The Trispectrum of the 4 Year COBE DMR Data

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THE TRISPECTRUM OF THE 4 YEAR *COBE* DMR DATA

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ABSTRACT

We propose an estimator for the trispectrum of a scalar random field on a sphere, discuss its geometrical and statistical properties, and outline its implementation. By estimating the trispectrum of the 4 yr *COBE* Differential Microwave Radiometer experiment data (in HEALPix pixelization), we find new evidence of a non-Gaussian signal associated with a known systematic effect. We find that by removing data from the sky maps for those periods of time perturbed by this effect, the amplitudes of the trispectrum coefficients become completely consistent with predictions for a Gaussian sky. These results reinforce the importance of statistical methods based in harmonic space for quantifying non-Gaussianity.

Subject headings: cosmic microwave background — cosmology: observations

1. INTRODUCTION

The cosmic microwave background (CMB) is the cleanest window on the origin of structure in the very early universe. A complete description of the statistical properties of cosmological fluctuations at a redshift of $z \simeq 1000$ affords us an essential insight into those processes that may have seeded the formation of galaxies. In a Gaussian theory of structure formation, such as the currently favored model of inflation, the power spectrum contains all possible information about the fluctuations. Any higher order moment can subsequently be described in terms of it. However, if the theory is non-Gaussian (as expected for structure formation theories due to local effects from primordial phase transitions or more generally from nonlinear processes), then there will be deviations from the simple Gaussian expressions for the higher order moments. Such behavior can serve as a powerful discriminator between different models of structure formation.

Most analyses of CMB data to date have focused on the angular power spectrum and its sensitivity to various parameters of cosmological theories. Some work has been done on the estimation of the three-point correlation function and its analog in spherical harmonic space, with intriguing results (Heavens 1998; Ferreira, Magueijo, & Górski 1998; Magueijo 2000; Banday, Zaroubi, & Górski 2000). It is the purpose of this Letter to propose a method for estimating the four-point spectrum, the *trispectrum*, and to apply it to the *COBE* Differential Microwave Radiometer (DMR) 4 yr data. This work complements the recent work of Hu (2001), where some of the properties of the angular trispectrum of the CMB are discussed.

The outline of this Letter is as follows: In § 2 we construct a set of orthonormal estimators and describe their properties for a Gaussian random field. In § 3 we apply the estimators to the *COBE* DMR 4 yr data. We show that we detect the non-Gaussian signal found in Ferreira et al. (1998), that it can be explained by the arguments presented in Banday et al. (2000), and in particular that this is a manifestation of a known systematic effect. We therefore conclude that the *COBE* 4 yr data is consistent with a Gaussian cosmological signal. In § 4 we summarize our results.

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2. THE ESTIMATOR

In this section we wish to construct a set of quantities for estimating the trispectrum of a random field on the sphere. The temperature anisotropy in a given direction on the celestial sphere, $T(\boldsymbol{n})$, can be expanded in terms of spherical harmonic functions, $Y_{lm}(\boldsymbol{n})$:

$$T(\boldsymbol{n}) = \sum_{lm} a_{lm} Y_{lm}(\boldsymbol{n}). \tag{1}$$

For any theory of structure formation, the a_{lm} coefficients are a set of random variables; we shall restrict ourselves to theories that are statistically homogeneous and isotropic. In this case, we can define the power spectrum C_l of the temperature anisotropies by $\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'}$.

We now seek to construct a set of tensors that are geometrically independent, describe their statistical properties for a Gaussian random field, and then discuss the practical issue of their implementation. Given a set of a_{lm}, we wish to find the index structure of the set of four-point correlators such that (1) they are rotationally invariant, (2) they form a complete basis (preferably orthonormal) of the whole space of admissible four-point correlators, and (3) they satisfy the appropriate symmetries under interchanges of m- and l-values. We shall restrict ourselves to the case in which $l_1 = l_2 = l_3 = l_4 = l$. Furthermore, throughout this section we keep l fixed. We determine the tensor $\mathcal{T}$ such that

$$\langle a_{lm_1} a_{lm_2} a_{lm_3} a_{lm_4} \rangle = \sum_{a=0}^{n} T_{l;\,a} \mathcal{T}^{a;\,l}_{m_1 m_2 m_3 m_4}, \tag{2}$$

where $n = \mathrm{int}(l/3)$ (due to reflection, permutation, and rotational symmetry). The $T_{l;\,a}$-values are then the components of the trispectrum that we wish to estimate. The explicit form of $\mathcal{T}$ is

$$\mathcal{T}^{a;\,l}_{m_1m_2m_3m_4} = \sum_{\alpha=0}^{l} L_l^{a\alpha}\bar{\mathcal{T}}^{\alpha;\,l}_{m_1m_2m_3m_4}, \tag{3}$$

$$\bar{\mathcal{T}}^{\alpha;\,l}_{m_1m_2m_3m_4} = \sum_{M=-2\alpha}^{2\alpha}(-1)^M\begin{pmatrix} l & l & 2\alpha \\ m_1 & m_2 & M\end{pmatrix} \times\begin{pmatrix} 2\alpha & l & l \\ -M & m_3 & m_4\end{pmatrix} + \text{inequivalent permutations}, \tag{4}$$

where the matrices in parentheses are the Wigner 3J symbols. The $\bar{\mathcal{T}}^{\alpha;\,l}$ are not orthogonal and satisfy

$$\bar{\mathcal{T}}^{\alpha;\,l}_{m_1m_2m_3m_4}\bar{\mathcal{T}}^{\alpha;\,l}_{m_1m_2m_3m_4} = \frac{3}{4\alpha+1}\delta_{\alpha\beta} + 6\begin{Bmatrix} l & l & 2\alpha \\ l & l & 2\beta\end{Bmatrix} \tag{5}$$

(where summation over the m_i is assumed), which has a rank of $n+1$. The matrix L_l in equation (3) is a rectangular matrix (with a triangular subblock) with $n+1$ columns and $l+1$ rows. It is constructed through a Gram-Schmidt procedure by subtracting for each α (starting from $\alpha = 0$) the projection onto all $a' < a$ and then normalizing the result. The $\alpha = 0$ (and hence $a = 0$) tensor is proportional to the Gaussian contribution. This can be easily seen given that for $\alpha = 0$ the Wigner 3J symbols are simply Kronecker δ symbols in the corresponding indices. The remaining $a > 0$ terms contain therefore no Gaussian signal and quantify the non-Gaussian part of the trispectrum.

The $\mathcal{T}$-values are orthonormal and can be used to construct an estimator for T_a from a realization of a_{lm}:

$$\hat{T}_{l;\,a} = \mathcal{T}^{a;\,l}_{m_1m_2m_3m_4}a_{lm_1}a_{lm_2}a_{lm_3}a_{lm_4}. \tag{6}$$

For a Gaussian random field we expect $\sigma^2[\hat{T}_{l;\,0}] \gg \sigma^2[\hat{T}_{l;\,a}]$ for $a > 0$, where $\sigma^2[A]$ denotes the variance of the random variable A and $\hat{T}_{l;\,0}$ is simply the square of the minimum variance estimator of the C_l. One finds that $\langle\hat{T}_{l;\,a}\rangle = 0$ and $\sigma^2[\hat{T}_{l;\,a}] = 24C_l^4$ for all $a > 0$.

To show that the $\hat{T}_{l;\,a}$ constitute a family of minimum variance estimators, we construct a linear combination of the estimators,

$$\mathcal{T}^{l}_{m_1m_2m_3m_4} = \sum_{a=0}^{n} c_a\mathcal{T}^{a;\,l}_{m_1m_2m_3m_4}, \tag{7}$$

and minimize the function

$$\begin{aligned}\sigma_l^2[c_a, \lambda] = {} & \langle(\mathcal{T}^{l}_{m_1m_2m_3m_4}a_{lm_1}a_{lm_2}a_{lm_3}a_{lm_4})^2\rangle \\ & - \langle\mathcal{T}^{\ell}_{m_1m_2m_3m_4}a_{lm_1}a_{lm_2}a_{lm_3}a_{lm_4}\rangle^2 \\ & - \lambda C_l^2(\mathcal{T}^{l}_{m_1m_2m_3m_4}\mathcal{T}^{l}_{m_1m_2m_3m_4} - 1),\end{aligned} \tag{8}$$

where summation over all m_i is implied. The last term, a Lagrange multiplier, ensures that $\mathcal{T}^l$ is normalized. We solve $\partial_{c_a}\sigma_l^2[c_a, \lambda] = \partial_\lambda\sigma_l^2[c_a, \lambda] = 0$ to find a set of two equations:

$$\begin{aligned}(24I + 72A_l)^{ab}c_b + \lambda c_b &= 0, \\ c^2 &= 1,\end{aligned} \tag{9}$$

where

$$A_l^{ab} = \mathcal{T}^{a;\,l}_{m_1m_2m_am_a}\mathcal{T}^{b;\,l}_{m_1m_2m_bm_b}. \tag{10}$$

This is an eigenvector equation where for a given eigenvector $\boldsymbol{c}$, the eigenvalue λ gives the expected variance of the estimator. Of the $n+1$ eigenvalues, one is large and has an eigenvector proportional to $\hat{T}_{l;\,a}$. The remaining eigenvalues have an amplitude of $\lambda = 24$, and each eigenvector is a $\hat{T}_{l;\,a}$ for $a > 0$.

Note that we can relate our parameterization to the one proposed in Hu (2001); if we reexpress equation (2) as

$$\langle a_{lm_1}a_{lm_2}a_{lm_3}a_{lm_4}\rangle = \sum_{\alpha=0}^{l}\bar{T}_{l;\,\alpha}\bar{\mathcal{T}}^{\alpha;\,l}_{m_1m_2m_3m_4}, \tag{11}$$

where $T_{l;\,a} = L_l^{a\alpha}\bar{T}_{l;\,\alpha}$, then Q_{ll}^{ll} as defined in equation (15) of Hu (2001) can be written as

$$Q_{ll}^{ll}(2\alpha) = \bar{T}_{l;\,\alpha} + 2(4\alpha+1)\sum_\beta\begin{Bmatrix} l & l & 2\alpha \\ l & l & 2\beta\end{Bmatrix}\bar{T}_{l;\,\beta}. \tag{12}$$

The numerical implementation of these estimators is more involved than for the bispectrum. If we omit the numerous symmetries, we have to consider for each l a set of up to $8l^3$ Wigner 3J symbols (compared to just one for the bispectrum). There are reasonably fast ways for constructing the Wigner 3J symbols (Schulten & Gordon 1976), but the number of operations per estimator scales as $O(l^6)$. For repeated computations of the estimators (e.g., in Monte Carlo studies), this can be partially avoided by storing the precomputed estimators in a look-up table, with the amount of memory required scaling as $O(l^4)$.

Clearly, to be able to estimate the trispectrum on small angular scales, approximate methods must be developed to make the procedure computationally feasible. However, the ability to constrain non-Gaussianity on large angular scales is in any case more important physically for two reasons: the ratio of the non-Gaussian to the Gaussian signal will in general be higher for lower moments, and the signal-to-noise ratio is better for low l. To understand these points, let us assume a source for non-Gaussianity that leads to approximately scale-invariant moments of the gravitational potential on arbitrary scales; i.e., $\langle\Phi(R)^N\rangle$ is constant for any R, where $\Phi(R)$ is the gravitational potential within a ball of radius R and $\langle\ldots\rangle$ denotes the ensemble average. This might be expected from a primordial source with no preferred scale such as inflation (Komatsu & Spergel 2000) or from an active source where the only scale is set by the horizon today (Durrer et al. 2000). Current observations of the CMB certainly favor such scale-invariant descriptions of the potential. One then expects the moment on the order of N of the a_{lm} to scale as $l^{2(1-N)}$. This signal will be competing against the fluctuations due to the disconnected (or Gaussian) part, which is proportional to $N!\,l^{-(2N+1)/2}$, the former therefore dominating for $N > 2$. Since the power spectrum for white noise has constant amplitude, the signal-to-noise ratio as a function of scale will have the same form as the scale-invariant power spectrum itself, therefore being larger for smaller l, i.e., larger angular scales.

3. RESULTS

As an application of the formalism described in § 2, we estimate the trispectrum of the co-added 53 and 90 GHz *COBE* DMR 4 yr sky maps in HEALPix format (Górski, Hivon, & Wandelt 1999). The resolution of the maps is $N_{\rm side} = 64$, or 49,152 pixels.

We do not extend our analysis beyond $l_{\rm max} = 20$ since the signal-to-noise ratio is poor for higher l. Hence, the maximal number of independent non-Gaussian estimators for the trispectrum is $\mathrm{int}(l_{\rm max}/3) = 6$. We set the pixels in the extended Galactic cut (Banday et al. 1997) to zero and subtract the residual monopole and dipole of the resulting map. After convolving the maps with spherical harmonics to extract a set of a_{lm}-values for $l \leq 20$, we then apply equation (6). To validate our software, we have estimated the bispectrum of the *COBE* DMR 4 yr sky data repixelized in the HEALPix format (denoted by EC for convenience) and reproduced the results of Ferreira et al. (1998), in particular the strong non-Gaussian signal present at $l = 16$. When an equivalent map, from which that part of the DMR time stream contaminated by the "eclipse effect"[5] is removed (denoted NEC), is subsequently analyzed, we also reproduce the results of Banday et al. (2000), namely, that the non-Gaussian signal is no longer detected. For our subsequent analysis we will present the trispectra of *both* the EC and NEC data.

One of our primary concerns is to compare our results with the assumption that the CMB sky measured by *COBE* DMR is Gaussian. To do so, we generate 10,000 full-sky maps at the same resolution using a scale-invariant power spectrum normalized to $Q_{\rm rms-PS} = 18\ \mu$K (Górski et al. 1998). We convolve each map with the DMR beam and add uncorrelated pixel noise with rms amplitude $\sigma_n = 15.95\ {\rm mK}/(N_{\rm obs})^{1/2}$, (where $N_{\rm obs}$ is the number of times a given pixel was observed); we then subject the synthetic map to the same procedure as the original data.

Figure 1 shows the trispectra of the DMR data together with Gaussian 95% confidence limits. Instead of the "raw" estimator (6), we prefer to use the normalized trispectrum, $\tau_l^{(a)} = \hat{T}_{l;\,a}/\hat{C}_l^2$ for $a > 1$ (where $\hat{C}_l = [1/(2l+1)] \sum_m |a_{lm}|^2$), thus effectively removing the dependence on the power spectrum. This prevents fluctuations in the power spectrum from introducing spurious signals and from masking real non-Gaussianities. Figure 1 shows that in this case, most values fall within the 95% confidence lines and demonstrate the scatter expected for a Gaussian random field.

Of particular interest is the value of the normalized $\tau^{(3)}$ at $l = 16$ in Figure 1. One finds that 99.9% of the Gaussian models in the EC case have a smaller $\tau^{(3)}$ than the measured one. This is clearly a manifestation of the non-Gaussianity found in Ferreira et al. (1998), which is highly localized in l space. However, if we estimate $\tau^{(3)}$ for the NEC we find that it falls comfortably within the 95% confidence limits. This leads us to believe that this detection of non-Gaussianity results from the eclipse effect, consistent with the hypothesis of Banday et al. (2000).

We construct a goodness of fit for our statistic. In Ferreira et al. (1998), a modified χ^2 was constructed that took into account the non-Gaussian distribution of each method: as above, the distribution of each estimator for a Gaussian sky

[5] The eclipse effect was an orbitally modulated signal that took place for approximately 2 months every year around the June solstice when the *COBE* spacecraft repeatedly flew through the Earth's shadow.

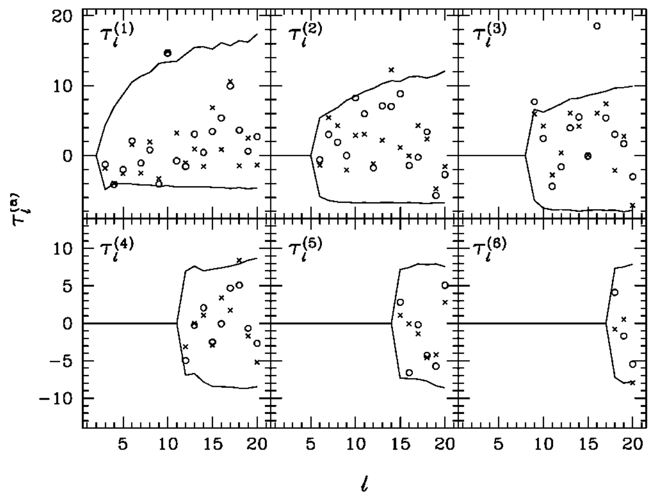

FIG. 1.—Six estimators of the normalized trispectrum applied to the EC data (*circles*) and NEC data (*crosses*); 95% of all simulated Gaussian skies lie within the solid lines. Although removing the eclipse data changes the noise properties, we find that the Gaussian confidence limits remain essentially unchanged.

was constructed and used as an approximate likelihood function to evaluate the goodness of fit. One shortcoming of such a method was that correlations between the estimates for different l-values were discarded. To include them, we use the Gaussian ensemble of data sets to derive the expectation values $\langle \ldots \rangle_{\rm G}$ and the covariance matrix C for both the power spectrum, C_l, and all seven trispectrum estimators, $\tau_l^{(0)}$ to $\tau_l^{(6)}$. We proceed to calculate the χ^2-value for the estimator $\mathcal{E}$ and the data set $\mathcal{D}$,

$$\chi^2[\mathcal{E}, \mathcal{D}] \equiv \sum_{l,\,l'} [\langle \mathcal{E}_l \rangle_{\rm G} - \mathcal{E}(\mathcal{D})_l] C^{-1}_{ll'} [\langle \mathcal{E}_{l'} \rangle_{\rm G} - \mathcal{E}(\mathcal{D})_{l'}], \qquad (13)$$

using as data sets the EC and NEC data. Finally, we use another 10,000 Gaussian realizations to estimate the expected distribution of the χ^2 for both the EC and the NEC data.

For all normalized non-Gaussian trispectrum estimators ($\tau^{(1)}$ to $\tau^{(6)}$), we find that 94% of the Gaussian models have a smaller χ^2 than the EC data, as can be seen in Figure 2. As expected, the main contribution to the χ^2 for the EC data stems from $\tau^{(3)}$ at $l = 16$; indeed, this is the only normalized trispectrum estimator that exhibits any significant non-Gaussianity, in this case at about 99.9%. If we use the NEC data, the detection vanishes. In this case, 60% of all Gaussian models have a lower χ^2 when computed over all six trispectrum estimators (83% for $\tau^{(3)}$ alone). Hence, the NEC data is compatible with Gaussianity.

4. DISCUSSION

In this Letter, we have derived an estimator for the trispectrum of a scalar random field on the sphere. Application of this estimator, normalized by the power spectrum (a procedure adopted in Ferreira et al. 1998 for the bispectrum; see also Komatsu et al. 2002 for a detailed discussion), to the *COBE* DMR data provides evidence for non-Gaussianity at the 94% confidence level. As in the case of the bispectrum, the signal is mainly present in the $l = 16$ multipole (and the $\tau^{(3)}$ estimator here). However, when data is excluded to correct for the eclipse effect, the non-Gaussian behavior is removed, allowing us to

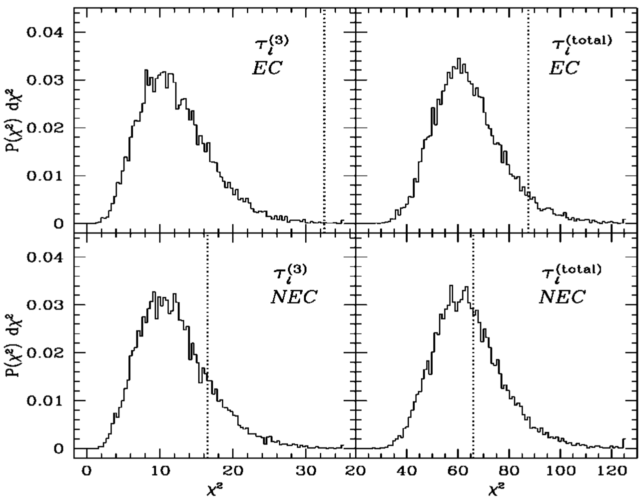

FIG. 2.—χ^2 distribution of the Gaussian models (*histogram*) and actual data value (*dotted line*) for the EC (*top row*) and NEC (*bottom row*) data sets. The left two graphs show $\tau^{(3)}$, which contains the main contribution to the non-Gaussian signal, while the right two graphs show the total χ^2 over all six non-Gaussian estimators, $\tau^{(1)}$ to $\tau^{(6)}$.

conclude that the non-Gaussianity present in the uncorrected sky maps is not cosmological in origin.

The detection of a signal that is so strongly localized in *l* space provides convincing support to our contention that the trispectrum is an important and sensitive probe of non-Gaussianity in the frequency (scale) domain. It affords complementary information to the bispectrum since it is an even moment and, despite the higher computational effort required, has the obvious advantage in that it can probe all values of *l*, not just the even ones.

Interestingly enough, from a theoretical perspective there may be some possible sources of non-Gaussianity for which the trispectrum provides a far more sensitive test than the bispectrum. In many cases, a given moment of the a_{lm}-values can be expressed as the projection of a cosmological field. If that field is vector-like in nature (as in the case of the Doppler effect or the Ostriker-Vishniac effect and its non-linear extensions), any odd moment may suffer from the Sunyaev-Kaiser cancellation, where the integral of a given wavenumber *k* over a smoothly varying projection function with width σ tends to suppress the moment by a factor on the order of $1/(\sigma k)^2$ (Sunyaev 1978; Kaiser 1984; Scannapieco 2000). For even moments one can always construct a scalar component that will not be subject to this cancellation. Such a tool will be of great use in the analysis of the data sets from the *Microwave Anisotropy Probe* and *Planck Surveyor* satellites.

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[6] See http://www.eso.org/science/healpix.

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Note added in proof.—E. Komatsu investigates the trispectrum of *COBE* DMR data in his Ph.D. thesis. His conclusions agree with ours, namely, that the *COBE* data is consistent with Gaussian initial fluctuations.