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THE TWISTING OPERATOR IN MULTI-VENEZIANO THEORY

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A B S T R A C T

Using the operatorial expression for the gauge conditions, we derive an explicit form for the twisting operator and the semi-twisting operator which takes the P states into V states. The expression for the last operator facilitates the proof of the V states factorization previously found by us. We show that the twisting operator proposed in the literature is inconsistent with multiple factorization, i.e., factorization of amplitudes with external spinning particles. The correct twisting operator depends on the integration variables of the twisted line, and this gives the spectrum a greater degeneracy than that previously obtained.

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1. INTRODUCTION

In a previous paper (I) [1], we saw how to factorize the multi-Veneziano amplitude in a new basis (called the V basis) which has very simple properties under twisting. This result was obtained analytically by a change of integration variables having a group property. We emphasized the relationship between this change of variables and the so-called gauge identities [2].

Having set up the formalism in Section 2, we devote Section 3 to rederiving and generalizing our results in a much simpler way, using the operator formalism [3], and in particular the group noticed by Gliozzi [4]. In Section 4 we consider multiple factorizations, and conclude that the twisting operator proposed by Caneschi, Schwimmer and Veneziano [5] is not the correct one to use in such situations, and in particular in loop calculations. In factorizing twisted graphs with external spinning particles, we see that the spectrum of internal particles is even larger than that found hitherto [2),3)]. As we shall show, this is because the previously found degeneracy [2] breeds a new degeneracy, which owing to the gauge conditions cannot be seen by looking at scalar particle amplitudes.

2. GAUGE AND TWISTING OPERATORS

We use the operator formalism for the multiparticle amplitude and define as usual [3] operators $a_\mu^{(n)}$ that satisfy [6]:

$$[a_\mu^{(m)}, a_\nu^{(n)}] = -g_{\mu\nu}\,\delta_{mn} \tag{2.1}$$

so that the N+M+4 particle Veneziano term of Fig. 1 is given by [7]:

$$B_{N+M+4} = \int_0^1 \Big[\prod_{i=1}^{N} d\mu(z_i)\Big]\Big[\prod_{j=1}^{M} d\mu(\bar z_j)\Big]\, d\mu(z)\, \times$$

$$\times \langle 0| V(q_M)\, \bar z_M^{\,L_0(\bar\Pi_M)} \cdots V(q_0)\, z^{L_0(\Pi)}\, V(p_0) \cdots V(p_N) |0\rangle \tag{2.2}$$

where

$$L_0(\Pi) = -\tfrac{1}{2}\Pi^2 + H = -\tfrac{1}{2}\Pi^2 - \sum_{n=1}^{\infty} n\, a^{(n)\dagger} \cdot a^{(n)} \tag{2.3}$$

$$\Pi = \sum_{i=0}^{N+1} p_i \qquad \Pi_e = \sum_{i=e}^{N+1} p_e \qquad \bar{\Pi}_e = -\sum_{i=e}^{M+1} q_e \tag{2.4}$$

$$d\mu(z) = dz\; z^{-(\alpha+1)}\,(1-z)^{\alpha-1} \tag{2.5}$$

$$V(p_i) = e^{-p_i\cdot\sum_{n=1}^{\infty}\frac{a^{(n)\dagger}}{\sqrt{n}}}\; e^{p_i\cdot\sum_{n=1}^{\infty}\frac{a^{(n)}}{\sqrt{n}}} \tag{2.6}$$

In what follows, the dependence of $L_0(\Pi)$ and similar operators on the momentum Π of the internal line will be suppressed when no confusion is possible.

We shall call the ket obtained by applying on the vacuum a chain of vertices and propagators (Fig.2) :

$$|p\rangle = \int_0^1 \Big[\prod_{i=1}^{N} d\mu(z_i)\Big]\, V(p_0)\, z_1^{L_0(\Pi_1)} \ldots\ldots\; z_N^{L_0(\Pi_N)}\, V(p_N)\,|0\rangle \tag{2.7}$$

a scalar ket $|p\rangle$ since it describes the coupling to scalar particles. Then the amplitude (2.2) can be factorized in the manner of Fubini and Veneziano [2] :

$$B_{N+M+4} = \int_0^1 d\mu(z)\;\langle q|\, z^{L_0}\,|p\rangle \tag{2.8}$$

Gliozzi [4] has noticed the very interesting fact that one can define, together with L_0, two operators L_+ and L_- associated with the momentum Π :

$$L_+ = L_-^{\dagger} = -\Pi\cdot a^{(1)\dagger} - \sum_{n=1}^{\infty}\sqrt{n(n+1)}\; a^{(n+1)\dagger}\cdot a^{(n)} \tag{2.9}$$

such that L_0, L_+, L_- satisfy the Lie algebra of the SU(1,1) group :

$$[L_+, L_0] = L_+ \qquad [L_-, L_0] = -L_- \qquad [L_-, L_+] = 2L_0 \tag{2.10}$$

This group is obviously not an invariance group; indeed, the Hamiltonian H is, according to Eq. (2.3), apart from the additive c number $-\Pi^2/2$, one of the generators. However, the theory is such that rotations around a fixed complex axis leave invariant the physical states (2.7). To see this, we define, following Gliozzi [4], the operator W :

$$W = L_0 - L_- \tag{2.11}$$

which has the property that its exponential leaves invariant the scalar kets :

$$e^{\alpha W} |p\rangle = |p\rangle \tag{2.12}$$

Equation (2.12) is an immediate consequence of the fact, proved by Gliozzi, that :

$$W |p\rangle = 0$$

The invariance under these rotations will prove to be extremely useful, because it will allow us to use the freedom in the definition of physical states to transform matrix element identities into operator identities.

Another interesting, although more technical, point about this SU(1,1) group, is the fact that many of the operators which we shall have to consider are obtained by exponentiating the generators L_0, L_-, L_+, or linear combinations of them. Therefore, they are elements of an (infinite-dimensional) unitary representation of SU(1,1), so that all identities between operators which depend only on the group law can be proved by looking at the two-dimensional representation of SU(1,1) [8]. For instance, we can work with the following representation of the infinitesimal generators :

$$L_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad L_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

obtaining for $e^{\alpha W}$:

$$e^{\alpha W} = \begin{pmatrix} e^{\alpha/2} & 0 \\ -2\sinh\alpha/2 & e^{-\alpha/2} \end{pmatrix} = e^{-L_-} \; e^{\alpha L_0} \; e^{L_-} \qquad (2.13)$$

After these preliminaries, let us now turn to the definition of the twisting operator Θ ; we wish to find Θ with the following properties :

$$\Theta\,|p\rangle = |\bar{p}\rangle \quad , \quad \Theta\,|\bar{p}\rangle = |p\rangle \quad \Rightarrow \quad \Theta^2 = 1 \qquad (2.14)$$

where $|\bar{p}\rangle$ is obtained from $|p\rangle$ by reversing the order of the momenta $p_0,\ldots,p_{N+1}$. The second property we require is

$$\Theta^\dagger \; z^{L_0} \; \Theta = z^{L_0} \qquad (2.15)$$

which is nothing but double twist invariance in operator form. A special form for the twisting operator was proposed [4),5)] (and explained further in the Appendix) :

$$\Omega = e^{L_+} \; (-1)^H = (-1)^H \; e^{-L_+} \qquad (2.16)$$

Ω , however satisfies only (2.14) and not (2.15). Double twist invariance is true between scalar states because of the gauge conditions. We shall explain in Section 4 why it is fundamental that (2.15) be satisfied operatorially.

From the preceding discussion, it is clear that

$$\Theta(\alpha) = \Omega\, e^{\alpha W}$$

satisfies

$$\Theta\,|p\rangle = |\bar{p}\rangle$$

for every α . In addition it is easy to check, using for instance Pauli matrices, as indicated by (2.13), that for every α , $\Theta(\alpha)^2 = 1$. We shall now determine α by demanding that (2.15) be satisfied. Again, e.g., by Pauli matrices, it can be checked that

$$(1-z)^{L_0-L_+}\ e^{-L_-}\ z^{L_0}\ e^{-L_+}\ (1-z)^{L_0-L_-} = z^{L_0}$$

so that $\alpha = \ln(1-z)$ and therefore [9]

$$\Theta = \Omega\,(1-z)^{W} \tag{2.17}$$

We can now define a chain with "signaturized" propagators, by adding to each propagator in the chain (2.2) its twisted counterpart, and obtain an amplitude for the N point function (Fig.3) :

$$A_N = \int_0^1 \Big[\prod_{i=3}^{N-1} d\mu(z_i)\Big] \langle 0|\,V(p_2)\, z_3^{L_0(\Pi_3)}\big[1+\Theta(\Pi_3)\big]\ldots$$
$$\ldots\ z_{N-1}^{L_0(\Pi_{N-1})}\big[1+\Theta(\Pi_{N-1})\big]\ V(p_{N-1})\,|0\rangle \tag{2.18}$$

where

$$\Pi_\ell = \sum_{i=\ell}^{N} p_i$$

The amplitude (2.18) represents the sum of all single Veneziano terms having all poles in the variables Π^2_ℓ, $\ell = 3,\ldots,N-1$.

We defer to Section 4 the discussion of the importance of the $(1-z)^W$ factor included in Θ . Let us only remark that, due to (2.17) :

$$z^{L_0}(1+\Theta) = (1+\Theta^\dagger)\, z^{L_0} = \tfrac{1}{2}(1+\Theta^\dagger)\, z^{L_0}(1+\Theta) \tag{2.19}$$

so that all three equivalent expressions can be used in the chain (2.18).

3. V-U FACTORIZATION AND THE V CHAIN

Before going further, we wish to show how the factorization of the N+M+4 point function in terms of the eigenstates of the twisting operator ("V states"), which was proved in (I) by a change of variables, can also be derived with the operator formalism.

Proceeding as before, let us define a transformation operator, τ , that applied to a state $|p\rangle$, transforms it into a state $|V\rangle$ defined by :

$$|V\rangle = \int_0^1 d\varphi(\rho_i \cdot p_i) \exp\left[-\sum_{n=1}^{\infty} \frac{V^{(n)} \cdot a^{(n)\dagger}}{\sqrt{n}}\right] |0\rangle \qquad (3.1)$$

where [2),10)]

$$V^{(n)} = \sum_{i=0}^{N+1} \left(\frac{1}{2} - \rho_i\right)^n p_i \qquad (3.2)$$

We thus have :

$$|V\rangle = \tau\, |p\rangle \qquad (3.3)$$

As noticed by Gliozzi [4)], a possible operator which makes such a transformation is

$$\Omega(1/2) = (-1)^H e^{-\frac{1}{2}L_+} = e^{\frac{1}{2}L_+} (-1)^H \qquad (3.4)$$

A simple derivation of (3.4) is given in the Appendix. As in the previous Section :

$$\tau = \Omega(1/2)\, e^{\beta W} \qquad (3.5)$$

has the same effect as $\Omega(\frac{1}{2})$ on a scalar state $|p\rangle$ and we are going to determine β in order to ensure V-U factorization.

Indeed, we want the states $|V\rangle$ to factorize the propagator, i.e., we want the propagator in the V basis to be of the form x^{L_0}, where f is some function of z [actually we shall use $(4f)^{L_0}$ to be consistent with our definitions in (I)]. We thus look for an operator τ such that :

$$\tau^{\dagger} (4f)^{L_0} \tau = z^{L_0} \tag{3.6}$$

By inserting (3.5) into (3.6), we find, after some manipulations of 2x2 matrices that [11] :

$$f(z) = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}} \qquad \text{or} \qquad z = \frac{4f}{(1+f)^2} \tag{3.7}$$

and

$$\beta = \ln\left(\frac{1-f}{1+f}\right) = \ln\sqrt{1-z} \tag{3.8}$$

We can then write τ as :

$$\tau = (-1)^H e^{-\frac{1}{2}L_+} \left(\frac{1-f}{1+f}\right)^W = (-1)^H e^{-\frac{1}{2}L_+} \left(\sqrt{1-z}\right)^W \tag{3.9}$$

Furthermore, by using the identities :

$$\Omega\, e^{\alpha W}\, \Omega = e^{-\alpha W} \qquad \Omega(1/2)\,\Omega = (-1)^H\, \Omega(1/2)$$

we obtain

$$\tau \, \Theta \, \tau^{-1} = (-1)^H \tag{3.10}$$

which shows that the twisting operator is diagonal in the V basis, as was expected.

Equation (3.6) shows directly the factorization of the multiparticle amplitude (2.2) in terms of the V states (3.3). Indeed, making in (2.2) the change of variables $z \to f$ we recover the V-U factorization of (I) :

$$B_{N+M+4} = \int_0^1 d\nu(f) \, \langle U | (4f)^{L_0} | V \rangle \tag{3.11}$$

$$d\nu(\rho) = d\rho \; 4^{-a} \rho^{-a-1} (1+\rho)(1-\rho)^{2a-1} \tag{3.12}$$

The generalization to multiple factorization is obvious. Starting from the "P chain" (2.2), we replace each $z_i^{L_0}$ by its value in (3.6) and make the change of variables $z_i \to f_i$. The propagator will be given by $(4f_i)^{L_0(\Pi_i)}$, but the new vertex $V'(p_i)$ will depend on the integration variables f_i and f_{i+1} :

$$V'(p_i) = \tau(\rho_i, \Pi_i) \, V(p_i) \, \tau^{\dagger}(\rho_{i+1}, \Pi_{i+1}) \tag{3.13}$$

The vertices at the two ends of the chain will of course depend only on one integration variables. The chain in the V basis will then be written as (Fig.3) :

$$\begin{aligned} B_N = \int_0^1 \Big[\prod_{i=3}^{N-1} d\nu(\rho_i)\Big] \langle 0| V'(p_2)(4\rho_3)^{L_0(\Pi_3)} V'(p_3) \ldots \\ \ldots (4\rho_{N-1})^{L_0(\Pi_{N-1})} V'(p_{N-1}) |0\rangle \end{aligned} \tag{3.14}$$

If we now wish to write the chain (2.18) with signaturized propagators, in the V representation, we remark that, from Eqs. (3.6) and (3.10) :

$$z^{L_0}(1+\Theta) = \tau^{\dagger} (4\rho)^{L_0} (1+(-1)^H) \, \tau \tag{3.15}$$

so that the signaturized propagator in the V representation is

$$(4\rho)^{L_0} (1+(-1)^H)$$

as noted in (I). We then have for the chain A_N :

$$\begin{aligned} A_N = \int_0^1 \Big[\prod_{i=3}^{N-1} d\nu(\rho_i)\Big] \langle 0| V'(p_2)(4\rho_3)^{L_0(\Pi_3)}(1+(-1)^H) \ldots \\ \ldots (4\rho_{N-1})^{L_0(\Pi_{N-1})}(1+(-1)^H) V'(p_{N-1}) |0\rangle \end{aligned} \tag{3.16}$$

4. DISCUSSION

Consider the expression for the chain of Fig. 2 :

$$\int_0^1 \left[\prod_{i=1}^{N} d\mu(z_i)\right] V(p_0)\, z_1^{L_0}(1+\Theta)V(p_1)\ldots V(p_N)\,|0\rangle \qquad (4.1a)$$

This is a linear combination of states corresponding to different permutations of the external particles, the different permutations being realized by the twists on the internal lines. For each of these states, and hence for the sum it was shown that :

$$\Theta\,|\;\rangle = \Omega\,|\;\rangle$$

Such a state we also called a "scalar ket". By the same argument, any or all of the Θ's in (4.1a) can be replaced by Ω .

Similarly in the expression :

$$\int_0^1 \left[\prod_{i=1}^{N} d\mu(z_i)\right] \langle 0|\, V(p_N)(1+\Theta^\dagger)\, z_N^{L_0}\ldots(1+\Theta^\dagger)\, z_1^{L_0}\, V(p_0) \qquad (4.1b)$$

it is possible to replace $\Theta^\dagger$ by $\Omega^\dagger$. We call such an expression a "scalar bra". Since :

$$z^{L_0}(1+\Theta) = (1+\Theta^\dagger)\, z^{L_0} = \tfrac{1}{2}(1+\Theta^\dagger)\, z^{L_0}\,(1+\Theta) \qquad (4.2)$$

the expression (see Fig.3) :

$$\int_0^1 \left[\prod_{i=3}^{N-1} d\mu(z_i)\right] \langle 0|\, V(p_2)\, z_3^{L_0}(1+\Theta)V(p_3)\ldots V(p_{N-1})\,|0\rangle \qquad (4.3)$$

has the property that if we break it on any intermediate line, the expression to the right of the propagator is a "scalar ket", while the expression to the left is a "scalar bra".

By starting at the left and working to the right, we can replace each Θ by Ω . Thus all propagators in (4.3) can be replaced by

$$z^{L_0}\ (1+\Omega) \tag{4.4}$$

as long as all the external particles are scalars. Similarly, starting at the right and working to the left, we can replace each propagator by :

$$(1+\Omega^{\dagger})\ z^{L_0} \tag{4.5}$$

Starting on a line in the middle, we can use for the propagator of this line

$$\frac{1}{z}\ (1+\Omega^{\dagger})\ z^{L_0}\ (1+\Omega) \tag{4.6}$$

Now, when (4.6) acts on a scalar ket, the result is no longer a scalar ket because of the $\Omega^{\dagger}$. Therefore, the only way we can eliminate the factors $(1-z)^W$ is to have propagators $z^{L_0}(1+\Omega)$ to the right, and $(1+\Omega^{\dagger})z^{L_0}$ to the left of the factor (4.6). In particular we can have no other line with propagator (4.6) because if it is on the right of the line previously mentioned, we can no longer replace $\Theta^{\dagger}$ by $\Omega^{\dagger}$, and similarly if it is on the left.

We have seen that for scalar particles, chains with signaturized propagators (4.2), (4.4) or (4.5) are all equivalent. On any given line, one cannot interchange (4.4) or (4.5) while all three expressions (4.2) are completely equivalent and always interchangeable, since the identities in (4.2) are operatorial identities. It looks as if we have an embarrassment of possible "signaturized propagators", but it is the last mentioned property that will lead us to argue that, contrarily to the original proposal [5], (4.4) or (4.5) are not satisfactory, and it is the equivalent forms (4.2) which must be used on each line.

The integrated chains described above are all equivalent and so give the same result if a complete set of occupation number states $|\{n\}\rangle$ is inserted beside any of the factors z^{L_0}. However, the meaning - as well as the individual contribution of each occupation number - can depend upon the chain chosen.

An individual term in the sum contains a product of two factors :

$$\langle 0| V(p_2) P(z_3) \ldots \; |\{n\}\rangle \langle \{n\}| \ldots \; P(z_{N-1}) V(p_{N-1}) |0\rangle$$

where $P(z)$ is the signaturized propagator. Now that the left-hand factor ends in $|\{n\}\rangle$, which is not a scalar ket, we cannot replace Θ by Ω in that chain, but <u>only</u> $\Theta^\dagger$ by $\Omega^\dagger$. Similarly, in the right-hand factor, we can only replace Θ by Ω and <u>not</u> $\Theta^\dagger$ by $\Omega^\dagger$. Thus propagators (4.4) and (4.5) cease to be equivalent. There are only two ways of achieving a correct factorization :

a) use (4.2) on each line with $\frac{1}{2}(1+\Theta^\dagger)z^{L_0}(1+\Theta)$ on the line to be factorized;

b) use (4.6) on the line to be factorized and (4.4) to the right, (4.5) to the left.

In this case the contributions are the same and correspond to combinations of P and $\overline{P}$ states [2)] on the right, and similar contributions on the left.

The second way of achieving this factorization, b), is obviously unsatisfactory, since when we make a second factorization, the result would depend upon the order in which we made the two factorizations.

Suppose in fact we make a second factorization to the right of the first one. Then the middle factor is :

$$\langle \{n\}| \ldots \; V z^{L_0} (1+\Omega) V \ldots \; |\{m\}\rangle \tag{4.7}$$

If we reverse the order of the factorizations, the middle factor is :

$$\langle \{n\}| \ldots \; V (1+\Omega^\dagger) z^{L_0} V \ldots \; |\{m\}\rangle \tag{4.8}$$

The chains (4.7) and (4.8) are <u>not</u> the same, since, by having occupation number states at both ends, we have "frozen" the propagators, and cannot re-introduce the Θ's at all.

Finally, try to make a third factorization, by introducing occupation number states in the middle of either (4.7) or (4.8). The result obviously does not factorize, in the sense that the factor on the right is not constructed in the Hermitian conjugate way of that on the left.

<u>The chain $z^{L_0}(1+\Theta)$ leads to no difficulties of this kind, and we conclude that this is the genuine factorizing chain</u>.

At first sight, this result would seem surprising, because the starting point - a chain describing scalar particles - is identical whether we use $z^{L_0}(1+\Theta)$ or $z^{L_0}(1+\Omega)$. It would seem that the same statement must be true for spinning particles, simply by taking the residues of the corresponding poles in a larger scalar amplitude. This would indeed be true if the states contributing to any pole were not degenerate in mass and in the form of the coupling. In this theory, we obviously have a degeneracy in mass, and the gauge conditions do lead to a degeneracy in coupling.

Having established the necessity of the chain with Θ's, we notice that this introduces an extra z dependence except in the case when there are no twists at all. We shall discuss the consequences of this z dependence later.

So far we have spoken only of the P representation. We have shown that [Eq. (3.5)] :

$$z^{L_0}(1+\Theta) = \tau^{\dagger}(4f)^{L_0}(1+(-1)^{H})\,\tau \tag{4.9}$$

and this corresponds to the V representation propagator. This time the z dependence appears in τ , if a complete set of intermediate states is inserted next to $(4f)^{L_0}$. The only difference is that in the V representation this extra z dependence arises even if we do not consider twists. On the other hand, the advantage of this representation is that introducing twists makes very little extra complications.

In calculating the loops, one first has to do a double factorization and, as we have seen, it is essential to use the signaturized propagator (4.2) rather than the one proposed by Caneschi, Schwimmer and Veneziano [5]. It is rewarding that the loop calculation, as the trace over all modes [7] of the chain, made with (4.9), is independent of whether it is calculated in the P or in the V representation, due to the operatorial identity (4.9).

We now investigate the consequence of the z (or f) dependence of the operators Θ and τ, by trying to factorize the chain of Fig. 4, where there is one occupation number state at each end. We write the amplitude in the P representation

$$A_{\{j\}\{i\}} = \frac{1}{2}\int_0^1 \left[\prod_{i=1}^{N} d\mu(z_i)\right]\left[\prod_{j=1}^{M} d\mu(\bar{z}_j)\right] d\mu(z) \langle\{j\}|(1+\Theta)V(q_M)$$
$$\ldots\ldots V(q_0)(1+\Theta^{\dagger})z^{L_0}(1+\Theta)V(p_0)\ldots\ldots V(p_N)(1+\Theta^{\dagger})\,|\{i\}\rangle \tag{4.10}$$

We then insert a complete set of occupation number states next to z^{L_0} and get :

$$A_{\{j\}\{i\}} = \frac{1}{2}\sum_{\{n\}}\int_0^1 \prod d\mu(z_i)\prod d\mu(\bar{z}_j)\, d\mu(z)\; z^{n-\alpha(s)-1}\,(1-z)^{a-1}$$
$$\langle\{j\}|(1+\Theta)V(q_M)\ldots(1+\Theta^{\dagger}(z))|\{n\}\rangle\langle\{n\}|(1+\Theta(z))V(p_0)\ldots\;|\{i\}\rangle \tag{4.11}$$

Since $\Theta(z)$ is analytic near $z=0$, we can expand both matrix elements as power series in z, e.g., as :

$$\int_0^1 \langle\{n\}|(1+\Theta(z))V(p_0)\ldots V(p_N)(1+\Theta^{\dagger}(z_N))\,|\{i\}\rangle\prod_{i=1}^{N} d\mu(z_i)$$
$$= \sum_{k=0}^{\infty} \Gamma_k^{\{n\}\{i\}}(p)\; z^k \tag{4.12}$$

and obtain [forgetting about the trivial factor $(1-z)^{a-1}$ which can be included by the device of the scalar mode [7]]

$$A_{\{j\}\{i\}} = \frac{1}{2}\sum_{\{n\}}\sum_{k,\ell} \frac{\Gamma_\ell^{\{n\}\{j\}*}(q)\;\Gamma_k^{\{n\}\{i\}}(p)}{\alpha(s)-(n+k+\ell)} \tag{4.13}$$

This last equation shows that a particular intermediate state characterized by a partition $\{n\}$ gives rise to poles at

$$\alpha(s) = n + N$$

where n is the eigenvalue of H $(= -\sum m\, a^{(m)\dagger}\cdot a^{(m)})$, i.e., $n = \sum_i i n_i$, and N is an arbitrary non-negative integer. This implies that the daughter trajectories are even more degenerate than those obtained by Fubini and Veneziano [2], while the dominant trajectory is still non-degenerate. Unfortunately, Eq. (4.13) is still unsuited for investigating the additional degeneracy, because it is not written in a factorized form. Anyhow, it is apparent that the spectrum has an extra degeneracy, and that the gauge conditions act in such a way that the contribution of these extra states sums to zero when the external particles are scalars.

We are presently investigating whether it is possible to write Eq. (4.13) in a satisfactory factorized form, so that we can characterize the extra degeneracy.

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APPENDIX

The gauge conditions can be proved by making a change of variables

$$\rho_i = \frac{(1-z)\,\sigma_i}{1-\sigma_i z} \tag{A.1}$$

and one finds :

$$\int d\varphi\ F(\rho)$$
$$= (1-z)^{-\Pi^2/2} \int d\varphi\ F\left[\frac{(1-z)\rho}{1-z\rho}\right]\ e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} P^{(n)}\cdot\Pi} \tag{A.2}$$

The notation is as in (I), except that we understand the off-mass shell particle to be on the left. The proof follows that of Ref. 9), and can also be obtained from Eq. (2.6) of (I), by re-expressing V's in terms of P's and replacing f by z, where :

$$\frac{1}{f} + \frac{2}{z} = 1$$

Just as in (I), the change of variables (A.1) forms a group with the following group law :

$$(1-z_3) = (1-z_1)(1-z_2) \tag{A.3}$$

Taking

$$F(\rho) = |p\rangle\rangle = e^{-\sum_n P^{(n)}\cdot a^{(n)+}/\sqrt{n}}\ |0\rangle$$

[so that : $|p\rangle = \int d\varphi\,(\rho,p)\,|p\rangle\rangle$], (A.2) becomes :

$$|p\rangle = (1-z)^{-\Pi^2/2} \int d\varphi\ e^{-\sum_n \frac{z^n}{n} P^{(n)}\cdot\Pi}\ e^{-\sum_n P'^{(n)}\cdot a^{(n)+}/\sqrt{n}}\ |0\rangle$$

where :

$$P'^{(n)}(\rho_i) = P^{(n)}\left[\frac{(1-z)\,\rho_i}{1-z\rho_i}\right]$$

One seeks an operator S such that :

$$S\,|p\rangle\rangle = (1-z)^{-\Pi^2/2}\, e^{-\sum_n \frac{z^n}{n} P^{(n)}.\Pi}\, e^{-\sum_n P'^{(n)}.a^{(n)\dagger}/\sqrt{n}}\,|0\rangle \qquad (A.4)$$

Chiu, Rebbi and Matsuda [9] obtained S in normal order form, but we wish to indicate how to derive the alternate form quoted by them, and used by us in the text of this paper.

Because of the group property (A.3) :

$$S = (1-z)^W$$

and by taking z small, so that $S = 1 - zW$, we find from (A.4), using the method of coherent states, that W is indeed given by (2.11).

We now show that Gliozzi's form [4] of Ω and $\Omega(\frac{1}{2})$ can be derived by a similar argument.

Caneschi, Schwimmer and Veneziano noted that Ω could be written

$$\Omega = e^{-\Pi.a^\dagger}\, e^{-\sum_{n>m} a^{(n)\dagger}\sqrt{\frac{m}{n}}\binom{n}{m}a^{(m)}}\,(-1)^H = \tilde{\Omega}\,(-1)^H \qquad (A.5)$$

where :

$$\tilde{\Omega}\,|p\rangle\rangle = |\tilde{p}\rangle\rangle$$

and, symbolically [2] :

$$\tilde{P}^{(n)} = (1+P)^n$$

The minus signs have been all put into the $(-1)^H$. Suppose now we seek an operator $\tilde{\Omega}(\lambda)$ such that :

$$\tilde{\Omega}(\lambda)\,|\,p\rangle\!\rangle = |\,\tilde{p}_\lambda\rangle \qquad \tilde{P}_\lambda^{(n)} = (\lambda + P)^n$$

Since

$$[(\tilde{P}_\lambda)_{\lambda'}]^{(n)} = (\lambda + \lambda' + P)^n = \tilde{P}_{\lambda+\lambda'}^{(n)}$$

we see, applying the operation twice, that :

$$\tilde{\Omega}(\lambda)\,\tilde{\Omega}(\lambda') = \tilde{\Omega}(\lambda + \lambda')$$

which indicates that we can write :

$$\tilde{\Omega}(\lambda) = e^{\lambda A} \tag{A.6}$$

Taking λ to be small, we see - as above - that $A = L_+$. Hence :

$$\Omega(\lambda) = e^{\lambda L_+}(-1)^H = (-1)^H e^{-\lambda L_+} \tag{A.7}$$

where $\Omega(\lambda)$ takes $P^{(n)}$ into $(\lambda - P)^n$. In particular, $\Omega = \Omega(1)$ is the twisting operator and $\Omega(\frac{1}{2})$ is the semi-twisting operator taking $P^{(n)}$ to $V^{(n)} = (\frac{1}{2} - P)^n$.

REFERENCES AND FOOTNOTES

1) D. Amati, M. Le Bellac and D. Olive, CERN preprint TH.1102 (1969) - to be published in Nuovo Cimento - referred to as (I).

2) S. Fubini and G. Veneziano, M.I.T. preprint (1969) - to be published in Nuovo Cimento;
K. Bardakçi and S. Mandelstam, Berkeley preprint (1969) - to be published in Phys.Rev.

3) S. Fubini, D. Gordon and G. Veneziano, Phys.Letters 29B, 679 (1969).

4) F. Gliozzi, University of Torino preprint (1969).

5) L. Caneschi, A. Schwimmer and G. Veneziano, Phys.Letters 30B, 351 (1969).

6) Our metric is $g_{oo} = -g_{ii} = +1$, so that the creation operator, $a_o^{(n)\dagger}$, acting on the vacuum, gives states of negative norm. The dot product between operators is defined by :

$$a^{(n)\dagger} \cdot a^{(n)} = a_\mu^{(n)\dagger} \, g_{\mu\nu} \, a_\nu^{(n)} .$$

7) D. Amati, C. Bouchiat and J.L. Gervais, Nuovo Cimento Letters 2, 399 (1969).

8) Since the two-dimensional representation of SU(1,1) is not unitary, one has to be careful in using always L_+ and L_-, and not their Hermitian conjugates. Moreover, since we have complex axes for the rotations, we are indeed dealing with a representation of the complex group (SL(2,C)).

9) This result has been previously obtained by C.B. Chiu, S. Matsuda and C. Rebbi, Caltech. preprint, by direct calculation. Further relation with their work is given in the Appendix.

10) We use the notations and conventions of (I) : $d\varphi(\rho, p)$, for instance, is the integrand of the Veneziano N point function. However, our $V^{(n)}$'s are divided by a factor 2^n, so that we come back to the original definition of Fubini and Veneziano [2].

11) In the preprint version of (I) there is an extra factor 4 in the first expression of (3.7) due to a misprint.

Figures

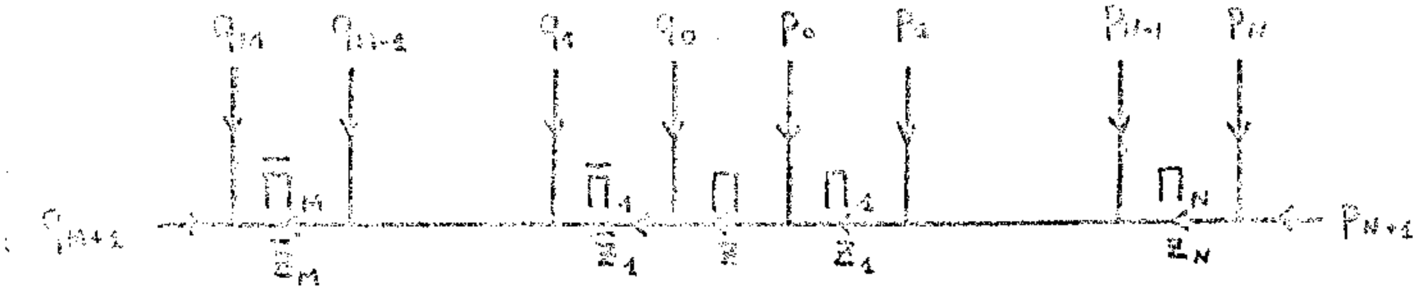

Fig. 1

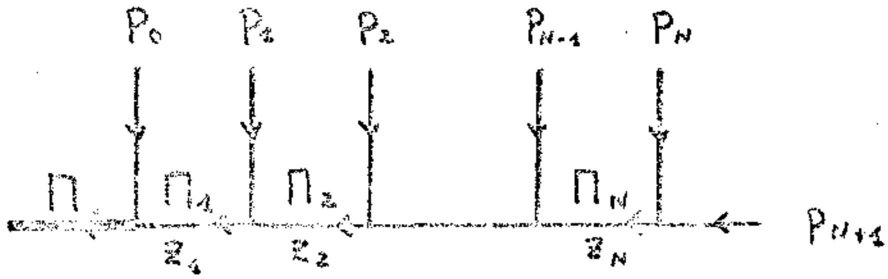

Fig. 2

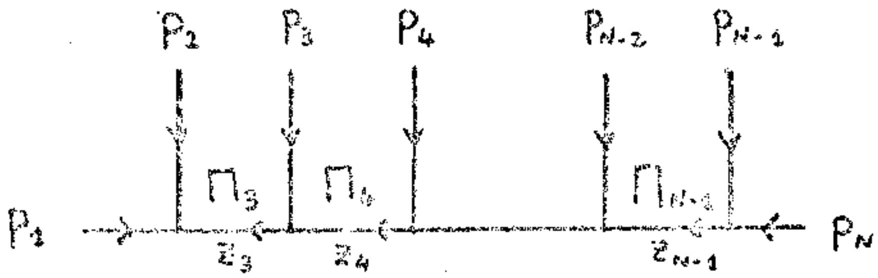

Fig. 3

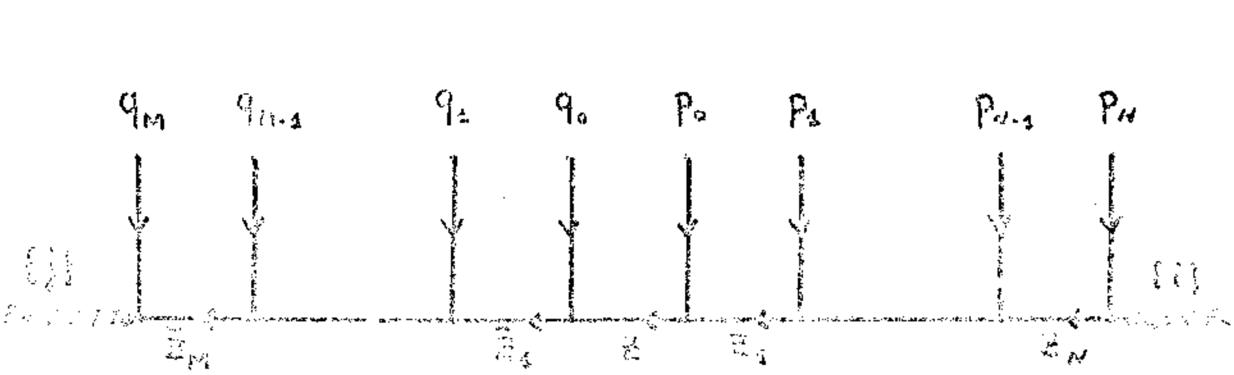

Fig. 4