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THE TRIANGULAR E-MODEL OF CHANCE-CONSTRAINED PROGRAMMING WITH STOCHASTIC A-MATRIX

by

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Abstract

The triangular model of chance-constrained programming with stochastic A-matrix and deterministic right hand side is considered. The use of conditional probabilities makes it possible to solve this problem for any type of distribution function of the elements of the A-matrix provided that there is only one decision variable at each stage. The extension of the model to several decision variables per stage is possible under certain conditions and for special distribution (stable distributions) of the elements of A.

1. Introduction.

The triangular E-model of chance constrained programming has been introduced by Charnes and Kirby [5] and later extended by [10] and especially [6]. This paper treats the same triangular E-model but rather than considering a random right-hand vector b in the chance constraints, we develop the case of randomness in the coefficient matrix A. The use of conditional probabilities makes it possible to solve this problem for any type of distribution function of the elements of the A-matrix provided that there is only one decision variable at each stage. The so-called 1-feasibility concept [14] or safety-first principle [6] will be used to avoid decision rules which could create inconsistent sample points whenever there exists a feasible decision rule. The extension of the model to several decision variables per stage is possible under certain conditions and for special distributions (stable distributions) of the elements of A.

Section 2 contains the statement of the problem and some notational conventions. An example from the field of production planning which can be reduced to this general model is given in Section 3. In Section 4 we indicate how the probabilistic constraints can be rewritten as deterministic constraints. The relationship between 1-feasibility and safety-first principles and the properties of the solution set and objective function are investigated in Section 5. It turns out that every sub-problem consists in maximizing (minimizing) a concave (convex) function over a convex set. The extension of several decision variables at each stage is treated in 6. Finally, Section 7 contains two detailed examples.

2. The Model and Notational Conventions

The formal model which we will be concerned with in this paper has been formulated as follows :

$$\max \; E\,(cx)$$

subject to :

$$\text{(2.1)} \qquad Pr\,(Ax \leqslant b) \geqslant \alpha$$

$$x \geqslant 0$$

where the following notation has been adopted :

$A = (a_{ij})$	is a lower triangular (nxn) matrix with random elements a_{ij} $(i,j) \in \{1,\dots,n\}^2$, $a_{ij} = 0$ for $j > i$.
a^i	is the i^{th} column of A and assumed to be known before the $(i+1)^{st}$ decision x_{i+1} has to be made.
c	1xn vector of random elements c_i.
b	nx1 vector of non-random elements b_i.
x	nx1 decision vector, the decision to be made at stage i is $x_i, i \in \{1,\dots,n\}$.
E(.)	expectation operator with respect to all random variables involved.
Pr(.)	probability operator which is assumed to apply in parallel.
$a^{(i)}$	the nxi matrix with elements $(a^1,\dots,a^i)$
$c_{(i)}$	the 1xi vector with elements $(c_1,\dots,c_i)$
d_i	$\equiv (a^i, c_i)$
$d_{(i)}$	$\equiv (a^{(i)}, c_{(i)})$
$F_{a_{ii} \mid d_{(i-1)}}(.)$	conditional distribution function of a_{ii} given the observations $d_{(i-1)}$
$E_{d_i \mid d_{(i-1)}}(.)$	conditional expectation operator with respect to (a^i, c_i) given the observations $(a^{(i-1)}, c_{(i-1)})$
α	vector of probability numbers, decided upon a priori and each $\alpha_i \in [0,1]$

3. An Example in Production Planning

To illustrate one possible application of the model (2.1) we consider an example taken from the field of production planning.

A firm has to make sequential decisions concerning the production of a good over a n-unit time period. The price of the product in period i π_i (i=1,...,n) is a random variable and the firm is a price-taker, i.e. it has no control over te prices. Also, the total production cost k_i (i=1,...,n) of one unit in period i is random. Assume further that demand prospects are such that oversupply of the market within the n periods in considered to be impossible, although stochastic demand constraints could be easily incorporated. Clearly the random variables π_i and k_i will depend upon $\pi_{(i-1)}$ and $k_{(i-1)}$. Indeed through observation of $\pi_{(i-1)}$ and $k_{(i-1)}$ we get new information about the market and the cost structure of the product. Hence the decision x_i as to how many units to produce in period i has to be chosen dependent upon the prices and production costs experienced in previous periods.

At the beginning of the decision process fixed amounts L_i (which can be made dependent upon previous observations) are budgeted for production. Budgets not fully used up in previous periods can be transferred to later periods and previous profits (losses) increase (decrease) funds available for production in period i by some fixed proportion. The reason for this is that during periods of large profits we want to create the possibility of heavier investment in production. Moreover, to allow for overspending in a period when prospects are particularly favorable, we want the budget constraints to hold with prescribed probabilities. By choice of $\alpha_n = 1$ in the n^{th} period constraint we can make sure that at the end of the time horizon the total budget ceiling is not exceeded.

As indicated earlier, we interpret the probability constraints to be conditional probabilities given the previous observations. The problem may then be formulated as follows :

$$\max \; E\left[\sum_{i=1}^{n} (\pi_i - k_i)\, x_i\right]$$

subject to :

$$\Pr\left\{ k_1 x_1 \leq L_1 \right\} \geq \alpha_1$$

$$(3.1) \quad \Pr\left\{ k_i x_i \leq L_i + \sum_{j=1}^{i-1} \left[L_j - k_j x_j + \beta_j(\pi_j - k_j)x_j\right] \Big| \; k_{(i-1)}, \pi_{(i-1)} \right\} \geq \alpha_i \qquad i \in \left\{2,\ldots,n\right\}$$

$$x_i \geq 0 \qquad i \in \left\{1,\ldots,n\right\}$$

where :

π_i price in period i (random variable depending on $\pi_{(i-1)}$.

k_i total production cost per unit in period i (random variable depending on $k_{(i-1)}$.

L_i part of the budget for period i which may or may not depend on $k_{(i-1)}$, $\pi_{(i-1)}$.

β_i determines the fraction of the profit (loss) in period i , $\beta_i(\pi_i - k_i)x_i$, that will be available for production from period i+1 on, $0 \leq \beta_i \leq 1$.

Pr conditional probability operator of k_i given $k_{(i-1)}$ and $\Pi_{(i-1)}$.

x_i amount to be produced in the i^{th} period , $i \in \{1,\ldots,n\}$.

Rewriting the problem we obtain the following equivalent formulation for (3.1) :

$$\max E \left[\sum_{i=1}^{n} (\Pi_i - k_i)x_i \right]$$

subject to :

$$\Pr\{ k_1 x_1 \leq L_1 \} \geq \alpha_1$$

(3.2) $$\Pr\left\{ \sum_{j=1}^{i-1} \left[k_j - \beta_j(\Pi_j - k_j)\right] x_j + k_i x_i \leq \sum_{j=1}^{i} L_j | k_{(i-1)}, \Pi_{(i-1)} \right\} \geq \alpha_i \qquad i \in \{2,\ldots,n\}$$

$$x_i \geq 0 \qquad i \in \{1,\ldots,n\} .$$

Defining

$$a_{ij} = \begin{cases} k_j - \beta_j(\Pi_j - k_j) & j < i \\ k_j & j = i \\ 0 & j > i \end{cases}$$

and $c_i = \Pi_i - k_i$ $\qquad i=1,\ldots,n$

$$b_i = \sum_{j=1}^{i} L_j \qquad i=1,\ldots,n$$

we see that the production model is of the general form described in Section 2.

The applicability of the general model is however not restricted to the above example. As will become clear upon inspection, the one-stock investment model of B. Näslund [13] can be brought to fit into the general framework developed here. This is of particular interest to note since Näslund's model so far has been studied only in the context of first-order decision rules whereas we shall use the more general sequential decision rules obtained from conditional probability constraints to solve the above model. The procedure developed here might also be useful for the study of chance-constrained capital budgeting problems as in [3] and [4] where solutions are presented in terms of zero order rules.

4. Deterministic Equivalents of the Chance Constraints

As indicated in the previous section, we consider the decision x_i to be a function of the previous observations $d_{(i-1)} = (a^{(i-1)}, c_{(i-1)})$. By a decision rule for problem (2.1) we mean a relation of the form

$$x_i = x_i(d_{(i-1)}) \tag{4.1}$$

mapping the observations $d_{(i-1)}$ into the reals. We observe that the first decision is independent of any random variable. Using the notations of Section 2 we can now write the conditional probability constraint of the i^{th} period as follows :

$$\Pr\left(\sum_{j=1}^{i} a_{ij}x_j \leq b_i/d_{(i-1)}\right) \geq \alpha_i$$

or equivalently

$$\Pr\left(a_{ii}x_i \leq b_i - \sum_{j=1}^{i-1} a_{ij}x_j/d_{(i-1)}\right) \geq \alpha_i \tag{4.2}$$

Let $x_i(d_{(i-1)}) \geq 0$ be any decision function for the i^{th} period satisfying the non-negativity constraint of problem (2.1) and define N to be the set

(4.3) $$N_x = \left\{ d_{(i-1)} / x_i(d_{(i-1)}) = 0 \right\}$$

Then (4.2) implies that

(4.4.a) $$0 \leq b_i - \sum_{j=1}^{i-1} a_{ij} x_j \qquad \forall\, d_{(i-1)} \in N_x$$

From the non-negativity of x_i we have that for $d_{(i-1)} \in \bar{N}_x$ where $\bar{N}_x$ is the complementary set of N_x in the space of all possible outcomes of $d_{(i-1)}$, the following inequality must hold :

$$\Pr\left(a_{ii} \leq \frac{1}{x_i} (b_i - \sum_{j=1}^{i-1} a_{ij} x_j) / d_{(i-1)} \right) \geq \alpha_i \qquad \forall\, d_{(i-1)} \in \bar{N}_x$$

or equivalently

(4.4.b) $$\sum_{j=1}^{i-1} a_{ij} x_j + x_i \, F^{-1}_{a_{ii}/d_{(i-1)}} (\alpha_i) \leq b_i \qquad \forall\, d_{(i-1)} \in \bar{N}_x$$

Now observe that for $d_{(i-1)} \in N_x$, (4.4.b) and (4.4.a) coincide ; hence the deterministic equivalent of (4.2) using the non-negativity of x_i is given by

(4.4) $$\sum_{j=1}^{i-1} a_{ij} x_j + x_i \, F^{-1}_{a_{ii}/d_{(i-1)}} (\alpha_i) \leq b_i \qquad \forall\, d_{(i-1)}$$

This leads us to define 1-feasibility (see [14]) of a sequential decision rule $\left\{ x_i(d_{(i-1)}) \right\}_{i=1,\ldots,n}$ as follows :

Definition 4.1 : A sequential decision rule $\{ x_i(d_{(i-1)}) \}$ $i=1,\ldots,n$ is called 1-feasible if

(i) $x_i(d_{(i-1)}) \geq 0$

(ii) (4.4) is fulfilled for all possible realizations $d_{(i-1)}$ and $i \in \{1,\ldots,n\}$.

For typographical reasons we set

$$(4.5)\quad f_i\,(\alpha_i, d_{(i-1)}) = F^{-1}_{a_{ii}/d_{(i-1)}}\,(\alpha_i) \quad \text{for all } i \in \{1,\ldots,n\}.$$

Note that for $i=1$ the right-hand side of (4.5) is the inverse of the (unconditional) marginal distribution of a_{11} evaluated at α_1. We can now state the deterministic equivalent of the constraint set of problem (2.1) as follows :

$$(4.6)\quad \sum_{j=1}^{i-1} a_{ij}x_j + x_i\, f_i(\alpha_i, d_{(i-1)}) \leq b_i\ ;\ x_i = x_i(d_{(i-1)}) \geq 0$$

for all $d_{(i-1)}$;

Remark 4.1. Suppose that $b_i \geq 0$, $i \in \{1,\ldots,n\}$, then $x_i(d_{i-1})=0$ for all $d_{(i-1)}$, $i \in \{1,\ldots,n\}$, is always a feasible decision rule. In particular, in this case there do not exist sample points which could create inconsistencies.

Remark 4.2. From the equivalent formulation (4.6) of the constraint set (2.1) we infer that for a finite sample space it is possible to derive a linear program as a deterministic equivalent, in the case where the coefficients in the objective function are non-random or stochastically independent random variables. The way to obtain the linear program is essentially the same as used in connection with programming under uncertainty, see [16] and [17] . One indexes the decisions to be made at stage i by the possible observations on $d_{(i-1)}$ and computes the objective function explicitly using the (known) probabilities of the possible combinations of the observations. Clearly for only modestly sized sample spaces the resulting problem becomes already very large. But one can expect that due to the triangularity of the stochastic matrix and the particular choice of the decisions to be dependent only upon prior observations the resulting linear programming problem has a special structure that can be exploited in computation.

5. Properties of Solution Set and Objective Function

Let $d_{(i-1)}$ for $i \in \{1,\ldots,n\}$ be any given observation on the random variables of the $(i-1)^{st}$ periods and denote by $d^{(i-1)}_{(k-1)}$ for $k \in \{i+1,\ldots,n\}$ any observation $d_{(k-1)}$ such that the $(i-1)$ first components equal the given $d_{(i-1)}$. Correspondingly let $d^{(i-1)}_{(j-1)}$ for $j \in \{1,\ldots,i\}$ denote the vector of observations obtained from $d_{(i-1)}$ by deleting the last $i-j$ elements of $d_{(i-1)}$. Note that $d^{(0)}_{(k-1)} = d_{(k-1)}$ for all $k \in \{2,\ldots,n\}$.

In a manner similar to that used in [6] , define the set $C_i^+(d_{(i-1)},x_{(i-1)})$ for given $d_{(i-1)}$ and $x_{(i-1)}$ recursively as follows :

$$
\begin{aligned}
(5.1) \qquad C_i^+ &= C_i^+(d_{(i-1)},x_{(i-1)}) \\
&= \{ x_i \geqslant 0 / (1) \ \forall\, d^{(i-1)}_{(k-1)} \ \exists\, \lambda_j(d_{(j-1)}) \geqslant 0 \ , \ j=i+1,\ldots,k \\
&\qquad \text{such that } (5.2.a) \text{ holds for all } k \in \{i+1,\ldots,n\} \\
&\qquad (2) \qquad (5.2.b) \text{ holds } \}
\end{aligned}
$$

where

$$
(5.2)\begin{cases}
(5.2.a) \quad a_{ki}x_i + \sum_{j=i+1}^{k-1} a_{kj}\lambda_j(d_{(j-1)}) + \lambda_k(d_{(k-1)})\, f_k(\alpha_k, d_{(k-1)}) \leqslant b_k - \sum_{j=1}^{i-1} a_{kj}x_j \\
(5.2.b) \quad f_i(\alpha_i, d_{(i-1)})x_i \leqslant b_i - \sum_{j=1}^{i-1} a_{ij}x_j
\end{cases}
$$

For these set definitions to be recursively meaningful, we have to choose x_j $(j=1,\ldots,i-1)$ such that :

$$(5.3) \quad x_j \in C_j^+ \,(d_{(j-1)}^{(i-1)}, x_{(j-1)}) \qquad j \in \{1,\ldots,i-1\}$$

The proof of the following remark is straightforward and will be omitted.

Remark 5.1.

1. There exists a 1 - feasible decision rule iff $C_1^+ \neq \emptyset$.
2. Let $\{x_i(d_{(i-1)})\}$ $i=1,\ldots,n$ be any decision rule. Then

   $\{x_i(d_{(i-1)})\}$ $i=1,\ldots,n$ is 1 - feasible iff $x_i(d_{i-1}) \in C_i^+ \,(d_{(i-1)})$

   for all $d_{(i-1)}$, $i \in \{1,\ldots,n\}$.

Theorem 5.1 : For any given, $d_{(i-1)}$ and $x_{(i-1)}$ the set $C_i^+(d_{(i-1)})$ is convex in x_i.

Proof :[1] Let $x_i^1, x_i^2 \in C_i^+ \,(d_{(i-1)})$, then for $\forall\, d_{(k-1)}^{(i-1)}$, there exists

$\lambda_j \geq 0$ and $\mu_j \geq 0$, $j \in \{i+1,\ldots,k\}$ such that

$$b_i - \sum_{j=1}^{i-1} a_{ij}x_j \geqslant x_i^1\, f_i(\alpha_i, d_{(i-1)})$$

[1] For ease of notation we will write $x_i, \lambda_j, \mu_j, \ldots$ whereas it should be $x_i(d_{(i-1)})$, $\lambda_j(d_{(j-1)})$, $\mu_j(d_{(j-1)}), \ldots$

$$(5.4)\quad b_k - \sum_{j=1}^{i-1} a_{kj}x_j - a_{ki}x_i^1 - \sum_{j=i+1}^{k-1} a_{kj}\lambda_j \geq \lambda_k f_k(\alpha_k, d_{(k-1)})$$

$$\text{for } k \in \{ i+1, \ldots, n \}$$

and

(5.5) similar expressions for x_i^2 where λ_j ($j \in \{ i+1, \ldots, k \}$) is replaced by μ_j, $j \in \{ i+1, \ldots, k \}$.

Multiplying the inequalities of (5.4) by $0 \leq \gamma \leq 1$ and those of (5.5) by $(1-\gamma)$ and adding the corresponding inequalities, we get :

$$b_i - \sum_{j=1}^{i-1} a_{ij}x_j \geq [\gamma x_i^1 + (1-\gamma)\, x_i^2]\; f_i\, (\alpha_i, d_{(i-1)})$$

$$b_k - \sum_{j=1}^{i-1} a_{ij}x_j - a_{ki}[\gamma x_i^1 + (1-\gamma)x_i^2] - \sum_{j=i+1}^{k-1} a_{kj}[\gamma \lambda_j + (1-\gamma)\mu_j] \geq$$

$$[\gamma \lambda_k + (1-\gamma)\mu_k]\; f_k(\alpha_k, d_{(k-1)}) \text{ for } \forall\, d_{(k-1)}^{(i-1)}, \; k \in \{ i+1, \ldots, n \}$$

which means that $[\gamma x_i^1 + (1-\gamma)\, x_i^2] \in C_i^+$ for $0 \leq \gamma \leq 1$. Q.E.D.

Let us now turn to the properties of the objective function. Using dynamic programming in the familiar backward manner [1], the objective function for the i^{th} stage, given a set of observations $d_{(i-1)}$ and decisions $x_{(i-1)}$, can be written as follows :

$$(5.6)\quad \Psi_i\, (d_{(i-1)}, x_{(i-1)}) = \max_{x_i \in C_i^+} \; \underset{d_i/d_{(i-1)}}{E} \; (\sum_{p=i}^{m} c_p x_p)$$

$$= \max_{x_i \in C_i^+} \; \underset{d_i/d_{(i-1)}}{E} \; \{ c_i x_i + \Psi_{i+1}(d_{(i)}, x_{(i)}) \}$$

$$= \max_{x_i \in C_i^+} \; H_i(x_i)$$

$$(5.7)\quad \text{where } H_i(x_i) = \underset{d_i/d_{(i-1)}}{E} \; \{ c_i x_i + \Psi_{i+1}\, (d_{(i)}, x_{(i)}) \}$$

Theorem 5.2 : For any fixed $d_{(i-1)}$ and $x_{(i-1)}$ the function $H_i(x_i)$ defined in (5.7) is concave over $x_i \in C_i^+$.

The proof goes by backward induction and is almost identical to the proof given in [6] . It will therefore be omitted.

Remark 5.2. :

In the general model it follows from (4.4) that the following relation must hold at stage i if $f_i(\alpha_i, d_{(i-1)}) \geq 0$, for all $d_{(i-1)}$:

$$\sum_{j=1}^{i-1} a_{ij} x_j \leq b_i \ .$$

In the two period problem we now show that the set C_1^+ can be characterized by linear constraints.

Conditions (5.2.b) and (5.2.a) become in that case :

(5.2.b) $f_1(\alpha_1, d_o)\, x_1 \leq b.$

(5.2.a) For all d_1, $\exists\ \lambda_2(d_1) \ni$

$$a_{21} x_1 + \lambda_2(d_1)\, f_2(\alpha_2, d_1) \leq b_2$$

In (5.2.a) , two cases are possible :

1) $f_2(\alpha_2, d_1) < 0$ for all possible $d_1 \Rightarrow$ (5.2.a) is void since $\lambda_2(d_1)$ can be taken large enough for the constraint to hold.

2) $f_2(\alpha_2, d_1) \geq 0$ for some d_1, $d_1 \in N_1$ say $\Rightarrow$ (5.2.a) becomes $(\sup_{d_1 \in N_1} a_{21}) x_1 \leq b_1$

since one can set $\lambda_2(d_1) = 0$ for $d_1 \in N_1$.

This result however, cannot be generalized to more than two periods.

6. Extension to Several Decisions Per Stage

Consider the more general problem :

$$\max \quad E \left[\sum_{i=1}^{n} \sum_{j=1}^{m_i} c_{ij} x_{ij} \right]$$

subject to :

$$\Pr \left(\sum_{j=1}^{m_1} a^1_{1j} x_{1j} \leq b_1 \right) \geq \alpha_1$$

(6.1)

$$\Pr \left(\sum_{j=1}^{m_1} a^2_{1j} x_{1j} + \sum_{j=1}^{m_2} a^2_{2j} x_{2j} \leq b_2 \right) \geq \alpha_2$$

$$\vdots$$

$$\Pr \left(\sum_{i=1}^{n-1} \sum_{j=1}^{m_i} a^n_{ij} x_{ij} + \sum_{j=1}^{m_n} a^n_{nj} x_{nj} \leq b_n \right) \geq \alpha_n$$

$x_{ij} \geq 0$ for all i, j.

Define :

$A^i = (a^p_{ij}) \qquad p \in \{ i, \ldots, n \}$

$\qquad\qquad\qquad j \in \{ 1, \ldots, m_i \}$

$A^{(i)} = A^1, A^2, \ldots, A^i$

$c_i = (c_{ij}), \qquad j \in \{ 1, \ldots, m_i \}$

$c_{(i)} = c_1, c_2, \ldots, c_i$

$D_i = (A^i, c_i)$

$D_{(i)} = (A^{(i)}, c_{(i)})$

$$x_i = (x_{ij}) \qquad j \in \{1,\ldots,m_i\}$$

$$x_{(i)} = x_1,\ldots,x_i$$

Again, observations on the random variables A^i are made after decisions $x_{ij}(j \in \{1,\ldots,m_i\})$ are selected and before decisions $x_{i+1,j}(j \in \{1,\ldots,m_{i+1}\})$.

Initially the following condition will be imposed on the random variables of the constraint set : the elements of A^i are conditionally independent random variables by which we mean that (some of) their parameters may depend upon previous observations A^j $(j \in \{1,\ldots,i-1\})$, however, once these parameters known the r.v. are independent.

The difficulty in this generalized problem is to find a deterministic equivalent for the chance constraints. We shall now show how this can be done for the i-th constraint when the a^i_{ij} $(j=1,\ldots,m_i)$ have independent symmetric stable distributions (1) , with the same characteristic exponent [see appendix.]

Introducing the following notation :

$y \sim S_y(\alpha, \delta, c, \beta)$: y is stable distributed with

characteristic exponent $0 < \alpha \leq 2$

location parameter δ

scale parameter $c \geq 0$

symmetry coefficient $|\beta| \leq 1$.

Let $a^i_{ij} \sim S(\alpha, \delta^i_{ij}, c^i_{ij}, 0)$ for $j \in \{1,\ldots,m_i\}$. From the properties of symmetric stable distributions it follows that :

---

(1) : The reason why this transformation cannot be extended to other types of distributions follows from the convolution property of stable distributions (see definition A.2) and the transformation (A.5) which make the cumulative distribution function independent of the vector x_i.

$$\sum_{j=1}^{m_i} a^i_{ij} x_{ij} \sim S \left(\alpha , \sum_{j=1}^{m_i} \delta^i_{ij} x_{ij} , \sum_{j=1}^{m_i} c^i_{ij} / x_{ij} |^{\alpha} , 0 \right)$$

or $\sim S \left(\alpha , \sum_{j=1}^{m_i} \delta^i_{ij} x_{ij}, \sum_{j=1}^{m_i} c^i_{ij} x^{\alpha}_{ij} , 0 \right)$ since $x_{ij} \geq 0$.

Hence

$$\frac{\sum_{j=1}^{m_i} a^i_{ij} x_{ij} - \sum_{j=1}^{m_i} \delta^i_{ij} x_{ij}}{\left(\sum_{j=1}^{m_i} c^i_{ij} x^{\alpha}_{ij} \right)^{1/\alpha}} \sim S \, (\alpha, 0, 1, 0)^{(1)}$$

Writing the i^{th} constraint as :

$$Pr \left(\sum_{j=1}^{m_i} a^i_{ij} x_{ij} \leq b_i - \sum_{k=1}^{i-1} \sum_{j=1}^{m_k} a^i_{kj} x_{kj} \right) \geq \alpha_i$$

or

$$F \left(\frac{b_i - \sum_{k=1}^{i-1} \sum_{j=1}^{m_k} a^i_{kj} x_{kj} - \sum_{j=1}^{m_i} \delta^i_{ij} x_{ij}}{\left(\sum_{j=1}^{m_i} c^i_{ij} x^{\alpha}_{ij} \right)^{1/\alpha}} \right) \geq \alpha_i$$

where F(.) is the cumulative distribution function for a standardized symmetric stable distribution with characteristic exponent α . The deterministic equivalent for the i^{th} chance constraint becomes :

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(1) Notice that for the normal distribution (α=2), the standardized variable has a variance =2.

(7.2) $$\sum_{j=1}^{m_i} \delta^i_{ij}\, x_{ij} + F^{-1}(\alpha_i)\left(\sum_{j=1}^{m_i} c^i_{ij}\, x^{\alpha}_{ij}\right)^{1/\alpha} \leqslant b_i - \sum_{k=1}^{i-1}\sum_{j=1}^{m_k} a^i_{kj} x_{kj}$$

where $F^{-1}(\alpha_i)$ can be found (using interpolation) in the tables given in [7] if $1 \leqslant \alpha \leqslant 2$.

<u>Theorem 7.2</u> : The function $\left(\sum_{j=1}^{m_i} c^i_{ij} x^{\alpha}_{ij}\right)^{1/\alpha}$

with $\begin{cases} c^i_{ij} \geqslant 0 \; ; \; j \in \{1,\dots,m_i\} \\ x_{ij} \geqslant 0 \end{cases}$ is

(1) convex in $x_i = (x_{ij})$, $j \in \{1,\dots,m_i\}$, if $1 \leqslant \alpha \leqslant 2$

(2) concave in x_i if $0 < \alpha \leqslant 1$.

<u>Proof</u> : To prove (1), we know from Minkowski's inequality that, for $u_j \geqslant 0$ $(j=1,\dots,m_i)$ and $v_j \geqslant 0$ $(j=1,\dots,m_i)$:

$$\left[\sum_{j=1}^{m_i} (u_j + v_j)^{\alpha}\right]^{1/\alpha} \leqslant \left(\sum_{j=1}^{m_i} u_j^{\alpha}\right)^{1/\alpha} + \left(\sum_{j=1}^{m_i} v_j^{\alpha}\right)^{1/\alpha} \quad \text{for } \alpha \geqslant 1.$$

Setting $u_j = \lambda (c^i_{ij})^{1/\alpha}\, x^1_{ij}$ with $0 \leqslant \lambda \leqslant 1$ and $c^i_{ij} \geqslant 0$

$\forall\, j \in \{1,\dots,m_i\}$

$$v_j = (1-\lambda)\,(c^i_{ij})^{1/\alpha}\, x^2_{ij} \qquad \forall\; j \in \{1,\dots,m_i\}$$

Then :

$$\left\{ \sum_{j=1}^{m_i} c_{ij}^i \left[\lambda x_{ij}^1 + (1-\lambda) x_{ij}^2 \right]^{\alpha} \right\}^{1/\alpha} \leq \lambda \left[\sum_{j=1}^{m_i} c_{ij}^i (x_{ij}^1)^{\alpha} \right]^{1/\alpha} + (1-\lambda) \left[\sum_{j=1}^{m_i} c_{ij}^i (x_{ij}^2)^{\alpha} \right]^{1/\alpha}$$

for $0 \leq \lambda \leq 1,\ 1 \leq \alpha \leq 2$ and $c_{ij}^i \geq 0\ \forall j$

The proof of (2) follows from the reversed Minkowski inequality when $0 < \alpha \leq 1$.

Q.E.D.

It follows from Theorem 7.1 that the set of points $x_{ij} \geq 0$, $j \in \{1,\ldots,m_i\}$ satisfying (7.2) is a convex set whenever :

(1) $\alpha_i \geq .5$ and $1 \leq \alpha \leq 2$

(2) $\alpha_i \leq .5$ and $0 < \alpha \leq 1$

The most interesting case from a computational viewpoint is clearly when the a_{ij}^i $(j=1,\ldots,m_i)$ are Cauchy-distributed ($\alpha = 1$) since inequality (7.2) becomes linear in x_{ij}, $\forall$ j. Similar to the procedure in Section 5, we define a set C_i^+ as :

$$C_i^+ = C_i^+ (D_{(i-1)}, x_{(i-1)})$$

$$= \left\{ x_i \geq 0 \mid (1)\ \forall D_{(k-1)}^{(i-1)}\ \exists \lambda_{jp}(D_{(j-1)}) \geq 0, \begin{cases} p = 1,\ldots,m_j \\ j = i+1,\ldots,k \end{cases} \right. \tag{7.3}$$

such that (7.4) holds for all $k = i+1,\ldots,n$

(2) (7.2) holds

where

$$\sum_{p=1}^{m_i} a_{ip}^k x_{ip} + \sum_{j=i+1}^{k-1} \sum_{p=1}^{m_j} a_{jp}^k \lambda_{jp} + \sum_{j=1}^{m_k} \delta_{kp}^k \lambda_{kp} + F^{-1}(\alpha_k) \left(\sum_{p=1}^{m_k} c_{kp}^k \lambda_{kp}^{\alpha} \right)^{1/\alpha} \tag{7.4}$$

$$\leq b_k - \sum_{j=1}^{i-1} \sum_{p=1}^{m_j} a_{jp}^k x_{jp} \qquad (1)$$

---

(1) For ease of notation we use abbreviated expressions as :

$\lambda_{jp} \equiv \lambda_{jp}(D_{(j-1)})$ $\qquad c_{kp}^k \equiv c_{kp}^k (D_{(k-1)})$

$\delta_{jp}^k \equiv \delta_{kp}^k (D_{(k-1)})$

However since stable distributions assign positive probability to any interval on the real line, we will generally be able to find a^k_{jp}, $p=1,\ldots,m_j$ and $j=i,\ldots,k$ such that $C^+_i = \emptyset$. Therefore to have a meaningful problem in practice we should only consider values of a^k_{jp} within a certain interval around δ^k_{jp}, $p \in \{1,\ldots,m_j\}$ and $j \in \{i,\ldots,k\}$. A procedure to follow in practice might be as follows : for $k=i+1$, we consider values of

a^{i+1}_{ip}, $p \in \{1,\ldots,m_i\}$, in a fairly broad interval around δ^{i+1}_{ip}.

As k grows larger, the intervals around the location parameters δ^k_{jp}, $p \in \{1,\ldots,m_j\}$ and $j \in \{1,\ldots,k-1\}$, can gradually be taken smaller.

Using the general procedure of Theorem 5.1 and Minkowski's inequality it is easy to prove that the set C^+_i is convex if

(1) $1 \leq \alpha \leq 2$ and $\alpha_j \geq .5$, $j \in \{i,\ldots,k\}$

(2) $0 < \alpha \leq 1$ and $\alpha_j \leq .5$, $j \in \{i,\ldots,k\}$. (1)

Let us now drop the assumption of conditional independence and assume that the vector $(a^i_{i1},\ldots,a^i_{im_i})$ has a multivariate symmetric stable distribution of order 1 [See Appendix] :

$$\mathcal{L}(a^i_{i1},\ldots,a^i_{im_i}) = S_{m_i}(1,\delta,\Omega,\alpha).$$

By (A.7) it follows that :

$$\mathcal{L}\left(\sum_{j=1}^{m_i} a^i_{ij}x_{ij}\right) = S_1(1,\delta x_i, x_i\Omega x_i, \alpha).$$

---

(1) Throughout the paper we have kept the characteristic coefficient the same for all the stable distributions involved in the model. The requirement however is that α be equal only for stable distributions regarding variables whose values will be known in the same period.

By the properties of characteristic functions, then :

$$\mathcal{L}\left(\frac{\sum_i a^i_{ij} x_{ij} - \delta x_i}{(x_i \Omega x_i)^{1/2}}\right) = S_1\,(1, 0, 1, \alpha)$$

so that the log characteristic function of $\dfrac{\sum_j a^i_{ij} x_{ij} - \delta x_i}{(x_i \Omega x_i)^{1/2}}$ is :

$$\log \phi(t) = -\frac{1}{2}\,|t|^{\alpha}$$

The deterministic equivalent of the i^{th} constraint can then be derived as :

$$(7.5) \quad \delta x_i + F^{-1}(\alpha_i)\,(x_i \Omega x_i)^{1/2} \leqslant b_i - \sum_{j=1}^{i-1} \sum_{p=1}^{m_j} a^i_{jp} x_{jp}$$

where $F^{-1}(\alpha_i)$ can be found again in the tables in [7] if $1 \leqslant \alpha \leqslant 2$.

In a similar way as in (7.3) we define a set C_i^+ using (7.5). Since $(x_i \Omega x_i)^{1/2}$ is a convex function (for a proof, see [9]), convexity of the set C_i^+ can be proved whenever $\alpha_j \geqslant .5$, $j \in i, \ldots, n$.

7. Numerical Examples

Example 1 : Consider the following two-period problem :

$$\max \; E\,(c_1 x_1 + c_2 x_2)$$

subject to : $\Pr\,(a_1 x_1 \leqslant b_1) \geqslant \alpha_1$

$$\Pr\,(a_1 x_1 + a_2 x_2 \leqslant b_2 \mid a_1, c_1) \geqslant \alpha_2$$

$$x_1, x_2 \geqslant 0$$

with $\quad c_1 \sim U(15,25)$

$$c_2 \sim U(c_1-6,\ c_1+4)$$

$$a_1 \sim U(100,200)$$

$$a_2 \sim U(75, 2a_1 - 75)$$

$$b_1, b_2 \geqslant 0$$

where $y \sim U(p,q)$ means that y is uniformly distributed over the interval $[p,q]$.

The second period maximization problem can be written :

$$\max_{x_2 \quad c_2/c_1} E\ (c_2 x_2) = \max_{x_2}\ (c_1 - 1)\ x_2$$

Since $c_1 - 1 > 0$ for all possible values of c_1, we will choose x_2 as large as possible :

$$x_2^* = \max \left\{ \frac{b_2 - a_1 x_1}{F^{-1}_{a_2\ a_1}(\alpha_2)} \ ,\ 0 \right\}$$

$$= \max \left\{ \frac{b_2 - a_1 x_1}{\alpha_2 (2a_1 - 150) + 75} \ ,\ 0 \right\}$$

By remark 5.2.:

$$b_2 - a_1 x_1 \geqslant 0 \qquad \text{for} \quad \forall\ a_1$$

It follows that :

$$x_2^{*} = \frac{b_2 - a_1 x_1^{*}}{\alpha_2(2a_1-150)+75}$$

The first period constraints are given by :

$$x_1 \leqslant \frac{b_1}{F_{a_1}^{-1}\ (\alpha_1)} = \frac{b_1}{10(1+\alpha_1)}$$

$$b_2 - a_1 x_1 \geqslant 0 \qquad \text{for } \forall\, a_1$$

which is equivalent to :

$$\begin{cases} x_1 \leqslant \dfrac{b_1}{10(1+\alpha_1)} \\ x_1 \leqslant \dfrac{b_2}{200} \end{cases}$$

The solution set of the first period problem follows then as :

$$C_1^{+} = \left\{ x_1 \geqslant 0 / x_1 \leqslant \min \left(\frac{b_1}{100(1+\alpha_1)} \ , \ \frac{b_2}{200} \right) \right\}$$

Now,

$$\Psi_1 = \max_{x_1 \in C_1^{+}} \ \underset{a_1, c_1}{E} \left\{ c_1 x_1 + (c_1 - 1) \left[\frac{b_2 - a_1 x_1}{\alpha_2(2a_1-150) + 75} \right] \right\}$$

$$= \max_{x_1 \in C_1^{+}} \left\{ 20\, x_1 - 19\, x_1 \ \underset{a_1}{E} \ \frac{a_1}{\alpha_2(2a_1-150) + 75} + K \right\}$$

where K is independent of x_1.

As can be verified ; for $\alpha_2 > 0$:

$$E_{a_1}\left[\frac{a_1}{\alpha_2(2a_1-150)+75}\right] = \frac{1}{2\alpha_2} + \frac{3}{8\alpha_2}\left(1 - \frac{1}{2\alpha_2}\right)\ln\frac{3+10\alpha_2}{3+2\alpha_2}$$

If $\mathcal{L} = 20 - \frac{19}{2\alpha_2} - \frac{57}{8\alpha_2}\left(1 - \frac{1}{2\alpha_2}\right)\ln\frac{3+10\alpha_2}{3+2\alpha_2} > 0$

$$\Rightarrow x_1^* = \min\left(\frac{b_1}{100(1+\alpha_1)}, \frac{b_2}{200}\right)$$

$$\mathcal{L} < 0 \Rightarrow x_1^* = 0$$

$\mathcal{L} = 0 \Rightarrow$ choose any $x_1^* \ni$

$$0 \leq x_1^* \leq \min\left(\frac{b_1}{100(1+\alpha_1)}, \frac{b_2}{200}\right)$$

Example 2 : Consider the following three period problem :

$$\max \sum_{i=1}^{3} x_i$$

s.t. $\Pr(a_1x_1 \leq b_1) \geq \alpha_1$

$$\Pr(a_1x_1 + a_2x_2 \leq b_2) \geq \alpha_2$$

$$\Pr(a_1x_1 + a_2x_2 + a_3x_3 \leq b_3) \geq \alpha_3$$

$$x_i \geq 0 \ , \quad i = 1, 2, 3$$

where

$a_1 \sim U(4,8)$

$a_2 \sim U(10-a_1, 10+a_1)$

$a_3 \sim U(-a_2, 3a_2)$

and $\alpha_2 = 1/2$; $\alpha_3 = 3/4$.

The third period problem :

$$\max x_3$$

$$\text{s.t.} \begin{cases} x_3 \, F^{-1}_{a_3|a_2} \, (\alpha_3) \leqslant b_3 - a_1 x_1 - a_2 x_2 \\ x_3 \geqslant 0. \end{cases}$$

Since $F^{-1}_{a_3|a_2}\ (\alpha_3) = (4\alpha_3 - 1)a_2 = 2a_2 > 0$ for all possible values of a_1 and a_2 , we have

$$x_3^{*}\ (d_{(2)} = \frac{b_3 - a_1 x_1^{*} - a_2 x_2^{*}}{2\ a_2}$$

since we have from the previous period constraint that

$$\exists\ \lambda_3 \geqslant 0 \ni a_1 x_1 + a_2 x_2 + \lambda_3 F^{-1}_{a_3|a_2}\ (\alpha_3) \leqslant b_3 \quad \text{for all } a_2$$

which implies $a_2 x_2 \leqslant b_1 - a_1 x_1$ for all a_2.

The second period problem :

$$\max_{x_2} \left\{ x_2 + \mathop{E}_{a_2|a_1} \left(\frac{b_3 - a_1 x_1 - a_2 x_2}{2\ a_2} \right) \right\}$$

$$\text{s.t.} \quad \left[10 + (2\alpha_2 - 1)\ a_1\right]\ x_2 \leqslant b_2 - a_1 x_1$$

$$\max_{a_2}\ x_2 \leqslant b_3 - a_1 x_1$$

$$x_2 \geqslant 0$$

or

$$x_2 \leqslant \min \left(\frac{b_2 - a_1 x_1}{10} \;,\; \frac{b_3 - a_1 x_1}{10 + a_1} \right)$$

$$x_2 \geqslant 0$$

Since the coefficient of x_2 (= 1/2) is positive, we take x_2 as large as possible , i.e.

$$x_2^* (d_1) = \min \left(\frac{b_2 - a_1 x_1}{10} \;,\; \frac{b_3 - a_1 x_1}{10 + a_1} \right) .$$

We again know that this minimum is non-negative from the first period 1-feasibility constraints. To facilitate computation we assume $b_2 \geqslant b_3$ so that

$$x_2^* (d_1) = \frac{b_3 - a_1 x_1}{10 + a_1}$$

The first period problem :

$$\max \left\{ x_1 + E \left[\frac{b_3 - a_1 x_1}{10 + a_1} + \underset{a_2/a_1}{E} \left(\frac{b_3 - a_1 x_1 - a_2 (b_3 - a_1 x_1 / 10 + a_1}{2\, a_2} \right) \right] \right\}$$

$$\text{s.t.} \qquad x_1 F^{-1}_{a_1} (\alpha_1) \leqslant b_1$$

$$\exists\, \lambda_2 (d_1) \geqslant 0 \quad \text{and} \quad \lambda_3 (d_2) \geqslant 0 \quad \text{such that}$$

$$a_1 x_1 + \lambda_2 (d_1) F^{-1}_{a_2/a_1} (1/2) \leqslant b_2, \ \forall\, a_1$$

$$a_1 x_1 + a_2 \lambda_2 (d_1) + \lambda_3 (d_{(2)}) F^{-1}_{a_3/a_2} (3/4) \leqslant b_3, \ \forall\, a_1, a_2$$

$$x_1 \geqslant 0$$

Since $F^{-1}_{a_2/a_1}\ (1/2) > 0\ ,\ \forall\ a_1$

$F^{-1}_{a_3/a_2}\ (3/4) > 0\ ,\ \forall\ a_1, a_2$

it is easily verified that the above constraints reduce to :

$$0 \leqslant x_1 \leqslant \min\left(\frac{b_1}{F^{-1}_{a_1}(\alpha_1)}\ ,\ \frac{b_2}{8}\ ,\ \frac{b_3}{8}\right) = \min\left(\frac{b_1}{F^{-1}_{a_1}(\alpha_1)}\ ,\ \frac{b_3}{8}\right)$$

The coefficient of x_1 in the objective function can be computed as :

$$\frac{3}{2} + \frac{17}{8}\log 7 - \frac{35}{8}\log 3 > 0$$

which means that

$$x_1^* = \min\left(\frac{b_1}{F^{-1}_{a_1}(\alpha_1)}\ ,\ \frac{b_3}{8}\right)$$

APPENDIX

1. Definitions and Properties

Def. A.1 : Two distribution functions F and G belong to the same type if they are connected by the following relation :

(A.1) $$G(x) = F\left(\frac{x-a}{b}\right) \text{ with } b > 0.$$

Def. A.2 : A distribution belongs to a stable type if its type is closed with respect to convolutions. (see [12] and [8]).

Properties of Stable Distributions

1. All distributions are absolutely continuous.
2. The log characteristic function of the most general form of a stable distribution is of the form :

(A.2) $$\log \quad (t) = i\delta t - c|t|^{\alpha}\left\{1 + i\beta \frac{t}{|t|} h(|t|,\alpha)\right\}$$

where the constants c, β, α satisfy $c \geq 0$

$|\beta| \leq 1$

$0 < \alpha \leq 2$ and α real.

$h(|t|, \alpha)$ is given by :

$$h(|t|,\alpha) = \text{tang } \alpha\pi/2 \quad \text{if } \alpha \neq 1$$
$$= 2/\pi \log |t| \quad \text{if } \alpha = 1$$

The distribution is called symmetric stable if $\beta = 0$.

3. All stable distributions are unimodal.
4. For $0 < \alpha \leq 1$, stable distributions have no first or higher order moments.
   $1 < \alpha < 2$, a first moment exists but no higher moments.
   $\alpha = 2$, all moments exist.

5. Stable distribution functions with exponent $0 < \alpha < 1$ and parameter $|\beta| = 1$ are one-sided distributions. They are bounded to the right if $\beta = +1$ and bounded to the left if $\beta = -1$.

6. The following special cases arise :

- for $\alpha = 2$, $\beta = 0$: $\log \phi(t) = i\delta t - ct^2$ corresponds to the log characteristic function of a normal distribution.
- for $\alpha = 1$, $\beta = 0$: $\log \phi(t) = i\delta t - c|t|$ corresponds to the log characteristic function of a Cauchy-distribution with density function :

$$p(x) = \frac{c}{\pi\left[c^2 + (x-\delta)^2\right]}, \quad -\infty < x < \infty, \quad c > 0.$$

- for $\alpha = 1/2$, $\beta = -1$, $c=1$, $\delta = 0$: $\log \phi(t) = -|t|^{1/2}\left\{1 - i\frac{t}{|t|}\right\}$

  corresponds to the log characteristic function of a one-sided distribution function with density :

$$\text{(A.3)} \quad p(x) = 0 \quad \text{if } x < 0$$
$$= (2\pi)^{-1/2} x^{-3/2} e^{-1/2x} \quad \text{if } x > 0.$$

Apart from these special cases, no stable distribution functions are known whose density functions are elementary functions.

2. Symmetric Stable Distribution Functions

Suppose x has a symmetric stable distribution with log characteristic function :

$$\text{(A.4)} \quad \log \phi_x(t) = i\delta t - c|t|^\alpha$$

It follows that the standardized variable

$$\text{(A.5)} \quad u = \frac{x - \delta}{c^{1/\alpha}}$$

has a log characteristic function :

$$\log \phi_u(t) = -|t|^{\alpha}$$

Using results of Bergstrom on series expansion to approximate densities [2], Fama and Roll [7] computed cumulative distribution functions and fractiles of standardized symmetric stable distributions for the characteristic exponent $1 \leq \alpha \leq 2$. They also discuss estimation procedures for the coefficients δ, c and α.

The univariate family of stable distributions has been extended to the multivariate case [11]. In the case of multivariate symmetric stable distributions, Press [15] considers the following family which has several interesting properties :

$$\text{(A.6)} \qquad \log \phi_x(t) = i\delta t - \frac{1}{2} \sum_{j=1}^{m} (t'\Omega_j t)^{\alpha/2}$$

where m is some integer ≥ 1. (m is called the order of the family)

$\delta = (\delta_1, \ldots, \delta_p)$ is an arbitrary p-vector

Ω_j : (pxp) positive semidefinite matrix, $\forall$ j

α characteristic exponent $0 < \alpha \leq 2$

$x = (x_1, \ldots, x_p)$

If a vector $x = (x_1, \ldots, x_p)$ belongs to the family with log characteristic function (A.6), we denote this by :

$$\mathcal{L}(x) = S_p(m, \delta, \Omega_i, \alpha)$$

We will use the following property (for a proof, see [15]). Suppose x: pxl and $\mathcal{L}(x) = S_p(m, \delta, \Omega_i, \alpha)$. Then if y: qxl and $y = Ax + b$, where A: qxp and b: qxl, $q \leq p$,

$$(A.7)\quad \mathcal{L}(y) = S_q\,(m, A\beta + b,\ A\Omega_i A', \alpha).$$

Press does not give estimation procedures for the parameters ; however one of his next papers will deal with this problem.

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