

$n - 3$ edges incident with u . Although $K_n(u)$ refers to a family of graphs, we will treat the family as a single graph for simplicity, since we will not be concerned with the number of deleted edges.

Given n_1 and n_2 odd integers, let $A = K_{n_1}$ and $B = K_{n_2}$, and take two additional points x_1 and x_2 . Form the graph $G(n_1, n_2)$ of even order $n_1 + n_2 + 2$ by joining x_1 and x_2 to all points of A and all points of B , and possibly to each other. Since we will not be concerned with whether or not x_1 and x_2 are joined, we use the same notation for both graphs.

Given n_1, \dots, n_k odd integers ≥ 3 , we form the "daisy" graph $D(n_1, \dots, n_k)$ of even order $n_1 + \dots + n_k - k$ as follows. Let $x_1, x_2, \dots, x_k, x_1$ be a k -cycle. Use x_1, x_2 and $n_1 - 2$ more points distinct from x_1, \dots, x_k to form a K_{n_1} . Similarly, construct a K_{n_i} on x_i and x_{i+1} for $i = 2, \dots, k$ (read subscripts modulo k). The resulting graph is $D(n_1, \dots, n_k)$; in this case the notation refers to a single graph.

Figure 3 illustrates the above notation.



If a block has even order, we will refer to it as an even block. Similarly an odd block is a block of odd order. We now characterize those graphs with tree-covering numbers as large as possible.

Theorem 5. Let G be connected of order n . Then $\varrho'(G) = \lceil n/2 \rceil$ if and only if

- 1) All odd blocks of G are complete;
- 2) G has at most one even block;

and

- 3) if G has an even block, that block is $K_m, K_m(u), G(n_1, n_2)$ or $D(n_1, \dots, n_k)$.

The proof essentially consists of a case by case analysis of the possibilities; it is available from the authors upon request.

References

- [1] *K. Chartrand and M. Behzad*: "Introduction to the Theory of Graphs," Allyn and Bacon, Boston, 1971.
- [2] *E. A. Nordhaus and J. W. Gaddum*: On complementary graphs, *Amer. Math. Monthly* 63 (1956), 175—177.
- [3] *E. J. Roberts*: The fully indecomposable matrix and its associated bipartite graph — An investigation of combinatorial and structural properties, NASA Technical Memorandum TMX-58037, January 1970.
- [4] *W. T. Tutte*: "Connectivity in Graphs," University of Toronto Press, Toronto, 1966.

Authors' address: Bell Laboratories, Murray Hill, New Jersey 07974, U.S.A.