

# THE TURN OF THE SCREW: OPTIMAL DESIGN OF AN ARCHIMEDES SCREW

By Chris Rorres<sup>1</sup>

**ABSTRACT:** The geometry of an Archimedes screw is governed by certain external parameters (its outer radius, length, and slope) and certain internal parameters (its inner radius, number of blades, and the pitch of the blades). The external parameters are usually determined by the location of the screw and how much water is to be lifted. The internal parameters, however, are free to be chosen to optimize the performance of the screw. In this paper the inner radius and pitch that maximize the volume of water lifted in one turn of the screw are found. The optimal parameter values found are compared with the values used in a screw described by the Roman architect and engineer Vitruvius in the first century B.C., and with values used in the design of modern Archimedes screws.

## INTRODUCTION

One of the oldest machines still in use is the Archimedes screw, a device for lifting water for irrigation and drainage purposes. Its invention has traditionally been credited to Archimedes (circa 287–212 B.C.). For example, Diodorus Siculus (Greek historian, circa first century B.C.) writes

men easily irrigate the whole of it [an island in the delta of the Nile] by means of a certain instrument conceived by Archimedes of Syracuse, and which gets its name [cochlias] because it has the form of a spiral or screw.

And from Athenaeus of Naucratis (Greek historian, circa A.D. 200):

The bilge-water [of the ship *Syracusia*], even when it became very deep, could easily be pumped out by one man with the aid of the screw, an invention of Archimedes.

(See references under Diodorus Siculus and Athenaeus of Naucratis.) Archimedes, however, made no reference to it in his extant works [cf. Heath (1897) and Dijksterhuis (1938)], and it may be that he simply transmitted its knowledge from Egypt (where it is believed he studied in Alexandria under the students of Euclid) to Syracuse (his native city in Sicily). On the other hand, in defense of Archimedes: no mention of the device exists before his time, its design involves the type of geometry in which he excelled, and his abilities as an inventor of mechanical and military machines are well documented.

The Roman engineer and architect Vitruvius gave a detailed and informative description of the construction of an Archimedes screw in his *De Architectura*, written in the first century B.C. (See reference under Vitruvius.) Vitruvius's description contributed greatly to keeping the device well known throughout the ages, and the particular screw he described will be used throughout this paper as a test case.

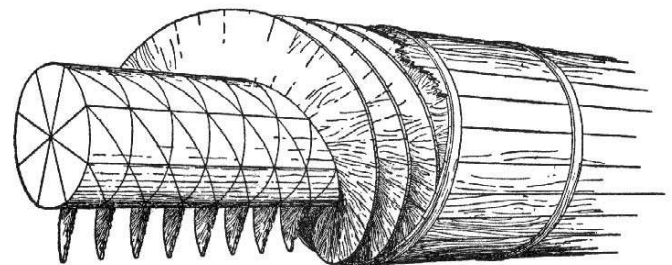
Vitruvius's screw began with a tree trunk shaped into a cylindrical core (the "inner cylinder") whose length is 16 times its diameter (Fig. 1). On this cylindrical core eight intertwined helical blades (also called "flights" or "starts") were constructed by nailing withes (slender flexible willow twigs) together up to a height equal to the radius of the core. The period

(or "pitch") of these blades was equal to the circumference of the cylindrical core. Finally, an outer cylindrical covering (the "outer cylinder") of wooden planks was nailed to the helical blades. Liquid pitch was smeared over all of the parts during their assembly to make the screw watertight. The rigid screw was mounted so that it could be rotated along its length (Fig. 2) and was tilted in the direction of the hypotenuse of a 3-4-5 triangle (a "Pythagorean right-angled triangle," as Vitruvius calls it). Its bottom end was immersed in a body of water (the "lower reservoir"), and through the rotation of the screw water was lifted to the top end of the screw into an "upper reservoir."

When an Archimedes screw is tilted, "buckets" that can trap water are formed between the blades. These buckets appear to move upward when the screw is rotated, carrying the water within them. The screw collects water from the lower reservoir, where the buckets are formed, and empties it into the upper reservoir, where the buckets are unformed. When operated manually it is rotated by a crank (Fig. 3) or by a man walking around the circumference of the outer cylinder in a treadmill manner.

In modern industrial screws, the outer cylinder is usually fixed and the blades attached to the inner cylinder are rotated within it (Fig. 4). This allows the top half of the outer cylinder to be eliminated so that a stationary trough is formed from the bottom half of the outer cylinder. Such a construction permits easy access to the interior of the screw, in order to remove debris and for routine maintenance. In addition, the stationary outer cylinder relieves the moving blades and inner cylinder of some of the weight of the water. A disadvantage of this design is that water can leak down through the small gap between the moving blades and the stationary trough. However, this leakage can be considered an advantage in that it allows the screw to drain when it stops rotating.

The Archimedes screw has had a resurgence in recent years because of its proven trouble-free design and its ability to lift wastewater and debris-laden water effectively. It has also



**FIG. 1. Vitruvius's Eight-Bladed Screw as Described in his *De Architectura*. Diagram by Morris Hicky Morgan (Vitruvius Reference)**

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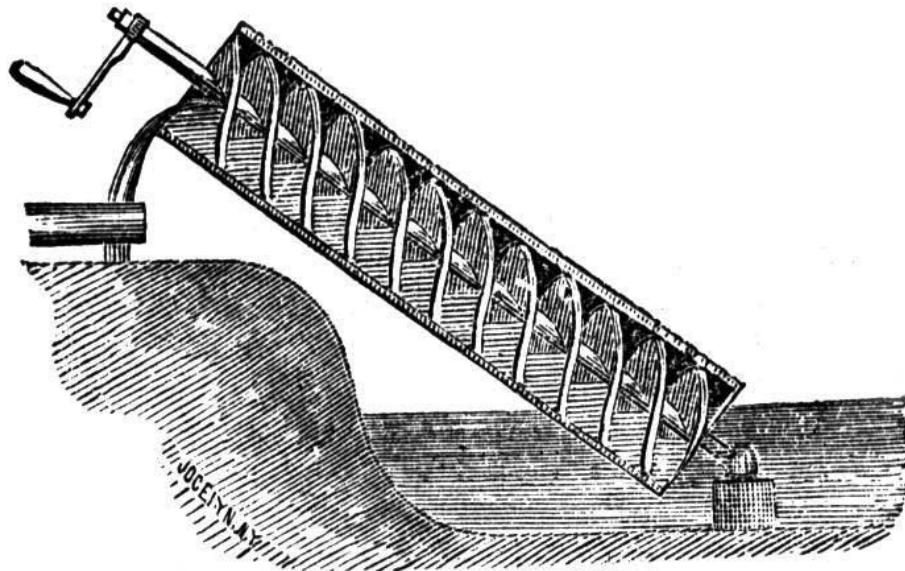


FIG. 2. Profile of Three-Bladed Archimedes Screw



FIG. 3. Two-Bladed Archimedes Screw in Use in Modern Times in Egypt to Irrigate Fields

proved valuable in installations where damage to aquatic life must be minimized.

The purpose of this paper is to examine the optimal design of an Archimedes screw. By this is meant that geometry that will maximize the amount of water delivered to the upper reservoir *in one turn of the screw*. This optimization will be performed under the assumption that the outer radius of the screw is specified, for clearly the amount of water lifted in each revolution can be continually increased by continually increasing this radius.

The amount of water lifted *per unit time* can also be increased by increasing the rotational velocity of the screw. However, there is a practical limit to how fast one can rotate the screw. A handbook on the design and operation of Archimedes screws (Nagel 1968, p. 37) states that, based on field experience, the rotational velocity of a screw in revolutions per minute should be no larger than  $50/D^{2/3}$ , where  $D$  is the diameter of the outer cylinder in meters. Thus a screw with an outside diameter of 1 m should have a maximum rotational velocity of 50 rpm. If the screw is rotated much faster, tur-



FIG. 4. Seven 96-Inch-Diameter, 3-Flight Screw Pumps That Lift Wastewater from Pretreatment Section to Balance of Treatment Plant in Memphis, Tenn. [Each Pump Has Capacity of 19,900 gal./min (Courtesy Lakeside Equipment Corp., Bartlett, Ill.)]

bulence and sloshing prevent the buckets from being filled and the screw simply churns the water in the lower reservoir rather than lifting it.

## PROBLEM FORMULATION

Fig. 5 is a profile view of a segment of an Archimedes screw showing how the buckets are formed between pairs of adjacent blades. The buckets move up the screw as it is rotated clockwise when viewed from the upper reservoir. (Throughout this paper the screws are oriented for clockwise rotation). The inner and outer edges of each blade determine two sinusoidal curves of the same period (or pitch) and phase. One has an amplitude equal to the inner radius of the screw and the other has an amplitude equal to the outer radius. The angle  $\theta$  that the screw makes with the horizontal determines its slope  $\tan \theta$ . Now defined are the following three “external” parameters:

$R_o$  = radius of screw’s outer cylinder (m)

$L$  = total length of screw (m)

$K$  = slope of screw (dimensionless)

These external parameters are usually determined by the site of the screw and the materials available for its construction. In this paper, these three parameters are taken as fixed. In addition to these parameters, the following three “internal” parameters are needed to completely specify the geometry of the screw:

$R_i$  = radius of screw’s inner cylinder (m) ( $0 \leq R_i \leq R_o$ )

$\Lambda$  = pitch (or period) of one blade (m) ( $0 \leq \Lambda \leq 2\pi R_o/K$ )

$N$  = number of blades (dimensionless),  $N = 1, 2, \dots$

By “one cycle of the screw” is meant a segment of the screw whose length is equal to one pitch of the screw. The volume of one cycle of the screw is  $\pi R_o^2 \Lambda$ .

By a “chute” is meant a region of the screw bounded by two adjacent blades and the inner and outer cylinders. The region between the inner and outer cylinders of the screw consists of  $N$  disjoint congruent chutes separated by the  $N$  blades. The volume of each chute is  $\pi(R_o^2 - R_i^2)L/N$ .

By a “bucket” is meant one of the maximally connected regions occupied by the trapped water within any one chute. Each bucket is filled with water at the lower reservoir and is emptied into the upper reservoir. Within one cycle of the screw the volume of the water within all the buckets is  $N$  times the

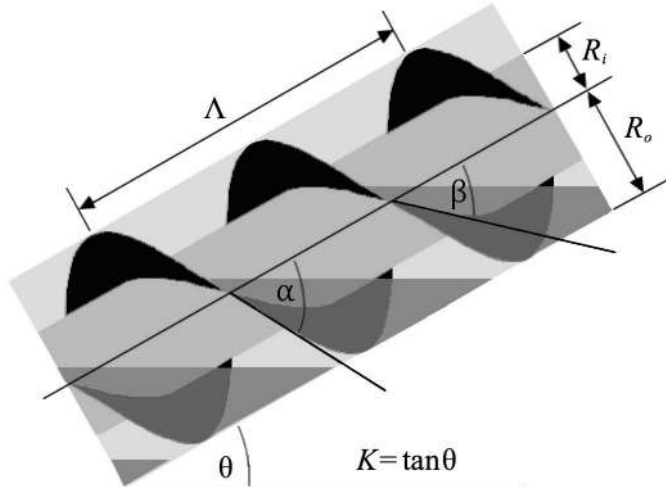


FIG. 5. Profile View of Segment of Two-Bladed Archimedes Screw

volume of one bucket. The volume of one bucket is denoted by  $V_B$ . It is a complicated function of  $N$ ,  $K$ ,  $\Lambda$ ,  $R_o$  and  $R_i$ .

The restriction  $\Lambda \leq 2\pi R_o/K$  given above on the pitch  $\Lambda$  of the screw needs some discussion. In order for water to be trapped in the screw, it is necessary that the sinusoidal curve in Fig. 5 defining the outer edge of a blade tilts downward as it crosses the axis of the screw. In terms of the angles  $\theta$  and  $\alpha$  in the figure, it is necessary that  $\theta \leq \alpha$  or, equivalently,  $\tan \theta \leq \tan \alpha$ . Now  $\tan \theta = K$  by definition, and  $\tan \alpha = R_o(2\pi/\Lambda)$  since the sinusoidal curve has amplitude  $R_o$  and period  $\Lambda$ . The condition  $\tan \theta \leq \tan \alpha$  is thus  $K \leq R_o(2\pi/\Lambda)$ , which is just  $\Lambda \leq 2\pi R_o/K$ .

Fig. 5 also shows the angle  $\beta$  that the sinusoidal curve defining the inner edge of a blade makes with axis of the screw. Because the inner sinusoidal has amplitude  $R_i$  and period  $\Lambda$ , it follows that  $\tan \beta = R_i(2\pi/\Lambda)$ . If  $\theta \leq \beta$ , so that the inner sinusoidal curve tilts downward as it crosses the axis of the screw, then the horizontal water level in a bucket is tangent to this inner sinusoidal curve. As above, the condition  $\theta \leq \beta$  can be expressed as  $\Lambda \leq 2\pi R_i/K$ .

If  $\Lambda$  lies in the interval  $(2\pi R_i/K, 2\pi R_o/K)$ , then the horizontal water level is right at the point where the inner and outer sinusoidal curves cross the axis of the screw. At that point the inner sinusoidal curve has positive slope (with respect to the horizontal) and the outer sinusoidal curve has negative slope. The buckets of water that form in this case are rather small and in, in fact, will not be in contact with the inner cylinder. Summarizing, the horizontal water level of a bucket is

1. Tangent to the inner sinusoidal curve in a profile view if  $\Lambda \in (0, 2\pi R_i/K)$  [Fig. 6(a)].
2. At the intersection of the outer and inner sinusoidal curves in a profile view if  $\Lambda \in (2\pi R_i/K, 2\pi R_o/K)$  [Fig. 6(b)].

Returning to our optimization problem, the following quantity is now defined:

$V_T$  = volume of water in one cycle of the screw ( $\text{m}^3$ )

This quantity is also the volume of water emptied into the upper reservoir with each turn of the screw—precisely the quantity to be maximized. Note also that  $V_T = NV_B$ , expressing the fact that  $N$  buckets of water are emptied into the upper reservoir with each turn of the screw. The basic problem of this paper can now be stated as follows:

**Problem:** Given  $N$ ,  $R_o$ , and  $K$ , find values of  $R_i$  and  $\Lambda$  that maximize  $V_T$ .

In words: Given the number of blades, outer radius, and slope

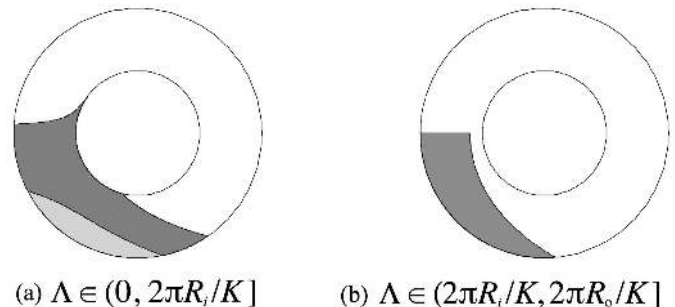


FIG. 6. Views of Single Bucket Looking Down Two Screws with Transparent Blades and Different Pitches (Both Screws Are 8-Bladed and Their Inner Radius is Half Their Outer Radius; Dark-Shaded Region Is Horizontal Water Surface and Light-Shaded Region Is Portion of Bucket in Contact with Blade Nearer the Top of Screw)

of an Archimedes screw, find the inner radius and pitch that will maximize the volume of water emptied into the upper reservoir with each turn of the screw.

In this paper it is assumed that the blades have negligible thickness. With this assumption the volume of water in one cycle monotonically increases with the number  $N$  of blades. If the blades had some nonnegligible thickness, however, they would occupy an increasing fraction of the volume of the screw as their number increased and a point of diminishing returns would be reached. One could then include the determination of an optimal  $N$  in the problem statement. But in modern screws the number of blades is usually 1, 2, or 3, because of manufacturing, weight, and cost constraints. It is thus assumed that the number of blades is predetermined, and their thickness is neglected in the computations.

## DIMENSIONLESS PARAMETERS

The analysis of the optimization problem is begun by defining a dimensionless parameter  $\nu$  as the ratio of  $V_T$  (the volume of water in one cycle of the screw) to  $\pi R_o^2 \Lambda$  (the total volume of one cycle of the screw):

$$\nu = \frac{V_T}{\pi R_o^2 \Lambda} = \text{volume ratio} \quad (1)$$

= fraction of the volume of one cycle of the screw occupied by water.

This volume ratio is also the fraction of the entire screw occupied by water if the ends where water is either entering or leaving the screw are neglected. By its definition  $\nu$  is a number between 0 and 1. Notice that maximizing  $V_T$  is not the same as maximizing  $\nu$ .

Two more dimensionless parameters are defined as follows:

$$\rho = \frac{R_i}{R_o} = \text{radius ratio} \quad (0 \leq \rho \leq 1) \quad (2)$$

and

$$\lambda = \frac{K\Lambda}{2\pi R_o} = \text{pitch ratio} \quad (0 \leq \lambda \leq 1) \quad (3)$$

Because  $R_o$  = maximum value of the inner radius  $R_i$ , the radius ratio  $\rho$  is the ratio of the actual inner radius to its maximum possible value. Similarly, because  $2\pi R_o/K$  = maximum pitch for which buckets will form, the pitch ratio  $\lambda$  is the ratio of the actual pitch to its maximum possible value. In the case of Vitruvius's screw:  $R_i = (1/2)R_o$ ,  $\Lambda = 2\pi R_i$ ,  $K = 3/4$ , and so  $\rho = 1/2$  and  $\lambda = 3/8$ .

A dimensional analysis shows that  $\nu$  depends only on  $N$ ,  $\rho$ , and  $\lambda$ . This quantity can be written as  $\nu(N, \rho, \lambda)$  to emphasize this fact. From the above three equations it follows that

$$V_T = \left( \frac{2\pi^2 R_o^3}{K} \right) \lambda \nu(N, \rho, \lambda) \quad (4)$$

Given  $N$ ,  $R_o$ , and  $K$ , the problem of maximizing  $V_T$  with respect to  $R_i$  and  $\Lambda$  can then be reduced to maximizing  $\lambda \nu(N, \rho, \lambda)$  with respect to  $\rho$  and  $\lambda$ , each restricted to the interval  $[0, 1]$ . Let the values of  $\rho$  and  $\lambda$  that maximize  $\lambda \nu(N, \rho, \lambda)$  be denoted by  $\rho^*$  and  $\lambda^*$ , respectively. Then the optimal values of  $R_i$ ,  $\Lambda$ , and  $V_T$  are given by

$$R_i^* = \rho^* R_o \quad (5)$$

$$\Lambda^* = \frac{2\pi R_o \lambda^*}{K} \quad (6)$$

and

$$V_T^* = \left( \frac{2\pi^2 R_o^3}{K} \right) \lambda^* \nu(N, \rho^*, \lambda^*) \quad (7)$$

## VOLUME OF A BUCKET

The algorithm for computing the volume ratio  $\nu(N, \rho, \lambda)$  for a specific set of screw parameters  $N$ ,  $\rho$ , and  $\lambda$  is described in Appendix I. As an example of this algorithm, consider the volume ratio for Vitruvius's 8-bladed screw. Since  $\rho = 1/2$  and  $\lambda = 3/8$  for his screw, the desired volume ratio is  $\nu(8, 1/2, 3/8)$ . The algorithm gave 0.1703 for this volume ratio, so that 17% of Vitruvius's screw was occupied by water when operating. With this value of  $\nu(8, 1/2, 3/8)$ , Eqs. (1–3) and the fact that  $K = 3/4$  leads to

$$V_T = 1.68 R_o^3 \quad (8)$$

In the next section this formula is compared with the corresponding formula that the optimization algorithm provides.

Fig. 7 is plot of  $\nu(N, \rho, \lambda)$  for  $N = 8$  and all values of  $\rho$  and  $\lambda$  between 0 and 1. Plots of  $\nu(N, \rho, \lambda)$  for other values of  $N$  are similar, with  $\nu(N, \rho, \lambda)$  monotonically increasing as  $N$  increases for each fixed  $\rho$  and  $\lambda$ .

Fig. 7 shows that the volume ratio has a global maximum for some value of  $\rho$  when  $\lambda = 0$ . The case  $\lambda = 0$  arises when either the slope or the pitch of the screw is zero. Although neither situation is practical, they may be considered as the limiting cases when the slope or the pitch approaches zero. Fig. 8 is a typical cross section of a screw when  $\lambda = 0$ , showing how the water level is tangent to the top of the inner cylinder. Every cross section of the screw has the same water profile when  $\lambda = 0$  and this is true regardless of the number

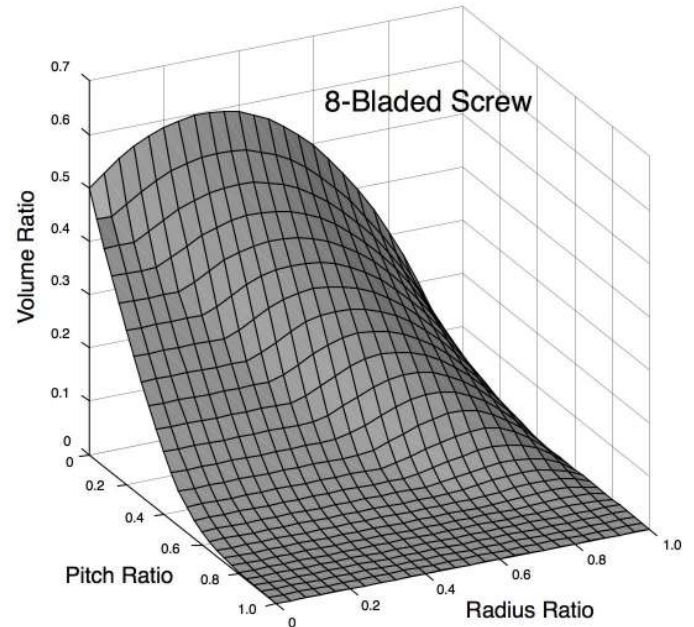


FIG. 7. Graph of  $\nu(N, \rho, \lambda)$  (Volume Ratio) versus  $\rho$  (Radius Ratio) and  $\lambda$  (Pitch Ratio) for 8-Bladed Screw ( $N = 8$ )

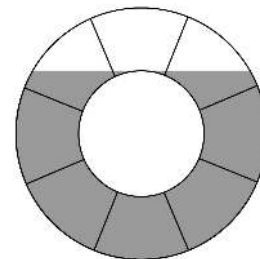


FIG. 8. Cross Section of 8-Bladed Screw with Zero Pitch Ratio ( $\lambda = 0$ )



of blades the screw has. A little geometry then shows that the fraction of the area of the cross section of the screw occupied by water is given by

$$A_0(\rho) = (1 - \rho^2) - \frac{1}{\pi} \cos^{-1}(\rho) + \frac{\rho}{\pi} \sqrt{1 - \rho^2} \quad (9)$$

Now, because each cross section of the screw is occupied by this same fraction of water, the volume ratio of the screw must also be given by this expression. Consequently, (9) is the formula for the curve obtained by slicing the surface in Fig. 7 with the plane  $\lambda = 0$ ; that is,  $A_0(\rho) = v(N, \rho, 0)$ . A little calculus shows that  $A_0(\rho)$  is maximized when  $\rho = 1/\sqrt{1 + \pi^2} = 0.30331 \dots$ . The corresponding value of  $v$  is  $0.598093 \dots$ . Thus no Archimedes screw can have more than 60% of its volume occupied by water, and this maximum value is approached when the slope or pitch is very small and the inner radius of the screw is about 30% of the outer radius.

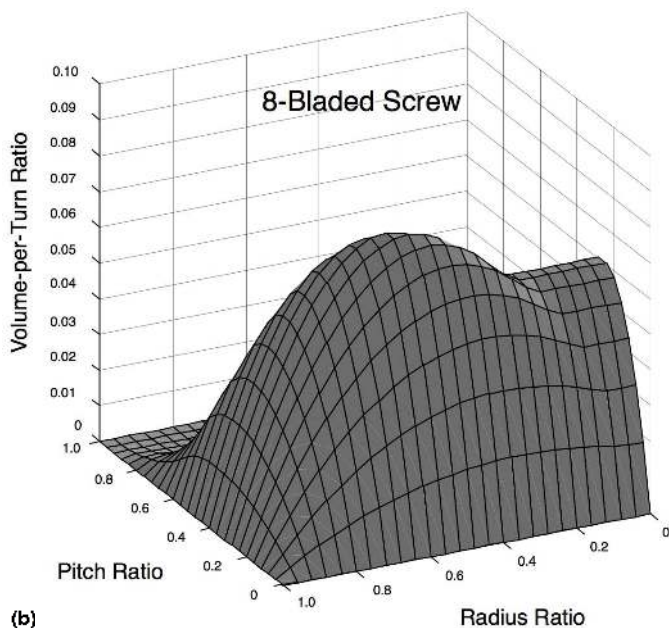
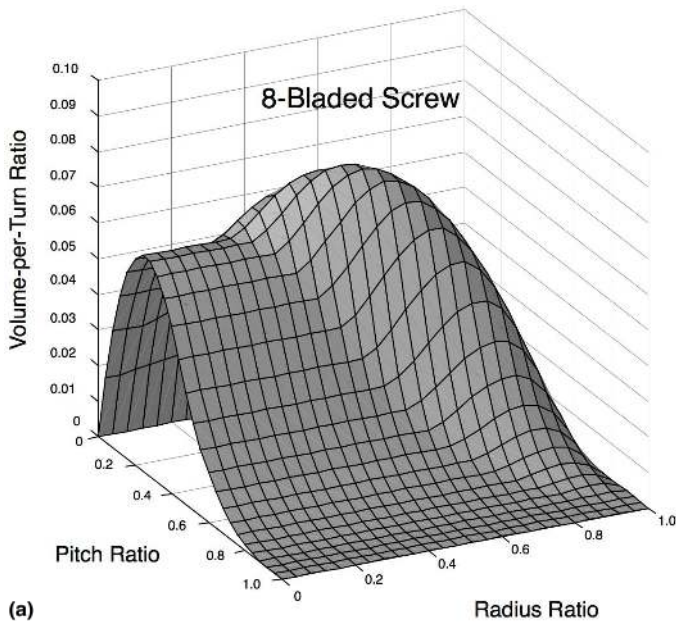


FIG. 9. Two views of Graph of  $\lambda v(N, \rho, \lambda)$  (Volume-per-Turn Ratio) versus  $\rho$  (Radius Ratio) and  $\lambda$  (Pitch Ratio) for 8-Bladed Screw ( $N = 8$ )

## OPTIMAL SCREW

The determination of the optimal screw is now considered. This is the problem of finding the values of  $\rho$  and  $\lambda$  in the interval  $[0, 1]$  that maximize the function of  $\lambda v(N, \rho, \lambda)$ . Fig. 9 is a plot  $\lambda v(N, \rho, \lambda)$  for an 8-bladed screw. As with  $v(N, \rho, \lambda)$ , plots of  $\lambda v(N, \rho, \lambda)$  for different values of  $N$  are similar to each other, with  $\lambda v(N, \rho, \lambda)$  monotonically increasing as  $N$  increases.

It can be seen from Fig. 9 that  $\lambda v(N, \rho, \lambda)$  has a unique maximum inside the unit square. This maximum was computed numerically for various  $N$  using the “fmin” function of MatLab (Hanselman and Littlefield 1995), which implements a Nelder-Mead simplex search (Nelder and Mead 1965). For an 8-bladed screw, the peak value of the volume-per-turn ratio was found to be 0.0771, attained when  $\rho^* = 0.5354$  and  $\lambda^* = 0.2957$ . Eq. (7) then shows that the volume per turn in the optimal case is given by

$$V_T^* = \frac{1.52R_o^3}{K} \quad (10)$$

Table 1 gives the optimal values of the pitch and radius ratios for screws with 1 to 25 blades, together with the corresponding optimal values of the volume-per-turn and the volume ratios. The last row gives the limiting value of these quantities as the number of blades approaches infinity. It provides an upper bound for the amount of water that can be lifted in one turn since the volume-per-turn ratio monotonically increases as  $N$  increases.

Fig. 10 represents the data in Table 1 in graphical form. Notice how  $\rho^*$  varies only in the interval  $(0.5352, 0.5394)$ , and so to two decimal places the optimal ratio is 0.54 for any number of blades. It may be that the optimal radius is the same for any number of blades and this slight variation of  $\rho^*$  may

TABLE 1. Optimal Ratio Parameters of Archimedes Screw for Various Numbers of Blades

Number of blades $N$	Optimal radius ratio $\rho^*$	Optimal pitch ratio $\lambda^*$	Optimal volume-per-turn ratio $\lambda^* v(N, \rho^*, \lambda^*)$	Optimal volume ratio $v(N, \rho^*, \lambda^*)$
(1)	(2)	(3)	(4)	(5)
1	0.5358	0.1285	0.0361	0.2811
2	0.5369	0.1863	0.0512	0.2747
3	0.5357	0.2217	0.0598	0.2697
4	0.5353	0.2456	0.0655	0.2667
5	0.5352	0.2630	0.0696	0.2647
6	0.5353	0.2763	0.0727	0.2631
7	0.5354	0.2869	0.0752	0.2619
8	0.5354	0.2957	0.0771	0.2609
9	0.5356	0.3029	0.0788	0.2601
10	0.5356	0.3092	0.0802	0.2592
11	0.5358	0.3145	0.0813	0.2586
12	0.5360	0.3193	0.0824	0.2580
13	0.5360	0.3234	0.0833	0.2574
14	0.5360	0.3270	0.0841	0.2571
15	0.5364	0.3303	0.0848	0.2567
16	0.5362	0.3333	0.0854	0.2562
17	0.5362	0.3364	0.0860	0.2556
18	0.5368	0.3380	0.0865	0.2559
19	0.5364	0.3404	0.0870	0.2555
20	0.5365	0.3426	0.0874	0.2551
21	0.5370	0.3440	0.0878	0.2553
22	0.5365	0.3465	0.0882	0.2544
23	0.5369	0.3481	0.0885	0.2543
24	0.5367	0.3500	0.0888	0.2538
25	0.5371	0.3507	0.0891	0.2542
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.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$\infty$	0.5394	0.3953	0.0977	0.2471

be a numerical artifact resulting from the approximate numerical integration needed to compute  $v(N, \rho, \lambda)$  and the approximate numerical optimization of  $\lambda v(N, \rho, \lambda)$ .

Table 1 also shows that  $v(N, \rho^*, \lambda^*)$  decreases from 0.2811 to 0.2471 as  $N$  increases from one to infinity. Thus, a decreasing fraction of the screw is occupied by water as the number of blades increases.

Fig. 11 shows a view of a single bucket looking down the screw for the optimal screws with 1, 2, 3, or 4 blades. Notice how the horizontal water surface is in one piece for a 1- or 2-bladed piece but is in two pieces for a 3- or 4-bladed screw. When the water surface is in one piece, there is a clear air passage from the top to the bottom of the screw if the screw has a complete watertight outer cylinder. However, when the water surface is broken into two pieces by the inner cylinder, the buckets of water close off the chutes and so air is trapped in the spaces between the buckets in any one chute.

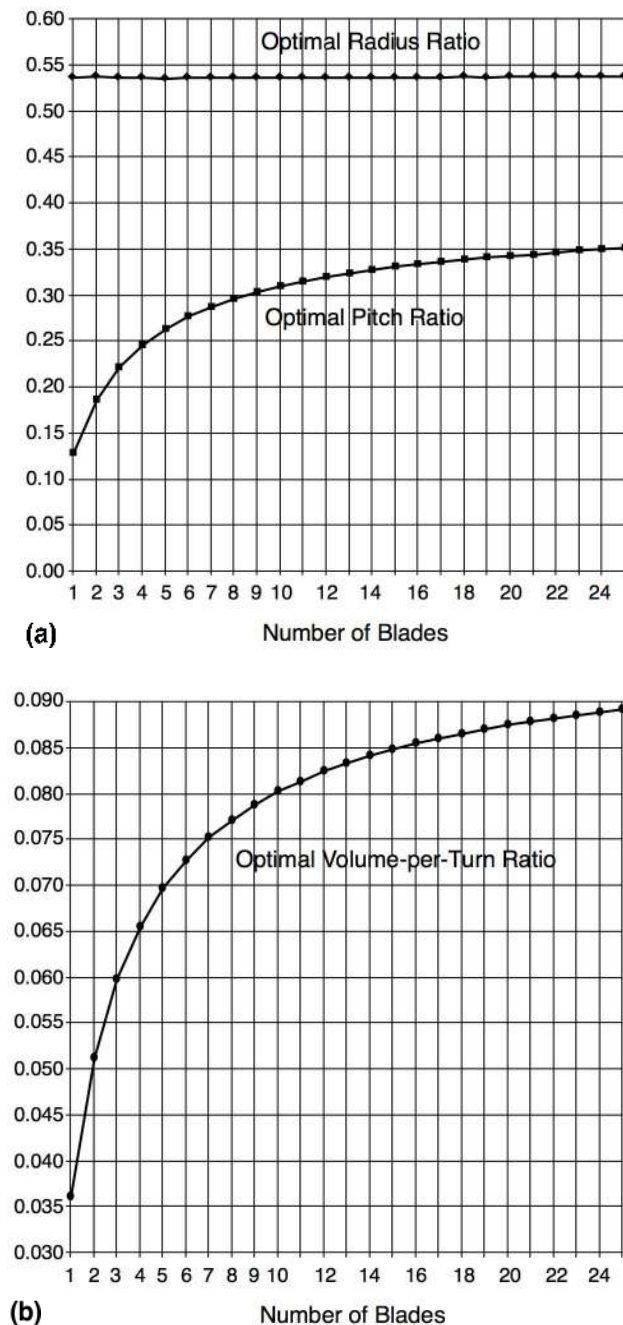


FIG. 10. Graphical Representation of Data in Table 1

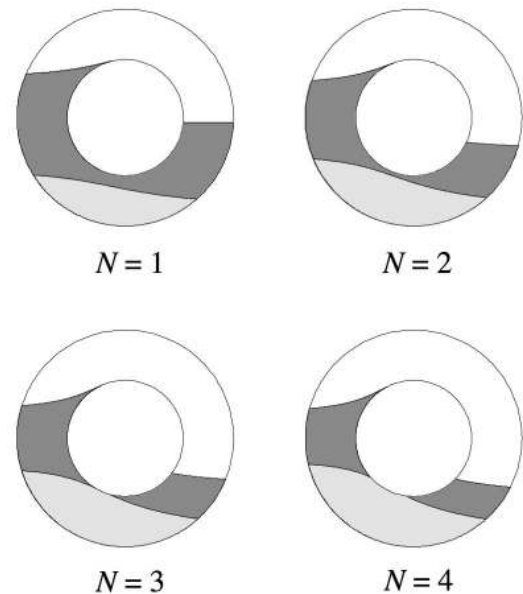


FIG. 11. Views Looking down Optimal 1-, 2-, 3-, and 4-Bladed Screw of Single Bucket Lying Between Two Adjacent Blades (Dark-Shaded Region Is Horizontal Water Surface and Light-Shaded Region Is Portion of bucket in Contact with Blade nearer Top of Screw)

## CONCLUSIONS

Vitruvius's screw configuration can be compared with its optimal configuration as determined by the calculations above. As shown in (8), the volume per turn of the Vitruvius screw is governed by  $V_T = 1.68R_o^3$ . The optimal volume per turn is governed by (10) with  $K = 3/4$ , which gives  $V_T^* = 2.03R_o^3$ . The optimal design thus results in a fractional increase in the water lifted per turn given by  $V_T^*/V_T = 2.03/1.68 = 1.21$ , or a percentage increase of 21%. Consequently, the output of Vitruvius's screw is fairly close to that of the optimal 8-bladed screw. In addition, its radius ratio is within 7% of the optimal value (0.5 versus 0.5354), its pitch ratio is within 27% of the optimal value (0.375 versus 0.2957), and the construction lines associated with its design are much simpler than those that would be needed to construct the optimal screw. No doubt many generations of experience went into the design of the screw that Vitruvius described.

Fig. 12 shows one bucket in a screw of Vitruvius's design and one bucket in a screw of the corresponding optimal design.

Ritz-Atro Pumpwerksbau of Nürnberg, Germany, a manufacturer of Archimedes screws, prepared an *Archimedean Screw Pump Handbook* in 1968 to "provide the technical information needed for the calculation, planning, construction, and operation of water pumping installations using Archimedean screw pumps" (Nagel 1968). This handbook gives certain rules of thumb for maximizing the volume of water raised with each turn of the screw based on heuristic arguments and field experience. In particular, it states that the ratio of the inner radius to the outer radius should be between 0.45 and 0.55 for most conditions (Nagel 1968, p. 30). The results of this paper show that it should be 0.54 under all circumstances when the corresponding optimal pitch is used.

For the pitch, the handbook gives the following heuristic rule for a screw tilted at an angle  $\theta$  (Nagel 1968, p. 31):

$$\Lambda = 2.4R_o \quad \text{if } \theta < 30^\circ$$

$$\Lambda = 2.0R_o \quad \text{if } \theta = 30^\circ$$

$$\Lambda = 1.6R_o \quad \text{if } \theta > 30^\circ$$

The result obtained in this paper is given in (6):  $\Lambda = (2\pi\lambda^* \cot \theta)R_o$ , where the numerical value of  $\lambda^*$  is given in Table 1

and depends on  $N$ . Fig. 13 is a graph of  $\Lambda$  as a function of  $\theta$  as given by this formula for the cases  $N = 1, 2$ , and  $3$ . Also in Fig. 13 is a graph of the piecewise constant formula determined by the handbook's heuristic rule. The heuristic rule can be seen to be a pretty good match for the exact relationship, especially for a 2-bladed screw. The handbook further states that to avoid a proliferation of screw designs and the concomitant manufacturing costs, the choice  $\Lambda = 2.0R_o$  should be made for all angles. Fig. 13 justifies this particular simplifying choice for 1-, 2-, and 3-bladed screws and for angles around  $30^\circ$ .

Finally, it seems appropriate to close this paper with the following words that Vitruvius used to close the description of his screw:

I have now described as clearly as I could, to make them better known, the principles on which wooden engines for raising water are constructed, and how they get their motion so that they may be of unlimited usefulness through their revolutions.

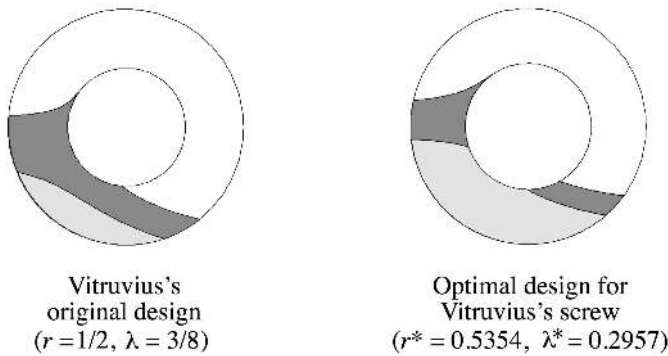


FIG. 12. Two Views Looking down Two 8-Bladed Screws of Single Bucket Lying Between Two Adjacent Transparent Blades (Dark-Shaded Region Is Horizontal Water Surface and Light-Shaded Region Is Portion of Bucket in Contact with Blade nearer Top of Screw)

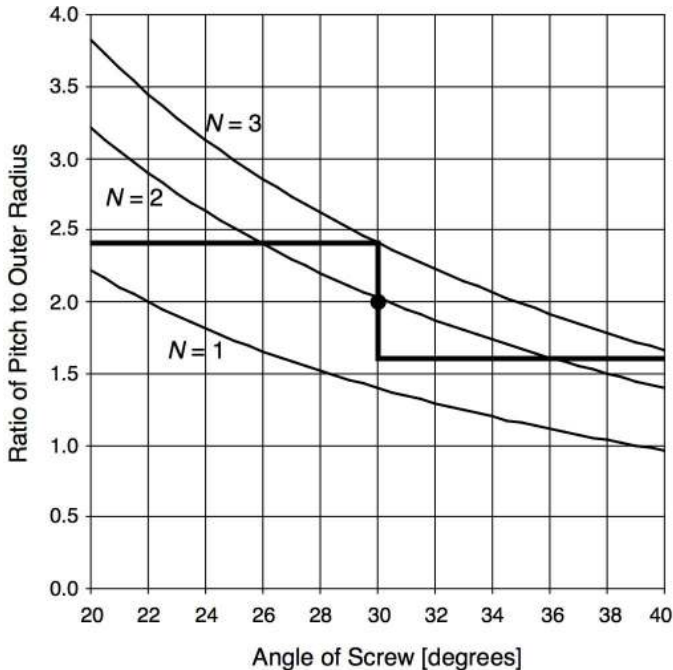


FIG. 13. Graph of Ratio of Pitch of Screw to Its Outer Radius versus Angle of Inclination of Screw When Number of Blades Is 1, 2, or 3 [Also Shown Is Piecewise Constant Approximation Discussed in *Archimedean Screw Pump Handbook* (Nagel 1968)]

## APPENDIX I. VOLUME OF A BUCKET (DETAILS)

This appendix describes the algorithm used in the paper for the computation of the volume ratio  $v(N, \rho, \lambda)$ . Run an  $x$ -axis along the axis of the screw, with  $x$  increasing going up the screw. Select any particular bucket of the screw at any particular time and set

$A_B(x)$  = Area ( $m^2$ ) of the water in the cross section of the screw at position  $x$  (m) in the selected bucket.

The volume of  $V_B$  of the bucket is the integral of  $A_B(x)$  over the extent of the bucket along the  $x$ -axis. Thus, if  $(x_0, x_1)$  is the interval along the  $x$ -axis over which the bucket is defined, then

$$V_B = \int_{x_0}^{x_1} A_B(x) dx \quad (11)$$

Next define a new (dimensionless) variable by

$$\phi = \frac{2\pi x}{\Lambda} \quad (12)$$

called the *blade angle*. The blade angle goes through  $2\pi$  rad over one cycle of the screw. With respect to the blade angle, (11) becomes

$$V_B = \frac{\Lambda}{2\pi} \int_{\phi_0}^{\phi_1} \hat{A}_B(\phi) d\phi \quad (13)$$

where  $\hat{A}_B(\phi)$  = function  $A_B(x)$  expressed as a function of  $\phi$ , and the new limits of integration are  $\phi_0 = 2\pi x_0/\Lambda$  and  $\phi_1 = 2\pi x_1/\Lambda$ . Next define the dimensionless parameter

$$\gamma_B(\phi) = \frac{\hat{A}_B(\phi)}{\pi R_o^2} = \text{area ratio} \quad (14)$$

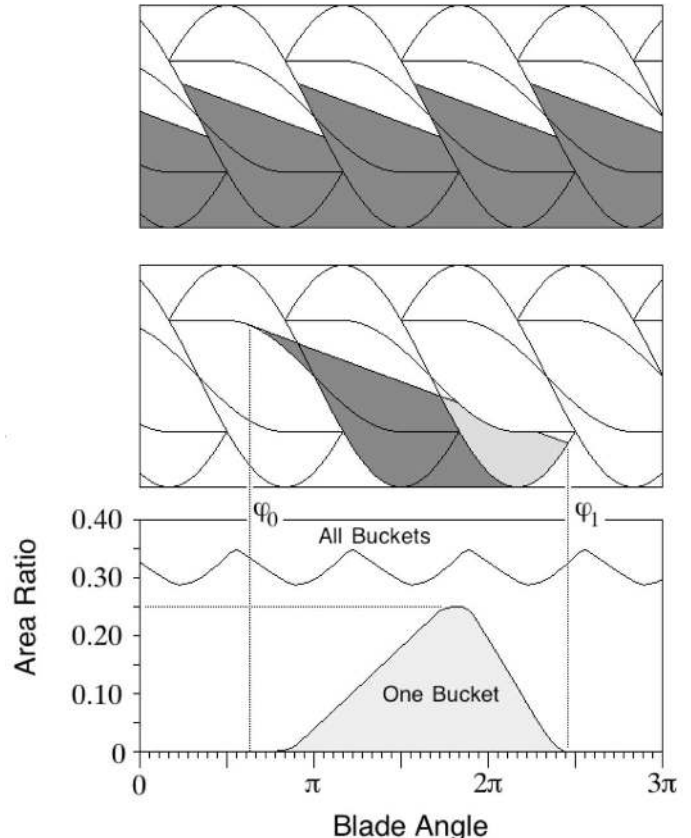


FIG. 14. Area Ratio of Typical Bucket of 3-Bladed Screw as Function of Blade Angle



which is the ratio of the cross-sectional area of the water in one bucket to the cross-sectional area of the screw. Combining (13) and (14) with (1), then, results in

$$\nu = \frac{V_T}{\pi R_o^2 \Lambda} = \frac{NV_B}{\pi R_o^2 \Lambda} = \frac{N}{2\pi} \int_{\phi_0}^{\phi_1} \gamma_B(\phi) d\phi \quad (15)$$

The above concepts are illustrated in Fig. 14 with a 3-bladed screw for which  $K = 3/8$ ,  $\Lambda = \pi R_o$ , and  $R_i = (1/2)R_o$  (thus,  $N = 3$ ,  $\rho = 1/2$ , and  $\lambda = 3/16$ ). The top diagram shows the profile of the screw with the bucket shaded. Of course, the screw is actually tilting upward with slope  $K = 3/8$  and the straight lines slanting downward in the diagram are the horizontal water levels in the buckets. The middle diagram shows a single bucket with the angles  $\phi_0$  and  $\phi_1$  indicated. The origin has been chosen along the  $\phi$ -axis so that the equation of the sine curve forming the outer edge of the lower blade of the bucket is  $R_o \sin \phi$ . The bottom diagram is a graph of  $\gamma_B(\phi)$  versus  $\phi$  of the particular bucket selected. (The top curve in the bottom graph is the cumulative area ratio of all of the buckets in the screw. It is a periodic curve with period  $2\pi/N$  and its average value is  $\nu$ .)

Fig. 15 shows cross sections of this 3-bladed screw for values of the blade angle from  $90^\circ$  to  $420^\circ$ . The chute containing the selected bucket is outlined heavily, and the cross section of the bucket itself is shaded darker than the buckets in the other two chutes. The cycle of cross sections in Fig. 15 repeats with each pitch of the screw, that is, for every  $360^\circ$  increase of the blade angle. In fact, the pattern of cross sections actually

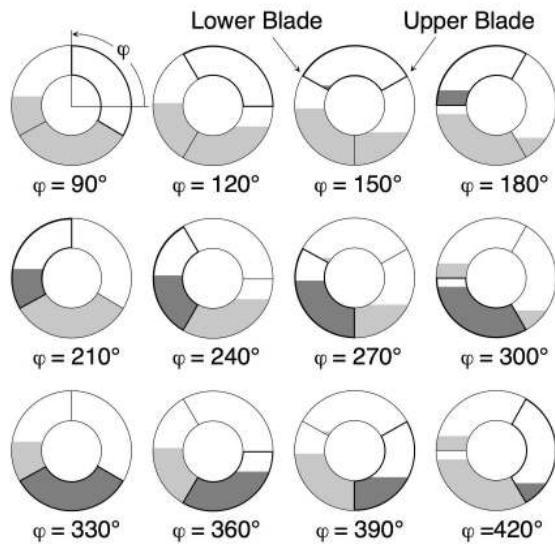


FIG. 15. Cross Section of Screw for Various Values of Blade Angle (Water in One Chute Is Darkened for Easier Identification)

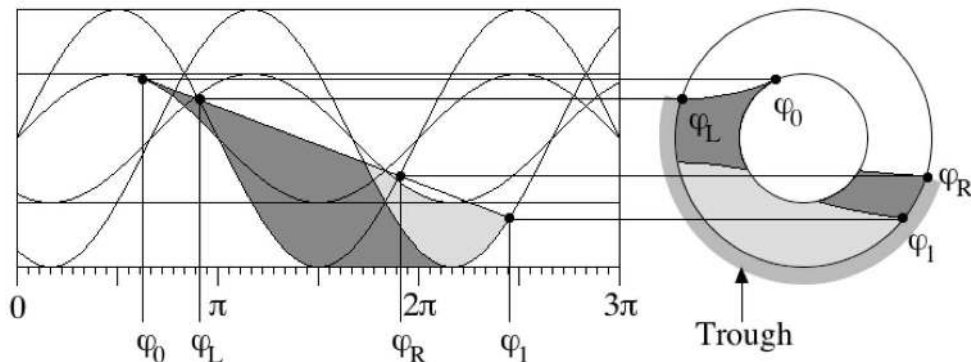


FIG. 16. Profile and Cross-Sectional Views of Screw in Figs. 14 and 15

repeats every  $120^\circ$  or, more generally, every  $2\pi/N$  radians for an  $N$ -bladed screw. Notice that for angles close to  $330^\circ$  the cross section of the bucket completely fills the cross section of the chute so that the value of  $\gamma_B$  is  $1/4$ , since for this screw the cross-sectional area of each chute is one-fourth the cross-sectional area of the entire screw.

Fig. 16 illustrates how certain angles associated with the geometry of a bucket are determined. In both the profiles and cross-sectional views only a single bucket is shown and only the two blades of the chute containing the bucket are shown. Additionally, the surfaces of the two blades and the inner and outer cylinders have been made transparent. Looking down into the screw in the cross-sectional view, the darker region is the top horizontal water surface of the bucket and the lighter region is the portion of the bucket in contact with the upper blade of the chute. Letting  $y$  denote distance above the axis of the screw in the profile view, the following curves in this figure have the following equations:

$$\text{Outer edge of the lower (left) blade: } y = R_o \sin \phi$$

$$\text{Inner edge of the lower (left) blade: } y = R_i \sin \phi$$

$$\text{Outer edge of the upper (right) blade: } y = R_o \sin(\phi - 2\pi/N)$$

$$\text{Inner edge of the upper (right) blade: } y = R_i \sin(\phi - 2\pi/N)$$

$$\text{Water level in the bucket: } y = -\frac{K\Lambda}{2\pi} (\phi - \phi_0) + R_i \sin \phi_0$$

The angle  $\phi_0$ , where the bucket begins, is where the water level is tangent to the inner edge of the lower blade. Thus

$$R_i \cos \phi_0 = -\frac{K\Lambda}{2\pi} \quad \text{or} \quad \cos \phi_0 = -\frac{\lambda}{\rho} \quad (16)$$

(This is assuming  $\lambda < \rho$ ; otherwise  $\phi_0 = \pi$ .) The angle  $\phi_1$  where the bucket ends is where the water level hits the outer edge of the upper blade for a third time. Thus

$$R_o \sin \left( \phi_1 - \frac{2\pi}{N} \right) = -\frac{K\Lambda}{2\pi} (\phi_1 - \phi_0) + R_i \sin \phi_0$$

$$\text{or} \quad \sin \left( \phi_1 - \frac{2\pi}{N} \right) = -\lambda(\phi_1 - \phi_0) + \rho \sin \phi_0 \quad (17)$$

The angles denoted  $\phi_L$  or  $\phi_R$  determine the portion of the outer cylinder in contact with the water. If the outer cylinder is an open trough, then they determine the minimum angular boundaries of the trough needed to contain the buckets of water. Both angles are determined by points of intersection of the water level and the outer edge of the lower blade, so that they are both solutions of the equation

$$R_o \sin \phi = -\frac{K\Lambda}{2\pi} (\phi - \phi_0) + R_i \sin \phi_0$$

$$\text{or} \quad \sin \phi = -\lambda(\phi - \phi_0) + \rho \sin \phi_0 \quad (18)$$



For Vitruvius's screw ( $\rho = 1/2$ ,  $\lambda = 3/16$ ) these angles turn out to  $\phi_0 = 112.02^\circ$ ,  $\phi_1 = 441.96^\circ$ ,  $\phi_L = 162.68^\circ$ , and  $\phi_R = 343.00^\circ$ . Depending on the parameters of the screw, there are many other possibilities as to how the water level can hit the four sinusoidal curves and the inner and outer cylinders in Fig. 16. One must be careful with the bookkeeping to make sure the correct angles are chosen in the correct ranges.

As seen in Fig. 15, each cross section of a bucket consists of a portion of the cross section of a chute cut off by a straight line representing the horizontal water level of the bucket in that cross section. The cross section of a chute, in turn, consists of a region bounded by two concentric circles and two rays. Basic geometry can be used to find exact formulas for the function  $\gamma_B(\phi)$ , although the formulas are quite messy because there are many ways that the horizontal water line can hit the two circular boundaries and two straight-line boundaries of a chute cross section.

Even though exact formulas for  $\gamma_B(\phi)$  for each  $\phi$  can be generated, it is not possible to obtain an exact expression for the integral in (15) that determines the volume ratio. This is because the integral in (15) must be written as a sum of sub-integrals, each for a different shape of the bucket cross section, and the angles needed as limits for these subintegrals (such as the angles  $\phi_1$ ,  $\phi_L$ , and  $\phi_R$ ) cannot be expressed in closed form. Consequently, a MatLab computer program was written to determine the volume ratio numerically. This program implemented Simpson's Rule to evaluate the integral in (15) using a blade-angle spacing of  $2^\circ$ .

## ACKNOWLEDGMENTS

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## APPENDIX III. NOTATION

The following symbols are used in this paper:

- $A_0$  = fraction of area of cross section of horizontal screw occupied by water (dimensionless);
- $A_B$  = area of water in cross section of one bucket ( $\text{m}^2$ );
- $K$  = slope of screw (dimensionless);
- $L$  = total length of screw (m);
- $N$  = number of blades in screw (dimensionless);
- $R_i$  = radius of screw's inner cylinder (m);
- $R_o$  = radius of screw's outer cylinder (m);
- $V_B$  = volume of one bucket of water ( $\text{m}^3$ );
- $V_t$  = volume of water lifted in one turn of screw ( $\text{m}^3$ );
- $x$  = position along axis of screw (m);
- $y$  = distance perpendicular to axis of screw (m);
- $\alpha$  = angle of incline of spiral intersection of blade and outer cylinder with respect to axis of screw (rad);
- $\beta$  = angle of incline of spiral intersection of blade and inner cylinder with respect to axis of screw (rad);
- $\theta$  = angle of incline of screw (rad);
- $\Lambda$  = pitch (or period) of blade (m);
- $\lambda$  = pitch ratio =  $K\Lambda/2\pi R_o$  (dimensionless);
- $\nu$  = fraction of volume of one cycle of screw occupied by water (volume ratio) (dimensionless);
- $\rho$  = radius ratio =  $R_i/R_o$  (dimensionless); and
- $\phi$  = blade angle =  $2\pi x/\Lambda$  (rad).

(Note: The optimal value of a parameter is denoted by an asterisk; e.g.,  $V_t^*$ .)