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# The Turán number of the graph $3P_4$

Abstract. Let $ex(n,G)$ denote the maximum number of edges in a graph on $n$ vertices which does not contain $G$ as a subgraph. Let $P_i$ denote a path consisting of $i$ vertices and let $mP_i$ denote $m$ disjoint copies of $P_i$. In this paper we count $ex(n,3P_4)$.

**1. Introduction.** Let $G=(V(G),E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The Turán number of the graph $G$, denoted by $ex(n,G)$, is the maximum number of edges in a graph on $n$ vertices which does not contain $G$ as a subgraph. Let $P_i$ denote a path consisting of $i$ vertices and let $mP_i$ denote $m$ disjoint copies of $P_i$. By $C_q$ we denote a cycle of order $q$. For two vertex disjoint graphs $G$ and $F$ by $G \cup F$ we denote the vertex disjoint union of $G$ and $F$, and by $G+F$ we denote the join of the graphs. By $\overline{G}$ we denote the complement of the graph $G$. For a vertex $x \in V(G)$ we define $N_G(x)=\{y \in V(G) | \{x,y\} \in E(G)\}$. Let $F$ be a subgraph of $G$. Let $\deg_F(x)=N_G(x)\cap V(F)$. Moreover, for $A \subseteq V(G)$ let $G|_A$ denote the subgraph of $G$ induced by $A$. The basic notions not defined in this paper can be found in [5]. First we present the following important lemma which is used to prove our main results.

**Lemma 1** (Erdős, Gallai [2]). *Suppose that $|V(G)|=n$. If the following inequality*

$$\frac{(n-1)(l-1)}{2}+1 \leq |E(G)|$$

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*is satisfied for some* $l \in \mathbf{N}$, *then there exists a cycle* $C_q$ *in* $G$ *for some* $q \geq l$.

We will use the following famous theorem.

**Theorem 1** (Faudree and Schelp [3]). *If* $G$ *is a graph with* $|V(G)| = kn + r$ $(0 \leq k, 0 \leq r < n)$ *and* $G$ *contains no* $P_{n+1}$, *then* $|E(G)| \leq kn(n-1)/2 + r(r-1)/2$ *with the equality if and only if* $G = kK_n \cup K_r$ *or* $G = tK_n \cup (K_{(n-1)/2} + \overline{K}_{(n+1)/2+(k-t-1)n+r})$ *for some* $0 \leq t < k$, *where* $n$ *is odd, and* $k > 0$, $r = (n \pm 1)/2$.

Gorgol [4] studied the Turán number for disjoint copies of graphs. She counted $ex(n, 2P_3)$ and $ex(n, 3P_3)$.

**Theorem 2** (Gorgol [4]).

$$ex(n, 2P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1, \text{ for } n \geq 9.$$

$$ex(n, 3P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 4, \text{ for } n \geq 14.$$

Moreover, she proved more general results concerning the properties of some extremal Turán graphs for disjoint copies of a given graph. Bushaw and Kettle [1] extended some of Gorgol's results as follows.

**Theorem 3** (Bushaw and Kettle [1]).

$$ex(n, kP_3) = \left\lfloor \frac{n-k+1}{2} \right\rfloor + (n-k+1)(k-1) + \binom{k-1}{2}, \text{ for } n \geq 7k.$$

$$ex(n, kP_t) = \left(n - k\left\lfloor \frac{t}{2} \right\rfloor + 1\right)\left(k\left\lfloor \frac{t}{2} \right\rfloor - 1\right) + \binom{k\lfloor \frac{t}{2} \rfloor - 1}{2} + \epsilon,$$

*for* $n \geq 2t\left(1 + k\left(\lceil \frac{t}{2} \rceil + 1\right)\binom{t}{\lfloor \frac{t}{2} \rfloor}\right)$, *where* $\epsilon = 1$ *for odd* $t$ *and* $\epsilon = 0$ *for even* $t$.

In particular, Bushaw and Kettle [1] counted $ex(n, 3P_4)$ for the case $n \geq 440$. We present $ex(n, 3P_4)$ for all positive integers $n$.

**2. Results.** First we prove the following result.

**Theorem 4.** *Let* $n \geq 15$. *Then*

$$ex(n, 3P_4) = 5n - 15. \tag{1}$$

**Proof.** First note that the graph $K_5 + \overline{K}_{n-5}$ does not contain $3P_4$ as a subgraph. Therefore, $ex(n, 3P_4) \geq 5n - 15$ and we would like to prove the opposite inequality. Suppose that there exists a graph $G$ with $|V(G)| = n \geq 15$ and $|E(G)| = 5n - 14$ without $3P_4$ as a subgraph. Applying Lemma 1 to the graph $G$, we obtain

$$\frac{(n-1)(l-1)}{2} + 1 \leq 5n - 14,$$

$$l \leq 11 - \frac{20}{n-1}.$$

By $n \geq 15$,

$$\frac{20}{n-1} < 2$$

and we conclude that $l \leq 9$. It means that the graph $G$ contains a cycle $C_q,\ q \geq 9$. Let $0, 1, 2, \ldots, q-1$ be the consecutive vertices in $C_q$. We should consider the following cases:

***Case 1.*** Let $q \geq 12$. We have $C_{12}$ in $G$, so $3P_4$ is a subgraph of $G$, a contradiction.

***Case 2.*** Let $q = 11$.

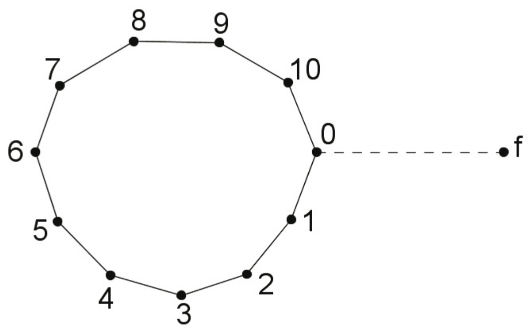


FIGURE 1. A graph $G$ with the cycle $C_{11}$.

Let $F = G - V(C_{11})$. Note that $C_{11}$ cannot be connected by an edge with $F$ (see Figure 1 for an illustration). The minimum number of edges in $F$ is equal to $5n - 14 - 55 = 5n - 69$. By Theorem 1 we know that

$$ex(k, P_4) = 3\left\lfloor \frac{k}{3} \right\rfloor + \binom{r}{2},\ k \equiv r \pmod 3$$

where $r$ is the rest from dividing $k$ by 3. We set $k = n - 11$. If

$$3\left\lfloor \frac{n-11}{3} \right\rfloor + \binom{r}{2} < 5n - 69$$

then it means that $P_4$ is a subgraph of $F$. We check this.

(a) $r = 0$

$$5n - 3\frac{n-11}{3} > 69$$
$$n > 14.$$

(b) $r = 1$

$$5n - 3\frac{n-12}{3} > 69$$
$$n > 14.$$

(c) $r = 2$

$$5n - 3\frac{n-13}{3} > 1 + 69$$
$$n > 14.$$

So we get $P_4$ in $F$, a contradiction.

***Case 3.*** Let $q = 10$.

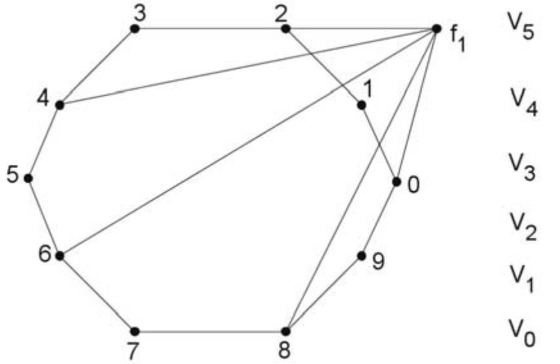


FIGURE 2. A graph $G$ with the cycle $C_{10}$.

Let $F = G - V(C_{10})$. Note that $|V(F)| = n - 10$. The set of edges containing a vertex of $F$ we can divide into:

- edges connecting $C_{10}$ and $F$, i.e. the edges $\{x, f\}$ with $x \in V(C_{10})$, $f \in V(F)$,
- edges connecting both vertices inside $F$, i.e. the edges $\{f_i, f_j\}$ with $f_i$, $f_j \in V(F)$, $i \neq j$.

Notice that if the edge $\{0, f_1\}$ exists for some $f_1 \in V(F)$, then there cannot exist edges $\{1, f_1\}$ and $\{9, f_1\}$, in the opposite case we obtain a longer cycle, i.e. $C_{11}$. So at most 5 vertices of $C_{10}$ can be adjacent to the vertex $f_1 \in V(F)$ (see Figure 2 for an illustration). Moreover, $\{j, f\} \notin E(G)$ for $f \in V(F) - \{f_1\}$ and $j \neq 2l + 1$, $l = 0, 1, 2, 3, 4$, $j \in V(C_{10})$, in the opposite case we get $3P_4$ in $G$. Let

$$V(F) = V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$$

be the partition of $V(F)$ such that each vertex from $V_i$ has exactly $i$ neighbors in $C_{10}$. Note that vertices from sets $V_i$, $i > 0$ cannot be connected between them and $\deg_F(u) = 0$ for each $u \in \bigcup_{i=1}^{5} V_i$. Vertices from the set $V_0$ can be connected only between them. So if $|V_0| = k$, then $|E(G|_{V_0})| \leq ex(k, P_4)$. First we show that $|E(G|_{V_0})| \leq |V_0|$. If $r \equiv 0 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k}{3} = k = |V_0|.$$

If $r \equiv 1 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k-1}{3} = k - 1 \leq |V_0|.$$

If $r \equiv 2 \pmod 3$, then

$$ex(k, P_4) = 3\frac{k-2}{3} + 1 = k - 1 \leq |V_0|.$$

We consider three subcases:

***Case 3.1.*** Let $V_5 \neq \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} - \binom{5}{2} + \sum_{i=1}^{5} i \cdot |V_i| + |E(V_0)| \leq 35 + \sum_{i=1}^{5} i \cdot |V_i| + |V_0|$$

$$\leq 35 + 5\sum_{i=1}^{5} |V_i| + 5|V_0| = 35 + 5(n - 10) = 5n - 15.$$

Recall that $|E(G)| = 5n - 14$. So we must add one more edge and we obtain $3P_4$ in $G$, a contradiction.

***Case 3.2.*** Let $V_5 = \emptyset$ and $V_4 \neq \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} - 5 + \sum_{i=1}^{4} i \cdot |V_i| + |V_0| \leq 45 - 5 + 4 \cdot \sum_{i=1}^{4} |V_i| + |V_0|$$
$$\leq 4n < 5n - 14$$

for $n \geq 15$. So again we must add one more edge which means that we get a $3P_4$ in $G$, a contradiction.

***Case 3.3.*** Let $V_5 = \emptyset$ and $V_4 = \emptyset$. Then

$$|E(G)| \leq \binom{10}{2} + 3\sum_{i=1}^{3} |V_i| + |V_0| \leq 45 + 3(n - 10) = 3n + 15 < 5n - 14$$

for $n \geq 15$. We obtain a contradiction.

***Case 4.*** Let $q = 9$. Let $F = G - V(C_9)$. If there does not exist any edge between $C_9$ and $F$, then $|E(F)| \geq 5n - 50$. So if $ex(n - 9, P_4) < 5n - 50$, then there exists a path $P_4$ in the graph $F$.

(a) $r = 0$

$$3\frac{n-9}{3} < 5n - 50,$$
$$n > 10.$$

(b) $r = 1$

$$3\frac{n-10}{3} < 5n - 50,$$
$$n > 10.$$

(c) $r = 2$

$$3\frac{n-11}{3} + 1 < 5n - 50,$$

$$n > 10.$$

In this case we obtain a contradiction.

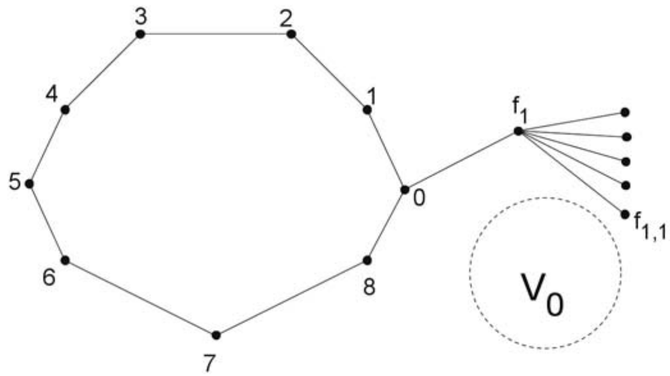


FIGURE 3. A graph $G$ with the cycle $C_9$.

Suppose that there exists an edge $\{0, f_1\}$ for some $f_1 \in V(F)$. Note that the vertex $f_1$ can be adjacent to another vertex $f_{1,1}$ from $F$ and we do not obtain $3P_4$ (see Figure 3 for an illustration). Now we cannot create other edges from $F - \{f_1, f_{1,1}\}$ to $C_9$, in the opposite case we obtain $3P_4$. Note that

$$\begin{aligned} |E(G)| &\leq \binom{9}{2} + 7 + |N_F(f_1)| + ex(n - 10 - |N_F(f_1)|, P_4) \\ &\leq 43 + |N_F(f_1)| + (n - 10 - |N_F(f_1)|) = n + 33 < 5n - 14 \end{aligned}$$

for $n \geq 12$.

So we have $3P_4$ in graph $G$, a contradiction. The proof is completed. □

**Remark 1.** Note that if $n \in \{1, \ldots, 11\}$, then $ex(n, 3P_4) = \binom{n}{2}$. It is clear because the total number of vertices does not exceed 12 and $K_n$ does not contain $3P_4$. Moreover, $\binom{n}{2} \geq 5n - 15$ for $n \in \{1, \ldots, 11\}$.

**Remark 2.** For $n = 12$ we have $ex(n, 3P_4) \geq \binom{11}{2} = 55$. It is clear because $K_{11} \cup K_1$ does not contain $3P_4$. Let $|E(G)| = 56$. Applying Lemma 1, we obtain that

$$\frac{11(l-1)}{2} + 1 \leq 56,$$

$$l \leq 11.$$

So there exists a cycle $C_q$, $q \geq 11$. It is clear that if $q = 12$, there exists $3P_4$.

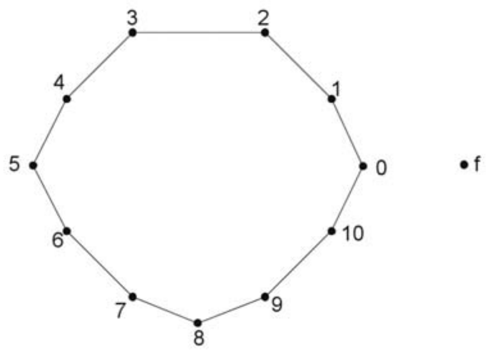


FIGURE 4. A graph with and $C_{11}$ for $ex(12, 3P_4)$.

If $q = 11$, then we have $\deg_{C_{11}}(f) = 0$ for $f \in V(F)$ and $|E(G|_{C_{11}})| \leq \binom{11}{2} = 55$ (see Figure 4 for an illustration). We get a contradiction.

For $n = 13$ we have $ex(n, 3P_4) \geq \binom{11}{2} + \binom{2}{2} = 56$. It follows from the fact that $K_{11} \cup K_2$ does not contain $3P_4$. Let $|E(G)| = 57$. Applying Lemma 1, we obtain that

$$\frac{12(l-1)}{2} + 1 \leq 57,$$

$$l \leq 10.$$

If $q \geq 12$, then there exists $3P_4$. If $q = 11$, then we have a cycle $C_{11}$ and a path $P_2$. But these two graphs cannot have edges between them and the total number of edges is equal to 56. So we must add one more edge and we obtain $3P_4$, a contradiction. If $q = 10$, then we have at most 45 edges in $G|_{V(C_{10})}$ and we need at least 12 more edges. We have 3 vertices outside $C_{10}$, say $f_1, f_2, f_3$. If $\{f_1, f_2\} \in E(G)$, then $N(f_i) \cap V(C_{10}) = \emptyset$ for $i = 1, 2$, in the opposite case we get $3P_4$. Thus $\deg_{C_{10}}(f_i) \geq 4$ for some $i = 1, 2, 3$. Note that $\deg_{C_{10}}(f_i) \leq 5$, $i = 1, 2, 3$, in the opposite case we get $C_{11}$. If $f_1$ has 4 edges with $C_{10}$, then we must delete at least $\binom{4}{2} = 6$ edges from $K_{10}$. So we need 14 more edges. So $\deg_{C_{10}}(f_i) > 5$ and we have a contradiction.

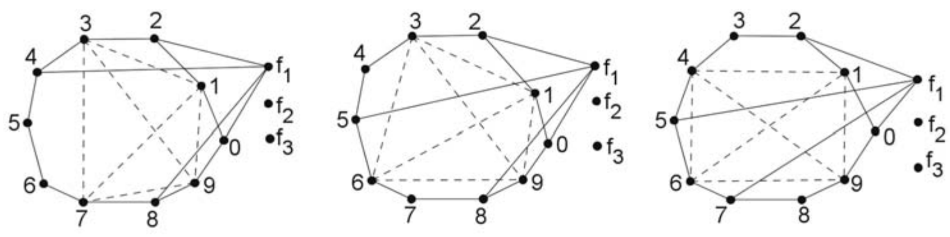

FIGURE 5. Graphs with $C_{10}$ for $ex(13, 3P_4)$.

Figure 5 presents a subgraph of $G$ with the cycle $C_{10}$. Dotted lines denote edges in $\overline{G}$, in the opposite case we get a longer cycle in $G$.

For $n = 14$ we have $ex(n, 3P_4) \geq \binom{11}{2} + \binom{3}{2} = 58$. It follows from the fact that $K_{11} \cup K_3$ does not contain $3P_4$. Let $|E(G)| = 59$. Applying Lemma 1, we have

$$\frac{13(l-1)}{2} + 1 \leq 59,$$

$$l \leq 9.$$

If $q \geq 12$, then we have $3P_4$ in $G$. If $q = 11$, then we have 55 edges in $K_{11}$ and 3 edges in $K_3$ and $K_{11}$ and $K_3$ must be disjoint. But we have 58 edges so we must add one more edge and we obtain $3P_4$, a contradiction. Let $q = 10$. We have 45 edges in $K_{10}$ and we need 14 more edges. We have 4 vertices outside $C_{10}$. So $\deg_{C_{10}}(f_i) > 3$ for some $i = 1, 2, 3, 4$. Moreover, $\deg_{C_{10}}(f_i) \leq 5$ for $i = 1, 2, 3, 4$, in the opposite case we get a cycle $C_{11}$. If $f_1$ creates 5 edges with the vertices of $C_{10}$, then we must delete 10 edges from $K_{10}$. So we need 19 more edges. But we have only 3 vertices in $V(F) - \{f_1\}$, so $\deg_{C_{10}}(f_i) > 5$ for some $i = 2, 3, 4$ and we have a contradiction. Similarly, if $\deg_{C_{10}}(f_1) = 4$ and $\deg_{C_{10}}(f_i) \leq 4$ for $i = 2, 3, 4$, then we must delete at least 6 edges from $K_{10}$. So we need 20 more edges. We have three vertices in $V(F) - \{f_1\}$. Hence $\deg_{C_{10}}(f_i) > 5$ for some $i = 2, 3, 4$ and we have a contradiction.

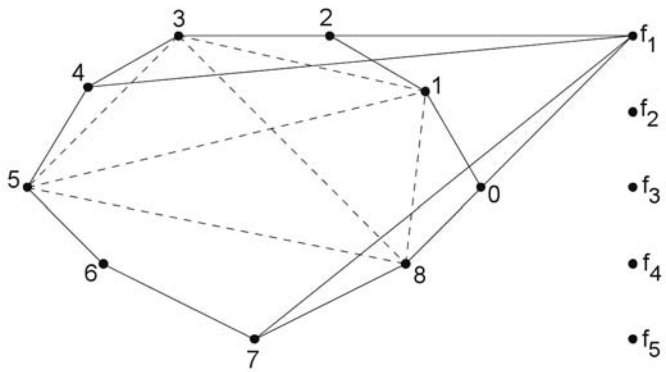


FIGURE 6. A graph with and $C_9$ for $ex(14, 3P_4)$.

Let $q = 9$. We have 36 edges in $K_9$. So we need at least 23 edges outside $G|_{C_9}$. We have 5 vertices outside the cycle $C_9$, i.e. in the graph $F$ (see Figure 6 for an illustration). Recall that $ex(5, P_4) = 4$. So we have at least 19 edges between $V(C_9)$ and $V(F)$. Thus there exists a vertex $f_i \in V(F)$, such that $\deg_{C_9}(f_i) \geq 4$. Note that $\deg_{C_9}(f_i) \leq 4$ for any $f_i \in V(F)$, in the opposite case we get a cycle $C_{10}$. Let $f_1$ be the vertex adjacent to four vertices of $C_9$. Then $G|_{C_9}$ is not isomorphic to $K_9$, in the opposite case we get a longer cycle. We must delete from $K_9$ at least $\binom{4}{2} = 6$ edges (see dotted lines in Figure 6). So now we need at least 21 edges between $V(C_9)$

and $V(F)$. But we have only 5 vertices in $F$. So there exists $f_i$ for which $\deg_{C_9}(f_i) > 4$ and we have a contradiction.

Summarizing, we collect results from above remarks in Theorem 5.

**Theorem 5.** *Let $n$ be a natural number and $n \le 14$. Then*

$$ex(n, 3P_4) = \binom{n}{2} \text{ for } n \le 11,$$

$$ex(n, 3P_4) = \binom{11}{2} = 55 \text{ for } n = 12,$$

$$ex(n, 3P_4) = \binom{11}{2} + \binom{2}{2} = 56 \text{ for } n = 13,$$

$$ex(n, 3P_4) = \binom{11}{2} + \binom{3}{2} = 58 \text{ for } n = 14.$$

Theorems 4 and 5 present the Turán number $ex(n, 3P_4)$ for all positive integers $n$.

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