

The two-loop hexagon Wilson loop in $\mathcal{N} = 4$ SYM

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ABSTRACT: In the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the conformal symmetry constrains multi-loop n -edged Wilson loops to be given in terms of the one-loop n -edged Wilson loop, augmented, for $n \geq 6$, by a function of conformally invariant cross ratios. That function is termed the remainder function. In a recent paper, we have displayed the first analytic computation of the two-loop six-edged Wilson loop, and thus of the corresponding remainder function, in terms of known mathematical functions. Although the calculation was performed in the quasi-multi-Regge kinematics of a pair along the ladder, the Regge exactness of the six-edged Wilson loop in those kinematics entails that the result is the same as in general kinematics. We show in detail how the most difficult of the integrals is computed, which contribute to the six-edged Wilson loop. Finally, the remainder function is given as a function of uniform transcendent weight four in terms of Goncharov polylogarithms. We consider also some asymptotic values of the remainder function, and the value when all the cross ratios are equal.

KEYWORDS: Supersymmetric gauge theory, Gauge Symmetry

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1 Introduction

In the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, Anastasiou, Bern, Dixon and Kosower [1] made an ansatz for the iterative structure of the colour-stripped two-loop scattering amplitude with an arbitrary number n of external legs in a maximally-helicity violating (MHV) configuration. Writing at any loop order L , the amplitude $M_n^{(L)}$ as the tree-level amplitude, $M_n^{(0)}$, which depends on the helicity configuration, times a scalar function, $m_n^{(L)}$, the proposed iteration formula for the two-loop MHV amplitude $m_n^{(2)}(\epsilon)$ was

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon). \quad (1.1)$$

Thus the two-loop amplitude is determined in terms of the one-loop MHV amplitude $m_n^{(1)}(\epsilon)$ evaluated through to $\mathcal{O}(\epsilon^2)$ in the dimensional-regularisation parameter $\epsilon = (4 - d)/2$, the constant $C^{(2)} = -\zeta_2^2/2$, and the function $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$, with $\zeta_i = \zeta(i)$ and $\zeta(z)$ the Riemann zeta function. In ref. [1], the iteration formula (1.1) was shown to be correct for the two-loop four-point amplitude, which previously had been evaluated analytically [2]. Eq. (1.1) was proven to be correct also for the two-loop five-point amplitude through numerical calculations [3, 4].¹

Subsequently, Bern, Dixon and one of the present authors extended the ansatz by proposing an all-loop resummation formula [9] for the colour-stripped n -point MHV amplitude, which implies a tower of iteration formulae, allowing one to determine the n -point amplitude at a given number of loops in terms of amplitudes with fewer loops, evaluated to higher orders of ϵ . In ref. [9], the ansatz was shown to be correct for the three-loop four-point amplitude, by evaluating analytically $m_4^{(3)}(\epsilon)$ through to finite terms, as well as $m_4^{(2)}(\epsilon)$ through to $\mathcal{O}(\epsilon^2)$ and $m_4^{(1)}(\epsilon)$ through to $\mathcal{O}(\epsilon^4)$.

However, eq. (1.1) fails for the two-loop six-point amplitude: in ref. [10] it was shown that the finite pieces of the parity-even part of $m_6^{(2)}(\epsilon)$ are incorrectly determined by

¹The one-loop five-point amplitude to $\mathcal{O}(\epsilon^2)$ has been computed analytically in the multi-Regge kinematics [5, 6]. The great computational complexity introduced in the amplitude by the higher orders in ϵ makes it desirable to devise regularisation schemes for which such higher-order terms are not present. A step in this direction has been made in refs. [7, 8] where a mass regulator has been introduced, which does not require, in the square of the amplitude, higher-order terms in the regulator.

eq. (1.1),² although the parity-odd part of $m_6^{(2)}(\epsilon)$ does fulfill eq. (1.1) [16]. In particular, it was shown numerically that the two-loop remainder function, defined as the difference between the two-loop amplitude and the ansatz for it,

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2} \left[m_n^{(1)}(\epsilon) \right]^2 - f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon), \quad (1.2)$$

is different from zero for $n = 6$, where $R_n^{(2)}$ is a function of the kinematical parameters of the n -point amplitude, but a constant with respect to ϵ . The analytic computation of $R_6^{(2)}$ has been performed recently in ref. [17].

In the strong-coupling limit, Alday and Maldacena [18] showed that planar scattering amplitudes exponentiate like in the ansatz, and suggested that the vacuum expectation value of the light-like n -edged Wilson loop could be related to the n -point amplitude in $\mathcal{N} = 4$ SYM. At weak coupling and at one-loop accuracy, a similar duality was found between the four-edged Wilson loop and the four-point MHV amplitude [19]. At one loop, the duality has been extended to Wilson loops and MHV amplitudes with an arbitrary number of points [20]. At two loops, the duality has been verified for the four-edged Wilson loop and the planar four-point MHV amplitude [21], and for the five-edged [22] and six-edged [12, 23] Wilson loops and the parity-even part of the corresponding planar MHV amplitudes.

Furthermore, it was shown that the L -loop light-like Wilson loop exhibits a conformal symmetry, and that the solution of the Ward identity for a special conformal boost is given by the ansatz, augmented, for $n \geq 6$, by a function $R_{n,WL}^{(L)}$ of conformally invariant cross ratios [22]. Because of the duality between Wilson loops and amplitudes at one and two loops, $R_{n,WL}^{(2)}$ can be identified as the remainder function of eq. (1.2).

In refs. [23, 24], the two-loop n -edged Wilson loop has been given in terms of Feynman-parameter-like integrals. Furthermore, in ref. [24] a numerical algorithm has been set up, which is valid for the two-loop n -edged Wilson loop and by which the two-loop seven-edged and eight-edged Wilson loops have been computed,³ although the corresponding planar two-loop MHV amplitudes are not known.⁴ Thus, also the remainder functions $R_{7,WL}^{(2)}$ and $R_{8,WL}^{(2)}$ of the Wilson loops are known numerically, and the numerical evidence [24] confirms that they are functions of conformally invariant cross ratios only. However, their analytic form is in general unknown.

In ref. [17], we gave a brief account of our analytic computation at weak coupling of the two-loop six-edged Wilson loop in general kinematics. The computation was performed in the Euclidean region in $D = 4 - 2\epsilon$ dimensions, where the result is real, and it was found

²There were hints of a failure from the strong-coupling limit of an amplitude with a large number of legs [11], from the two-loop six-edged Wilson loop [12], from the six-point amplitude analysed in the multi-Regge kinematics in a Minkowski region [13–15].

³In fact, in a particular kinematic setup for which only $2n$ -edged regular polygons are allowed [25], and for which the simplest non-trivial remainder function is the one of the two-loop eight-edged Wilson loop, the remainder function has been computed numerically through the algorithm of ref. [24] for Wilson loops with up to 30 edges [26].

⁴The parity-even part of the planar two-loop n -point MHV amplitudes has been given in terms of scalar Feynman integrals, yet to be evaluated [27].

in agreement with the numerical evaluation of ref. [24]. For $n = 6$, $R_{6,WL}^{(2)}$ is a function of the three conformally invariant cross ratios, u_1, u_2, u_3 . However, it is sufficient to compute the two-loop six-edged Wilson loop in any kinematical limit which does not modify the analytic dependence of $R_{6,WL}^{(2)}$ on u_1, u_2, u_3 [28]. Among such limits are the ones which feature an exact Regge factorisation of the Wilson loop [19]. In ref. [17], we showed that the exact Regge factorisation is exhibited by the L -loop Wilson loops $w_n^{(L)}$ in the quasi-multi-Regge kinematics (QMRK) of a cluster of $(n-4)$ particles along a Regge-limit ladder,⁵ thus in particular by $w_6^{(L)}$ in the QMRK of a pair along the ladder [29, 30], by $w_7^{(L)}$ in the QMRK of three-of-a-kind along the ladder [31], by $w_8^{(L)}$ in the QMRK of four-of-a-kind along the ladder [32]. Then, we illustrated briefly how the Feynman-parameter-like integrals of the two-loop six-edged Wilson loop have been computed in the QMRK of a pair along the ladder, and commented on the type of functions which appear in the final result. Because of the exact Regge factorisation, the ensuing remainder function is valid in general kinematics. It can be expressed as a linear combination of Goncharov polylogarithms of uniform transcendental weight four. In ref. [17], the remainder function was presented in an electronic form at www.arxiv.org where a text file containing the *Mathematica* expression was provided. Furthermore, the remainder function for $u_1 = u_2 = u_3 = u$ was computed, and compared to the numerical values quoted in ref. [24] and to the analytic expression in the strong coupling proposed in ref. [33].

In this paper, we provide a detailed account of the most difficult integral we had to evaluate in the analytic computation of the two-loop six-edged Wilson loop, as well as the explicit expression of the remainder function $R_{6,WL}^{(2)}$ as a function of the cross ratios u_1, u_2, u_3 , and for $u_1 = u_2 = u_3 = u$. Furthermore, we compute the asymptotic values of $R_{6,WL}^{(2)}$ for large or small values of the cross ratios. Finally, we briefly comment on the exact Regge factorisation of the Wilson loop in the QMRK of a pair along the ladder in backward scattering, and on the possibility of computing the remainder function in that kinematic setup.

In section 2, we write the two-loop Wilson loop in terms of the one-loop Wilson loop plus a remainder function $R_{n,WL}^{(2)}$. Then we write the six-edged Wilson loop in terms of Feynman-parameter-like integrals [24], and derive Mellin-Barnes representations for all of them. Finally, we exploit the Regge exactness of the Wilson loop, and extract the leading behaviour of the integrals in the QMRK of a pair along the ladder. In that fashion, the Mellin-Barnes integrals are reduced to one threefold integral plus several twofold and onefold integrals. In section 3 and appendix C, we describe the evaluation of the diagram which generates the threefold integral. The full expression of the remainder function $R_{6,WL}^{(2)}$ is rather lengthy and is given in appendix H as a function of uniform transcendental weight four in terms of Goncharov polylogarithms. In section 4, we consider some asymptotic values of the remainder function when the conformal cross ratios are either large or small and in section 5, we evaluate the remainder function when all the cross ratios are equal, $u_1 = u_2 = u_3 = u$ and consider some special values of it. Our conclusions are given in

⁵Note that the corresponding L -loop amplitudes do not exhibit exact Regge factorisation because the one-loop amplitude to higher orders in ϵ is not Regge exact [5, 6].

section 6. Definitions of harmonic sums and Goncharov polylogarithms are recalled in appendix A and B. The multi-Regge and collinear limits of the remainder function are discussed in appendix D. Appendices E, F and G collect relations between Goncharov multiple polylogarithms and (harmonic) polylogarithms for several special values of the arguments.

2 The two-loop Wilson loop

2.1 Definitions

The Wilson loop is defined through the path-ordered exponential,

$$W[\mathcal{C}_n] = \text{Tr } \mathcal{P} \exp \left[ig \oint d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right], \tag{2.1}$$

computed on a closed contour \mathcal{C}_n . In what follows, the closed contour is a light-like n -edged polygonal contour [18]. The contour is such that labelling the n vertices of the polygon as x_1, \dots, x_n , the distance between any two contiguous vertices, i.e., the length of the edge in between, is given by the momentum of a particle in the corresponding colour-ordered scattering amplitude,

$$p_i = x_i - x_{i+1}, \tag{2.2}$$

with $i = 1, \dots, n$. Because the n momenta add up to zero, $\sum_{i=1}^n p_i = 0$, the n -edged contour closes, provided we make the identification $x_1 = x_{n+1}$.

In the weak-coupling limit, the Wilson loop can be computed as an expansion in the coupling. The expansion of eq. (2.1) is done through the non-abelian exponentiation theorem [34, 35], which gives the vacuum expectation value of the Wilson loop as an exponential,

$$\langle W[\mathcal{C}_n] \rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)}, \tag{2.3}$$

where the coupling is defined as

$$a = \frac{g^2 N}{8\pi^2}. \tag{2.4}$$

For the first two loop orders, one obtains

$$w_n^{(1)} = W_n^{(1)}, \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left(W_n^{(1)} \right)^2. \tag{2.5}$$

The one-loop coefficient $w_n^{(1)}$ was evaluated in refs. [19, 20], where it was given in terms of the one-loop n -point MHV amplitude,

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n \frac{\zeta_2}{2} + \mathcal{O}(\epsilon), \tag{2.6}$$

where the amplitude is a sum of one-loop *two-mass-easy* box functions [36],

$$m_n^{(1)} = \sum_{p,q} F^{2\text{me}}(p, q, P, Q), \tag{2.7}$$

where p and q are two external momenta corresponding to two opposite massless legs, while the two remaining legs P and Q are massive. The two-loop coefficient $w_n^{(2)}$ has been computed analytically for $n = 4$ [21], $n = 5$ [22] and $n = 6$ [17], and numerically for $n = 6$ [23] and $n = 7, 8$ [24].

In ref. [22] it was established that the Wilson loop fulfils a special conformal Ward identity, whose solution is the BDS ansatz plus, for $n \geq 6$, an arbitrary function of the conformally invariant cross ratios, defined in eq. (2.11). Thus, the two-loop coefficient $w_n^{(2)}$ can be written as

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon) w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon), \quad (2.8)$$

where the constant is the same as in eq. (1.1), $C_{WL}^{(2)} = C^{(2)} = -\zeta_2^2/2$, and the function $f_{WL}^{(2)}(\epsilon)$ is [21, 24, 37],

$$f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2. \quad (2.9)$$

With the two-loop coefficient $w_n^{(2)}$ given by eqs. (2.8) and (2.9) and the planar two-loop MHV amplitude given by eqs. (1.1) and (1.2), the duality between Wilson loops and amplitudes is expressed by the equality of their remainder functions [24],

$$R_{n,WL}^{(2)} = R_n^{(2)}. \quad (2.10)$$

Defining the conformally invariant cross ratios as,

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}, \quad (2.11)$$

for $n = 6$, they are [22],

$$u_{36} = u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2}, \quad u_{14} = u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}, \quad u_{25} = u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}, \quad (2.12)$$

where $x_{ij}^2 = (x_i - x_j)^2$, and using eq. (2.2) one sees that $x_{i,i+2}^2 = s_{i,i+1}$ and $x_{i,i+3}^2 = s_{i,i+1,i+2}$, where the labels are understood to be modulo 6.

2.2 The two-loop six-edged Wilson loop

The diagrams that enter the computation of the two-loop six-edged Wilson loop have been spelled out in ref. [24].⁶ In terms of those diagrams, we write the two-loop six-edged Wilson

⁶Note that there is a factor 1/8 missing in eqs. (4.4) and (B.1) of ref. [24].

loop as,

$$\begin{aligned}
w_6^{(2)} = & \mathcal{C} [f_H(p_1, p_2, p_3; 0, p_4 + p_5 + p_6, 0) + f_H(p_1, p_2, p_4; p_3, p_5 + p_6, 0) \\
& + f_H(p_1, p_2, p_5; p_3 + p_4, p_6, 0) + (1/3)f_H(p_1, p_3, p_5; p_4, p_6, p_2) \\
& + f_C(p_1, p_2, p_3; 0, p_4 + p_5 + p_6, 0) + f_C(p_1, p_2, p_4; p_3, p_5 + p_6, 0) \\
& + f_C(p_1, p_2, p_5; p_3 + p_4, p_6, 0) + f_C(p_1, p_2, p_6; p_3 + p_4 + p_5, 0, 0) \\
& + f_C(p_1, p_3, p_4; 0, p_5 + p_6, p_2) + f_C(p_1, p_3, p_5; p_4, p_6, p_2) \\
& + f_C(p_1, p_3, p_6; p_4 + p_5, 0, p_2) + f_C(p_1, p_4, p_5; 0, p_6, p_2 + p_3) \\
& + f_C(p_1, p_4, p_6; p_5, 0, p_2 + p_3) + f_C(p_1, p_5, p_6; 0, 0, p_2 + p_3 + p_4) \\
& + f_X(p_1, p_2; p_3 + p_4 + p_5 + p_6, 0) + f_Y(p_1, p_2; p_3 + p_4 + p_5 + p_6, 0) \\
& + f_Y(p_2, p_1; 0, p_3 + p_4 + p_5 + p_6) + f_X(p_1, p_3; p_4 + p_5 + p_6, p_2) \\
& + f_Y(p_1, p_3; p_4 + p_5 + p_6, p_2) + f_Y(p_3, p_1; p_2, p_4 + p_5 + p_6) \\
& + (1/2)f_X(p_1, p_4; p_5 + p_6, p_2 + p_3) + f_Y(p_1, p_4; p_5 + p_6, p_2 + p_3) \\
& + (-1/2)f_P(p_1, p_3; p_4 + p_5 + p_6, p_2) f_P(p_2, p_4; p_1 + p_5 + p_6, p_3) \\
& + (-1/2)f_P(p_1, p_3; p_4 + p_5 + p_6, p_2) f_P(p_2, p_5; p_1 + p_6, p_3 + p_4) \\
& + (-1/4)f_P(p_1, p_4; p_5 + p_6, p_2 + p_3) f_P(p_2, p_5; p_1 + p_6, p_3 + p_4) \\
& + \text{cyclic permutations of } (p_1, p_2, p_3, p_4, p_5, p_6)] , \tag{2.13}
\end{aligned}$$

where, in the terminology of ref. [24], f_H stands for a hard diagram, f_C for a curtain diagram, f_X for a cross diagram, f_Y for a Y diagram plus half a self-energy diagram, f_P for a factorised cross diagram. Furthermore,

$$\mathcal{C} = 2a^2 \mu^{4\epsilon} [\Gamma(1 + \epsilon) e^{\gamma\epsilon}]^2 , \tag{2.14}$$

and the scale μ^2 is given in terms of the Wilson loop scale, $\mu_{WL}^2 = \pi e^\gamma \mu^2$.

The six-edged Wilson loop is a function of the six external momenta p_i , $1 \leq i \leq 6$. Imposing momentum conservation and on-shellness reduces the number of independent multi-particle invariants to nine.⁷ As the basic kinematic invariants, we choose $s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{342}, s_{345}$, all the other kinematic invariants being related to them by the following relations,

$$\begin{aligned}
s_{13} &= -s_{12} + s_{123} - s_{23}, & s_{14} &= -s_{123} + s_{23} - s_{234} + s_{56}, \\
s_{15} &= -s_{16} + s_{234} - s_{56}, & s_{24} &= -s_{23} + s_{234} - s_{34}, \\
s_{25} &= s_{16} - s_{234} + s_{34} - s_{345}, & s_{26} &= -s_{12} - s_{16} + s_{345}, \\
s_{35} &= -s_{34} + s_{345} - s_{45}, & s_{36} &= s_{12} - s_{123} - s_{345} + s_{45}, \\
s_{46} &= s_{123} - s_{45} - s_{56} . & & \tag{2.15}
\end{aligned}$$

We use the parametric representations of the Wilson loop diagrams given in ref. [24] and we derive appropriate Mellin-Barnes (MB) representations for all of them. In multi-loop

⁷Note that in four dimensions we could also impose the Gram determinant constraint. However, we do not impose this constraint and treat the nine invariants as independent variables.

calculations it is sometimes difficult to find an optimal choice for the MB representation. However, in our case the MB representations are introduced in a straightforward way using the basic formula

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z) \Gamma(\lambda+z) \frac{A^z}{B^{\lambda+z}}. \quad (2.16)$$

where the contour is chosen such as to separate the poles in $\Gamma(\dots - z)$ from the poles in $\Gamma(\dots + z)$. Note that in our case λ is in general an integer plus an off-set corresponding to the dimensional regulator ϵ . In order to resolve the singularity structures in ϵ , we apply the strategy based on the MB representation and given in refs. [38–41]. To this effect, we apply the codes `MB` [42] and `MBresolve` [43] and obtain a set of MB integrals which can be safely expanded in ϵ under the integration sign. After applying these codes, all the integration contours are straight vertical lines. Then we proceed and simplify the computation by exploiting the Regge exactness of the Wilson loop [17] and extract the leading quasi-multi-Regge behaviour by applying `MBasymptotics` [44]. Finally, we apply `barnesroutines` [45] to perform integrations that can be done by corollaries of Barnes lemmas.

To illustrate this procedure, let us consider the hard diagram $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ of (2.13). A parametric representation of this diagram was given in eqs. (B.1)–(B.5) of ref. [24]. We consider separately the nine terms originating from the decomposition of the numerator (B.1). In particular, for the first part of the first line of (B.1), $1/2s_{13}s_{15}\alpha_1\alpha_2(1-\tau_1)$, we have the following parametric integral:

$$\begin{aligned} F = & \frac{\Gamma(2-2\epsilon)(s_{12}-s_{123}+s_{23})(s_{16}-s_{234}+s_{56})}{2\Gamma(1-\epsilon)^2} \\ & \times \int_0^1 \dots \int_0^1 \left(\prod_{i=1}^3 d\alpha_i \right) \left(\prod_{i=1}^3 d\tau_i \right) \delta \left(\sum_{i=1}^3 \alpha_i - 1 \right) (1-\tau_1)\alpha_1^{1-\epsilon}\alpha_2^{1-\epsilon}\alpha_3^{-\epsilon} \\ & \times [-s_{12}\alpha_1\alpha_2(1-\tau_1)(1-\tau_2) - s_{123}\alpha_1\alpha_2(1-\tau_1)\tau_2 - s_{23}\alpha_1\alpha_2\tau_1\tau_2 - s_{16}\alpha_1\alpha_3\tau_1\tau_3 \\ & - s_{234}\alpha_1\alpha_3\tau_1(1-\tau_3) - s_{56}\alpha_1\alpha_3(1-\tau_1)(1-\tau_3) - s_{34}\alpha_2\alpha_3(1-\tau_2)(1-\tau_3) \\ & - s_{345}\alpha_2\alpha_3(1-\tau_2)\tau_3 - s_{45}\alpha_2\alpha_3\tau_2\tau_3]^{2\epsilon-2}. \end{aligned} \quad (2.17)$$

Separating different terms of the function in the square brackets using eq. (2.16), we obtain the following eightfold MB representation:

$$\begin{aligned} F = & \frac{(s_{12}-s_{123}+s_{23})(s_{16}-s_{234}+s_{56})}{2\Gamma(1-\epsilon)^2\Gamma(\epsilon+1)} \int_{-i\infty}^{+i\infty} \dots \int_{-i\infty}^{+i\infty} \left(\prod_{i=1}^8 \frac{dz_i}{2\pi i} \Gamma(-z_i) \right) \\ & \times (-s_{12})^{z_1} (-s_{123})^{z_7} (-s_{16})^{z_6} (-s_{23})^{z_2} (-s_{234})^{z_8} (-s_{34})^{z_3} (-s_{45})^{z_4} \\ & \times (-s_{56})^{z_5} (-s_{345})^{2\epsilon-z_1-z_2-z_3-z_4-z_5-z_6-z_7-z_8-2} \\ & \times \Gamma(\epsilon-z_1-z_2-z_7-1)\Gamma(2\epsilon-z_2-z_4-z_5-z_6-z_7-z_8-1) \\ & \times \Gamma(2\epsilon-z_1-z_2-z_3-z_5-z_7-z_8-1)\Gamma(-\epsilon+z_1+z_2+z_5+z_6+z_7+z_8+2) \\ & \times \Gamma(\epsilon-z_5-z_6-z_8)\Gamma(-2\epsilon+z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8+2) \\ & \times \frac{\Gamma(z_1+z_5+z_7+2)\Gamma(z_2+z_4+z_7+1)\Gamma(z_2+z_6+z_8+1)\Gamma(z_3+z_5+z_8+1)}{\Gamma(2\epsilon-z_1-z_2-z_7)\Gamma(2\epsilon-z_5-z_6-z_8)\Gamma(z_1+z_2+z_5+z_6+z_7+z_8+3)}. \end{aligned} \quad (2.18)$$

Then we apply the codes `MB` [42] and `MBresolve` [43] to resolve the singularity structure in ϵ .

It might seem that we have made the situation more complicated because, instead of the fivefold integral in eq. (2.17) (one of the six integrations is performed using the δ function), we have now the eightfold integral (2.18). However, eq. (2.18) as well as the MB representations of the other contributions to eq. (2.13) is much more convenient for taking various limits. In fact, the cornerstone of our approach is to expand eq. (2.8) in some limit such that for $n = 6$ the computation of the remainder function is considerably simplified. Explicitly, we rewrite eq. (2.8) as

$$R_{6,WL}^{(2)} = \left[w_6^{(2)}(\epsilon) - f_{WL}^{(2)}(\epsilon) w_6^{(1)}(2\epsilon) \right] \Big|_{\epsilon=0} - C_{WL}^{(2)}, \quad (2.19)$$

and look for a limit in which (i) the cross ratios the remainder function depends upon take non-trivial values and (ii) the two-loop hexagon Wilson loop $w_6^{(2)}(\epsilon)$ is as simple as possible. The simplest variant of such a limit is the quasi-multi-Regge limit (QMRK) of a pair along the ladder [29, 30]. In those kinematics, the outgoing gluons are strongly ordered in rapidity, except for a central pair of gluons along the ladder, while their transverse momenta are all of the same size. In the physical region, defining 1 and 2 as the incoming gluons, with momenta $p_2 = (p_2^+/2, 0, 0, p_2^+/2)$ and $p_1 = (p_1^-/2, 0, 0, -p_1^-/2)$, and 3, 4, 5, 6 as the outgoing gluons, the ordering can be chosen as

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|, \quad (2.20)$$

where the particle momentum p is parametrised in terms of the rapidity y and the azimuthal angle ϕ , $p = (|p_\perp| \cosh y, |p_\perp| \cos \phi, |p_\perp| \sin \phi, |p_\perp| \sinh y)$. We shall work in the Euclidean region, where the Wilson loop is real. There the Mandelstam invariants are taken as all negative, and in the QMRK of a pair along the ladder they are ordered as follows,

$$-s_{12} \gg -s_{34}, -s_{56}, -s_{345}, -s_{123} \gg -s_{23}, -s_{45}, -s_{61}, -s_{234}. \quad (2.21)$$

Introducing a parameter $\lambda \ll 1$, the hierarchy above is equivalent to the rescaling

$$\{s_{34}, s_{56}, s_{123}, s_{345}\} = \mathcal{O}(\lambda), \quad \{s_{23}, s_{45}, s_{61}, s_{234}\} = \mathcal{O}(\lambda^2). \quad (2.22)$$

It is easy to see that in this limit the three cross ratios (2.12) do not take trivial limiting values [28],

$$\begin{aligned} u_1 &\rightarrow u_1^{QMRK} = \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)} = \mathcal{O}(1), \\ u_2 &\rightarrow u_2^{QMRK} = \frac{|p_{3\perp}|^2 p_5^+ p_6^-}{(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)(p_4^+ + p_5^+) p_6^-} = \mathcal{O}(1), \\ u_3 &\rightarrow u_3^{QMRK} = \frac{|p_{6\perp}|^2 p_3^+ p_4^-}{p_3^+ (p_4^- + p_5^-)(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-)} = \mathcal{O}(1). \end{aligned} \quad (2.23)$$

Taking this limit on eq. (2.19), its right-hand side simplifies. However, the Regge exactness of the Wilson loop allows us to take, one after the other, not only this limit

but also the five limits obtained from the first one by cyclic permutations of the external momenta p_1, \dots, p_6 [17]. For example, the second limit in this series is,

$$\{s_{45}, s_{61}, s_{234}, s_{123}\} = \mathcal{O}(\lambda), \quad \{s_{34}, s_{56}, s_{12}, s_{345}\} = \mathcal{O}(\lambda^2). \quad (2.24)$$

While taking these consecutive limits, we keep in each case the leading power asymptotics (including all the logarithms), a step which is fully automatized by the code `MBasymptotics` [44]. We also apply the code `barnesroutines` [45] whenever possible to perform integrations that can be done by corollaries of Barnes lemmas.

Finally, we arrive at a set of multiple MB integrals of a much simpler type than the original ones. After applying our procedure, all integrals are at most threefold and all of them are explicitly dependent on the cross ratios only.⁸ We checked numerically that the sum of the MB integrals in the QMRK equals the sum of all the original parametric integrals, the latter being evaluated numerically using `FIESTA` [46, 47]. In particular, for the diagram $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$, the eightfold integral of eq. (2.18) reduces to a combination of one threefold integral, 51 twofold integrals and 22 onefold integrals and a term without any integration left. Note that, after taking the six consecutive limits described above, this diagram is the only one that involves a threefold integral, all other contributions to eq. (2.13) involving at most twofold integrals. The threefold contribution to $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ reads,

$$-\frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{z_1} u_2^{z_2} u_3^{z_3} \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1), \quad (2.25)$$

where the contours are straight vertical lines such that,

$$\text{Re}(z_1) = -\frac{1}{3}, \quad \text{Re}(z_2) = -\frac{1}{4}, \quad \text{Re}(z_3) = -\frac{1}{5}. \quad (2.26)$$

The explicit evaluation of this integral is reviewed in the next section, whereas the full analytic expression for the remainder function is given in appendix H and is also available in electronic form at www.arXiv.org.

3 Evaluation of the hard diagram

In this section we review the computation of the MB integrals we derived in the previous section. Apart from the threefold integral contributing to $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$, eq. (2.25), all the integrals are at most twofold and can be computed by closing the integration contours at infinity and summing up residues using the standard techniques. Therefore, in this paper we only concentrate on the case of the hard diagram and present in detail the analytic computation of the integral in eq. (2.25).

⁸Note that the coefficients of the integrals do not only depend on the cross ratios, but on logarithms of Mandelstam invariants as well. This is to be expected since the contribution to $w_6^{(2)}$ depends on such quantities.

We rewrite eq. (2.25) in the form,

$$-\frac{1}{8} (F(u_1, u_2, u_3) + F(u_2, u_3, u_1) + F(u_3, u_1, u_2)) , \quad (3.1)$$

where we define,

$$F(u_1, u_2, u_3) = \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} z_3 u_1^{z_1} u_2^{z_2} u_3^{z_3} \\ \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1+z_2+1) \Gamma(z_1+z_3) \Gamma(z_2+z_3) . \quad (3.2)$$

Note that this function is symmetric in its first two arguments,

$$F(u_1, u_2, u_3) = F(u_2, u_1, u_3) , \quad (3.3)$$

so that the expression in eq. (3.1) is totally symmetric in the three cross ratios. We start with the change of variable $z_3 = z'_3 + 1$. This also shifts the corresponding contour, $\text{Re}(z_3) = -1/5 \rightarrow \text{Re}(z'_3) = -6/5$. Shifting the contour back to $\text{Re}(z'_3) = -1/5$, we arrive at the expression,

$$F(u_1, u_2, u_3) = \tilde{F}(u_1, u_2, u_3) + R_{-1}(u_1, u_2, u_3) + R_{-1-z_1}(u_1, u_2, u_3) + R_{-1-z_2}(u_1, u_2, u_3) , \quad (3.4)$$

where the threefold integral \tilde{F} is given by

$$\tilde{F}(u_1, u_2, u_3) = \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} \frac{1}{1+z_3} u_1^{z_1} u_2^{z_2} u_3^{z_3+1} \\ \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1+z_2+1) \Gamma(z_1+z_3+1) \Gamma(z_2+z_3+1) , \quad (3.5)$$

and we made the relabelling $z'_3 \rightarrow z_3$. The integration contours are given by eq. (2.26). The functions $R_j(u_1, u_2, u_3)$ arise from taking the residues of the poles in $z'_3 = j$ that we crossed when shifting the contour from $\text{Re}(z'_3) = -6/5$ to $\text{Re}(z'_3) = -1/5$,

$$R_{-1}(u_1, u_2, u_3) = - \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} u_1^{z_1} u_2^{z_2} \Gamma(-z_1)^2 \Gamma(z_1) \Gamma(-z_2)^2 \Gamma(z_2) \\ \times \Gamma(z_1+z_2+1) , \\ R_{-1-z_1}(u_1, u_2, u_3) = \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} u_1^{z_1} u_2^{z_2} u_3^{-z_1} \frac{1}{z_1} \Gamma(-z_1)^2 \Gamma(z_1+1)^2 \\ \times \Gamma(-z_2)^2 \Gamma(z_2-z_1) \Gamma(z_1+z_2+1) , \\ R_{-1-z_2}(u_1, u_2, u_3) = \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} u_1^{z_1} u_2^{z_2} u_3^{-z_2} \frac{1}{z_2} \Gamma(-z_2)^2 \Gamma(z_2+1)^2 \\ \times \Gamma(-z_1)^2 \Gamma(z_1-z_2) \Gamma(z_1+z_2+1) . \quad (3.6)$$

Note that we have the relation,

$$R_{-1-z_2}(u_1, u_2, u_3) = R_{-1-z_1}(u_2, u_1, u_3) , \quad (3.7)$$

where we assumed that the contours on both sides are chosen according to eq. (2.26). The computation of $\tilde{F}(u_1, u_2, u_3)$ is detailed in this section, whereas the computation of the

residues $R_j(u_1, u_2, u_3)$ is discussed in appendix C. We start by writing $\tilde{F}(u_1, u_2, u_3)$ as the integral of the derivative,

$$\tilde{F}(u_1, u_2, u_3) = \tilde{F}(u_1, u_2, 0) + \int_0^{u_3} du \frac{\partial}{\partial u} \tilde{F}(u_1, u_2, u). \quad (3.8)$$

The value for $u_3 = 0$ can be easily obtained by expanding around small values of u_3 using MBasymptotics. We find,

$$\tilde{F}(u_1, u_2, 0) = 0. \quad (3.9)$$

Next, we follow the procedure used in ref. [5] and we replace the MB integrations over z_1 , z_2 and z_3 by Euler integrations using the formula (see, e.g., ref. [41]),

$$\begin{aligned} & \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z_1) \Gamma(c - z_1) \Gamma(b + z_1) \Gamma(c + z_1) X^{z_1} \\ &= \Gamma(a) \Gamma(b + c) \int_0^1 dv v^{b-1} (1 - v)^{a+c-1} (1 - (1 - X)v)^{-a}. \end{aligned} \quad (3.10)$$

This leaves us with a fourfold Euler integral,

$$\begin{aligned} \tilde{F}(u_1, u_2, u_3) &= \int_0^1 dv_1 \int_0^1 dv_2 \int_0^1 dv_3 \int_0^{u_3} du (1 - (1 - u_1)v_1)^{-1} \\ &\times \left(1 - v_2 \left(1 - \frac{u_2(1 - v_1)}{1 - (1 - u_1)v_1} \right) \right)^{-1} (1 - v_3(1 - uv_1v_2))^{-1}. \end{aligned} \quad (3.11)$$

Some comments are in order: Firstly, eq. (3.10) is only valid if the contour separates the poles in $\Gamma(\dots - z_i)$ from the poles in $\Gamma(\dots + z_i)$. It is easy to observe that our contours, eq. (2.26), fulfill this requirement. Secondly, we tacitly exchanged the order of the integrations in deriving eq. (3.11). We checked numerically that this operation is allowed in the present case.

The integrals over u and v_3 in eq. (3.11) can be done very easily, resulting in the following twofold integral,

$$\begin{aligned} \tilde{F}(u_1, u_2, u_3) &= - \int_0^1 dv_1 \int_0^1 dv_2 v_1^{-1} v_2^{-1} \left(\frac{\pi^2}{6} - \text{Li}_2(1 - v_1 v_2 u_3) \right) \\ &\times (-u_2 v_2 + v_1(u_1(v_2 - 1) + (u_2 - 1)v_2 + 1) + v_2 - 1)^{-1}. \end{aligned} \quad (3.12)$$

The remaining twofold integral can be computed in terms of Goncharov multiple polylogarithms [48, 49], defined recursively by,

$$G(a, \vec{w}; z) = \int_0^z \frac{dt}{t - a} G(\vec{w}; t), \quad G(a; z) = \ln \left(1 - \frac{z}{a} \right). \quad (3.13)$$

If all indices are zero we define

$$G(\vec{0}_n; z) = \int_1^z \frac{dt}{t} G(\vec{0}_{n-1}; t) = \frac{1}{n!} \ln^n z. \quad (3.14)$$

In particular cases the Goncharov polylogarithms can be expressed in terms of ordinary logarithms and polylogarithms, e.g.,

$$G(\vec{a}_n; z) = \frac{1}{n!} \ln^n \left(1 - \frac{z}{a} \right), \quad G(\vec{0}_{n-1}, a; z) = -\text{Li}_n \left(\frac{z}{a} \right). \quad (3.15)$$

We define the weight of the function $G(\vec{w}; z)$ as the number of elements in the vector \vec{w} . The Goncharov polylogarithms form a shuffle algebra, i.e., a product of two G functions of weight w_1 and w_2 can be expressed as a linear combination of functions of weight $w = w_1 + w_2$,

$$G(\vec{w}_1; z) G(\vec{w}_2; z) = \sum_{\vec{w}=\vec{w}_1 \uplus \vec{w}_2} G(\vec{w}; z), \tag{3.16}$$

where $\vec{w}_1 \uplus \vec{w}_2$ denotes all the mergings of the vectors \vec{w}_1 and \vec{w}_2 , i.e., all possible concatenations of \vec{w}_1 and \vec{w}_2 in which relative orderings of \vec{w}_1 and \vec{w}_2 are preserved. Furthermore, if the rightmost element of the weight vector \vec{w} is non zero, the polylogarithms are invariant under a rescaling of the arguments,

$$G(k \vec{w}; k z) = G(\vec{w}; z). \tag{3.17}$$

A more detailed review of Goncharov polylogarithms and of their properties and special values is presented in appendix B, E, F and G.

Using the definitions we just introduced, we can rewrite the numerator of the integrand in the form,

$$\begin{aligned} \frac{\pi^2}{6} - \text{Li}_2(1 - v_1 v_2 u_3) &= G(1, 0; v_1 v_2 u_3) \\ &= G(1; v_1 v_2 u_3) G(0; v_1 v_2 u_3) - G(0, 1; v_1 v_2 u_3) \\ &= G\left(\frac{1}{v_1 u_3}; v_2\right) \left[G(0; v_1) + G(0; v_2) + G(0; u_3)\right] - G\left(0, \frac{1}{v_1 u_3}; v_2\right) \\ &= \left[G(0; v_1) + G(0; u_3)\right] G\left(\frac{1}{v_1 u_3}; v_2\right) + G\left(\frac{1}{v_1 u_3}, 0; v_2\right). \end{aligned} \tag{3.18}$$

After partial fractioning the integrand in v_2 , we arrive at,

$$\begin{aligned} \tilde{F}(u_1, u_2, u_3) &= - \int_0^1 dv_1 \int_0^1 dv_2 v_1^{-1} (u_1 v_1 - v_1 + 1)^{-1} \\ &\times \left\{ \left[G(0; v_1) + G(0; u_3)\right] G\left(\frac{1}{v_1 u_3}; v_2\right) + G\left(\frac{1}{v_1 u_3}, 0; v_2\right) \right\} \\ &\times \left[\left(v_2 - \frac{u_1 v_1 - v_1 + 1}{u_1 v_1 + u_2 v_1 - u_2 - v_1 + 1}\right)^{-1} - v_2^{-1} \right]. \end{aligned} \tag{3.19}$$

We see that the integral over v_2 can be reduced to a sum of four terms, each of them consisting of a Goncharov polylogarithm divided by a linear function of v_2 . This form matches precisely the recursive definition (3.13) of the polylogarithms, and we can easily perform the integral in v_2 in terms of those special functions, e.g.,

$$\begin{aligned} &\int_0^1 dv_2 G\left(\frac{1}{v_1 u_3}, 0; v_2\right) \left(v_2 - \frac{u_1 v_1 - v_1 + 1}{u_1 v_1 + u_2 v_1 - u_2 - v_1 + 1}\right)^{-1} \\ &= G\left(\frac{u_1 v_1 - v_1 + 1}{u_1 v_1 + u_2 v_1 - u_2 - v_1 + 1}, \frac{1}{v_1 u_3}, 0; 1\right). \end{aligned} \tag{3.20}$$

All other terms in eq. (3.19) can be integrated in the same way.

We now turn to the remaining integral in v_1 . From eq. (3.20) it is clear that the integration in v_2 has produced an integrand which depends on Goncharov polylogarithms whose weight vectors are rather complicated functions of v_1 . In order to perform the integration over v_1 in the same way as we did for v_2 , we need all polylogarithms to be of the form $G(\vec{w}; v_1)$, where \vec{w} is independent of v_1 . In appendix B we describe an algorithm that allows us to rewrite all the terms in the required form, e.g.,

$$\begin{aligned}
 & G\left(\frac{u_1 v_1 - v_1 + 1}{u_1 v_1 + u_2 v_1 - u_2 - v_1 + 1}, \frac{1}{v_1 u_3}, 0; 1\right) = \\
 & -G(1, 0; u_2) G\left(u_{123}^{(+)}; v_1\right) + \frac{1}{6} \pi^2 G\left(u_{123}^{(+)}; v_1\right) + G(0; u_2) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}; v_1\right) - \\
 & G(0; u_2) G\left(0, u_{123}^{(-)}; v_1\right) - G(0; u_2) G\left(0, u_{123}^{(+)}; v_1\right) + \frac{1}{6} G\left(u_{123}^{(-)}; v_1\right) (\pi^2 - 6G(1, 0; u_2)) + \\
 & G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}; v_1\right) \left(G(1, 0; u_2) - \frac{\pi^2}{6}\right) - G(0; u_2) G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; v_1\right) + \\
 & G(0; u_2) G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; v_1\right) + G(0; u_2) G\left(u_{123}^{(-)}, \frac{1}{1 - u_1}; v_1\right) - \\
 & G(0; u_2) G\left(u_{123}^{(-)}, \frac{u_2 - 1}{u_1 + u_2 - 1}; v_1\right) + G(0; u_2) G\left(u_{123}^{(+)}, \frac{1}{1 - u_1}; v_1\right) - \\
 & G(0; u_2) G\left(u_{123}^{(+)}, \frac{u_2 - 1}{u_1 + u_2 - 1}; v_1\right) - 2G\left(0, 0, \frac{1}{u_3}; v_1\right) - G\left(0, 1, \frac{1}{u_3}; v_1\right) + \\
 & G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; v_1\right) - G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; v_1\right) - G\left(0, u_{123}^{(-)}, 1; v_1\right) + \\
 & G\left(0, u_{123}^{(-)}, \frac{1}{1 - u_1}; v_1\right) + G\left(0, u_{123}^{(-)}, \frac{1}{u_3}; v_1\right) - G\left(0, u_{123}^{(+)}, 1; v_1\right) + \\
 & G\left(0, u_{123}^{(+)}, \frac{1}{1 - u_1}; v_1\right) + G\left(0, u_{123}^{(+)}, \frac{1}{u_3}; v_1\right) - G\left(1, 0, \frac{1}{u_3}; v_1\right) - \\
 & G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, 1; v_1\right) + G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, \frac{1}{1 - u_1}; v_1\right) + \\
 & G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; v_1\right) - G\left(\frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; v_1\right) + \\
 & G\left(u_{123}^{(-)}, 0, \frac{1}{u_3}; v_1\right) + G\left(u_{123}^{(-)}, \frac{1}{1 - u_1}, 1; v_1\right) - G\left(u_{123}^{(-)}, \frac{1}{1 - u_1}, \frac{1}{1 - u_1}; v_1\right) - \\
 & G\left(u_{123}^{(-)}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; v_1\right) + G\left(u_{123}^{(-)}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; v_1\right) + G\left(u_{123}^{(+)}, 0, \frac{1}{u_3}; v_1\right) + \\
 & G\left(u_{123}^{(+)}, \frac{1}{1 - u_1}, 1; v_1\right) - G\left(u_{123}^{(+)}, \frac{1}{1 - u_1}, \frac{1}{1 - u_1}; v_1\right) - G\left(u_{123}^{(+)}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; v_1\right) + \\
 & G\left(u_{123}^{(+)}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; v_1\right),
 \end{aligned} \tag{3.21}$$

where we defined,

$$u_{jkl}^{(\pm)} = \frac{1 - u_j - u_k + u_l \pm \sqrt{(u_j + u_k - u_l - 1)^2 - 4(1 - u_j)(1 - u_k)u_l}}{2(1 - u_j)u_l}. \tag{3.22}$$

A comment is in order about the square roots in eq. (3.22): It turns out that the square roots become complex for certain values of the cross ratios inside the unit cube, but they

always come in pairs such that the sum of the two contributions is real. To emphasize this property, we introduce the following notation,

$$\mathcal{G}(\dots, u_{ijk}, \dots; z) = G(\dots, u_{ijk}^{(+)}, \dots; z) + G(\dots, u_{ijk}^{(-)}, \dots; z). \quad (3.23)$$

Note that this definition follows the same spirit as the definition of Clausen's function, defined as the real or imaginary parts of the ordinary polylogarithms,

$$\text{Cl}_n(\theta) = \begin{cases} \frac{1}{2} [\text{Li}_n(e^{i\theta}) + \text{Li}_n(e^{-i\theta})], & n \text{ odd,} \\ \frac{1}{2i} [\text{Li}_n(e^{i\theta}) - \text{Li}_n(e^{-i\theta})], & n \text{ even.} \end{cases} \quad (3.24)$$

All the integrations can now be done very easily using eq. (3.13), and we find,

$$\begin{aligned} \tilde{F}(u_1, u_2, u_3) = & \\ & G(0; u_2) G(0; u_3) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) - G(1, 0; u_2) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) + \\ & \frac{1}{6} \pi^2 G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) - G(0; u_2) G(0; u_3) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) + \\ & G(1, 0; u_2) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) - \frac{1}{6} \pi^2 G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) - \\ & G(0; u_3) G\left(0, 1, \frac{1}{u_3}; 1\right) + G(0; u_2) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0; 1\right) + \\ & G(0; u_3) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; 1\right) + G(0; u_2) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; 1\right) - \\ & G(0; u_3) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; 1\right) - G(0; u_2) G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) + \\ & G(0; u_3) G\left(\frac{1}{1 - u_1}, 1, \frac{1}{u_3}; 1\right) - G(0; u_2) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0; 1\right) - \\ & G(0; u_3) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; 1\right) - G(0; u_2) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; 1\right) + \\ & G(0; u_3) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; 1\right) + \\ & G(0; u_2) G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) - G\left(0, 1, \frac{1}{u_3}, 0; 1\right) + \\ & G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0, 1; 1\right) - G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0, \frac{1}{1 - u_1}; 1\right) + \\ & G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1, 0; 1\right) - G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, 0; 1\right) + \\ & G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, 1; 1\right) - G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}, \frac{1}{1 - u_1}; 1\right) - \\ & G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1; 1\right) + G\left(0, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, \frac{1}{1 - u_1}; 1\right) + \\ & G\left(\frac{1}{1 - u_1}, 1, \frac{1}{u_3}, 0; 1\right) - G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0, 1; 1\right) + \\ & G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 0, \frac{1}{1 - u_1}; 1\right) - G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}, 1, 0; 1\right) + \end{aligned} \quad (3.25)$$

$$\begin{aligned}
 & G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 0; 1\right) - G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 1; 1\right) + \\
 & G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) - \\
 & G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) - G(0; u_2) G(0; u_3) \mathcal{G}(0, u_{123}; 1) + \\
 & G(1, 0; u_2) \mathcal{G}(0, u_{123}; 1) - \frac{1}{6}\pi^2 \mathcal{G}(0, u_{123}; 1) + G(0; u_2) G(0; u_3) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}; 1\right) - \\
 & G(1, 0; u_2) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}; 1\right) + \frac{1}{6}\pi^2 \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}; 1\right) - G(0; u_2) \mathcal{G}(0, u_{123}, 0; 1) - \\
 & G(0; u_3) \mathcal{G}(0, u_{123}, 1; 1) - G(0; u_2) \mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}; 1\right) + G(0; u_3) \mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}; 1\right) + \\
 & G(0; u_2) \mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}; 1\right) + G(0; u_3) \mathcal{G}\left(0, u_{123}, \frac{1}{u_3}; 1\right) + \\
 & G(0; u_2) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0; 1\right) + G(0; u_3) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 1; 1\right) + \\
 & G(0; u_2) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}; 1\right) - G(0; u_3) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}; 1\right) - \\
 & G(0; u_2) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}; 1\right) - G(0; u_3) \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{u_3}; 1\right) - \\
 & \mathcal{G}(0, u_{123}, 0, 1; 1) + \mathcal{G}\left(0, u_{123}, 0, \frac{1}{1-u_1}; 1\right) - \mathcal{G}(0, u_{123}, 1, 0; 1) + \\
 & \mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, 0; 1\right) - \mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, 1; 1\right) + \mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + \\
 & \mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) - \mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) + \\
 & \mathcal{G}\left(0, u_{123}, \frac{1}{u_3}, 0; 1\right) + \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0, 1; 1\right) - \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0, \frac{1}{1-u_1}; 1\right) + \\
 & \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 1, 0; 1\right) - \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, 0; 1\right) + \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, 1; 1\right) - \\
 & \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) + \\
 & \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) - \mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{u_3}, 0; 1\right).
 \end{aligned}$$

Note that all the terms in this expression are of uniform transcendental weight four, as expected. Then $F(u_1, u_2, u_3)$ in eq. (3.4) is obtained by combining eq. (3.25) with the residues $R_j(u_1, u_2, u_3)$ computed in appendix C.

4 Asymptotic values of the remainder function

In this section we study the asymptotic behaviour of the remainder function in various limits. For the sake of simplicity, we exclusively studied strongly ordered limits, i.e., limits where any ratio of conformal cross ratios is either small or large. Note that since the remainder function is completely symmetric in its arguments, it is enough to study the strongly ordered limits for a specific ordering, all other orderings being obtained by symmetry. The technique described in this section to compute the asymptotic behaviour in the

various limits can easily be extended to non-strongly ordered limits. In the next section we briefly comment on such limits when all cross ratios are equal.

We start with the limit where all cross ratios are small, $u_1 \ll u_2 \ll u_3 \ll 1$. We can easily obtain the leading contribution by using `MBasymptotics`. We find,

$$\begin{aligned} & \lim_{u_1 \ll u_2 \ll u_3 \ll 1} R_{6,WL}^{(2)}(u_1, u_2, u_3) \\ &= \frac{\pi^2}{24} (\ln u_1 \ln u_2 + \ln u_2 \ln u_3 + \ln u_3 \ln u_1) + \frac{17\pi^2}{1440} + \mathcal{O}(u_i). \end{aligned} \tag{4.1}$$

In exactly the same way, we can find the asymptotic behaviour when some of the cross ratios are equal to unity and all the others are small,

$$\begin{aligned} & \lim_{u_1 \ll u_2 \ll 1} R_{6,WL}^{(2)}(u_1, u_2, 1) = 0, \\ & \lim_{u_1 \ll 1} R_{6,WL}^{(2)}(u_1, 1, 1) = \frac{1}{2}\zeta_3 \ln u_1 - \frac{\pi^4}{96} + \mathcal{O}(u_1). \end{aligned} \tag{4.2}$$

Note that the limit $u_1 \ll u_2 \ll 1$, with $u_3 = 1$, corresponds to the multi-Regge limit (D.3).

We now repeat the previous analysis in the limit where the cross ratios are large, $u_1 \gg u_2 \gg u_3 \gg 1$. Using again `MBasymptotics` to extract the leading behaviour, we find,

$$\begin{aligned} & \lim_{u_1 \gg u_2 \gg u_3 \gg 1} R_{6,WL}^{(2)}(u_1, u_2, u_3) = \\ & -\frac{1}{96} \ln^4 \frac{u_1}{u_2 u_3} - \frac{5}{48} \pi^2 \ln^2 \frac{u_1}{u_2 u_3} - \frac{1}{2} \zeta_3 \ln \frac{u_1}{u_2 u_3} - \frac{157\pi^4}{1440} + \mathcal{O}(1/u_i). \end{aligned} \tag{4.3}$$

Similarly, for the case where some of the cross ratios are equal to unity, we find,

$$\begin{aligned} & \lim_{u_1 \gg u_2 \gg 1} R_{6,WL}^{(2)}(u_1, u_2, 1) = -\frac{1}{96} \ln^4 \frac{u_1}{u_2} - \frac{1}{12} \pi^2 \ln^2 \frac{u_1}{u_2} - \frac{3\pi^4}{40} + \mathcal{O}(1/u_i), \\ & \lim_{u_1 \gg 1} R_{6,WL}^{(2)}(u_1, 1, 1) = -\frac{1}{96} \ln^4 u_1 - \frac{1}{16} \pi^2 \ln^2 u_1 + \frac{1}{2} \zeta_3 \ln u_1 - \frac{23\pi^4}{480} + \mathcal{O}(1/u_1). \end{aligned} \tag{4.4}$$

5 The remainder function for all cross ratios equal

In this section we discuss the form of the remainder function in the special case when all the cross ratios are equal, $u_1 = u_2 = u_3 = u$. In ref. [17] several special values were presented for this case. We start by briefly reviewing how these values were obtained and present some additional special values. At the end of this section we give the analytic form of $R_{6,WL}^{(2)}(u, u, u)$ for arbitrary u .

In the special case where $u = 1$, which corresponds to a regular hexagon [26, 33], most of the integrations are easily done using Barnes lemmas and their corollaries, leaving us with at most onefold integrals. Note that some of these integrals involve Γ functions with poles in half integer values which lead to multiple binomial sums [50, 51], but all these contributions cancel out when combining all the pieces. Applying this strategy to our integrals, we immediately find the value quoted in ref. [17],

$$R_{6,WL}^{(2)}(1, 1, 1) = -\frac{\pi^4}{36} \simeq -2.70581\dots \tag{5.1}$$

Note that this value agrees with the value conjectured in ref. [24]. The asymptotic behaviour of $R_{6,WL}^{(2)}(u, u, u)$ for $u \rightarrow 0$ can be obtained in a similar way using `MBasymptotics`, which leaves us with at most trivial onefold integrals. The result is

$$R_{6,WL}^{(2)}(u, u, u) = \frac{\pi^2}{8} \ln^2 u + \frac{17\pi^4}{1440} + \mathcal{O}(u). \quad (5.2)$$

Finally, the asymptotic value for large u is obtained in exactly the same way. We can perform a rescaling $u \rightarrow \lambda^{-1} u$ and expand around small values of λ using `MBasymptotics`. We find

$$\lim_{u \rightarrow \infty} R_{6,WL}^{(2)}(u, u, u) = -\frac{\pi^4}{144} + \mathcal{O}(1/u) \simeq -0.67645\dots, \quad (5.3)$$

in very good agreement with the numerical value quoted in ref. [24].

For $u = 1/2$, the denominator in eq. (3.12) drastically simplifies. Repeating the derivation of section 3, we obtain,

$$R_{6,WL}^{(2)}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = -\frac{105}{64} \zeta_3 \ln 2 - \frac{5}{64} \ln^4 2 + \frac{5}{64} \pi^2 \ln^2 2 - \frac{15}{8} \text{Li}_4\left(\frac{1}{2}\right) + \frac{17\pi^4}{2304} \quad (5.4)$$

$$\simeq -1.26609\dots$$

Let us now turn to the generic case where all three cross ratios are equal but they still take generic values. In this limit it is easy to see that eqs. (3.22) and (C.16) reduce to

$$u_{ijk}^{(\pm)} \rightarrow \mu^{(\pm)} = \frac{1 \pm \sqrt{1-4u}}{2u}, \quad v_{ijk}^{(\pm)} \rightarrow \nu^{(\pm)} = \pm \frac{1}{\sqrt{1-u}}. \quad (5.5)$$

We can massage the resulting expression and apply the reduction algorithm of appendix B to simplify the expression as much as possible. In particular, we can remove all the dependence on $\nu^{(\pm)}$. As regards $\mu^{(\pm)}$, we observe that similar arguments have already been found in the strong coupling case [33].⁹ Note that for $u = 1/4$ the square roots in eq. (5.5) vanish. This value corresponds to a regular hexagon in a space with a (2, 2) signature [33]. Using the relations of appendix E and F, we find,

$$R_{6,WL}^{(2)}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = 3\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - \frac{9}{2} \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - \frac{567}{4} \text{Li}_3\left(\frac{1}{3}\right) \ln 2$$

$$+ \frac{543}{4} \text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + \frac{567}{8} \text{Li}_3\left(\frac{1}{3}\right) \ln 3 - \frac{567}{4} \text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \frac{1323}{16} \zeta_3 \ln 2$$

$$+ \frac{945}{32} \zeta_3 \ln 3 - \frac{39}{32} \ln^4 2 - \frac{257}{64} \ln^4 3 + \frac{173}{8} \ln 3 \ln^3 2 + \frac{189}{8} \ln^3 3 \ln 2 - \frac{543}{16} \ln^2 3 \ln^2 2$$

$$- \frac{63}{16} \pi^2 \ln^2 2 - \frac{181}{64} \pi^2 \ln^2 3 + \frac{189}{2} \text{Li}_4\left(\frac{1}{2}\right) + \frac{1701}{8} \text{Li}_4\left(\frac{1}{3}\right) - \frac{543}{16} \text{Li}_4\left(-\frac{1}{3}\right)$$

$$+ \frac{555}{2} \text{Li}_4\left(-\frac{1}{2}\right) - \frac{9}{2} \text{Li}_2\left(\frac{1}{3}\right)^2 - \frac{567}{16} S_{2,2}\left(-\frac{1}{3}\right) - \frac{567}{4} S_{2,2}\left(-\frac{1}{2}\right) - \frac{2123\pi^4}{2880}$$

$$\simeq 1.08917\dots \quad (5.6)$$

⁹We are grateful to Paul Heslop for pointing out that

$$(1-u) \mu^{(+)} = 1 + \frac{\mu}{x_\epsilon}, \quad (1-u) \mu^{(-)} = 1 + \frac{1}{\mu x_\epsilon},$$

where μ and x_ϵ are defined in [33].

Finally, let us turn to the expression for generic values of u . Using the notation introduced in eq. (3.23) as well as the corresponding one for harmonic polylogarithms,

$$\mathcal{H}(\vec{w}; 1/\mu) = H(\vec{w}; 1/\mu^{(+)}) + H(\vec{w}; 1/\mu^{(-)}), \quad (5.7)$$

the final answer for the remainder function reads, when all three cross ratios are equal,

$$\begin{aligned} R_{6,WL}^{(2)}(u, u, u) = & \quad (5.8) \\ & \frac{1}{4}\pi^2 G\left(1, \frac{1}{2}; u\right) + \frac{1}{8}\pi^2 G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}; 1\right) - 3G\left(0, 1, 0, \frac{1}{2}; u\right) - 3G\left(0, 1, \frac{1}{2}, 0; u\right) + \\ & \frac{3}{4}G\left(0, \frac{u-1}{2u-1}, 0, \frac{1}{1-u}; 1\right) + \frac{3}{4}G\left(0, \frac{u-1}{2u-1}, \frac{1}{1-u}, 0; 1\right) - \frac{3}{4}G\left(0, \frac{u-1}{2u-1}, \frac{1}{1-u}, 1; 1\right) + \\ & \frac{3}{4}G\left(0, \frac{u-1}{2u-1}, \frac{1}{1-u}, \frac{1}{1-u}; 1\right) - \frac{3}{4}G\left(0, \frac{u-1}{2u-1}, \frac{u-1}{2u-1}, \frac{1}{1-u}; 1\right) - 6G\left(1, 0, 0, \frac{1}{2}; u\right) - \\ & 3G\left(1, 0, \frac{1}{2}, 0; u\right) + 3G\left(1, \frac{1}{2}, 0, 0; u\right) - 3G\left(1, \frac{1}{2}, 1, 0; u\right) - \frac{3}{4}G\left(\frac{1}{1-u}, 1, \frac{1}{u}, 0; 1\right) + \\ & \frac{3}{2}G\left(\frac{1}{1-u}, \frac{1}{1-u}, 1, \frac{1}{1-u}; 1\right) + \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, 0, 1; 1\right) - \\ & \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, 0, \frac{1}{1-u}; 1\right) + \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, 1, 0; 1\right) - \\ & \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{1}{1-u}, 0; 1\right) + \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{1}{1-u}, 1; 1\right) - \\ & \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{1}{1-u}, \frac{1}{1-u}; 1\right) - \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{u-1}{2u-1}, 1; 1\right) + \\ & \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{u-1}{2u-1}, \frac{1}{1-u}; 1\right) - \frac{1}{8}\pi^2 \mathcal{G}\left(\frac{1}{1-u}, \mu; 1\right) - \frac{3}{4}\mathcal{G}\left(0, \mu, 0, \frac{1}{1-u}; 1\right) - \\ & \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{1}{1-u}, 0; 1\right) + \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{1}{1-u}, 1; 1\right) - \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{1}{1-u}, \frac{1}{1-u}; 1\right) - \\ & \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{1}{u}, 0; 1\right) - \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{u-1}{2u-1}, 1; 1\right) + \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{u-1}{2u-1}, \frac{1}{1-u}; 1\right) - \\ & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, 0, 1; 1\right) + \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, 0, \frac{1}{1-u}; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, 1, 0; 1\right) + \\ & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{1}{1-u}, 0; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{1}{1-u}, 1; 1\right) + \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{1}{1-u}, \frac{1}{1-u}; 1\right) + \\ & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{1}{u}, 0; 1\right) + \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{u-1}{2u-1}, 1; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{u-1}{2u-1}, \frac{1}{1-u}; 1\right) - \\ & \frac{3}{4}G\left(\frac{1}{1-u}, 1, \frac{1}{u}; 1\right) H(0; u) + \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, 0; 1\right) H(0; u) + \\ & \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, 1; 1\right) H(0; u) - \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}, \frac{u-1}{2u-1}; 1\right) H(0; u) - \\ & \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{1}{u}; 1\right) H(0; u) - \frac{3}{4}\mathcal{G}\left(0, \mu, \frac{u-1}{2u-1}; 1\right) H(0; u) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, 0; 1\right) H(0; u) - \\ & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, 1; 1\right) H(0; u) + \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{1}{u}; 1\right) H(0; u) + \\ & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu, \frac{u-1}{2u-1}; 1\right) H(0; u) + \frac{3}{2}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}; 1\right) H(0, 0; u) - \end{aligned}$$

$$\begin{aligned}
 & \frac{3}{2}\mathcal{G}\left(\frac{1}{1-u}, \mu; 1\right) H(0, 0; u) - \frac{1}{8}\pi^2 H(0, 0; u) + 3H(0, 0; u)H(0, 1; (2u)) + \frac{1}{4}\pi^2 H(0, 1; (2u)) + \\
 & \frac{3}{2}H(0, 0; u)H\left(0, 1; \frac{2u-1}{u-1}\right) - \frac{1}{8}\pi^2 H\left(0, 1; \frac{2u-1}{u-1}\right) + \frac{3}{4}G\left(\frac{1}{1-u}, \frac{u-1}{2u-1}; 1\right) H(1, 0; u) - \\
 & \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u}, \mu; 1\right) H(1, 0; u) + 3H(0, 1; (2u))H(1, 0; u) + \frac{3}{4}H\left(0, 1; \frac{2u-1}{u-1}\right) H(1, 0; u) + \\
 & \frac{1}{8}\pi^2 H(1, 0; u) + \frac{1}{8}\pi^2 H(1, 1; u) - 6H(0; u)H(0, 0, 1; (2u)) - 3H(0; u)H\left(0, 0, 1; \frac{2u-1}{u-1}\right) - \\
 & \frac{3}{4}H(0; u)H\left(1, 0, 1; \frac{2u-1}{u-1}\right) - \frac{3}{2}H(0, 0, 0, 0; u) + 9H(0, 0, 0, 1; (2u)) + \\
 & \frac{9}{2}H\left(0, 0, 0, 1; \frac{2u-1}{u-1}\right) + \frac{15}{4}H(0, 0, 1, 0; u) - \frac{9}{2}H(0, 1, 0, 0; u) + \frac{3}{2}H\left(0, 1, 0, 1; \frac{2u-1}{u-1}\right) + \\
 & \frac{3}{2}H(0, 1, 1, 0; u) - \frac{3}{4}H\left(0, 1, 1, 1; \frac{2u-1}{u-1}\right) + 3H\left(1, 0, 0, 1; \frac{2u-1}{u-1}\right) + \frac{15}{4}H(1, 0, 1, 0; u) - \\
 & 3H(1, 1, 0, 0; u) + \frac{3}{4}H\left(1, 1, 0, 1; \frac{2u-1}{u-1}\right) + \frac{3}{2}H(1, 1, 1, 0; u) - \frac{1}{8}\pi^2 H(0; u)\mathcal{H}\left(1; \frac{1}{\mu}\right) - \\
 & \frac{3}{2}H(0, 0; u)\mathcal{H}\left(0, 1; \frac{1}{\mu}\right) - \frac{3}{4}H(1, 0; u)\mathcal{H}\left(0, 1; \frac{1}{\mu}\right) + \frac{1}{8}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{\mu}\right) + \\
 & 3H(0; u)\mathcal{H}\left(0, 0, 1; \frac{1}{\mu}\right) + \frac{3}{4}H(0; u)\mathcal{H}\left(0, 1, 1; \frac{1}{\mu}\right) + \frac{3}{4}H(0; u)\mathcal{H}\left(1, 0, 1; \frac{1}{\mu}\right) - \\
 & \frac{9}{2}\mathcal{H}\left(0, 0, 0, 1; \frac{1}{\mu}\right) - \frac{3}{2}\mathcal{H}\left(0, 0, 1, 1; \frac{1}{\mu}\right) - \frac{3}{2}\mathcal{H}\left(0, 1, 0, 1; \frac{1}{\mu}\right) - \frac{3}{2}\mathcal{H}\left(1, 0, 0, 1; \frac{1}{\mu}\right) + \\
 & 3H(0; u)\zeta_3 + \frac{3}{2}\mathcal{H}\left(1; \frac{1}{\mu}\right)\zeta_3 + \frac{89\pi^4}{1440}.
 \end{aligned}$$

6 Conclusion

In this paper, we have given details on the first analytic calculation of the remainder function of the two-loop six-edged Wilson loop in the Euclidean space in arbitrary kinematics, which we recently performed [17]. By displaying in detail how the most difficult of the integrals is computed, we have shown nonetheless how the whole calculation is greatly simplified by exploiting the Regge exactness of the six-edged Wilson loop in the quasi-multi-Regge kinematics of a pair along the ladder.

The remainder function is given as a combination of Goncharov polylogarithms of uniform transcendental weight four. The expression we have obtained is very lengthy. At present, we do not know whether, and if so to what extent, this expression can be further simplified by using some other kinematic limit that leaves the conformal cross ratios unchanged. Such a setup is for example found in backward scattering. Let us consider the physical region in which two gluons undergo a backward scattering. In a $2 \rightarrow 2$ scattering process, backward scattering may be obtained from forward scattering by crossing the t and u channels. In a $2 \rightarrow 4$ scattering process, we may choose the kinematics in which 1 and 2 are the incoming gluons, with momenta $p_2 = (p_2^+/2, 0, 0, p_2^+/2)$ and $p_1 = (p_1^-/2, 0, 0, -p_1^-/2)$, and 3, 4, 5, 6 are the outgoing gluons, with ordering

$$y_3 \ll y_4 \simeq y_5 \ll y_6; \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|. \quad (6.1)$$

We may term this the *backward quasi-multi-Regge kinematics of a pair along the ladder*. In the Euclidean region, where the Mandelstam invariants are all negative, the ordering (6.1) entails that

$$-s_{12}, -s_{23}, -s_{61}, -s_{234} \gg -s_{34}, -s_{56}, -s_{345}, -s_{123} \gg -s_{45}. \quad (6.2)$$

Introducing a parameter $\lambda \gg 1$, the hierarchy above is equivalent to the rescaling

$$\{s_{34}, s_{56}, s_{123}, s_{345}\} = \mathcal{O}(\lambda), \quad \{s_{12}, s_{23}, s_{61}, s_{234}\} = \mathcal{O}(\lambda^2), \quad (6.3)$$

whereas s_{45} is $\mathcal{O}(1)$. Note that this scaling is equivalent to the scaling of the Mandelstam invariants in the limit where three of the points of the Wilson loop are at infinity, as considered in ref. [33]. It is easy to see that in the limit (6.2) with the rescaling (6.3) the cross ratios (2.12) do not take trivial limiting values, and thus the six-edged Wilson loop is Regge exact in the backward QMRK of a pair along the ladder. We could hence repeat our computation in this limit with the hope that the ensuing analytic expression be simpler. Even though we have not performed the full evaluation of the remainder function in this limit, we have examined the diagram $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ in eq. (2.13) in the six limits obtained from eq. (6.2). We observe that, just like in section 2.2, the most complicated hard diagram reduces to a combination of one threefold integral, plus twofold and onefold integrals. Since this kinematic limit leads to the same threefold integral as the standard Regge limit, the result will be expressed in terms of the same functions and thus we expect it to be of similar complexity.

Even though the analytic form of the remainder function is very lengthy, the expression greatly simplifies when considering various limits. In section 4 we considered the remainder function in various strongly-ordered limits, and we presented in each case the leading term in the limit where the conformal cross ratios are either large or small. In section 5 we also considered the remainder function where all three conformal ratios take equal values, and computed explicitly the value of $R_{6,WL}^{(2)}(u, u, u)$ for $u = 1/4, 1/2, 1$ as well as the leading behaviour in the limit of large and small values of u .

The techniques described throughout the paper are generic, and not restricted to the case of a hexagon Wilson loop. In principle they can be applied to the computation of a polygon with an arbitrary number of edges, but in that case the set-up is complicated by the fact that the number of cross ratios grows with the number of edges, giving rise to multiple polylogarithms depending *a priori* on all those cross ratios. However, these techniques could be useful in the computation of special classes of regular polygons where the cross ratios take special values [26, 33, 52]. This is currently under investigation.

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A Nested harmonic sums

The nested harmonic sums are defined by [53],

$$\begin{aligned}
 S_i(n) &= Z_i(n) = H_n^{(i)} = \sum_{k=1}^n \frac{1}{k^i}, \\
 S_{i\bar{j}}(n) &= \sum_{k=1}^n \frac{S_{\bar{j}}(k)}{k^i} \quad \text{and} \quad Z_{i\bar{j}}(n) = \sum_{k=1}^n \frac{Z_{\bar{j}}(k-1)}{k^i}.
 \end{aligned}
 \tag{A.1}$$

The S and Z sums form an algebra. Let us illustrate this on a simple example,

$$\begin{aligned}
 S_i(n)S_j(n) &= \sum_{k_1=1}^n \sum_{k_2=1}^n \frac{1}{k_1^i k_2^j} \\
 &= \sum_{k_1=1}^n \frac{1}{k_1^i} \sum_{k_2=1}^{k_1} \frac{1}{k_2^j} + \sum_{k_2=1}^n \frac{1}{k_2^j} \sum_{k_1=1}^{k_2} \frac{1}{k_1^i} - \sum_{k=1}^n \frac{1}{k^{i+j}} \\
 &= S_{ij}(n) + S_{ji}(n) - S_{i+j}(n).
 \end{aligned}
 \tag{A.2}$$

A similar result can be obtained for the Z sums,¹⁰

$$Z_i(n)Z_j(n) = Z_{ij}(n) + Z_{ji}(n) + Z_{i+j}(n).
 \tag{A.3}$$

For sums of higher weight, a recursive application of the above procedure leads then to the reduction of any product of S or Z sums to a linear combination of those sums. Furthermore, S and Z sums can be interchanged, e.g.,

$$\begin{aligned}
 Z_{11}(n) &= \sum_{k=1}^n \frac{Z_1(k-1)}{k} = \sum_{k=1}^n \frac{1}{k} \sum_{\ell=1}^{k-1} \frac{1}{\ell} = \sum_{k=1}^n \frac{1}{k} \left(-\frac{1}{k} + \sum_{\ell=1}^k \frac{1}{\ell} \right) \\
 &= -\sum_{k=1}^n \frac{1}{k^2} + \sum_{k=1}^n \frac{1}{k} S_1(k) = -S_2(n) + S_{11}(n).
 \end{aligned}
 \tag{A.4}$$

For $n \rightarrow \infty$, the Euler-Zagier sums converge to multiple zeta values,

$$\lim_{n \rightarrow \infty} Z_{m_1 \dots m_k}(n) = \zeta(m_k, \dots, m_1).
 \tag{A.5}$$

¹⁰Note the sign difference with respect to eq. (A.2).

In ref. [54] generalisations of the S and Z sums were introduced to make them dependent on some variables,

$$\begin{aligned}
 S_i(n; x) &= Z_i(n; x) = \sum_{k=1}^n \frac{x^k}{k^i}, \\
 S_{ij}(n; x_1, \dots, x_\ell) &= \sum_{k=1}^n \frac{x_1^k}{k^i} S_j(k; x_2, \dots, x_n), \\
 Z_{ij}(n; x_1, \dots, x_\ell) &= \sum_{k=1}^n \frac{x_1^k}{k^i} Z_j(k-1; x_2, \dots, x_n).
 \end{aligned}
 \tag{A.6}$$

Those sums naturally share all the properties of the corresponding number sums introduced in the previous paragraph, e.g., they also form an algebra and the S and Z sums can be interchanged. In ref. [54] several algorithms were derived that allow one to express certain classes of nested sums as linear combinations of S and/or Z sums, and those algorithms are implemented in the FORM code `XSummer` [55]. Furthermore, for $n \rightarrow \infty$, the Z sums converge to Goncharov multiple polylogarithms,

$$\lim_{n \rightarrow \infty} Z_{m_1 \dots m_k}(n; x_1, \dots, x_k) = \text{Li}_{m_k, \dots, m_1}(x_k, \dots, x_1).
 \tag{A.7}$$

which are reviewed in the next section.

B Goncharov multiple polylogarithms

B.1 Definition

Let us define [48, 49]

$$\Omega(w) = \frac{dt}{t-w},
 \tag{B.1}$$

and iterated integrations by

$$\int_0^z \Omega(w_1) \circ \dots \circ \Omega(w_n) = \int_0^z \frac{dt}{t-w_1} \int_0^t \Omega(w_2) \circ \dots \circ \Omega(w_n).
 \tag{B.2}$$

Goncharov multiple polylogarithms can be defined by the iterated integration,

$$G(w_1, \dots, w_n; z) = \int_0^z \Omega(w_1) \circ \dots \circ \Omega(w_n),
 \tag{B.3}$$

and in the special case where all the w_i 's are zero, we define,

$$G(\vec{0}_n; z) = \frac{1}{n!} \ln^n z.
 \tag{B.4}$$

The vector $\vec{w} = (w_1, \dots, w_n)$ is called the weight vector of the polylogarithm and the number of elements in the weight vector is called the weight w of the polylogarithm. Iterated integrals form a shuffle algebra, and hence we can immediately write,

$$G(\vec{w}_1; z) G(\vec{w}_2; z) = \sum_{\vec{w} = \vec{w}_1 \uplus \vec{w}_2} G(\vec{w}; z).
 \tag{B.5}$$

Note that all terms in this equation have the same weight $w_1 + w_2$. The algebra properties of the Goncharov polylogarithms imply that not all the G functions are independent, but there must be (polynomial) relations among them. In particular, we can choose a basis where the rightmost index of all the weight vectors is non zero (apart from objects of the form $G(\vec{0}_n; z)$), e.g.,

$$\begin{aligned} G(a, 0, 0; z) &= G(0, 0; z) G(a; z) - G(0, 0, a; z) - G(0, a, 0; z) \\ &= G(0, 0; z) G(a; z) - G(0, 0, a; z) - (G(0, a; z) G(0; z) - 2G(0, 0, a; z)) \quad (\text{B.6}) \\ &= G(0, 0; z) G(a; z) + G(0, 0, a; z) - G(0, a; z) G(0; z). \end{aligned}$$

If the rightmost index of a G function is non zero, then the function is invariant under a rescaling of all its arguments,

$$G(k \vec{w}; k z) = G(\vec{w}; z). \quad (\text{B.7})$$

From the definition (B.3) it is easy to see that,

$$\lim_{w_i \rightarrow \infty} G(\vec{w}; z) = 0. \quad (\text{B.8})$$

Goncharov multiple polylogarithms can also be represented as multiple nested sums,

$$\text{Li}_{m_k, \dots, m_1}(x_k, \dots, x_1) = \sum_{n_1=1}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \dots \sum_{n_k=1}^{n_{k-1}-1} \frac{x_k^{n_k}}{n_k^{m_k}} = Z_{m_1, \dots, m_k}(\infty; x_1, \dots, x_k). \quad (\text{B.9})$$

Since the Li functions are the values at infinity of the Z sums introduced in the previous section, they share all the algebra properties of the Z sums. The G and Li functions define in fact the same class of function and are related by,

$$\text{Li}_{m_k, \dots, m_1}(x_k, \dots, x_1) = (-1)^k G\left(\underbrace{0, \dots, 0}_{m_1-1}, \frac{1}{x_1}, \dots, \underbrace{0, \dots, 0}_{m_k-1}, \frac{1}{x_1 \dots x_k}; 1\right). \quad (\text{B.10})$$

B.2 Special values

In some cases it is possible to express Goncharov multiple polylogarithms in terms of other functions, e.g.,

$$\begin{aligned} G(\vec{0}_n; z) &= \frac{1}{n!} \ln^n z, & G(\vec{a}_n; z) &= \frac{1}{n!} \ln^n \left(1 - \frac{z}{a}\right), \\ G(\vec{0}_{n-1}, a; z) &= -\text{Li}_n\left(\frac{z}{a}\right), & G(\vec{0}_n, \vec{a}_p; z) &= (-1)^p S_{n,p}\left(\frac{z}{a}\right), \end{aligned} \quad (\text{B.11})$$

and in the special case where the elements of the weight vector only take values in the set $\{-1, 0, +1\}$, Goncharov polylogarithms can be expressed in terms of the harmonic polylogarithms introduced by Remiddi and Vermaseren [56],

$$G(\vec{w}; z) = (-1)^k H(\vec{w}; z), \quad (\text{B.12})$$

where k is the number of elements in \vec{w} equal to $(+1)$.

Furthermore, up to weight two, Goncharov polylogarithms can be completely expressed in terms of ordinary logarithms and dilogarithms. In particular, if a and b are non zero, we find,

$$G(a, b; z) = \text{Li}_2\left(\frac{b-z}{b-a}\right) - \text{Li}_2\left(\frac{b}{b-a}\right) + \ln\left(1 - \frac{z}{b}\right) \ln\left(\frac{z-a}{b-a}\right). \quad (\text{B.13})$$

More special values of Goncharov multiple polylogarithms are presented in appendix **F** and **G**.

B.3 Reduction of polylogarithms of the form $G(\vec{w}(z); 1)$

In this section we present the algorithm used to express a polylogarithm of the form $G(\vec{w}(z); 1)$, where w is a rational function of z , as a linear combination of polylogarithms of the form $G(\vec{w}'; z)$, where \vec{w}' is independent of z . This algorithm is a generalisation of the corresponding algorithms described in refs. [5, 57, 58]. We start by writing $G(\vec{w}(z); 1)$ as the integral of the derivative,

$$G(\vec{w}(z); 1) = G(\vec{w}(z_0); 1) + \int_{z_0}^z dt \frac{\partial}{\partial t} G(\vec{w}(t); 1), \quad (\text{B.14})$$

where z_0 is arbitrary (provided that $G(\vec{w}(z_0); 1)$ exists). We now carry out the derivative on the integral representation of $G(\vec{w}(z); 1)$,

$$G(\vec{w}(z); 1) = G(\vec{w}(z_0); 1) + \int_{z_0}^z dt \sum_{i=1}^w \int_0^1 \Omega(w_1(t)) \circ \dots \circ \frac{\partial}{\partial t} \Omega(w_i(t)) \circ \dots \circ \Omega(w_w(t)), \quad (\text{B.15})$$

with

$$\frac{\partial}{\partial t} \Omega(w_i(t)) = \frac{dt_i}{(t_i - w_i(t))^2} \frac{\partial}{\partial t} w_i(t). \quad (\text{B.16})$$

The integrals over the t_i variables are easily performed using partial fractioning and integration by parts. At the end of this procedure, we are left with an integral over t whose integrand is a linear combination (with rational coefficients) of Goncharov polylogarithms of the form $G(\vec{w}_1(t); 1)$, with $w_1 = w - 1$. At this point we know recursively how to express these functions in terms of polylogarithms of the form $G(\vec{w}'_1; t)$ where \vec{w}'_1 is independent of t . The last integration is now done using partial fractioning and integration by parts, and since the upper integration limit is z , we end up with polylogarithms of the form $G(\vec{w}'; z)$.

C Evaluation of the additional residues of $F(u_1, u_2, u_3)$

C.1 Evaluation of $R_{-1}(u_1, u_2, u_3)$

In this appendix we give the details on the computations of the additional residues defined in eq. (3.6). For convenience let us start by introducing the definition that $\mathcal{R}_j(u_1, u_2, u_3; z_1, z_2)$ is the integrand of $R_j(u_1, u_2, u_3)$, i.e., we define,

$$R_j(u_1, u_2, u_3) \equiv \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \mathcal{R}_j(u_1, u_2, u_3; z_1, z_2). \quad (\text{C.1})$$

We now turn to the evaluation of $R_{-1}(u_1, u_2, u_3)$. We close the contours to the right, and take residues in $z_i = n_i$, $n_i \in \mathbb{N}$. We obtain,

$$\begin{aligned}
 R_{-1}(u_1, u_2, u_3) &= -\text{Res}_{z_1=0} \text{Res}_{z_2=0} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) \\
 &- \sum_{n=1}^{\infty} \left(\text{Res}_{z_1=n} \text{Res}_{z_2=0} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) + \text{Res}_{z_1=0} \text{Res}_{z_2=n} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) \right) \\
 &- \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \text{Res}_{z_1=n_1} \text{Res}_{z_2=n_2} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2).
 \end{aligned} \tag{C.2}$$

The first line in eq. (C.2) is trivial,

$$\begin{aligned}
 &-\text{Res}_{z_1=0} \text{Res}_{z_2=0} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) = \\
 &\zeta_3 \ln u_1 + \zeta_3 \ln u_2 - \frac{1}{4} \ln^2 u_2 \ln^2 u_1 - \frac{1}{6} \pi^2 \ln^2 u_1 - \frac{1}{6} \pi^2 \ln^2 u_2 - \frac{1}{6} \pi^2 \ln u_2 \ln u_1 - \frac{17\pi^4}{120}
 \end{aligned} \tag{C.3}$$

The single sum in the second line can be expressed in terms of S -sums and the sum can be performed using the algorithms A and B of refs. [54, 55],¹¹

$$\begin{aligned}
 &-\sum_{n=1}^{\infty} \left(\text{Res}_{z_1=n} \text{Res}_{z_2=0} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) + \text{Res}_{z_1=0} \text{Res}_{z_2=n} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) \right) \\
 &= \sum_{n=1}^{\infty} \left(\frac{u_1^n S_{1,1}(n)}{n^2} + \frac{u_2^n S_{1,1}(n)}{n^2} + \frac{u_1^n S_{1,2}(n)}{n} + \frac{u_1^n S_{2,1}(n)}{n} + \frac{u_2^n S_{1,2}(n)}{n} + \frac{u_2^n S_{2,1}(n)}{n} \right. \\
 &\quad - \frac{u_1^n \ln u_1 S_{1,1}(n)}{n} - \frac{u_2^n \ln u_2 S_{1,1}(n)}{n} - \frac{S_2(n) u_1^n}{n^2} - \frac{S_2(n) u_2^n}{n^2} + \frac{S_1(n) u_1^n \ln u_2}{n^2} \\
 &\quad + \frac{S_1(n) u_2^n \ln u_1}{n^2} + \frac{\pi^2 u_1^n}{3n^2} + \frac{\pi^2 u_2^n}{3n^2} + \frac{u_1^n \ln^2 u_2}{2n^2} + \frac{u_2^n \ln^2 u_1}{2n^2} - \frac{\pi^2 S_1(n) u_1^n}{6n} - \frac{2S_3(n) u_1^n}{n} \\
 &\quad - \frac{\pi^2 S_1(n) u_2^n}{6n} - \frac{2S_3(n) u_2^n}{n} - \frac{S_1(n) u_1^n \ln u_1 \ln u_2}{n} + \frac{S_2(n) u_1^n \ln u_1}{n} + \frac{S_2(n) u_1^n \ln u_2}{n} \\
 &\quad - \frac{S_1(n) u_2^n \ln u_1 \ln u_2}{n} + \frac{S_2(n) u_2^n \ln u_1}{n} + \frac{S_2(n) u_2^n \ln u_2}{n} + \frac{\zeta_3 u_1^n}{n} + \frac{\zeta_3 u_2^n}{n} \\
 &\quad - \frac{u_1^n \ln u_1 \ln^2 u_2}{2n} - \frac{u_2^n \ln^2 u_1 \ln u_2}{2n} - \frac{\pi^2 u_1^n \ln u_1}{3n} - \frac{\pi^2 u_1^n \ln u_2}{6n} - \frac{\pi^2 u_2^n \ln u_1}{6n} \\
 &\quad \left. - \frac{\pi^2 u_2^n \ln u_2}{3n} \right)
 \end{aligned} \tag{C.4}$$

¹¹Note that we used a private implementation of these algorithms in Mathematica.

$$\begin{aligned}
 &= \zeta_3 H(1; u_1) + \zeta_3 H(1; u_2) - \frac{1}{6} \pi^2 H(0; u_2) H(1; u_1) - \frac{1}{6} \pi^2 H(0; u_1) H(1; u_2) \\
 &\quad - \frac{1}{6} \pi^2 H(0, 1; u_1) - \frac{1}{6} \pi^2 H(0, 1; u_2) - H(0, 0; u_2) H(1, 0; u_1) - \frac{1}{3} \pi^2 H(1, 0; u_1) \\
 &\quad - H(0, 0; u_1) H(1, 0; u_2) - \frac{1}{3} \pi^2 H(1, 0; u_2) - \frac{1}{6} \pi^2 H(1, 1; u_1) - \frac{1}{6} \pi^2 H(1, 1; u_2) \\
 &\quad - H(0; u_2) H(0, 1, 0; u_1) - H(0; u_1) H(0, 1, 0; u_2) - H(0; u_2) H(1, 1, 0; u_1) \\
 &\quad - H(0; u_1) H(1, 1, 0; u_2) - H(0, 1, 1, 0; u_1) - H(0, 1, 1, 0; u_2) - H(1, 1, 1, 0; u_1) \\
 &\quad - H(1, 1, 1, 0; u_2) ,
 \end{aligned}$$

where $H(\vec{w}; x)$ denote the standard harmonic polylogarithms of Remiddi and Vermaseren [56]. The double sum in the third line is rewritten as a nested sum by letting $n = n_1 + n_2$,

$$\begin{aligned}
 &- \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \text{Res}_{z_1=n_1} \text{Res}_{z_2=n_2} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) \\
 &= \sum_{n=1}^{\infty} \sum_{n_1=1}^{n-1} \binom{n}{n_1} \left\{ \frac{\ln u_1 u_1^{n_1} u_2^{n-n_1}}{n^2 n_1} - \frac{\ln u_1 u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)} + \frac{\ln u_1 u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)^2} - \frac{\ln u_1 \ln u_2 u_1^{n_1} u_2^{n-n_1}}{n n_1} \right. \\
 &\quad + \frac{\ln u_1 \ln u_2 u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} + \frac{\ln u_2 u_1^{n_1} u_2^{n-n_1}}{n^2 n_1} - \frac{\ln u_2 u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)} + \frac{\ln u_2 u_1^{n_1} u_2^{n-n_1}}{n n_1^2} \\
 &\quad - \frac{\ln u_1 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n n_1} + \frac{\ln u_1 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} - \frac{\ln u_2 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n n_1} \\
 &\quad + \frac{\ln u_2 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} + \frac{2 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n^2 n_1} - \frac{2 S_1(n) u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)} + \frac{S_1(n) u_1^{n_1} u_2^{n-n_1}}{n n_1^2} \\
 &\quad + \frac{S_1(n) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)^2} + \frac{\ln u_1 S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1} - \frac{\ln u_1 S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} \\
 &\quad + \frac{S_1(n) S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1} - \frac{S_1(n) S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} - \frac{S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n^2 n_1} \\
 &\quad + \frac{S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)} - \frac{S_1(n - n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1^2} + \frac{\ln u_2 S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1} \\
 &\quad - \frac{\ln u_2 S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1 - n} + \frac{S_1(n) S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1} - \frac{S_1(n) S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} \\
 &\quad - \frac{S_1(n - n_1) S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n n_1} + \frac{S_1(n - n_1) S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} - \frac{S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n^2 n_1} \\
 &\quad + \frac{S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)} - \frac{S_1(n_1) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)^2} + \frac{2 S_2(n) u_1^{n_1} u_2^{n-n_1}}{n n_1} - \frac{2 S_2(n) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} \\
 &\quad - \frac{2 S_{1,1}(n) u_1^{n_1} u_2^{n-n_1}}{n n_1} + \frac{2 S_{1,1}(n) u_1^{n_1} u_2^{n-n_1}}{n (n_1 - n)} - \frac{2 u_1^{n_1} u_2^{n-n_1}}{n^3 n_1} + \frac{2 u_1^{n_1} u_2^{n-n_1}}{n^3 (n_1 - n)} - \frac{u_1^{n_1} u_2^{n-n_1}}{n^2 n_1^2} \\
 &\quad \left. - \frac{u_1^{n_1} u_2^{n-n_1}}{n^2 (n_1 - n)^2} - \frac{\pi^2 u_1^{n_1} u_2^{n-n_1}}{6 n n_1} + \frac{\pi^2 u_1^{n_1} u_2^{n-n_1}}{6 n (n_1 - n)} \right\} .
 \end{aligned} \tag{C.5}$$

All the sums in this expression can be performed using the algorithms C and D of refs. [54]. With the help of xSummer [55] we find,

$$\begin{aligned}
 & - \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \text{Res}_{z_1=n_1} \text{Res}_{z_2=n_2} \mathcal{R}_{-1}(u_1, u_2, u_3; z_1, z_2) = \\
 & - \ln u_1 \ln u_2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{6}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{6}\pi^2 G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \\
 & 2G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - 2G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + 4G\left(0, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) + \\
 & 4G\left(0, 0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) - 8G\left(0, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) + 2G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_2}; 1\right) - \\
 & 2G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) + \\
 & 2G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) - 2G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - \\
 & G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) + 2G\left(0, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) + 2G\left(0, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) - \\
 & 4G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) + 2G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1}; 1\right) + 4G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_2}; 1\right) - \\
 & 2G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) - \\
 & G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) + 4G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1}; 1\right) + \\
 & 2G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_2}; 1\right) - 2G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - \\
 & G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) - G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) + \\
 & \frac{1}{3}\pi^2 H(0, 1; u_1) + \frac{1}{3}\pi^2 H(0, 1; u_2) - \frac{1}{3}\pi^2 H(0, 1; u_1+u_2) + \frac{1}{6}\pi^2 H(1, 1; u_1) + \frac{1}{6}\pi^2 H(1, 1; u_2) + \\
 & 12H(0, 0, 0, 1; u_1) + 12H(0, 0, 0, 1; u_2) - 12H(0, 0, 0, 1; u_1+u_2) - 2H(0, 0, 1, 1; u_1) - \\
 & 2H(0, 0, 1, 1; u_2) + 8H(0, 0, 1, 1; u_1+u_2) - H(0, 1, 1, 1; u_1) - H(0, 1, 1, 1; u_2) - \\
 & 4H(0, 1, 1, 1; u_1+u_2) - H(1, 0, 1, 1; u_1) - H(1, 0, 1, 1; u_2) - H(1, 1, 0, 1; u_1) - \\
 & H(1, 1, 0, 1; u_2) + G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 - \\
 & 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 + 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + \\
 & G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 - 2G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) \ln u_1 + \\
 & G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 - 4H(0, 0, 1; u_1) \ln u_1 - \\
 & 4H(0, 0, 1; u_2) \ln u_1 + 4H(0, 0, 1; u_1+u_2) \ln u_1 + H(0, 1, 1; u_1) \ln u_1 - H(0, 1, 1; u_2) \ln u_1 - \\
 & 2H(0, 1, 1; u_1+u_2) \ln u_1 - H(1, 0, 1; u_1) \ln u_1 - H(1, 0, 1; u_2) \ln u_1 + H(1, 1, 1; u_1) \ln u_1 + \\
 & G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 + G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 - 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 + \\
 & 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 - 2G\left(\frac{1}{u_1}, 0, \frac{1}{u_2}; 1\right) \ln u_2 + G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_2 +
 \end{aligned}
 \tag{C.6}$$

$$\begin{aligned}
& G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 + G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_2 + \\
& G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 - 4H(0, 0, 1; u_1) \ln u_2 - 4H(0, 0, 1; u_2) \ln u_2 + \\
& 4H(0, 0, 1; u_1+u_2) \ln u_2 - H(0, 1, 1; u_1) \ln u_2 + H(0, 1, 1; u_2) \ln u_2 - \\
& 2H(0, 1, 1; u_1+u_2) \ln u_2 - H(1, 0, 1; u_1) \ln u_2 - H(1, 0, 1; u_2) \ln u_2 + H(1, 1, 1; u_2) \ln u_2 - \\
& G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 \ln u_2 + 2H(0, 1; u_1) \ln u_1 \ln u_2 + 2H(0, 1; u_2) \ln u_1 \ln u_2 - \\
& 2H(0, 1; u_1+u_2) \ln u_1 \ln u_2 + H(1, 1; u_1) \ln u_1 \ln u_2 + H(1, 1; u_2) \ln u_1 \ln u_2 .
\end{aligned}$$

C.2 Evaluation of $R_{-1-z_1}(u_1, u_2, u_3)$

We now turn to the evaluation of the residue, $R_{-1-z_1}(u_1, u_2, u_3)$, in $z_3 = -1 - z_1$ by exchanging the MB integrations with the Euler integration, eq. (3.10). Changing the integration variables from z_2 to $-z_2$ and shifting the integration contours, we arrive at the following representation for $R_{-1-z_1}(u_1, u_2, u_3)$,

$$\begin{aligned}
R_{-1-z_1}(u_1, u_2, u_3) &= \tilde{R}_{-1-z_1}^{(1)}(u_1, u_2, u_3) + \tilde{R}_{-1-z_1}^{(2)}(u_1, u_2, u_3) \\
&- \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} \text{Res}_{z_1=0} \mathcal{R}_{-1-z_1}(u_1, u_2, u_3; z_1, z_2),
\end{aligned} \tag{C.7}$$

where we defined,

$$\begin{aligned}
\tilde{R}_{-1-z_1}^{(1)}(u_1, u_2, u_3) &= \int_{-2/3-i\infty}^{-2/3+i\infty} \frac{dz_1}{2\pi i} \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_2}{2\pi i} z_2 u_1^{-z_2} u_2^{z_1+1} u_3^{z_2} \\
&\times \Gamma(-z_1-1) \Gamma(-z_1) \Gamma(z_1-z_2+1) \Gamma(-z_2)^2 \Gamma(z_2)^2 \Gamma(z_1+z_2+1), \\
\tilde{R}_{-1-z_1}^{(2)}(u_1, u_2, u_3) &= - \int_{-2/3-i\infty}^{-2/3+i\infty} \frac{dz_1}{2\pi i} \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_2}{2\pi i} \frac{z_2^2}{z_1+1} u_1^{-z_2} u_2^{z_1+1} u_3^{z_2} \\
&\times \Gamma(-z_1-1) \Gamma(-z_1) \Gamma(z_1-z_2+1) \Gamma(-z_2)^2 \Gamma(z_2)^2 \Gamma(z_1+z_2+1).
\end{aligned} \tag{C.8}$$

In each case, the integration contours are straight lines and their position is explicitly indicated for each integral. The residue appearing in eq. (C.7) arises from shifting the integration contours¹² and is given by the onefold integral,

$$\begin{aligned}
& \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} \text{Res}_{z_1=0} \mathcal{R}_{-1-z_1}(u_1, u_2, u_3; z_1, z_2) \\
&= \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} z_2 u_1^{-z_2} u_3^{z_2} \Gamma(-z_2)^3 \Gamma(z_2)^3 \\
&\times (z_2 \ln u_2 + 2\gamma_E z_2 + z_2 \psi(-z_2) + z_2 \psi(z_2) - 1),
\end{aligned} \tag{C.9}$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ denotes the digamma function and γ_E is the Euler-Mascheroni constant, $\gamma_E = -\psi(1)$. Closing the integration contour to the right and summing up

¹²Note that there are in principle two more non-zero residues coming from crossing the poles in $z_1 = -z_2$ and $z_2 = -z_1$, but it turns out that these two contributions cancel mutually.

residues in $z_2 = n_2 \in \mathbb{N}^*$, we obtain,

$$\begin{aligned}
 & \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} \text{Res}_{z_1=0} \mathcal{R}_{-1-z_1}(u_1, u_2, u_3; z_1, z_2) \\
 &= -\frac{1}{6} \ln^3 u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{6} \ln^3 u_3 H\left(1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \ln^2 u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) \\
 &+ \ln^2 u_1 H\left(1, 1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \ln u_2 \ln^2 u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \ln u_3 \ln^2 u_1 H\left(1; -\frac{u_3}{u_1}\right) \\
 &- \frac{1}{2} \ln^2 u_3 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \ln^2 u_3 H\left(0, 1; -\frac{u_3}{u_1}\right) + \ln^2 u_3 H\left(1, 1; -\frac{u_3}{u_1}\right) \\
 &+ \frac{1}{2} \ln u_2 \ln^2 u_3 H\left(1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \pi^2 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) - \ln u_1 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) \\
 &+ 2 \ln u_1 H\left(0, 1, 1; -\frac{u_3}{u_1}\right) + 2 \ln u_1 H\left(1, 0, 1; -\frac{u_3}{u_1}\right) + \ln u_2 \ln u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) \\
 &+ \ln u_3 \ln u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_3 \ln u_1 H\left(1, 1; -\frac{u_3}{u_1}\right) \\
 &- \ln u_2 \ln u_3 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \pi^2 \ln u_2 H\left(1; -\frac{u_3}{u_1}\right) + \ln u_2 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) \\
 &+ \frac{1}{2} \pi^2 \ln u_3 H\left(1; -\frac{u_3}{u_1}\right) + \ln u_3 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_3 H\left(0, 1, 1; -\frac{u_3}{u_1}\right) \\
 &- 2 \ln u_3 H\left(1, 0, 1; -\frac{u_3}{u_1}\right) - \ln u_2 \ln u_3 H\left(0, 1; -\frac{u_3}{u_1}\right) - 2\zeta_3 H\left(1; -\frac{u_3}{u_1}\right) \\
 &- \frac{1}{2} \pi^2 H\left(0, 1; -\frac{u_3}{u_1}\right) + \pi^2 H\left(1, 1; -\frac{u_3}{u_1}\right) - H\left(0, 0, 0, 1; -\frac{u_3}{u_1}\right) \\
 &+ 2H\left(0, 0, 1, 1; -\frac{u_3}{u_1}\right) + 2H\left(0, 1, 0, 1; -\frac{u_3}{u_1}\right) + 2H\left(1, 0, 0, 1; -\frac{u_3}{u_1}\right).
 \end{aligned} \tag{C.10}$$

Let us now turn to the computation of $\tilde{R}_{-1-z_1}^{(1)}(u_1, u_2, u_3)$. We apply eq. (3.10) and we obtain,

$$\begin{aligned}
 \tilde{R}_{-1-z_1}^{(1)}(u_1, u_2, u_3) &= \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_2}{2\pi i} \text{Res}_{z_1=-1} \tilde{\mathcal{R}}_{-1-z_1}^{(1)}(u_1, u_2, u_3; z_1, z_2) \\
 &+ \int_0^1 dv \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_2}{2\pi i} u_2 z_2^2 u_1^{-z_2} u_3^{z_2} (1-v)^{z_2-1} v^{-z_2} (1-(1-u_2)v)^{-z_2-1} \\
 &\times \Gamma(-z_2)^3 \Gamma(z_2)^3.
 \end{aligned} \tag{C.11}$$

We tacitly exchanged again the MB and Euler integrations, having checked numerically that this operation is allowed. The residue in eq. (C.11) comes from the fact that in eq. (3.10) the contour must be such that it separates the poles in $\Gamma(\dots + z)$ from the poles

in $\Gamma(\dots - z)$. The computation of the residue is trivial,

$$\begin{aligned}
& \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_2}{2\pi i} \text{Res}_{z_1=-1} \tilde{\mathcal{R}}_{-1-z_1}^{(1)}(u_1, u_2, u_3; z_1, z_2) \\
&= -\frac{1}{2} \ln^2 u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \ln^2 u_3 H\left(0, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_1 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) \\
&+ \ln u_3 \ln u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) + 2 \ln u_3 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \pi^2 H\left(0, 1; -\frac{u_3}{u_1}\right) \\
&- 3H\left(0, 0, 0, 1; -\frac{u_3}{u_1}\right).
\end{aligned} \tag{C.12}$$

The MB integration in the second term is also trivial and yields

$$\begin{aligned}
& \int_0^1 dv \frac{u_2}{2(1-v)(1-(1-u_2)v)} \left\{ 2 \text{Li}_3\left(\frac{(v-1)u_3}{vu_1(1-v(1-u_2))}\right) \right. \\
&+ \text{Li}_2\left(\frac{(v-1)u_3}{vu_1(1-v(1-u_2))}\right) (2 \ln(1-(1-u_2)v) + 2 \ln u_1 - 2 \ln u_3 - 2 \ln(1-v) + 2 \ln v) \\
&- (\ln^2(1-(1-u_2)v) - 2 \ln u_1 \ln(1-v) - 2 \ln(1-v) \ln(1-(1-u_2)v) + 2 \ln u_3 \ln(1-v) \\
&+ 2 \ln u_1 \ln v + 2 \ln v \ln(1-(1-u_2)v) + 2 \ln u_1 \ln(1-(1-u_2)v) - 2 \ln u_3 \ln v \\
&- 2 \ln u_3 \ln(1-(1-u_2)v) + \ln^2 u_1 + \ln^2 u_3 - 2 \ln u_1 \ln u_3 + \ln^2(1-v) + \ln^2 v \\
&\left. - 2 \ln v \ln(1-v) + \pi^2) \ln\left(1 - \frac{u_3(v-1)}{u_1v(1-(1-u_2)v)}\right) \right\}.
\end{aligned} \tag{C.13}$$

The remaining integration over v can be done in a similar algorithmic way as for $\tilde{F}(u_1, u_2, u_3)$, so we will be brief on its derivation. We start by expressing the polylogarithms appearing in the integrand of eq. (C.13) in terms of Goncharov multiple polylogarithms,

$$\text{Li}_n\left(\frac{(v-1)u_3}{vu_1(1-v(1-u_2))}\right) = -G\left(\vec{0}_{n-1}, \frac{vu_1(1-v(1-u_2))}{(v-1)u_3}; 1\right). \tag{C.14}$$

Using the algorithm of appendix B, we can express the Goncharov polylogarithms in the right-hand side of eq. (C.13) as a linear combination of Goncharov polylogarithms of the form $G(\dots; v)$, e.g.,

$$\begin{aligned}
& G\left(0, 0, \frac{vu_1(1-v(1-u_2))}{(v-1)u_3}; 1\right) = \\
& - \ln u_1 \ln^2 u_2 + \ln u_3 \ln^2 u_2 - G\left(0, \frac{1}{1-u_2}; 1\right) \ln u_2 - G\left(\frac{1}{1-u_2}, 0; 1\right) \ln u_2 - \\
& G\left(\frac{1}{1-u_2}, 1; 1\right) \ln u_2 - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_2 + \mathcal{G}(0, v_{213}; 1) \ln u_2 + \mathcal{G}(v_{213}, 1; 1) \ln u_2 + \\
& \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) \ln u_2 + G(0; v) \ln u_1 \ln u_2 - G(1; v) \ln u_1 \ln u_2 + G\left(\frac{1}{1-u_2}; v\right) \ln u_1 \ln u_2 -
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
& G(0; v) \ln u_3 \ln u_2 + G(1; v) \ln u_3 \ln u_2 - G\left(\frac{1}{1-u_2}; v\right) \ln u_3 \ln u_2 + \frac{1}{6}\pi^2 \ln u_2 - \frac{1}{6}\pi^2 G(0; v) + \\
& \frac{1}{6}\pi^2 G(1; v) - \frac{1}{6}\pi^2 G\left(\frac{1}{1-u_2}; v\right) + G(0; v)G\left(0, \frac{1}{1-u_2}; 1\right) - G(1; v)G\left(0, \frac{1}{1-u_2}; 1\right) + \\
& G\left(\frac{1}{1-u_2}; v\right)G\left(0, \frac{1}{1-u_2}; 1\right) + G(0; v)G\left(\frac{1}{1-u_2}, 0; 1\right) - G(1; v)G\left(\frac{1}{1-u_2}, 0; 1\right) + \\
& G\left(\frac{1}{1-u_2}; v\right)G\left(\frac{1}{1-u_2}, 0; 1\right) + G(0; v)G\left(\frac{1}{1-u_2}, 1; 1\right) - G(1; v)G\left(\frac{1}{1-u_2}, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}; v\right)G\left(\frac{1}{1-u_2}, 1; 1\right) + G(0; v)G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \\
& G(1; v)G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}; v\right)G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G(0, 0, 0; v) + \\
& G\left(0, 0, \frac{1}{1-u_2}; 1\right) - G\left(0, 0, \frac{1}{1-u_2}; v\right) + G(0, 1, 0; v) + G\left(0, 1, \frac{1}{1-u_2}; v\right) + \\
& G\left(0, \frac{1}{1-u_2}, 0; 1\right) - G\left(0, \frac{1}{1-u_2}, 0; v\right) + G\left(0, \frac{1}{1-u_2}, 1; 1\right) + \\
& G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; v\right) + G(1, 0, 0; v) + G\left(1, 0, \frac{1}{1-u_2}; v\right) - \\
& G(1, 1, 0; v) - G\left(1, 1, \frac{1}{1-u_2}; v\right) + G\left(1, \frac{1}{1-u_2}, 0; v\right) + G\left(1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; v\right) + \\
& G\left(\frac{1}{1-u_2}, 0, 0; 1\right) - G\left(\frac{1}{1-u_2}, 0, 0; v\right) + G\left(\frac{1}{1-u_2}, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) - \\
& G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; v\right) + G\left(\frac{1}{1-u_2}, 1, 0; v\right) + G\left(\frac{1}{1-u_2}, 1, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; v\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; v\right) + \\
& G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; v\right) - \\
& G(0; v)\mathcal{G}(0, v_{213}; 1) + G(1; v)\mathcal{G}(0, v_{213}; 1) - G\left(\frac{1}{1-u_2}; v\right)\mathcal{G}(0, v_{213}; 1) - G(0; v)\mathcal{G}(v_{213}, 1; 1) + \\
& G(1; v)\mathcal{G}(v_{213}, 1; 1) - G\left(\frac{1}{1-u_2}; v\right)\mathcal{G}(v_{213}, 1; 1) - G(0; v)\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) + \\
& G(1; v)\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) - G\left(\frac{1}{1-u_2}; v\right)\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) - \mathcal{G}(0, 0, v_{213}; 1) + \\
& \mathcal{G}(0, 0, v_{213}; v) - \mathcal{G}(0, 1, v_{213}; v) - \mathcal{G}(0, v_{213}, 1; 1) - \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) + \\
& \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; v\right) - \mathcal{G}(1, 0, v_{213}; v) + \mathcal{G}(1, 1, v_{213}; v) - \mathcal{G}\left(1, \frac{1}{1-u_2}, v_{213}; v\right) - \\
& \mathcal{G}(v_{213}, 1, 1; 1) - \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) + \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; v\right) - \mathcal{G}\left(\frac{1}{1-u_2}, 1, v_{213}; v\right) - \\
& \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1; 1\right) - \mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) + \mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; v\right) - \\
& G(0, 0; v) \ln u_1 + G(0, 1; v) \ln u_1 + G\left(0, \frac{1}{1-u_2}; 1\right) \ln u_1 - G\left(0, \frac{1}{1-u_2}; v\right) \ln u_1 + \\
& G(1, 0; v) \ln u_1 - G(1, 1; v) \ln u_1 + G\left(1, \frac{1}{1-u_2}; v\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 0; 1\right) \ln u_1 -
\end{aligned}$$

$$\begin{aligned}
 & G\left(\frac{1}{1-u_2}, 0; v\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 1; v\right) \ln u_1 + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_1 - \\
 & G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; v\right) \ln u_1 + G(0, 0; v) \ln u_3 - G(0, 1; v) \ln u_3 - G\left(0, \frac{1}{1-u_2}; 1\right) \ln u_3 + \\
 & G\left(0, \frac{1}{1-u_2}; v\right) \ln u_3 - G(1, 0; v) \ln u_3 + G(1, 1; v) \ln u_3 - G\left(1, \frac{1}{1-u_2}; v\right) \ln u_3 - \\
 & G\left(\frac{1}{1-u_2}, 0; 1\right) \ln u_3 + G\left(\frac{1}{1-u_2}, 0; v\right) \ln u_3 - G\left(\frac{1}{1-u_2}, 1; v\right) \ln u_3 - \\
 & G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_3 + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; v\right) \ln u_3,
 \end{aligned}$$

where we introduced the quantity,

$$v_{jkl}^{(\pm)} = \frac{u_k - u_l \pm \sqrt{-4u_j u_k u_l + 2u_k u_l + u_k^2 + u_l^2}}{2(1-u_j)u_k}. \quad (\text{C.16})$$

Finally, the integration over v is done using the recursive definition of Goncharov polylogarithms, eq. (3.13), and the result reads,

$$\begin{aligned}
 & \frac{1}{2}G\left(\frac{1}{1-u_2}; 1\right) \ln^3 u_1 + \frac{1}{2}G\left(\frac{1}{1-u_2}, 0; 1\right) \ln^2 u_1 + \frac{1}{2}G\left(\frac{1}{1-u_2}, 1; 1\right) \ln^2 u_1 + \\
 & \frac{1}{2}G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln^2 u_1 - \frac{1}{2}\mathcal{G}(v_{213}, 1; 1) \ln^2 u_1 - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) \ln^2 u_1 + \\
 & G\left(\frac{1}{1-u_2}; 1\right) \ln u_2 \ln^2 u_1 - \frac{3}{2}G\left(\frac{1}{1-u_2}; 1\right) \ln u_3 \ln^2 u_1 - \frac{1}{12}\pi^2 \ln^2 u_1 + \\
 & G\left(\frac{1}{1-u_2}; 1\right) \ln^2 u_2 \ln u_1 + \frac{3}{2}G\left(\frac{1}{1-u_2}; 1\right) \ln^2 u_3 \ln u_1 + \frac{1}{3}\pi^2 G\left(\frac{1}{1-u_2}; 1\right) \ln u_1 + \\
 & G\left(0, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 + G\left(0, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 0, 0; 1\right) \ln u_1 + \\
 & G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 1, 0; 1\right) \ln u_1 - 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_1 + \\
 & 2G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) \ln u_1 + \\
 & 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_1 - \\
 & \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_1 - \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_1 - \mathcal{G}(v_{213}, 0, 1; 1) \ln u_1 - \\
 & \mathcal{G}(v_{213}, 1, 0; 1) \ln u_1 + 2\mathcal{G}(v_{213}, 1, 1; 1) \ln u_1 - 2\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 - \\
 & 2\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 - \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) \ln u_1 - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0; 1\right) \ln u_1 - \\
 & 2\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_1 - 2\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_1 - \\
 & G\left(\frac{1}{1-u_2}, 0; 1\right) \ln u_3 \ln u_1 - G\left(\frac{1}{1-u_2}, 1; 1\right) \ln u_3 \ln u_1 - \\
 & G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_3 \ln u_1 + \mathcal{G}(v_{213}, 1; 1) \ln u_3 \ln u_1 +
 \end{aligned}$$

$$\begin{aligned}
& \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) \ln u_3 \ln u_1 - 2G\left(\frac{1}{1-u_2}; 1\right) \ln u_2 \ln u_3 \ln u_1 + \frac{1}{6}\pi^2 \ln u_3 \ln u_1 - \\
& \zeta_3 \ln u_1 - \frac{1}{2}G\left(\frac{1}{1-u_2}; 1\right) \ln^3 u_3 + \frac{1}{2}G\left(\frac{1}{1-u_2}, 0; 1\right) \ln^2 u_3 + \frac{1}{2}G\left(\frac{1}{1-u_2}, 1; 1\right) \ln^2 u_3 + \\
& \frac{1}{2}G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln^2 u_3 - \frac{1}{2}\mathcal{G}(v_{213}, 1; 1) \ln^2 u_3 - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) \ln^2 u_3 + \\
& G\left(\frac{1}{1-u_2}; 1\right) \ln u_2 \ln^2 u_3 - \frac{1}{12}\pi^2 \ln^2 u_3 + \frac{1}{2}\pi^2 G\left(\frac{1}{1-u_2}, 0; 1\right) + \frac{1}{2}\pi^2 G\left(\frac{1}{1-u_2}, 1; 1\right) + \\
& \frac{1}{2}\pi^2 G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(0, 0, 1, \frac{1}{1-u_2}; 1\right) + G\left(0, 0, \frac{1}{1-u_2}, 1; 1\right) - \\
& 2G\left(0, 0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(0, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G\left(0, 1, 1, \frac{1}{1-u_2}; 1\right) + \\
& G\left(0, 1, \frac{1}{1-u_2}, 0; 1\right) - 2G\left(0, 1, \frac{1}{1-u_2}, 1; 1\right) + G\left(0, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \\
& G\left(0, \frac{1}{1-u_2}, 0, 1; 1\right) - 2G\left(0, \frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) + G\left(0, \frac{1}{1-u_2}, 1, 0; 1\right) - \\
& 2G\left(0, \frac{1}{1-u_2}, 1, 1; 1\right) - 2G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) - G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) - \\
& 3G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{1-u_2}, 0, 0, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{1-u_2}, 0, 1, 1; 1\right) - \\
& G\left(\frac{1}{1-u_2}, 0, 1, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, 0; 1\right) - 2G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, 1; 1\right) - \\
& 2G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 1, 0, 0; 1\right) - 2G\left(\frac{1}{1-u_2}, 1, 0, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G\left(\frac{1}{1-u_2}, 1, 1, 0; 1\right) + 3G\left(\frac{1}{1-u_2}, 1, 1, 1; 1\right) - \\
& 3G\left(\frac{1}{1-u_2}, 1, 1, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 0; 1\right) - 3G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, 0; 1\right) - \\
& 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, 1; 1\right) - 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) - \\
& 3G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, 1; 1\right) - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) - \\
& 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) - 3G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) - \\
& 3G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \frac{1}{2}\pi^2 \mathcal{G}(v_{213}, 1; 1) - \frac{1}{2}\pi^2 \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) + \\
& \mathcal{G}\left(0, 0, v_{213}, \frac{1}{1-u_2}; 1\right) + \mathcal{G}\left(0, 0, \frac{1}{1-u_2}, v_{213}; 1\right) + \mathcal{G}\left(0, v_{213}, 1, \frac{1}{1-u_2}; 1\right) + \\
& \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, 1; 1\right) + \mathcal{G}\left(0, \frac{1}{1-u_2}, 0, v_{213}; 1\right) + \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}, 1; 1\right) + \\
& \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) + 2\mathcal{G}\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) - \mathcal{G}(v_{213}, 0, 0, 1; 1) - \\
& \mathcal{G}(v_{213}, 0, 1, 0; 1) + 2\mathcal{G}(v_{213}, 0, 1, 1; 1) - \mathcal{G}\left(v_{213}, 0, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(v_{213}, 0, \frac{1}{1-u_2}, 1; 1\right) -
\end{aligned}$$

$$\begin{aligned}
& \mathcal{G}(v_{213}, 1, 0, 0; 1) + 2\mathcal{G}(v_{213}, 1, 0, 1; 1) - \mathcal{G}\left(v_{213}, 1, 0, \frac{1}{1-u_2}; 1\right) + 2\mathcal{G}(v_{213}, 1, 1, 0; 1) - \\
& 3\mathcal{G}(v_{213}, 1, 1, 1; 1) + 3\mathcal{G}\left(v_{213}, 1, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, 0; 1\right) + \\
& 3\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, 1; 1\right) - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 0, 1; 1\right) - \\
& \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 0; 1\right) + 3\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 1; 1\right) - \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) - \\
& \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) + \mathcal{G}\left(\frac{1}{1-u_2}, 0, 0, v_{213}; 1\right) + \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}, 1; 1\right) + \\
& \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}, \frac{1}{1-u_2}; 1\right) + 2\mathcal{G}\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, v_{213}; 1\right) - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, 0; 1\right) + \\
& \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, 1; 1\right) - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, \frac{1}{1-u_2}; 1\right) + \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1, 0; 1\right) + \\
& 2\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 0; 1\right) + \\
& 2\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 1; 1\right) - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \\
& 2\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, v_{213}; 1\right) + 2\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}, 1; 1\right) + \\
& \mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) + 3\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) + \\
& H(0, 0, 0, 1, 1-u_2) - \frac{1}{6}\pi^2 G\left(\frac{1}{1-u_2}; 1\right) \ln u_2 + 2G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_2 + \\
& 2G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) \ln u_2 + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_2 + \\
& 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) \ln u_2 + 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) \ln u_2 + \\
& 3G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_2 - \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_2 - \\
& \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_2 - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) \ln u_2 - \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_2 - \\
& \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) \ln u_2 - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1; 1\right) \ln u_2 - \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_2 - \\
& 2\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_2 - G\left(\frac{1}{1-u_2}; 1\right) \ln^2 u_2 \ln u_3 - \frac{1}{3}\pi^2 G\left(\frac{1}{1-u_2}; 1\right) \ln u_3 - \\
& G\left(0, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 - G\left(0, \frac{1}{1-u_2}, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, 0, 0; 1\right) \ln u_3 - \\
& G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, 1, 0; 1\right) \ln u_3 + 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_3 - \\
& 2G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) \ln u_3 - \\
& 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_3 + \\
& \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_3 + \mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_3 + \mathcal{G}(v_{213}, 0, 1; 1) \ln u_3 +
\end{aligned}$$

$$\begin{aligned} & \mathcal{G}(v_{213}, 1, 0; 1) \ln u_3 - 2\mathcal{G}(v_{213}, 1, 1; 1) \ln u_3 + 2\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 + \\ & 2\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_3 + \mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) \ln u_3 + \mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0; 1\right) \ln u_3 + \\ & 2\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_3 + 2\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) \ln u_3 + \ln u_3 \zeta_3 - \frac{11\pi^4}{90}. \end{aligned}$$

We now turn to the evaluation of $\tilde{R}_{-1-z_1}^{(2)}(u_1, u_2, u_3)$. The computation follows the same lines as for $\tilde{R}_{-1-z_1}^{(1)}(u_1, u_2, u_3)$, with a slight complication coming from the denominator in the integrand. To get rid of the denominator, we rewrite $\tilde{R}_{-1-z_1}^{(2)}(u_1, u_2, u_3)$ as the integral of the derivative with respect to u_2 ,

$$\tilde{R}_{-1-z_1}^{(2)}(u_1, u_2, u_3) = \tilde{R}_{-1-z_1}^{(2)}(u_1, 1, u_3) + \int_1^{u_2} du \frac{\partial}{\partial u} \tilde{R}_{-1-z_1}^{(2)}(u_1, u, u_3). \quad (\text{C.18})$$

Let us start with the first term in eq. (C.18). The value for $u_2 = 1$ is easily obtained by applying Barnes lemmas, by means of which one of the two integrations can be performed. This leaves us with a onefold integral trivial to compute,

$$\begin{aligned} \tilde{R}_{-1-z_1}^{(2)}(u_1, 1, u_3) &= \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} z_2^2 u_1^{-z_2} u_3^{z_2} \Gamma(-z_2)^3 \Gamma(z_2)^3 (\psi(-z_2) + \psi(z_2) + 2\gamma) \\ &= -\frac{1}{6} \ln^3 u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{6} \ln^3 u_3 H\left(1; -\frac{u_3}{u_1}\right) + \ln^2 u_1 H\left(1, 1; -\frac{u_3}{u_1}\right) \\ &+ \frac{1}{2} \ln u_3 \ln^2 u_1 H\left(1; -\frac{u_3}{u_1}\right) - \frac{1}{2} \ln^2 u_3 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) + \ln^2 u_3 H\left(1, 1; -\frac{u_3}{u_1}\right) \\ &- \frac{1}{2} \pi^2 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) + \ln u_1 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) + 2 \ln u_1 H\left(0, 1, 1; -\frac{u_3}{u_1}\right) \\ &+ 2 \ln u_1 H\left(1, 0, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_3 \ln u_1 H\left(1, 1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \pi^2 \ln u_3 H\left(1; -\frac{u_3}{u_1}\right) \\ &- \ln u_3 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_3 H\left(0, 1, 1; -\frac{u_3}{u_1}\right) - 2 \ln u_3 H\left(1, 0, 1; -\frac{u_3}{u_1}\right) \\ &- 2\zeta_3 H\left(1; -\frac{u_3}{u_1}\right) + \pi^2 H\left(1, 1; -\frac{u_3}{u_1}\right) + 2H\left(0, 0, 0, 1; -\frac{u_3}{u_1}\right) \\ &+ 2H\left(0, 0, 1, 1; -\frac{u_3}{u_1}\right) + 2H\left(0, 1, 0, 1; -\frac{u_3}{u_1}\right) + 2H\left(1, 0, 0, 1; -\frac{u_3}{u_1}\right). \end{aligned} \quad (\text{C.19})$$

The remaining term is again computed by exchanging one of the MB integrations with an Euler integration. Shifting the contours such that they satisfy the assumptions underlying

eq. (3.10) introduces an additional residue of the form,

$$\begin{aligned}
& \int_1^{u_2} du \frac{\partial}{\partial u} \text{Res}_{z_1=-1} \mathcal{R}_{-1-z_1}^{(2)}(u_1, u, u_3; z_1, z_2) \\
&= \int_1^{u_2} du \frac{\partial}{\partial u} \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} z_2^2 u_1^{-z_2} u_3^{z_2} \Gamma(-z_2)^3 \Gamma(z_2)^3 u^{-1} \\
&= \int_{+1/5-i\infty}^{+1/5+i\infty} \frac{dz_2}{2\pi i} z_2^2 u_1^{-z_2} u_3^{z_2} \Gamma(-z_2)^3 \Gamma(z_2)^3 \ln u_2 \\
&= \frac{1}{2} \ln u_2 \ln^2 u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \ln u_2 \ln^2 u_3 H\left(1; -\frac{u_3}{u_1}\right) + \ln u_2 \ln u_1 H\left(0, 1; -\frac{u_3}{u_1}\right) \\
&\quad - \ln u_2 \ln u_3 \ln u_1 H\left(1; -\frac{u_3}{u_1}\right) + \frac{1}{2} \pi^2 \ln u_2 H\left(1; -\frac{u_3}{u_1}\right) + \ln u_2 H\left(0, 0, 1; -\frac{u_3}{u_1}\right) \\
&\quad - \ln u_2 \ln u_3 H\left(0, 1; -\frac{u_3}{u_1}\right).
\end{aligned} \tag{C.20}$$

After inserting an Euler integral for the integral over z_1 in the second term of eq. (C.18) and exchanging the Euler integration and the integration over z_2 , the MB integral is trivial and can be performed by closing the contour and summing up residues,

$$\begin{aligned}
& \int_0^1 dv \int_1^{u_2} du \frac{u_3}{2} (uv - v + 1)^{-1} (-u_1 v^2 + u_1 u v^2 + u_1 v - u_3 v + u_3)^{-1} \\
&\quad \times \left\{ \ln^2((u-1)v+1) - 2 \ln u_1 \ln(1-v) - 2 \ln(1-v) \ln((u-1)v+1) \right. \\
&\quad + 2 \ln u_3 \ln(1-v) + 2 \ln u_1 \ln v + 2 \ln v \ln((u-1)v+1) + 2 \ln u_1 \ln((u-1)v+1) \\
&\quad - 2 \ln u_3 \ln v - 2 \ln u_3 \ln((u-1)v+1) + \ln^2 u_1 + \ln^2 u_3 - 2 \ln u_1 \ln u_3 + \ln^2(1-v) \\
&\quad \left. + \ln^2 v - 2 \ln v \ln(1-v) + \pi^2 \right\},
\end{aligned} \tag{C.21}$$

and the integration over u can be done easily in terms of ordinary polylogarithms, yielding,

$$\begin{aligned}
& \int_0^1 dv \frac{1}{6} v^{-1} (1-v)^{-1} \left\{ \ln^3(v(u_2-1)+1) - 3 \ln(1-v) \ln^2(v(u_2-1)+1) + \right. \\
&\quad \left. 3 \ln v \ln^2(v(u_2-1)+1) + 3 \ln u_1 \ln^2(v(u_2-1)+1) - 3 \ln u_3 \ln^2(v(u_2-1)+1) - \right. \\
&\quad \left. 3 \ln \left(-\frac{u_1(u_2-1)v^2 + (u_1-u_3)v + u_3}{(v-1)u_3} \right) \ln^2(v(u_2-1)+1) + \right. \\
&\quad \left. 3 \ln^2(1-v) \ln(v(u_2-1)+1) + 3 \ln^2 v \ln(v(u_2-1)+1) + 3 \ln^2 u_1 \ln(v(u_2-1)+1) + \right. \\
&\quad \left. 3 \ln^2 u_3 \ln(v(u_2-1)+1) - 6 \ln(1-v) \ln v \ln(v(u_2-1)+1) - \right. \\
&\quad \left. 6 \ln(1-v) \ln u_1 \ln(v(u_2-1)+1) + 6 \ln v \ln u_1 \ln(v(u_2-1)+1) + \right. \\
&\quad \left. 6 \ln(1-v) \ln u_3 \ln(v(u_2-1)+1) - 6 \ln v \ln u_3 \ln(v(u_2-1)+1) - \right. \\
&\quad \left. 6 \ln u_1 \ln u_3 \ln(v(u_2-1)+1) + \right.
\end{aligned} \tag{C.22}$$

$$\begin{aligned}
 & 6\ln(1-v)\ln\left(-\frac{u_1(u_2-1)v^2+(u_1-u_3)v+u_3}{(v-1)u_3}\right)\ln(v(u_2-1)+1)- \\
 & 6\ln v\ln\left(-\frac{u_1(u_2-1)v^2+(u_1-u_3)v+u_3}{(v-1)u_3}\right)\ln(v(u_2-1)+1)- \\
 & 6\ln u_1\ln\left(-\frac{u_1(u_2-1)v^2+(u_1-u_3)v+u_3}{(v-1)u_3}\right)\ln(v(u_2-1)+1)+ \\
 & 6\ln u_3\ln\left(-\frac{u_1(u_2-1)v^2+(u_1-u_3)v+u_3}{(v-1)u_3}\right)\ln(v(u_2-1)+1)- \\
 & 6\text{Li}_2\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)\ln(v(u_2-1)+1)+3\pi^2\ln(v(u_2-1)+1)- \\
 & 3\ln^2(1-v)\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 3\ln^2 v\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 3\ln^2 u_1\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 3\ln^2 u_3\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)+ \\
 & 6\ln(1-v)\ln v\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)+ \\
 & 6\ln(1-v)\ln u_1\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 6\ln v\ln u_1\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 6\ln(1-v)\ln u_3\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)+ \\
 & 6\ln v\ln u_3\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)+ \\
 & 6\ln u_1\ln u_3\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)- \\
 & 3\pi^2\ln(u_1(u_2-1)v^2+(u_1-u_3)v+u_3)+3\ln^2(1-v)\ln(vu_1-vu_3+u_3)+ \\
 & 3\ln^2 v\ln(vu_1-vu_3+u_3)+3\ln^2 u_1\ln(vu_1-vu_3+u_3)+3\ln^2 u_3\ln(vu_1-vu_3+u_3)- \\
 & 6\ln(1-v)\ln v\ln(vu_1-vu_3+u_3)-6\ln(1-v)\ln u_1\ln(vu_1-vu_3+u_3)+ \\
 & 6\ln v\ln u_1\ln(vu_1-vu_3+u_3)+6\ln(1-v)\ln u_3\ln(vu_1-vu_3+u_3)- \\
 & 6\ln v\ln u_3\ln(vu_1-vu_3+u_3)-6\ln u_1\ln u_3\ln(vu_1-vu_3+u_3)+ \\
 & 3\pi^2\ln(vu_1-vu_3+u_3)-6\ln(1-v)\text{Li}_2\left(\frac{vu_1}{(v-1)u_3}\right)+6\ln v\text{Li}_2\left(\frac{vu_1}{(v-1)u_3}\right)+ \\
 & 6\ln u_1\text{Li}_2\left(\frac{vu_1}{(v-1)u_3}\right)-6\ln u_3\text{Li}_2\left(\frac{vu_1}{(v-1)u_3}\right)+6\ln(1-v)\text{Li}_2\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)- \\
 & 6\ln v\text{Li}_2\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)-6\ln u_1\text{Li}_2\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)+ \\
 & 6\ln u_3\text{Li}_2\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)-6\text{Li}_3\left(\frac{vu_1}{(v-1)u_3}\right)+6\text{Li}_3\left(\frac{vu_1(v(u_2-1)+1)}{(v-1)u_3}\right)\Big\}.
 \end{aligned}$$

We are thus left with only the integration over v to be done. We proceed in the by now usual way by converting all the polylogarithms in eq. (C.22) into Goncharov polylogarithms using the algorithm of appendix B and then perform the integration over v using the recursive definition of the G -function. At the end of this procedure we find,

$$\begin{aligned}
 & \frac{1}{2}G\left(0, \frac{1}{1-u_2}; 1\right)\ln^2 u_1 + \frac{1}{2}G\left(0, \frac{1}{1-\frac{u_1}{u_3}}; 1\right)\ln^2 u_1 + \\
 & \frac{1}{2}G\left(\frac{1}{1-u_2}, 1; 1\right)\ln^2 u_1 + \frac{1}{2}G\left(-\frac{u_3}{u_1-u_3}, 1; 1\right)\ln^2 u_1 - \frac{1}{2}\mathcal{G}(0, v_{213}; 1)\ln^2 u_1 - \\
 & \frac{1}{2}\mathcal{G}(v_{213}, 1; 1)\ln^2 u_1 + G\left(0, 0, \frac{1}{1-u_2}; 1\right)\ln u_1 + G\left(0, 1, \frac{1}{1-u_2}; 1\right)\ln u_1 +
 \end{aligned} \tag{C.23}$$

$$\begin{aligned}
& G\left(0, \frac{1}{1-u_2}, 0; 1\right) \ln u_1 + G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_1 + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 0; 1\right) \ln u_1 - \\
& G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 0, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 1, 0; 1\right) \ln u_1 - \\
& 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 + \\
& G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0; 1\right) \ln u_1 - 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 1; 1\right) \ln u_1 - \\
& \mathcal{G}(0, v_{213}, 0; 1) \ln u_1 + \mathcal{G}(0, v_{213}, 1; 1) \ln u_1 - \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_1 - \mathcal{G}(v_{213}, 0, 1; 1) \ln u_1 - \\
& \mathcal{G}(v_{213}, 1, 0; 1) \ln u_1 + 2\mathcal{G}(v_{213}, 1, 1; 1) \ln u_1 - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 - \\
& \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 - G\left(0, \frac{1}{1-u_2}; 1\right) \ln u_3 \ln u_1 - G\left(0, \frac{1}{1-\frac{u_1}{u_3}}; 1\right) \ln u_3 \ln u_1 - \\
& G\left(\frac{1}{1-u_2}, 1; 1\right) \ln u_3 \ln u_1 - G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_3 \ln u_1 + \mathcal{G}(0, v_{213}; 1) \ln u_3 \ln u_1 + \\
& \mathcal{G}(v_{213}, 1; 1) \ln u_3 \ln u_1 + \frac{1}{2}G\left(0, \frac{1}{1-u_2}; 1\right) \ln^2 u_3 + \frac{1}{2}G\left(0, \frac{1}{1-\frac{u_1}{u_3}}; 1\right) \ln^2 u_3 + \\
& \frac{1}{2}G\left(\frac{1}{1-u_2}, 1; 1\right) \ln^2 u_3 + \frac{1}{2}G\left(-\frac{u_3}{u_1-u_3}, 1; 1\right) \ln^2 u_3 - \frac{1}{2}\mathcal{G}(0, v_{213}; 1) \ln^2 u_3 - \\
& \frac{1}{2}\mathcal{G}(v_{213}, 1; 1) \ln^2 u_3 + \frac{1}{2}\pi^2 G\left(0, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\pi^2 G\left(0, \frac{1}{1-\frac{u_1}{u_3}}; 1\right) + \frac{1}{2}\pi^2 G\left(\frac{1}{1-u_2}, 1; 1\right) + \\
& \frac{1}{2}\pi^2 G\left(-\frac{u_3}{u_1-u_3}, 1; 1\right) + G\left(0, 0, 0, \frac{1}{1-u_2}; 1\right) + G\left(0, 0, \frac{1}{1-u_2}, 0; 1\right) + \\
& G\left(0, 0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(0, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G\left(0, 1, 1, \frac{1}{1-u_2}; 1\right) + \\
& G\left(0, 1, \frac{1}{1-u_2}, 0; 1\right) - 2G\left(0, 1, \frac{1}{1-u_2}, 1; 1\right) + G\left(0, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \\
& G\left(0, \frac{1}{1-u_2}, 0, 0; 1\right) + G\left(0, \frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) - G\left(0, \frac{1}{1-u_2}, 1, 1; 1\right) + \\
& G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) + G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 0, 0; 1\right) - \\
& G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 0, 1; 1\right) - G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1, 0; 1\right) + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, 0, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, 0, 1, 0; 1\right) - 2G\left(\frac{1}{1-u_2}, 0, 1, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, 0, 1, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, 1; 1\right) + G\left(\frac{1}{1-u_2}, 1, 0, 0; 1\right) - \\
& 2G\left(\frac{1}{1-u_2}, 1, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G\left(\frac{1}{1-u_2}, 1, 1, 0; 1\right) + \\
& 3G\left(\frac{1}{1-u_2}, 1, 1, 1; 1\right) - 2G\left(\frac{1}{1-u_2}, 1, 1, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 0; 1\right) -
\end{aligned}$$

$$\begin{aligned}
& 2G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 1; 1\right) + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \\
& G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, 0; 1\right) - 2G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, 1; 1\right) + \\
& G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) + \\
& G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 0, 1; 1\right) + G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1, 0; 1\right) - 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1, 1; 1\right) + \\
& G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0, 0; 1\right) - 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0, 1; 1\right) - 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 1, 0; 1\right) + \\
& 3G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 1, 1; 1\right) - \frac{1}{2}\pi^2\mathcal{G}(0, v_{213}; 1) - \frac{1}{2}\pi^2\mathcal{G}(v_{213}, 1; 1) - \mathcal{G}(0, v_{213}, 0, 0; 1) + \\
& \mathcal{G}(0, v_{213}, 0, 1; 1) - \mathcal{G}\left(0, v_{213}, 0, \frac{1}{1-u_2}; 1\right) + \mathcal{G}(0, v_{213}, 1, 0; 1) - \mathcal{G}(0, v_{213}, 1, 1; 1) + \\
& \mathcal{G}\left(0, v_{213}, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, 0; 1\right) + \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, 1; 1\right) - \\
& \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \mathcal{G}(v_{213}, 0, 0, 1; 1) - \mathcal{G}(v_{213}, 0, 1, 0; 1) + 2\mathcal{G}(v_{213}, 0, 1, 1; 1) - \\
& \mathcal{G}\left(v_{213}, 0, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(v_{213}, 0, \frac{1}{1-u_2}, 1; 1\right) - \mathcal{G}(v_{213}, 1, 0, 0; 1) + 2\mathcal{G}(v_{213}, 1, 0, 1; 1) - \\
& \mathcal{G}\left(v_{213}, 1, 0, \frac{1}{1-u_2}; 1\right) + 2\mathcal{G}(v_{213}, 1, 1, 0; 1) - 3\mathcal{G}(v_{213}, 1, 1, 1; 1) + 2\mathcal{G}\left(v_{213}, 1, 1, \frac{1}{1-u_2}; 1\right) - \\
& \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, 0; 1\right) + 2\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, 1; 1\right) - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \\
& \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 0, 1; 1\right) - \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 0; 1\right) + 2\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 1; 1\right) - \\
& \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) - \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) - G\left(0, 0, \frac{1}{1-u_2}; 1\right) \ln u_3 - \\
& G\left(0, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 - G\left(0, \frac{1}{1-u_2}, 0; 1\right) \ln u_3 - G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \ln u_3 - \\
& G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 0; 1\right) \ln u_3 + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, 0, 1; 1\right) \ln u_3 - \\
& G\left(\frac{1}{1-u_2}, 1, 0; 1\right) \ln u_3 + 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 - \\
& G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0; 1\right) \ln u_3 + \\
& 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 1; 1\right) \ln u_3 + \mathcal{G}(0, v_{213}, 0; 1) \ln u_3 - \mathcal{G}(0, v_{213}, 1; 1) \ln u_3 + \\
& \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) \ln u_3 + \mathcal{G}(v_{213}, 0, 1; 1) \ln u_3 + \mathcal{G}(v_{213}, 1, 0; 1) \ln u_3 - \\
& 2\mathcal{G}(v_{213}, 1, 1; 1) \ln u_3 + \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) \ln u_3 + \mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_3.
\end{aligned}$$

D Limits of the remainder function

D.1 Multi-Regge limits

In this section we discuss the behaviour of the remainder function in the multi-Regge kinematics. These are defined by requiring the final-state gluons to be strongly ordered in rapidity while having comparable transverse momentum. If we choose the gluons 3 and 4 as incoming, this implies the hierarchy of scales,

$$-s_{34} \gg -s_{234}, -s_{345} \gg -s_{56}, -s_{61}, -s_{12} \gg -s_{45}, -s_{456}, -s_{23}. \quad (\text{D.1})$$

Introducing a parameter $\lambda \ll 1$, the above hierarchy is equivalent to a rescaling,

$$\{s_{234}, s_{345}\} = \mathcal{O}(\lambda), \quad \{s_{56}, s_{61}, s_{12}\} = \mathcal{O}(\lambda^2), \quad \{s_{45}, s_{456}, s_{23}\} = \mathcal{O}(\lambda^3), \quad (\text{D.2})$$

whereas s_{34} is $\mathcal{O}(1)$. In this limit all three conformal cross ratios take limiting values [28, 59, 60],

$$u_1 = \mathcal{O}(\lambda), \quad u_2 = \mathcal{O}(\lambda), \quad u_3 = 1 + \mathcal{O}(\lambda), \quad (\text{D.3})$$

and it was shown that in the Euclidean region the remainder function must vanish.

As a consistency check of our computation, we computed the leading behaviour of the remainder function in the multi-Regge limit. Apart from $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$, the sums of all the other terms in eq. (2.13) is expressed in terms of harmonic polylogarithms, thus we can directly expand the harmonic polylogarithms in the scaling parameter λ and only keep the leading term. However, $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ is expressed in terms of Goncharov polylogarithms whose arguments are complicated functions of the conformal cross ratios. It is therefore easier to compute $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ from scratch in the limit under consideration. In what follows, this technique is described on the example of the threefold contribution to $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ presented in eq. (2.25).

Defining a quantity \bar{u}_3 by $u_3 = 1 + \bar{u}_3$, we can reformulate the problem as finding the leading behaviour in the limit $\lambda \rightarrow 0$ of the integral

$$\begin{aligned} & -\frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) \lambda^{z_1+z_2} u_1^{z_1} u_2^{z_2} (1 + \lambda \bar{u}_3)^{z_3} \\ & \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1+z_2) \Gamma(z_2+z_3) \Gamma(z_3+z_1). \end{aligned} \quad (\text{D.4})$$

The code `MBasymptotics` allows us to extract the leading behaviour for $\lambda \rightarrow 0$ of MB integrals of the form

$$\int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \left(\prod_i \frac{dz_i}{2\pi i} \right) \lambda^{P(z_i)} f(z_i), \quad (\text{D.5})$$

where $P(z_i)$ is a polynomial in the integration variables, and $f(z_i)$ denotes a function independent of λ . However, the integral in eq. (D.4) violates this form. We can bring eq. (D.4) into the desired form to the price of introducing an additional MB integral by

applying eq. (2.16) to the term $(1 + \lambda \bar{u}_3)^{z_3}$. Eq. (D.4) now takes the form,

$$\begin{aligned}
 & -\frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} \frac{dz}{2\pi i} (z_2 z_3 + z_1 z_2 + z_1 z_3) \lambda^{z_1+z_2+z} u_1^{z_1} u_2^{z_2} \bar{u}_3^z \\
 & \quad \times \Gamma(-z) \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(z_1+z_2) \Gamma(z-z_3) \Gamma(-z_3) \Gamma(z_1+z_3) \Gamma(z_2+z_3) \\
 & -\frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_2 z_3 + z_1 z_2 + z_1 z_3) \lambda^{z_1+z_2} u_1^{z_1} u_2^{z_2} \\
 & \quad \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(z_1+z_2) \Gamma(-z_3)^2 \Gamma(z_1+z_3) \Gamma(z_2+z_3) \\
 & -\frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_2 z_3 + z_1 z_2 + z_1 z_3) \lambda^{z_1+z_2+1} u_1^{z_1} u_2^{z_2} \bar{u}_3^{z_3} \\
 & \quad \times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(z_1+z_2) \Gamma(-z_3)^2 \Gamma(z_1+z_3) \Gamma(z_2+z_3),
 \end{aligned} \tag{D.6}$$

where the contours for the integrations over z_i , $i = 1, 2, 3$, are given in eq. (2.26) and the contour for the integral over z is a straight vertical line with $\text{Re}(z) = +\frac{3}{2}$. The three-fold contributions arise when shifting the z contour from the form required by eq. (2.16) to a straight line. All the integrals in eq. (D.6) match precisely the form required by MBasymptotics, and we find,

$$\begin{aligned}
 & -\frac{1}{4} \zeta_3 \ln(u_1 u_2) - \frac{1}{192} \ln^4 \frac{u_1}{u_2} - \frac{5}{96} \pi^2 \ln^2 \frac{u_1}{u_2} - \frac{137\pi^4}{2880} \\
 & + \frac{1}{4} \int_{+1/3-i\infty}^{+1/3+i\infty} \frac{dz_1}{2\pi i} z_1^2 (u_1^{z_1} u_2^{-z_1} + u_1^{-z_1} u_2^{z_1}) \Gamma(-z_1)^4 \Gamma(z_1)^2 \Gamma(2z_1) \\
 & - \frac{1}{24} \int_{-1/5-i\infty}^{-1/5+i\infty} \frac{dz_3}{2\pi i} z_3 \Gamma(-z_3)^2 \Gamma(z_3)^2 \left(6(\ln u_1 + \ln u_2 + 2\gamma_E) \psi(z_3) + 6\gamma_E \ln u_2 \right. \\
 & \quad \left. + 6 \ln u_1 (\ln u_2 + \gamma_E) + 6\psi(z_3)^2 + \pi^2 + 6\gamma_E^2 \right).
 \end{aligned} \tag{D.7}$$

The integral over z_3 can be evaluated in terms of harmonic polylogarithms in the usual way by closing the contour to the right and summing up residues. The integral over z_1 is more special, because it involves poles in half-integer values of the Γ function. Summing up the tower of residues leads to multiple binomial sums [50, 51]. However, we observe that this contribution cancels against similar contributions coming from the twofold contributions to $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$, so we do not discuss this issue further. Finally, combining all the contributions, eq. (2.13), we find that the remainder function vanishes in the multi-Regge limit (D.1),

$$\lim_{\lambda \rightarrow 0} R_6^{(2)}(\lambda u_1, \lambda u_2, 1 + \lambda \bar{u}_3) = 0. \tag{D.8}$$

Note that there are five more ways in which we could have defined the multi-Regge limit, corresponding to the five cyclic permutations of the indices in eq. (D.1). It is clear that in terms of conformal cross ratios the six different ways of taking the multi-Regge limit are equivalent pairwise, and we have checked that our expression satisfies

$$\lim_{\lambda \rightarrow 0} R_6^{(2)}(1 + \lambda \bar{u}_1, \lambda u_2, \lambda u_3) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} R_6^{(2)}(\lambda u_1, 1 + \lambda \bar{u}_2, \lambda u_3) = 0, \tag{D.9}$$

Therefore, our result has the correct behaviour in all the multi-Regge limits.

D.2 Collinear limits

In this section we compute the remainder function in collinear kinematics. If the momenta of two external particles, say 1 and 2, become collinear, then the conformal cross ratios take the particular values,

$$u_1 \rightarrow 0, \quad u_2 \rightarrow (1-z) \frac{s_{56}}{s_{234}}, \quad u_3 \rightarrow 1-u_2, \quad (\text{D.10})$$

and the remainder function must vanish in this limit, i.e.,

$$\lim_{\lambda \rightarrow 0} R_6^{(2)}(\lambda u_1, u_2, 1-u_2) = 0, \quad \forall u_2. \quad (\text{D.11})$$

We proceed in a similar way to the multi-Regge limit, and we again only discuss here the case of the threefold contribution to $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$. Using `MBasymptotics` we obtain the leading behaviour of eq. (2.25) in the limit $\lambda \rightarrow 0$

$$\begin{aligned} & \frac{1}{4} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (1-u_2)^{z_3} u_2^{z_2} \Gamma(-z_2)^2 \Gamma(z_2) \Gamma(-z_3)^2 \Gamma(z_3) \Gamma(z_2+z_3) \\ & \times (z_3 z_2 \ln u_1 + 2\gamma_E z_3 z_2 + z_2 + z_3 + z_3 z_2 \psi(z_2) + z_3 z_2 \psi(z_3)), \end{aligned} \quad (\text{D.12})$$

with the integration contours given by eq. (2.26). Closing the integration contours to the right and taking residues results in binomial nested sums, similar to the ones encountered in eq. (C.5). All the sums can hence be done using `XSummer` and result in polylogarithms. Since the expression is rather lengthy and does not add anything new to the discussion, we do not show the result explicitly. Finally, combining this result with the contributions coming from the other contributions in eq. (2.13) we see that our expression for the remainder function satisfies eq. (D.11).

Similarly to the multi-Regge case, we could have defined five additional collinear limits obtained by cyclic permutations of the external momenta. The six limits one obtains in this way are again equivalent pairwise at the level of the remainder function (up to redefining, e.g., $u_2 \rightarrow 1-u_3$ in eq. (D.11)), and we checked explicitly that our result does not only satisfy the constraint (D.11) but also

$$\lim_{\lambda \rightarrow 0} R_6^{(2)}(u_1, \lambda u_2, 1-u_1) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} R_6^{(2)}(u_1, 1-u_1, \lambda u_3) = 0, \quad \forall u_1. \quad (\text{D.13})$$

E Special values of ordinary and harmonic polylogarithms

In this appendix we present several special values of polylogarithms up to weight four we encountered throughout our computation. All the identities of this section were obtained either using the PSLQ algorithm [61, 62] or using the HPL package [63]. The question whether a given transcendental number can be expressed as a polynomial with rational coefficients of other transcendental numbers, i.e., the problem of finding a basis in the space of transcendental numbers, is an open mathematical problem, and we must therefore make an *a priori* choice for our basis. Our choice consists in monomials in the following transcendental numbers:

- weight one: $\ln 2, \ln 3,$
- weight two: $\pi^2, \text{Li}_2(1/3),$
- weight three: $\zeta_3, \text{Li}_3(1/3), \text{Li}_3(-1/2),$
- weight four: $\text{Li}_4(1/2), \text{Li}_4(-1/2), \text{Li}_4(1/3), \text{Li}_4(-1/3), S_{2,2}(-1/2), S_{2,2}(-1/3).$

Note that the values of harmonic polylogarithms in $1/2$ presented in this appendix are sufficient to obtain all harmonic polylogarithms in $1/2$ up to weight four. Furthermore, as the space of harmonic polylogarithms is closed under the transformations $x \rightarrow (1-x)/(1+x)$ and $x \rightarrow -x$, these values are at the same time sufficient to construct all harmonic polylogarithms up to weight four in $-1/2$ and $\pm 1/3$, and we have hence proved at the same time that all harmonic polylogarithms up to weight four in these values can be expressed completely in the basis we just defined.

E.1 Polylogarithms of weight two

$$\text{Li}_2(-8) = 3 \ln^2 3 - 6 \ln 2 \ln 3 + 6 \text{Li}_2\left(\frac{1}{3}\right) - \frac{\pi^2}{2} \tag{E.1}$$

$$\text{Li}_2(-3) = -\ln^2 3 - 2 \text{Li}_2\left(\frac{1}{3}\right) \tag{E.2}$$

$$\text{Li}_2(-2) = \frac{1}{2} \ln^2 3 - \ln 2 \ln 3 + \text{Li}_2\left(\frac{1}{3}\right) - \frac{\pi^2}{6} \tag{E.3}$$

$$\text{Li}_2\left(-\frac{1}{2}\right) = -\frac{1}{2} \ln^2 2 - \frac{1}{2} \ln^2 3 + \ln 3 \ln 2 - \text{Li}_2\left(\frac{1}{3}\right) \tag{E.4}$$

$$\text{Li}_2\left(-\frac{1}{3}\right) = \frac{1}{2} \ln^2 3 + 2 \text{Li}_2\left(\frac{1}{3}\right) - \frac{\pi^2}{6} \tag{E.5}$$

$$\text{Li}_2\left(-\frac{1}{8}\right) = -\frac{9}{2} \ln^2 2 - 3 \ln^2 3 + 6 \ln 3 \ln 2 - 6 \text{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{3} \tag{E.6}$$

$$\text{Li}_2\left(\frac{1}{4}\right) = -2 \ln^2 2 - \ln^2 3 + 2 \ln 3 \ln 2 - 2 \text{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} \tag{E.7}$$

$$\text{Li}_2\left(\frac{2}{3}\right) = -\ln^2 3 + \ln 2 \ln 3 - \text{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} \tag{E.8}$$

$$\text{Li}_2\left(\frac{2}{3}\right) = -\ln^2 3 + \ln 2 \ln 3 - \text{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} \tag{E.9}$$

$$\text{Li}_2\left(\frac{3}{4}\right) = -2 \ln^2 2 + \ln^2 3 + 2 \text{Li}_2\left(\frac{1}{3}\right) \tag{E.10}$$

E.2 Polylogarithms of weight three

$$\text{Li}_3(-8) = -3 \ln^3 2 - \frac{3}{2} \pi^2 \ln 2 + 18 \text{Li}_3\left(-\frac{1}{2}\right) + \frac{49 \zeta_3}{4} \tag{E.11}$$

$$\text{Li}_3(-3) = -\frac{1}{3} \ln^3 3 + 2 \text{Li}_3\left(\frac{1}{3}\right) - \frac{13 \zeta_3}{6} \tag{E.12}$$

$$\text{Li}_3(-2) = -\frac{1}{6} \ln^3 2 - \frac{1}{6} \pi^2 \ln 2 + \text{Li}_3\left(-\frac{1}{2}\right) \quad (\text{E.13})$$

$$\text{Li}_3\left(-\frac{1}{3}\right) = -\frac{1}{6} \ln^3 3 + \frac{1}{6} \pi^2 \ln 3 + 2\text{Li}_3\left(\frac{1}{3}\right) - \frac{13\zeta_3}{6} \quad (\text{E.14})$$

$$\text{Li}_3\left(-\frac{1}{8}\right) = \frac{3}{2} \ln^3 2 - \pi^2 \ln 2 + 18\text{Li}_3\left(-\frac{1}{2}\right) + \frac{49\zeta_3}{4} \quad (\text{E.15})$$

$$\text{Li}_3\left(\frac{1}{4}\right) = \frac{2}{3} \ln^3 2 - \frac{1}{3} \pi^2 \ln 2 + 4\text{Li}_3\left(-\frac{1}{2}\right) + \frac{7\zeta_3}{2} \quad (\text{E.16})$$

$$\begin{aligned} \text{Li}_3\left(\frac{2}{3}\right) &= \frac{1}{6} \ln^3 2 + \frac{1}{3} \ln^3 3 - \frac{1}{2} \ln^2 3 \ln 2 + \frac{1}{6} \pi^2 \ln 2 - \frac{1}{6} \pi^2 \ln 3 - \text{Li}_3\left(\frac{1}{3}\right) \\ &\quad - \text{Li}_3\left(-\frac{1}{2}\right) + \zeta_3 \end{aligned} \quad (\text{E.17})$$

$$\text{Li}_3\left(\frac{3}{4}\right) = 2 \ln^3 2 + \frac{1}{3} \ln^3 3 - 2 \ln 3 \ln^2 2 - 2\text{Li}_3\left(\frac{1}{3}\right) - 4\text{Li}_3\left(-\frac{1}{2}\right) - \frac{\zeta_3}{3} \quad (\text{E.18})$$

E.3 Polylogarithms of weight four

$$\text{Li}_4(-8) = -4 \ln^4 2 - \frac{1}{2} \pi^2 \ln^2 2 - 42\text{Li}_4\left(\frac{1}{2}\right) - 54\text{Li}_4\left(-\frac{1}{2}\right) - \frac{3\pi^4}{40} \quad (\text{E.19})$$

$$\text{Li}_4(-3) = -\frac{1}{24} \ln^4 3 - \frac{1}{12} \pi^2 \ln^2 3 - \text{Li}_4\left(-\frac{1}{3}\right) - \frac{7\pi^4}{360} \quad (\text{E.20})$$

$$\text{Li}_4(-2) = -\frac{1}{24} \ln^4 2 - \frac{1}{12} \pi^2 \ln^2 2 - \text{Li}_4\left(-\frac{1}{2}\right) - \frac{7\pi^4}{360} \quad (\text{E.21})$$

$$\text{Li}_4\left(-\frac{1}{8}\right) = \frac{5}{8} \ln^4 2 - \frac{1}{4} \pi^2 \ln^2 2 + 42\text{Li}_4\left(\frac{1}{2}\right) + 54\text{Li}_4\left(-\frac{1}{2}\right) + \frac{\pi^4}{18} \quad (\text{E.22})$$

$$\text{Li}_4\left(\frac{1}{9}\right) = 8\text{Li}_4\left(-\frac{1}{3}\right) + 8\text{Li}_4\left(\frac{1}{3}\right) \quad (\text{E.23})$$

$$\text{Li}_4\left(\frac{1}{4}\right) = 8\text{Li}_4\left(-\frac{1}{2}\right) + 8\text{Li}_4\left(\frac{1}{2}\right) \quad (\text{E.24})$$

$$\begin{aligned} \text{Li}_4\left(\frac{2}{3}\right) &= -\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + \text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \zeta_3 \ln 2 - \zeta_3 \ln 3 + \frac{1}{12} \ln^4 2 \\ &\quad - \frac{1}{12} \ln^4 3 - \frac{1}{6} \ln 3 \ln^3 2 + \frac{1}{6} \ln^3 3 \ln 2 + \frac{1}{12} \pi^2 \ln^2 2 + \frac{1}{12} \pi^2 \ln^2 3 - \frac{1}{6} \pi^2 \ln 3 \ln 2 \\ &\quad - \text{Li}_4\left(\frac{1}{3}\right) - \text{Li}_4\left(-\frac{1}{2}\right) + S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{90} \end{aligned} \quad (\text{E.25})$$

$$\begin{aligned} \text{Li}_4\left(\frac{3}{4}\right) &= 4\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 2\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - \frac{19}{3} \zeta_3 \ln 2 + \frac{19}{6} \zeta_3 \ln 3 - \frac{4}{3} \ln^4 2 \\ &\quad + \frac{1}{4} \ln^4 3 + \frac{4}{3} \ln 3 \ln^3 2 - \frac{2}{3} \ln^3 3 \ln 2 + \frac{1}{3} \pi^2 \ln^2 2 - \frac{1}{12} \pi^2 \ln^2 3 - 8\text{Li}_4\left(\frac{1}{2}\right) \\ &\quad - \text{Li}_4\left(-\frac{1}{3}\right) - 8\text{Li}_4\left(-\frac{1}{2}\right) + S_{2,2}\left(-\frac{1}{3}\right) + \frac{\pi^4}{90} \end{aligned} \quad (\text{E.26})$$

$$\text{Li}_4\left(\frac{8}{9}\right) = -36\text{Li}_3\left(\frac{1}{3}\right) \ln 2 + 18\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 57\zeta_3 \ln 2 - \frac{57}{2} \zeta_3 \ln 3 + \frac{16}{3} \ln^4 2 \quad (\text{E.27})$$

$$\begin{aligned}
 & -\frac{77}{24}\ln^4 3 + 10\ln^3 3\ln 2 - 9\ln^2 3\ln^2 2 - \frac{5}{6}\pi^2\ln^2 2 + \frac{11}{6}\pi^2\ln^2 3 - 3\pi^2\ln 3\ln 2 \\
 & + 74\text{Li}_4\left(\frac{1}{2}\right) - 8\text{Li}_4\left(\frac{1}{3}\right) + 10\text{Li}_4\left(-\frac{1}{3}\right) + 54\text{Li}_4\left(-\frac{1}{2}\right) - 9S_{2,2}\left(-\frac{1}{3}\right) - \frac{11\pi^4}{120} \\
 S_{2,2}\left(-\frac{1}{8}\right) = & -36\text{Li}_3\left(\frac{1}{3}\right)\ln 2 + 54\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 + 18\text{Li}_3\left(\frac{1}{3}\right)\ln 3 \tag{E.28}
 \end{aligned}$$

$$\begin{aligned}
 & -36\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{363}{4}\zeta_3\ln 2 - 51\zeta_3\ln 3 + \frac{89}{24}\ln^4 2 - \frac{15}{8}\ln^4 3 + 6\ln 3\ln^3 2 \\
 & + 6\ln^3 3\ln 2 - 9\ln^2 3\ln^2 2 - \frac{29}{6}\pi^2\ln^2 2 + \frac{3}{2}\pi^2\ln^2 3 + 116\text{Li}_4\left(\frac{1}{2}\right) + 18\text{Li}_4\left(-\frac{1}{3}\right) \\
 & + 108\text{Li}_4\left(-\frac{1}{2}\right) - 9S_{2,2}\left(-\frac{1}{3}\right) - \frac{17\pi^4}{360}
 \end{aligned}$$

$$\begin{aligned}
 S_{2,2}\left(\frac{1}{3}\right) = & -\text{Li}_3\left(\frac{1}{3}\right)\ln 2 + \text{Li}_3\left(-\frac{1}{2}\right)\ln 2 + \text{Li}_3\left(\frac{1}{3}\right)\ln 3 - \text{Li}_3\left(-\frac{1}{2}\right)\ln 3 \tag{E.29} \\
 & -\frac{1}{24}\ln^4 2 - \frac{1}{24}\ln^4 3 + \frac{1}{6}\ln 3\ln^3 2 + \frac{1}{6}\ln^3 3\ln 2 - \frac{1}{4}\ln^2 3\ln^2 2 + 2\text{Li}_4\left(\frac{1}{3}\right) \\
 & + 2\text{Li}_4\left(-\frac{1}{2}\right) - S_{2,2}\left(-\frac{1}{2}\right)
 \end{aligned}$$

$$S_{2,2}\left(\frac{1}{2}\right) = -\frac{1}{8}\zeta_3\ln 2 + \frac{1}{24}\ln^4 2 + \frac{\pi^4}{720} \tag{E.30}$$

$$\begin{aligned}
 S_{2,2}\left(\frac{2}{3}\right) = & -\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - \text{Li}_3\left(\frac{1}{3}\right)\ln 3 + \zeta_3\ln 2 - \zeta_3\ln 3 + \frac{1}{24}\ln^4 2 + \frac{1}{8}\ln^4 3 \tag{E.31} \\
 & -\frac{1}{6}\ln^3 3\ln 2 - 2\text{Li}_4\left(\frac{1}{3}\right) - 2\text{Li}_4\left(-\frac{1}{2}\right) + S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{360}
 \end{aligned}$$

$$\begin{aligned}
 S_{2,2}\left(\frac{3}{4}\right) = & -8\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - 2\text{Li}_3\left(\frac{1}{3}\right)\ln 3 - 9\zeta_3\ln 2 + \frac{19}{6}\zeta_3\ln 3 + \frac{2}{3}\ln^4 2 \tag{E.32} \\
 & + \frac{5}{24}\ln^4 3 - \frac{4}{3}\ln 3\ln^3 2 + \frac{2}{3}\pi^2\ln^2 2 - \frac{1}{6}\pi^2\ln^2 3 - 16\text{Li}_4\left(\frac{1}{2}\right) - 2\text{Li}_4\left(-\frac{1}{3}\right) \\
 & - 16\text{Li}_4\left(-\frac{1}{2}\right) + S_{2,2}\left(-\frac{1}{3}\right) + \frac{\pi^4}{360}
 \end{aligned}$$

$$\begin{aligned}
 S_{2,2}\left(\frac{8}{9}\right) = & -36\text{Li}_3\left(\frac{1}{3}\right)\ln 2 - 6\text{Li}_3\left(\frac{1}{3}\right)\ln 3 - 36\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + 57\zeta_3\ln 2 \tag{E.33} \\
 & -\frac{107}{3}\zeta_3\ln 3 + \frac{4}{3}\ln^4 2 + \frac{35}{24}\ln^4 3 + 6\ln 3\ln^3 2 + 2\ln^3 3\ln 2 - 9\ln^2 3\ln^2 2 - \frac{4}{3}\pi^2\ln^2 2 \\
 & + \frac{1}{6}\pi^2\ln^2 3 + 32\text{Li}_4\left(\frac{1}{2}\right) - 16\text{Li}_4\left(\frac{1}{3}\right) + 2\text{Li}_4\left(-\frac{1}{3}\right) - 9S_{2,2}\left(-\frac{1}{3}\right) - \frac{7\pi^4}{45}
 \end{aligned}$$

$$\begin{aligned}
 H\left(-1, 1, -1, 1; \frac{1}{2}\right) = & -\text{Li}_2\left(\frac{1}{3}\right)\ln^2 2 + 2\text{Li}_2\left(\frac{1}{3}\right)\ln^2 3 - 4\text{Li}_3\left(\frac{1}{3}\right)\ln 2 \tag{E.34} \\
 & + 8\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - 2\text{Li}_2\left(\frac{1}{3}\right)\ln 3\ln 2 + 4\text{Li}_3\left(\frac{1}{3}\right)\ln 3 + \frac{181}{12}\zeta_3\ln 2 - \frac{73}{12}\zeta_3\ln 3 \\
 & + \frac{11}{8}\ln^4 2 + \frac{1}{12}\ln^4 3 + \frac{5}{6}\ln 3\ln^3 2 - \frac{1}{3}\ln^3 3\ln 2 - \frac{1}{2}\ln^2 3\ln^2 2 - \frac{23}{24}\pi^2\ln^2 2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4}\pi^2 \ln^2 3 - \frac{1}{12}\pi^2 \ln 3 \ln 2 + 32\text{Li}_4\left(\frac{1}{2}\right) + 4\text{Li}_4\left(-\frac{1}{3}\right) + 32\text{Li}_4\left(-\frac{1}{2}\right) + 2\text{Li}_2\left(\frac{1}{3}\right)^2 \\
& - \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{6} - 2S_{2,2}\left(-\frac{1}{3}\right) + \frac{\pi^4}{1440} \\
H\left(0, -1, -1, 1; \frac{1}{2}\right) &= -\frac{1}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 + \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 9\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \tag{E.35} \\
& - 7\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 - 4\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 8\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{115}{24}\zeta_3 \ln 2 \\
& - \frac{5}{3}\zeta_3 \ln 3 - \frac{1}{2}\ln^4 2 + \frac{5}{12}\ln^4 3 - \frac{5}{6}\ln 3 \ln^3 2 - 2\ln^3 3 \ln 2 + \frac{9}{4}\ln^2 3 \ln^2 2 + \frac{7}{24}\pi^2 \ln^2 2 \\
& + \frac{1}{8}\pi^2 \ln^2 3 + \frac{1}{12}\pi^2 \ln 3 \ln 2 - 9\text{Li}_4\left(\frac{1}{2}\right) - 12\text{Li}_4\left(\frac{1}{3}\right) + 2\text{Li}_4\left(-\frac{1}{3}\right) - 22\text{Li}_4\left(-\frac{1}{2}\right) \\
& + \text{Li}_2\left(\frac{1}{3}\right)^2 - \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{12} + 2S_{2,2}\left(-\frac{1}{3}\right) + 8S_{2,2}\left(-\frac{1}{2}\right) + \frac{7\pi^4}{240} \\
H\left(0, -1, 1, -1; \frac{1}{2}\right) &= -\frac{1}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 12\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \tag{E.36} \\
& + 12\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + 2\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + 6\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 12\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 \\
& + \frac{95}{8}\zeta_3 \ln 2 - \frac{19}{8}\zeta_3 \ln 3 + \frac{7}{12}\ln^4 2 - \frac{19}{32}\ln^4 3 + \frac{5}{2}\ln 3 \ln^3 2 + 3\ln^3 3 \ln 2 - \frac{17}{4}\ln^2 3 \ln^2 2 \\
& - \frac{17}{24}\pi^2 \ln^2 2 - \frac{1}{48}\pi^2 \ln^2 3 - \frac{1}{12}\pi^2 \ln 3 \ln 2 + 20\text{Li}_4\left(\frac{1}{2}\right) + \frac{27\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{3\text{Li}_4\left(-\frac{1}{3}\right)}{4} \\
& + 35\text{Li}_4\left(-\frac{1}{2}\right) - \text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{12} - 3S_{2,2}\left(-\frac{1}{3}\right) - 8S_{2,2}\left(-\frac{1}{2}\right) - \frac{7\pi^4}{240} \\
H\left(0, -1, 1, 1; \frac{1}{2}\right) &= \frac{1}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - 4\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{7}{2}\zeta_3 \ln 2 - \frac{1}{6}\ln^4 2 \tag{E.37} \\
& - \frac{1}{2}\ln 3 \ln^3 2 + \frac{1}{4}\ln^2 3 \ln^2 2 + \frac{1}{3}\pi^2 \ln^2 2 - 9\text{Li}_4\left(\frac{1}{2}\right) - 12\text{Li}_4\left(-\frac{1}{2}\right) - \frac{17\pi^4}{1440} \\
H\left(0, 0, -1, 1; \frac{1}{2}\right) &= -\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{3}{4}\zeta_3 \ln 2 - \frac{5}{24}\ln^4 2 + \frac{1}{8}\pi^2 \ln^2 2 \tag{E.38} \\
& - 2\text{Li}_4\left(\frac{1}{2}\right) - 6\text{Li}_4\left(-\frac{1}{2}\right) - \frac{31\pi^4}{1440} \\
H\left(0, 0, 1, -1; \frac{1}{2}\right) &= \text{Li}_3\left(\frac{1}{3}\right) \ln 2 - \text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{1}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 \tag{E.39} \\
& + \text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{53}{24}\zeta_3 \ln 2 + \frac{17}{12}\zeta_3 \ln 3 + \frac{1}{12}\ln^4 2 + \frac{5}{96}\ln^4 3 - \frac{1}{6}\ln 3 \ln^3 2 - \frac{1}{6}\ln^3 3 \ln 2 \\
& + \frac{1}{4}\ln^2 3 \ln^2 2 + \frac{1}{24}\pi^2 \ln^2 2 - \frac{1}{24}\pi^2 \ln^2 3 - 2\text{Li}_4\left(\frac{1}{2}\right) - \frac{\text{Li}_4\left(-\frac{1}{3}\right)}{2} + 2\text{Li}_4\left(-\frac{1}{2}\right) \\
& + \frac{S_{2,2}\left(-\frac{1}{3}\right)}{4} + S_{2,2}\left(-\frac{1}{2}\right) + \frac{11\pi^4}{480} \\
H\left(0, 1, -1, -1; \frac{1}{2}\right) &= 3\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 3\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{3}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 \tag{E.40}
\end{aligned}$$

$$\begin{aligned}
 & + 3\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 - \frac{107}{24}\zeta_3\ln 2 + \frac{25}{12}\zeta_3\ln 3 - \frac{1}{3}\ln^4 2 + \frac{11}{96}\ln^4 3 - \frac{1}{2}\ln 3\ln^3 2 - \frac{1}{2}\ln^3 3\ln 2 \\
 & + \frac{3}{4}\ln^2 3\ln^2 2 + \frac{7}{24}\pi^2\ln^2 2 - \frac{1}{24}\pi^2\ln^2 3 - 8\text{Li}_4\left(\frac{1}{2}\right) - 2\text{Li}_4\left(\frac{1}{3}\right) - \frac{\text{Li}_4\left(-\frac{1}{3}\right)}{2} \\
 & - 11\text{Li}_4\left(-\frac{1}{2}\right) + \frac{3S_{2,2}\left(-\frac{1}{3}\right)}{4} - \frac{\pi^4}{1440} \\
 H\left(0, 1, -1, 1; \frac{1}{2}\right) & = -7\text{Li}_3\left(\frac{1}{3}\right)\ln 2 + 10\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 + \frac{7}{2}\text{Li}_3\left(\frac{1}{3}\right)\ln 3 \tag{E.41} \\
 & - 7\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{77}{12}\zeta_3\ln 2 + \frac{35}{24}\zeta_3\ln 3 + \frac{17}{24}\ln^4 2 - \frac{7}{48}\ln^4 3 + \frac{7}{6}\ln 3\ln^3 2 + \frac{7}{6}\ln^3 3\ln 2 \\
 & - \frac{7}{4}\ln^2 3\ln^2 2 - \frac{7}{12}\pi^2\ln^2 2 - \frac{7}{48}\pi^2\ln^2 3 + 17\text{Li}_4\left(\frac{1}{2}\right) + \frac{21\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{7\text{Li}_4\left(-\frac{1}{3}\right)}{4} \\
 & + 32\text{Li}_4\left(-\frac{1}{2}\right) - \frac{7S_{2,2}\left(-\frac{1}{3}\right)}{4} - 7S_{2,2}\left(-\frac{1}{2}\right) - \frac{\pi^4}{240}
 \end{aligned}$$

$$\begin{aligned}
 H\left(0, 1, 0, -1; \frac{1}{2}\right) & = 2\text{Li}_3\left(\frac{1}{3}\right)\ln 2 - 2\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - \text{Li}_3\left(\frac{1}{3}\right)\ln 3 \tag{E.42} \\
 & + 2\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{25}{12}\zeta_3\ln 2 - \frac{11}{3}\zeta_3\ln 3 + \frac{5}{24}\ln^4 2 - \frac{1}{48}\ln^4 3 - \frac{1}{3}\ln 3\ln^3 2 - \frac{1}{3}\ln^3 3\ln 2 \\
 & + \frac{1}{2}\ln^2 3\ln^2 2 - \frac{1}{8}\pi^2\ln^2 2 + \frac{1}{6}\pi^2\ln^2 3 + 3\text{Li}_4\left(\frac{1}{2}\right) - 6\text{Li}_4\left(\frac{1}{3}\right) + 2\text{Li}_4\left(-\frac{1}{3}\right) \\
 & - 3\text{Li}_4\left(-\frac{1}{2}\right) + \frac{S_{2,2}\left(-\frac{1}{3}\right)}{2} + 2S_{2,2}\left(-\frac{1}{2}\right) + \frac{3\pi^4}{160}
 \end{aligned}$$

$$\begin{aligned}
 H\left(0, 1, 1, -1; \frac{1}{2}\right) & = 8\text{Li}_3\left(\frac{1}{3}\right)\ln 2 - 8\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - 4\text{Li}_3\left(\frac{1}{3}\right)\ln 3 \tag{E.43} \\
 & + 8\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 - \frac{14}{3}\zeta_3\ln 2 - \frac{5}{3}\zeta_3\ln 3 - \frac{1}{3}\ln^4 2 + \frac{1}{6}\ln^4 3 - \frac{4}{3}\ln 3\ln^3 2 - \frac{4}{3}\ln^3 3\ln 2 \\
 & + 2\ln^2 3\ln^2 2 + \frac{1}{3}\pi^2\ln^2 2 + \frac{1}{6}\pi^2\ln^2 3 - 10\text{Li}_4\left(\frac{1}{2}\right) - 12\text{Li}_4\left(\frac{1}{3}\right) + 2\text{Li}_4\left(-\frac{1}{3}\right) \\
 & - 23\text{Li}_4\left(-\frac{1}{2}\right) + 2S_{2,2}\left(-\frac{1}{3}\right) + 8S_{2,2}\left(-\frac{1}{2}\right) + \frac{19\pi^4}{720}
 \end{aligned}$$

E.4 Polylogarithms of complex arguments

$$\text{Li}_3\left(\frac{1}{2} + \frac{i}{2}\right) + \text{Li}_3\left(\frac{1}{2} - \frac{i}{2}\right) = \frac{1}{24}\ln^3 2 - \frac{5}{96}\pi^2\ln 2 + \frac{35}{32}\zeta_3 \tag{E.44}$$

$$\text{Li}_4\left(\frac{1}{2} + \frac{i}{2}\right) + \text{Li}_4\left(\frac{1}{2} - \frac{i}{2}\right) = \frac{1}{48}\ln^4 2 - \frac{5}{384}\pi^2\ln^2 2 + \frac{5\text{Li}_4\left(\frac{1}{2}\right)}{8} + \frac{343\pi^4}{46080} \tag{E.45}$$

F Special values of Goncharov multiple polylogarithms

In this appendix we present special values of Goncharov multiple polylogarithms that we encountered throughout our computation. The results are expressed in the same transcendental basis as in appendix E. All the expressions have been derived using the reduction

algorithm of appendix B. Note that we present here only values for polylogarithms of the form $G(\vec{w}; 1)$. It is very easy to extend this list to other types of polylogarithms by applying eq. (B.7).

F.1 Polylogarithms of weight two

$$G\left(\frac{4}{3}, 1; 1\right) = \ln^2 3 + 2\text{Li}_2\left(\frac{1}{3}\right) \tag{F.1}$$

$$G\left(\frac{3}{2}, 1; 1\right) = -\frac{1}{2}\ln^2 3 + \ln 2 \ln 3 - \text{Li}_2\left(\frac{1}{3}\right) + \frac{\pi^2}{6} \tag{F.2}$$

$$G(2, 1; 1) = \frac{\pi^2}{12} \tag{F.3}$$

F.2 Polylogarithms of weight three

$$G\left(-\frac{1}{2}, -\frac{1}{3}, -1; 1\right) = 4\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \text{Li}_2\left(\frac{1}{3}\right) \ln 3 + \frac{1}{2}\ln^3 2 - \frac{7}{6}\ln^3 3 - \ln 3 \ln^2 2 \tag{F.4}$$

$$+ 2\ln^2 3 \ln 2 - \frac{1}{6}\pi^2 \ln 2 + \frac{2}{3}\pi^2 \ln 3 + 10\text{Li}_3\left(\frac{1}{3}\right) - 3\text{Li}_3\left(-\frac{1}{2}\right) - 10\zeta_3$$

$$G\left(-\frac{1}{2}, -\frac{1}{3}, 0; 1\right) = \text{Li}_2\left(\frac{1}{3}\right) \ln 3 + \frac{1}{6}\ln^3 2 - \frac{1}{4}\ln^3 3 + \frac{1}{6}\pi^2 \ln 2 - \frac{7}{12}\pi^2 \ln 3 \tag{F.5}$$

$$- \text{Li}_3\left(\frac{1}{3}\right) + \frac{11\text{Li}_3\left(-\frac{1}{3}\right)}{2} - \text{Li}_3\left(-\frac{1}{2}\right) + \frac{17\zeta_3}{4}$$

$$G\left(-\frac{1}{2}, 0, -1; 1\right) = \text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \frac{1}{2}\ln^3 2 - \frac{1}{24}\ln^3 3 - \ln 3 \ln^2 2 + \frac{1}{2}\ln^2 3 \ln 2 \tag{F.6}$$

$$+ \frac{1}{12}\pi^2 \ln 2 + \frac{1}{8}\pi^2 \ln 3 + \frac{\text{Li}_3\left(\frac{1}{3}\right)}{2} - \frac{\text{Li}_3\left(-\frac{1}{3}\right)}{4} - 3\text{Li}_3\left(-\frac{1}{2}\right) - \frac{23\zeta_3}{8}$$

$$G\left(-\frac{1}{3}, -\frac{1}{2}, -1; 1\right) = -8\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - \frac{19}{6}\ln^3 2 + \frac{1}{3}\ln^3 3 + \frac{13}{2}\ln 3 \ln^2 2 \tag{F.7}$$

$$- 4\ln^2 3 \ln 2 - \frac{7}{12}\pi^2 \ln 2 - \frac{1}{6}\pi^2 \ln 3 - 2\text{Li}_3\left(\frac{1}{3}\right) + 19\text{Li}_3\left(-\frac{1}{2}\right) + \frac{121\zeta_3}{8}$$

$$G\left(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}; 1\right) = -14\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - \frac{17}{3}\ln^3 2 + 14\ln 3 \ln^2 2 - 7\ln^2 3 \ln 2 \tag{F.8}$$

$$- \frac{4}{3}\pi^2 \ln 2 + 34\text{Li}_3\left(-\frac{1}{2}\right) + \frac{49\zeta_3}{2}$$

$$G\left(-\frac{1}{3}, -\frac{1}{2}, 1; 1\right) = -10\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - 3\ln^3 2 - \frac{1}{6}\ln^3 3 + 6\ln 3 \ln^2 2 - \frac{7}{2}\ln^2 3 \ln 2 \tag{F.9}$$

$$- \frac{5}{6}\pi^2 \ln 2 + \frac{1}{3}\pi^2 \ln 3 + \text{Li}_3\left(\frac{1}{3}\right) + 18\text{Li}_3\left(-\frac{1}{2}\right) + \frac{45\zeta_3}{4}$$

$$G\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}; 1\right) = 2\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \frac{17}{6}\ln^3 2 - 4\ln 3 \ln^2 2 + \ln^2 3 \ln 2 + \pi^2 \ln 2 \tag{F.10}$$

$$- 17\text{Li}_3\left(-\frac{1}{2}\right) - \frac{49\zeta_3}{4}$$

$$G\left(-\frac{1}{3}, 1, -\frac{1}{2}; 1\right) = 10\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - 6\text{Li}_2\left(\frac{1}{3}\right) \ln 3 + \frac{25}{6} \ln^3 2 - \frac{2}{3} \ln^3 3 \quad (\text{F.11})$$

$$- 8 \ln 3 \ln^2 2 + 5 \ln^2 3 \ln 2 + \pi^2 \ln 2 - \frac{2}{3} \pi^2 \ln 3 - 14\text{Li}_3\left(\frac{1}{3}\right) - 25\text{Li}_3\left(-\frac{1}{2}\right) - \frac{77\zeta_3}{12}$$

$$G(0, -2, -8; 1) = -9\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + 6\text{Li}_2\left(\frac{1}{3}\right) \ln 3 - \frac{11}{6} \ln^3 2 + \frac{7}{6} \ln^3 3 \quad (\text{F.12})$$

$$+ 5 \ln 3 \ln^2 2 - 5 \ln^2 3 \ln 2 + \frac{1}{6} \pi^2 \ln 2 + \frac{1}{3} \pi^2 \ln 3 + 11\text{Li}_3\left(\frac{1}{3}\right) + 4\text{Li}_3\left(-\frac{1}{2}\right) - \frac{31\zeta_3}{6}$$

$$G(0, -2, 1; 1) = -\frac{1}{6} \ln^3 2 + \frac{1}{24} \ln^3 3 + \frac{1}{2} \ln 3 \ln^2 2 - \frac{1}{2} \ln^2 3 \ln 2 - \frac{1}{6} \pi^2 \ln 2 \quad (\text{F.13})$$

$$+ \frac{7}{24} \pi^2 \ln 3 + \frac{\text{Li}_3\left(\frac{1}{3}\right)}{2} - \frac{3\text{Li}_3\left(-\frac{1}{3}\right)}{4} + \text{Li}_3\left(-\frac{1}{2}\right) - \frac{13\zeta_3}{8}$$

$$G(0, 1, 2; 1) = \frac{1}{6} \pi^2 \ln 2 - \frac{5\zeta_3}{8} \quad (\text{F.14})$$

$$G(0, 2, 1; 1) = \frac{13\zeta_3}{8} - \frac{1}{4} \pi^2 \ln 2 \quad (\text{F.15})$$

$$G(0, 3, -1; 1) = \text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \frac{1}{2} \ln^3 2 - \frac{5}{24} \ln^3 3 - \frac{3}{2} \ln 3 \ln^2 2 + \frac{3}{2} \ln^2 3 \ln 2 \quad (\text{F.16})$$

$$- \frac{1}{6} \pi^2 \ln 2 + \frac{1}{24} \pi^2 \ln 3 + \frac{\text{Li}_3\left(\frac{1}{3}\right)}{2} + \frac{3\text{Li}_3\left(-\frac{1}{3}\right)}{4} - 3\text{Li}_3\left(-\frac{1}{2}\right) - \frac{9\zeta_3}{8}$$

$$G(0, 4, -2; 1) = -\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \text{Li}_2\left(\frac{1}{3}\right) \ln 3 - \frac{5}{6} \ln^3 2 + \frac{1}{2} \ln^3 3 + \frac{5}{2} \ln 3 \ln^2 2 \quad (\text{F.17})$$

$$- \frac{3}{2} \ln^2 3 \ln 2 - \frac{1}{6} \pi^2 \ln 2 - \frac{1}{6} \pi^2 \ln 3 + 4\text{Li}_3\left(-\frac{1}{2}\right) + \frac{7\zeta_3}{2}$$

$$G\left(\frac{4}{3}, 1, 4; 1\right) = 6\text{Li}_2\left(\frac{1}{3}\right) \ln 3 - 4\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \frac{2}{3} \ln^3 3 - 2 \ln 2 \ln^2 3 + \frac{2}{3} \pi^2 \ln 3 \quad (\text{F.18})$$

$$+ 14\text{Li}_3\left(\frac{1}{3}\right) - \frac{65\zeta_3}{6}$$

$$G\left(\frac{4}{3}, \frac{4}{3}, 1; 1\right) = -4\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - \frac{2}{3} \ln^3 2 + 2 \ln 3 \ln^2 2 - 2 \ln^2 3 \ln 2 + 4\text{Li}_3\left(-\frac{1}{2}\right) \quad (\text{F.19})$$

$$+ \frac{5\zeta_3}{2}$$

$$G\left(\frac{4}{3}, \frac{3}{2}, 1; 1\right) = 4\text{Li}_2\left(\frac{1}{3}\right) \ln 2 + \frac{5}{3} \ln^3 2 - \frac{5}{12} \ln^3 3 - 4 \ln 3 \ln^2 2 + 2 \ln^2 3 \ln 2 \quad (\text{F.20})$$

$$+ \frac{1}{2} \pi^2 \ln 2 + \frac{1}{12} \pi^2 \ln 3 + 3\text{Li}_3\left(\frac{1}{3}\right) - \frac{\text{Li}_3\left(-\frac{1}{3}\right)}{2} - 10\text{Li}_3\left(-\frac{1}{2}\right) - \frac{19\zeta_3}{2}$$

$$G\left(\frac{4}{3}, 2, 1; 1\right) = -2\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - \frac{2}{3} \ln^3 2 - \frac{1}{3} \ln^3 3 + 2 \ln 3 \ln^2 2 - \ln^2 3 \ln 2 \quad (\text{F.21})$$

$$- \frac{1}{12} \pi^2 \ln 2 + 2\text{Li}_3\left(\frac{1}{3}\right) + 4\text{Li}_3\left(-\frac{1}{2}\right) + \frac{47\zeta_3}{24}$$

$$G(2, 0, 1; 1) = \frac{1}{4} \pi^2 \ln 2 - \zeta_3 \quad (\text{F.22})$$

$$G(2, 1, 0; 1) = -\frac{5\zeta_3}{8} \tag{F.23}$$

$$G(2, 1, 1; 1) = -\frac{3\zeta_3}{4} \tag{F.24}$$

$$G(2, 1, 2; 1) = \frac{\zeta_3}{4} - \frac{1}{12}\pi^2 \ln 2 \tag{F.25}$$

$$G(2, 2, 1; 1) = -\frac{\zeta_3}{8} \tag{F.26}$$

$$G(3, 0, -1; 1) = \text{Li}_2\left(\frac{1}{3}\right) \ln 2 - 2\text{Li}_2\left(\frac{1}{3}\right) \ln 3 + \frac{1}{6} \ln^3 2 - \frac{5}{8} \ln^3 3 - \frac{1}{2} \ln 3 \ln^2 2 \tag{F.27}$$

$$+ \frac{1}{2} \ln^2 3 \ln 2 + \frac{1}{12} \pi^2 \ln 2 + \frac{1}{8} \pi^2 \ln 3 - \frac{\text{Li}_3\left(\frac{1}{3}\right)}{2} - \frac{7\text{Li}_3\left(-\frac{1}{3}\right)}{4} - \text{Li}_3\left(-\frac{1}{2}\right) - \frac{3\zeta_3}{2}$$

$$G(4, 0, -2; 1) = 3\text{Li}_2\left(\frac{1}{3}\right) \ln 2 - 2\text{Li}_2\left(\frac{1}{3}\right) \ln 3 + \frac{5}{6} \ln^3 2 - \frac{19}{24} \ln^3 3 - \frac{5}{2} \ln 3 \ln^2 2 \tag{F.28}$$

$$+ 2 \ln^2 3 \ln 2 + \frac{1}{12} \pi^2 \ln 2 + \frac{7}{24} \pi^2 \ln 3 + \frac{\text{Li}_3\left(\frac{1}{3}\right)}{2} - \frac{7\text{Li}_3\left(-\frac{1}{3}\right)}{4} - 6\text{Li}_3\left(-\frac{1}{2}\right) - \frac{23\zeta_3}{4}$$

F.3 Polylogarithms of weight four

$$G\left(-\frac{1}{2}, 0, -1, -1; 1\right) = \frac{1}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 + 8\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 11\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \tag{F.29}$$

$$- 4\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 8\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{35}{8}\zeta_3 \ln 2 - \frac{37}{24}\zeta_3 \ln 3 + \frac{5}{24} \ln^4 2 + \frac{1}{6} \ln^4 3$$

$$- \frac{11}{6} \ln 3 \ln^3 2 - \frac{4}{3} \ln^3 3 \ln 2 + \frac{9}{4} \ln^2 3 \ln^2 2 + \frac{7}{24} \pi^2 \ln^2 2 + \frac{1}{6} \pi^2 \ln^2 3 - 8\text{Li}_4\left(\frac{1}{2}\right)$$

$$- 12\text{Li}_4\left(\frac{1}{3}\right) + 2\text{Li}_4\left(-\frac{1}{3}\right) - 23\text{Li}_4\left(-\frac{1}{2}\right) + 2S_{2,2}\left(-\frac{1}{3}\right) + 8S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{480}$$

$$G\left(-\frac{1}{2}, 0, -\frac{1}{2}, -1; 1\right) = \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 17\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 19\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \tag{F.30}$$

$$- \text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 - \frac{15}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 19\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{13}{4}\zeta_3 \ln 2 - \frac{31}{4}\zeta_3 \ln 3$$

$$+ \frac{13}{24} \ln^4 2 + \frac{13}{32} \ln^4 3 - \frac{19}{6} \ln 3 \ln^3 2 - \frac{10}{3} \ln^3 3 \ln 2 + \frac{19}{4} \ln^2 3 \ln^2 2 + \frac{1}{4} \pi^2 \ln^2 2$$

$$+ \frac{7}{12} \pi^2 \ln^2 3 - \frac{1}{6} \pi^2 \ln 3 \ln 2 - 6\text{Li}_4\left(\frac{1}{2}\right) - 30\text{Li}_4\left(\frac{1}{3}\right) + \frac{15\text{Li}_4\left(-\frac{1}{3}\right)}{2} - 38\text{Li}_4\left(-\frac{1}{2}\right)$$

$$+ \text{Li}_2\left(\frac{1}{3}\right)^2 - \frac{\pi^2 \text{Li}_2\left(\frac{1}{3}\right)}{12} + \frac{15S_{2,2}\left(-\frac{1}{3}\right)}{4} + 19S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{16}$$

$$G\left(-\frac{1}{3}, -1, -\frac{1}{2}, 0; 1\right) = -\frac{7}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 15\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 19\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \tag{F.31}$$

$$+ 3\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 - \frac{21}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 11\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{293}{12}\zeta_3 \ln 2 + \frac{45}{8}\zeta_3 \ln 3$$

$$- \frac{31}{12} \ln^4 2 - \frac{1}{6} \ln^4 3 - \frac{11}{6} \ln 3 \ln^3 2 - \ln^3 3 \ln 2 + \frac{11}{4} \ln^2 3 \ln^2 2 + \frac{23}{24} \pi^2 \ln^2 2 - \frac{1}{48} \pi^2 \ln^2 3$$

$$\begin{aligned}
 & + \frac{1}{3}\pi^2 \ln 3 \ln 2 - 57\text{Li}_4\left(\frac{1}{2}\right) - \frac{33\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{13\text{Li}_4\left(-\frac{1}{3}\right)}{4} - 70\text{Li}_4\left(-\frac{1}{2}\right) - \frac{7\text{Li}_2\left(\frac{1}{3}\right)^2}{2} \\
 & + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{2} + \frac{21S_{2,2}\left(-\frac{1}{3}\right)}{4} + 11S_{2,2}\left(-\frac{1}{2}\right) + \frac{41\pi^4}{720} \\
 G\left(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}, -1; 1\right) & = -12\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 + \frac{11}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 \tag{F.32} \\
 & - 123\text{Li}_3\left(\frac{1}{3}\right) \ln 2 + 195\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + 3\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + \frac{123}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 \\
 & - 157\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \frac{249}{4}\zeta_3 \ln 2 + \frac{477}{8}\zeta_3 \ln 3 - \frac{73}{12} \ln^4 2 - \frac{1}{16} \ln^4 3 + \frac{229}{6} \ln 3 \ln^3 2 \\
 & + 22 \ln^3 3 \ln 2 - \frac{181}{4} \ln^2 3 \ln^2 2 - 5\pi^2 \ln^2 2 - \frac{85}{16}\pi^2 \ln^2 3 + \frac{7}{3}\pi^2 \ln 3 \ln 2 + 112\text{Li}_4\left(\frac{1}{2}\right) \\
 & + \frac{477\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{231\text{Li}_4\left(-\frac{1}{3}\right)}{4} + 365\text{Li}_4\left(-\frac{1}{2}\right) + \frac{11\text{Li}_2\left(\frac{1}{3}\right)^2}{2} - \pi^2\text{Li}_2\left(\frac{1}{3}\right) - \frac{123S_{2,2}\left(-\frac{1}{3}\right)}{4} \\
 & - 157S_{2,2}\left(-\frac{1}{2}\right) - \frac{397\pi^4}{720}
 \end{aligned}$$

$$\begin{aligned}
 G\left(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}, 0; 1\right) & = \frac{9}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 30\text{Li}_3\left(\frac{1}{3}\right) \ln 2 + 64\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \tag{F.33} \\
 & + 3\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + 14\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 62\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{47}{3}\zeta_3 \ln 2 + \frac{239}{6}\zeta_3 \ln 3 \\
 & - \frac{37}{24} \ln^4 2 + \frac{5}{3} \ln^4 3 + \frac{31}{3} \ln 3 \ln^3 2 + \frac{13}{2} \ln^3 3 \ln 2 - \frac{31}{2} \ln^2 3 \ln^2 2 - \frac{1}{12}\pi^2 \ln^2 2 - \frac{8}{3}\pi^2 \ln^2 3 \\
 & + \pi^2 \ln 3 \ln 2 + 18\text{Li}_4\left(\frac{1}{2}\right) + 96\text{Li}_4\left(\frac{1}{3}\right) - 34\text{Li}_4\left(-\frac{1}{3}\right) + 139\text{Li}_4\left(-\frac{1}{2}\right) + \frac{9\text{Li}_2\left(\frac{1}{3}\right)^2}{2} \\
 & + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{3} - 7S_{2,2}\left(-\frac{1}{3}\right) - 62S_{2,2}\left(-\frac{1}{2}\right) - \frac{61\pi^4}{360}
 \end{aligned}$$

$$\begin{aligned}
 G\left(-\frac{1}{3}, -\frac{1}{2}, 1, 1; 1\right) & = -12\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 + 6\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 15\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \tag{F.34} \\
 & + 15\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + \frac{7}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 19\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \frac{149}{4}\zeta_3 \ln 2 - \frac{589}{24}\zeta_3 \ln 3 \\
 & - \frac{5}{12} \ln^4 2 + \frac{5}{12} \ln^4 3 + \frac{37}{6} \ln 3 \ln^3 2 - 2 \ln^3 3 \ln 2 - \frac{11}{4} \ln^2 3 \ln^2 2 - \frac{3}{2}\pi^2 \ln^2 2 + \frac{71}{48}\pi^2 \ln^2 3 \\
 & - \frac{5}{6}\pi^2 \ln 3 \ln 2 + 52\text{Li}_4\left(\frac{1}{2}\right) - \frac{59\text{Li}_4\left(\frac{1}{3}\right)}{2} + \frac{63\text{Li}_4\left(-\frac{1}{3}\right)}{4} + 17\text{Li}_4\left(-\frac{1}{2}\right) + 6\text{Li}_2\left(\frac{1}{3}\right)^2 \\
 & - \frac{7S_{2,2}\left(-\frac{1}{3}\right)}{4} + 19S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{40}
 \end{aligned}$$

$$\begin{aligned}
 G\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}, -1; 1\right) & = -\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - 2\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 25\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \tag{F.35} \\
 & - 55\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{31}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 31\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{123}{4}\zeta_3 \ln 2 - \frac{155}{24}\zeta_3 \ln 3 \\
 & - 3 \ln^4 2 + \frac{7}{48} \ln^4 3 - \frac{16}{3} \ln 3 \ln^3 2 - \frac{25}{6} \ln^3 3 \ln 2 + \frac{29}{4} \ln^2 3 \ln^2 2 + \frac{17}{6}\pi^2 \ln^2 2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{13}{16}\pi^2 \ln^2 3 - \frac{1}{2}\pi^2 \ln 3 \ln 2 - 101\text{Li}_4\left(\frac{1}{2}\right) - \frac{93\text{Li}_4\left(\frac{1}{3}\right)}{2} + \frac{31\text{Li}_4\left(-\frac{1}{3}\right)}{4} - 160\text{Li}_4\left(-\frac{1}{2}\right) \\
 & - 2\text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{3} + \frac{31S_{2,2}\left(-\frac{1}{3}\right)}{4} + 31S_{2,2}\left(-\frac{1}{2}\right) + \frac{13\pi^4}{180} \\
 G\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}, 1; 1\right) & = 2\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - 6\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 15\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \\
 & - 18\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - \frac{7}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 18\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{145}{4}\zeta_3 \ln 2 + \frac{565}{24}\zeta_3 \ln 3 \\
 & - \frac{25}{8} \ln^4 2 - \frac{13}{24} \ln^4 3 - \frac{5}{3} \ln 3 \ln^3 2 + \frac{5}{2} \ln^3 3 \ln 2 - \frac{5}{4} \ln^2 3 \ln^2 2 + \frac{13}{6} \pi^2 \ln^2 2 \\
 & - \frac{71}{48} \pi^2 \ln^2 3 + \pi^2 \ln 3 \ln 2 - 94\text{Li}_4\left(\frac{1}{2}\right) + \frac{57\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{71\text{Li}_4\left(-\frac{1}{3}\right)}{4} - 71\text{Li}_4\left(-\frac{1}{2}\right) \\
 & - 6\text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{7S_{2,2}\left(-\frac{1}{3}\right)}{4} - 18S_{2,2}\left(-\frac{1}{2}\right) - \frac{4\pi^4}{45}
 \end{aligned} \tag{F.36}$$

$$\begin{aligned}
 G\left(-\frac{1}{3}, 0, -3, 1; 1\right) & = 8\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 28\text{Li}_3\left(\frac{1}{3}\right) \ln 2 + 8\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \\
 & - 8\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + 26\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 8\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + 57\zeta_3 \ln 2 - \frac{199}{6}\zeta_3 \ln 3 + 3 \ln^4 2 \\
 & - \frac{5}{8} \ln^4 3 - \frac{4}{3} \ln 3 \ln^3 2 + \frac{2}{3} \ln^3 3 \ln 2 - \frac{4}{3} \pi^2 \ln^2 2 + \frac{3}{2} \pi^2 \ln^2 3 - \frac{4}{3} \pi^2 \ln 3 \ln 2 + 64\text{Li}_4\left(\frac{1}{2}\right) \\
 & + 6\text{Li}_4\left(\frac{1}{3}\right) + 22\text{Li}_4\left(-\frac{1}{3}\right) + 32\text{Li}_4\left(-\frac{1}{2}\right) + 8\text{Li}_2\left(\frac{1}{3}\right)^2 - \frac{2\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{3} - 13S_{2,2}\left(-\frac{1}{3}\right) \\
 & - 8S_{2,2}\left(-\frac{1}{2}\right) - \frac{9\pi^4}{40}
 \end{aligned} \tag{F.37}$$

$$\begin{aligned}
 G\left(-\frac{1}{3}, 0, -\frac{1}{2}, 1; 1\right) & = 4\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 - 5\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 29\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \\
 & + 20\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 + 3\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + \frac{9}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 40\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{93}{4}\zeta_3 \ln 2 \\
 & + \frac{263}{8}\zeta_3 \ln 3 - \frac{49}{24} \ln^4 2 - \frac{29}{48} \ln^4 3 + \frac{8}{3} \ln 3 \ln^3 2 + \frac{19}{3} \ln^3 3 \ln 2 - \frac{27}{4} \ln^2 3 \ln^2 2 + \frac{1}{2} \pi^2 \ln^2 2 \\
 & - \frac{101}{48} \pi^2 \ln^2 3 + \frac{4}{3} \pi^2 \ln 3 \ln 2 - 38\text{Li}_4\left(\frac{1}{2}\right) + \frac{129\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{97\text{Li}_4\left(-\frac{1}{3}\right)}{4} + 29\text{Li}_4\left(-\frac{1}{2}\right) \\
 & - \frac{7\text{Li}_2\left(\frac{1}{3}\right)^2}{2} + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{2} - \frac{9S_{2,2}\left(-\frac{1}{3}\right)}{4} - 40S_{2,2}\left(-\frac{1}{2}\right) - \frac{131\pi^4}{720}
 \end{aligned} \tag{F.38}$$

$$G(0, 0, 1, 2; 1) = \frac{1}{8}\zeta_3 \ln 2 - \frac{1}{24} \ln^4 2 + \frac{1}{24} \pi^2 \ln^2 2 - \text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{288} \tag{F.39}$$

$$G(0, 0, 2, 1; 1) = \frac{1}{8} \ln^4 2 + 3\text{Li}_4\left(\frac{1}{2}\right) - \frac{11\pi^4}{720} \tag{F.40}$$

$$G(0, 1, 0, 2; 1) = -\frac{1}{4}\zeta_3 \ln 2 - \frac{1}{24} \ln^4 2 - \frac{1}{24} \pi^2 \ln^2 2 - \text{Li}_4\left(\frac{1}{2}\right) + \frac{19\pi^4}{1440} \tag{F.41}$$

$$G(0, 1, 1, 2; 1) = \frac{3}{4}\zeta_3 \ln 2 + \frac{1}{12} \ln^4 2 - \frac{1}{12} \pi^2 \ln^2 2 + 2\text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{60} \quad (\text{F.42})$$

$$G(0, 1, 2, 0; 1) = \frac{1}{8} \ln^4 2 - \frac{1}{24} \pi^2 \ln^2 2 + 3\text{Li}_4\left(\frac{1}{2}\right) - \frac{29\pi^4}{1440} \quad (\text{F.43})$$

$$G(0, 1, 2, 1; 1) = -\frac{\pi^4}{480} \quad (\text{F.44})$$

$$G(0, 1, 2, 2; 1) = \frac{5}{8}\zeta_3 \ln 2 - \frac{1}{12} \pi^2 \ln^2 2 - \frac{\pi^4}{480} \quad (\text{F.45})$$

$$G(0, 2, 0, 1; 1) = -\frac{1}{8} \ln^4 2 - \frac{1}{8} \pi^2 \ln^2 2 - 3\text{Li}_4\left(\frac{1}{2}\right) + \frac{7\pi^4}{288} \quad (\text{F.46})$$

$$G(0, 2, 1, 0; 1) = -\frac{1}{8} \ln^4 2 + \frac{1}{8} \pi^2 \ln^2 2 - 3\text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{160} \quad (\text{F.47})$$

$$G(0, 2, 1, 1; 1) = -\frac{1}{12} \ln^4 2 + \frac{1}{12} \pi^2 \ln^2 2 - 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{180} \quad (\text{F.48})$$

$$G(0, 2, 1, 2; 1) = \zeta_3 \ln 2 + \frac{1}{6} \ln^4 2 + \frac{1}{12} \pi^2 \ln^2 2 + 4\text{Li}_4\left(\frac{1}{2}\right) - \frac{5\pi^4}{144} \quad (\text{F.49})$$

$$G(0, 2, 2, 1; 1) = -\frac{1}{12} \ln^4 2 - \frac{1}{24} \pi^2 \ln^2 2 - 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{80} \quad (\text{F.50})$$

$$G\left(\frac{4}{3}, \frac{3}{2}, \frac{3}{2}, 0; 1\right) = \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 30\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 42\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \quad (\text{F.51})$$

$$\begin{aligned} & - \text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 - 3\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 42\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \frac{79}{6}\zeta_3 \ln 2 - \frac{149}{4}\zeta_3 \ln 3 \\ & + \frac{23}{12} \ln^4 2 - \ln^4 3 - 7 \ln 3 \ln^3 2 - \frac{31}{6} \ln^3 3 \ln 2 + 10 \ln^2 3 \ln^2 2 - \frac{1}{6} \pi^2 \ln^2 2 + \frac{21}{8} \pi^2 \ln^2 3 \\ & - \frac{5}{6} \pi^2 \ln 3 \ln 2 + 4\text{Li}_4\left(\frac{1}{2}\right) - 68\text{Li}_4\left(\frac{1}{3}\right) + \frac{49\text{Li}_4\left(-\frac{1}{3}\right)}{2} - 84\text{Li}_4\left(-\frac{1}{2}\right) + \frac{\text{Li}_2\left(\frac{1}{3}\right)^2}{2} \\ & - \frac{\pi^2 \text{Li}_2\left(\frac{1}{3}\right)}{6} + \frac{3S_{2,2}\left(-\frac{1}{3}\right)}{2} + 42S_{2,2}\left(-\frac{1}{2}\right) + \frac{\pi^4}{15} \end{aligned}$$

$$G\left(\frac{4}{3}, \frac{3}{2}, \frac{3}{2}, 1; 1\right) = -2\text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 + 13\text{Li}_3\left(\frac{1}{3}\right) \ln 2 - 23\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 \quad (\text{F.52})$$

$$\begin{aligned} & - \frac{13}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 + 23\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 + \frac{29}{12}\zeta_3 \ln 2 - \frac{305}{24}\zeta_3 \ln 3 + \frac{17}{24} \ln^4 2 - \frac{1}{2} \ln^4 3 \\ & - \frac{23}{6} \ln 3 \ln^3 2 - \frac{13}{6} \ln^3 3 \ln 2 + \frac{21}{4} \ln^2 3 \ln^2 2 + \frac{1}{4} \pi^2 \ln^2 2 + \frac{17}{16} \pi^2 \ln^2 3 - \frac{2}{3} \pi^2 \ln 3 \ln 2 \\ & - 6\text{Li}_4\left(\frac{1}{2}\right) - \frac{69\text{Li}_4\left(\frac{1}{3}\right)}{2} + \frac{47\text{Li}_4\left(-\frac{1}{3}\right)}{4} - 46\text{Li}_4\left(-\frac{1}{2}\right) - 2\text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{\pi^2 \text{Li}_2\left(\frac{1}{3}\right)}{6} \\ & + \frac{13S_{2,2}\left(-\frac{1}{3}\right)}{4} + 23S_{2,2}\left(-\frac{1}{2}\right) + \frac{13\pi^4}{180} \end{aligned}$$

$$G\left(\frac{4}{3}, \frac{3}{2}, 2, 0; 1\right) = \frac{15}{2}\text{Li}_2\left(\frac{1}{3}\right) \ln^2 2 + \text{Li}_2\left(\frac{1}{3}\right) \ln^2 3 - 3\text{Li}_3\left(\frac{1}{3}\right) \ln 2 \quad (\text{F.53})$$

$$- 27\text{Li}_3\left(-\frac{1}{2}\right) \ln 2 - 2\text{Li}_2\left(\frac{1}{3}\right) \ln 3 \ln 2 + \frac{5}{2}\text{Li}_3\left(\frac{1}{3}\right) \ln 3 - 5\text{Li}_3\left(-\frac{1}{2}\right) \ln 3 - \frac{283}{12}\zeta_3 \ln 2$$

$$\begin{aligned}
 & + \frac{25}{24} \zeta_3 \ln 3 + \frac{3}{2} \ln^4 2 + \frac{7}{32} \ln^4 3 - 7 \ln 3 \ln^3 2 - \frac{1}{2} \ln^3 3 \ln 2 + \frac{7}{2} \ln^2 3 \ln^2 2 + \frac{4}{3} \pi^2 \ln^2 2 \\
 & - \frac{3}{8} \pi^2 \ln^2 3 + \frac{5}{6} \pi^2 \ln 3 \ln 2 - 38 \text{Li}_4 \left(\frac{1}{2} \right) + 9 \text{Li}_4 \left(\frac{1}{3} \right) - \text{Li}_4 \left(-\frac{1}{3} \right) - 39 \text{Li}_4 \left(-\frac{1}{2} \right) \\
 & + \text{Li}_2 \left(\frac{1}{3} \right)^2 - \frac{7\pi^2 \text{Li}_2 \left(\frac{1}{3} \right)}{12} - \frac{5S_{2,2} \left(-\frac{1}{3} \right)}{4} - 5S_{2,2} \left(-\frac{1}{2} \right) - \frac{13\pi^4}{480} \\
 G \left(\frac{4}{3}, 2, 0, 0; 1 \right) & = 2 \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 2 - \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 3 - 6 \text{Li}_3 \left(\frac{1}{3} \right) \ln 2 \tag{F.54} \\
 & + 6 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 2 + 2 \text{Li}_3 \left(\frac{1}{3} \right) \ln 3 - 8 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 3 + \frac{1}{3} \zeta_3 \ln 2 + \frac{29}{6} \zeta_3 \ln 3 + \frac{1}{24} \ln^4 2 \\
 & - \frac{1}{6} \ln^4 3 + \frac{2}{3} \ln 3 \ln^3 2 + \ln^3 3 \ln 2 - \ln^2 3 \ln^2 2 - \frac{1}{6} \pi^2 \ln^2 2 - \frac{1}{4} \pi^2 \ln^2 3 + 12 \text{Li}_4 \left(\frac{1}{3} \right) \\
 & - 3 \text{Li}_4 \left(-\frac{1}{3} \right) + 14 \text{Li}_4 \left(-\frac{1}{2} \right) - \text{Li}_2 \left(\frac{1}{3} \right)^2 - S_{2,2} \left(-\frac{1}{3} \right) - 8S_{2,2} \left(-\frac{1}{2} \right) - \frac{7\pi^4}{360} \\
 G \left(\frac{4}{3}, 2, \frac{4}{3}, 1; 1 \right) & = \frac{1}{2} \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 2 + 3 \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 3 + 9 \text{Li}_3 \left(\frac{1}{3} \right) \ln 2 \tag{F.55} \\
 & - 5 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 2 - \frac{3}{2} \text{Li}_3 \left(\frac{1}{3} \right) \ln 3 + 3 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 3 - \frac{29}{6} \zeta_3 \ln 2 - \frac{5}{8} \zeta_3 \ln 3 - \frac{1}{12} \ln^4 2 \\
 & + \frac{13}{16} \ln^4 3 - \frac{1}{3} \ln 3 \ln^3 2 - \frac{3}{2} \ln^3 3 \ln 2 + \ln^2 3 \ln^2 2 + \frac{1}{6} \pi^2 \ln^2 2 - \frac{1}{48} \pi^2 \ln^2 3 + \frac{1}{6} \pi^2 \ln 3 \ln 2 \\
 & - 6 \text{Li}_4 \left(\frac{1}{2} \right) - \frac{9 \text{Li}_4 \left(\frac{1}{3} \right)}{2} + \frac{3 \text{Li}_4 \left(-\frac{1}{3} \right)}{4} - 13 \text{Li}_4 \left(-\frac{1}{2} \right) + 3 \text{Li}_2 \left(\frac{1}{3} \right)^2 - \frac{\pi^2 \text{Li}_2 \left(\frac{1}{3} \right)}{6} \\
 & + \frac{3S_{2,2} \left(-\frac{1}{3} \right)}{4} + 3S_{2,2} \left(-\frac{1}{2} \right) - \frac{11\pi^4}{720} \\
 G \left(\frac{4}{3}, 2, \frac{3}{2}, 1; 1 \right) & = -\frac{1}{2} \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 2 + \frac{3}{2} \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 3 - 15 \text{Li}_3 \left(\frac{1}{3} \right) \ln 2 \tag{F.56} \\
 & + 27 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 2 - \text{Li}_2 \left(\frac{1}{3} \right) \ln 3 \ln 2 + \frac{21}{2} \text{Li}_3 \left(\frac{1}{3} \right) \ln 3 - 11 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 3 + \frac{377}{12} \zeta_3 \ln 2 \\
 & - \frac{45}{8} \zeta_3 \ln 3 + \frac{5}{4} \ln^4 2 - \frac{1}{3} \ln^4 3 + \frac{7}{3} \ln 3 \ln^3 2 + 2 \ln^3 3 \ln 2 - 3 \ln^2 3 \ln^2 2 - \frac{41}{24} \pi^2 \ln^2 2 \\
 & + \frac{13}{48} \pi^2 \ln^2 3 - \frac{1}{2} \pi^2 \ln 3 \ln 2 + 57 \text{Li}_4 \left(\frac{1}{2} \right) + \frac{33 \text{Li}_4 \left(\frac{1}{3} \right)}{2} + \frac{13 \text{Li}_4 \left(-\frac{1}{3} \right)}{4} + 70 \text{Li}_4 \left(-\frac{1}{2} \right) \\
 & + \frac{3 \text{Li}_2 \left(\frac{1}{3} \right)^2}{2} - \frac{21 S_{2,2} \left(-\frac{1}{3} \right)}{4} - 11 S_{2,2} \left(-\frac{1}{2} \right) - \frac{61 \pi^4}{720} \\
 G \left(\frac{4}{3}, 2, 4, 0; 1 \right) & = -2 \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 2 - 5 \text{Li}_2 \left(\frac{1}{3} \right) \ln^2 3 + 17 \text{Li}_3 \left(\frac{1}{3} \right) \ln 2 \tag{F.57} \\
 & - 5 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 2 + 6 \text{Li}_2 \left(\frac{1}{3} \right) \ln 3 \ln 2 - \frac{23}{2} \text{Li}_3 \left(\frac{1}{3} \right) \ln 3 + 3 \text{Li}_3 \left(-\frac{1}{2} \right) \ln 3 - \frac{385}{12} \zeta_3 \ln 2 \\
 & + \frac{365}{24} \zeta_3 \ln 3 - \frac{23}{24} \ln^4 2 - \frac{1}{24} \ln^4 3 - \frac{7}{6} \ln 3 \ln^3 2 + \frac{1}{6} \ln^3 3 \ln 2 - \frac{1}{4} \ln^2 3 \ln^2 2 + \frac{31}{24} \pi^2 \ln^2 2 \\
 & - \frac{9}{16} \pi^2 \ln^2 3 + \frac{2}{3} \pi^2 \ln 3 \ln 2 - 17 \text{Li}_4 \left(\frac{1}{2} \right) - \frac{3 \text{Li}_4 \left(\frac{1}{3} \right)}{2} - \frac{39 \text{Li}_4 \left(-\frac{1}{3} \right)}{4} - 2 \text{Li}_4 \left(-\frac{1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 & -5\text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{2} + \frac{23S_{2,2}\left(-\frac{1}{3}\right)}{4} + 3S_{2,2}\left(-\frac{1}{2}\right) + \frac{59\pi^4}{720} \\
 G\left(\frac{3}{2}, 1, 2, 0; 1\right) &= 7\text{Li}_3\left(\frac{1}{3}\right)\ln 2 - 4\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 - \frac{7}{2}\text{Li}_3\left(\frac{1}{3}\right)\ln 3 \quad (\text{F.58}) \\
 & + 7\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 - \frac{85}{12}\zeta_3\ln 2 + \frac{5}{4}\zeta_3\ln 3 - \frac{5}{24}\ln^4 2 + \frac{19}{96}\ln^4 3 - \frac{7}{6}\ln 3\ln^3 2 - \frac{7}{6}\ln^3 3\ln 2 \\
 & + \frac{7}{4}\ln^2 3\ln^2 2 + \frac{3}{8}\pi^2\ln^2 2 + \frac{1}{24}\pi^2\ln^2 3 - 7\text{Li}_4\left(\frac{1}{2}\right) - 8\text{Li}_4\left(\frac{1}{3}\right) + \frac{\text{Li}_4\left(-\frac{1}{3}\right)}{2} \\
 & - 10\text{Li}_4\left(-\frac{1}{2}\right) + \frac{7S_{2,2}\left(-\frac{1}{3}\right)}{4} + 4S_{2,2}\left(-\frac{1}{2}\right) + \frac{47\pi^4}{720}
 \end{aligned}$$

$$\begin{aligned}
 G\left(\frac{3}{2}, 2, 0, 1; 1\right) &= -\text{Li}_2\left(\frac{1}{3}\right)\ln^2 3 + 4\text{Li}_3\left(\frac{1}{3}\right)\ln 2 - 8\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 \quad (\text{F.59}) \\
 & + 2\text{Li}_2\left(\frac{1}{3}\right)\ln 3\ln 2 - 2\text{Li}_3\left(\frac{1}{3}\right)\ln 3 + 4\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{5}{3}\zeta_3\ln 2 - \frac{137}{24}\zeta_3\ln 3 + \frac{5}{24}\ln^4 2 \\
 & - \frac{25}{96}\ln^4 3 - \frac{2}{3}\ln 3\ln^3 2 + \frac{1}{3}\ln^3 3\ln 2 + \frac{1}{4}\pi^2\ln^2 2 + \frac{29}{48}\pi^2\ln^2 3 - \frac{2}{3}\pi^2\ln 3\ln 2 - 4\text{Li}_4\left(\frac{1}{2}\right) \\
 & - \frac{21\text{Li}_4\left(\frac{1}{3}\right)}{2} + \frac{13\text{Li}_4\left(-\frac{1}{3}\right)}{4} - 15\text{Li}_4\left(-\frac{1}{2}\right) - \text{Li}_2\left(\frac{1}{3}\right)^2 + \frac{2\pi^2\text{Li}_2\left(\frac{1}{3}\right)}{3} + S_{2,2}\left(-\frac{1}{3}\right) \\
 & + 8S_{2,2}\left(-\frac{1}{2}\right) - \frac{\pi^4}{90}
 \end{aligned}$$

$$\begin{aligned}
 G\left(\frac{3}{2}, 2, 1, 0; 1\right) &= -3\text{Li}_3\left(\frac{1}{3}\right)\ln 2 + 4\text{Li}_3\left(-\frac{1}{2}\right)\ln 2 + \frac{3}{2}\text{Li}_3\left(\frac{1}{3}\right)\ln 3 \quad (\text{F.60}) \\
 & - 3\text{Li}_3\left(-\frac{1}{2}\right)\ln 3 + \frac{3}{4}\zeta_3\ln 2 + \frac{67}{24}\zeta_3\ln 3 - \frac{1}{24}\ln^4 2 - \frac{1}{48}\ln^4 3 + \frac{1}{2}\ln 3\ln^3 2 + \frac{1}{2}\ln^3 3\ln 2 \\
 & - \frac{3}{4}\ln^2 3\ln^2 2 - \frac{7}{48}\pi^2\ln^2 3 + 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{13\text{Li}_4\left(\frac{1}{3}\right)}{2} - \frac{7\text{Li}_4\left(-\frac{1}{3}\right)}{4} + 9\text{Li}_4\left(-\frac{1}{2}\right) \\
 & - \frac{3S_{2,2}\left(-\frac{1}{3}\right)}{4} - 4S_{2,2}\left(-\frac{1}{2}\right) - \frac{3\pi^4}{160}
 \end{aligned}$$

$$\begin{aligned}
 G(2, 0, 0, 1; 1) &= \frac{7}{8}\zeta_3\ln 2 + \frac{1}{24}\ln^4 2 + \frac{1}{12}\pi^2\ln^2 2 + \text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{80} \quad (\text{F.61}) \\
 G(2, 0, 1, 0; 1) &= -\frac{7}{4}\zeta_3\ln 2 + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2 + \text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{1440} \quad (\text{F.62}) \\
 G(2, 0, 1, 1; 1) &= \frac{19\pi^4}{1440} - \frac{7}{4}\zeta_3\ln 2 \quad (\text{F.63}) \\
 G(2, 0, 1, 2; 1) &= -\frac{13}{8}\zeta_3\ln 2 - \frac{1}{6}\ln^4 2 - \frac{1}{12}\pi^2\ln^2 2 - 4\text{Li}_4\left(\frac{1}{2}\right) + \frac{7\pi^4}{180} \quad (\text{F.64}) \\
 G(2, 0, 2, 1; 1) &= -\frac{21}{8}\zeta_3\ln 2 + \frac{1}{4}\pi^2\ln^2 2 + \frac{7\pi^4}{720} \quad (\text{F.65}) \\
 G(2, 1, 0, 0; 1) &= \frac{7}{8}\zeta_3\ln 2 + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2 + \text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{288} \quad (\text{F.66}) \\
 G(2, 1, 0, 1; 1) &= \frac{7}{2}\zeta_3\ln 2 + \frac{1}{6}\ln^4 2 - \frac{1}{6}\pi^2\ln^2 2 + 4\text{Li}_4\left(\frac{1}{2}\right) - \frac{71\pi^4}{1440} \quad (\text{F.67})
 \end{aligned}$$

$$G(2, 1, 0, 2; 1) = \frac{13}{4}\zeta_3 \ln 2 + \frac{1}{6} \ln^4 2 - \frac{1}{8} \pi^2 \ln^2 2 + 4\text{Li}_4\left(\frac{1}{2}\right) - \frac{11\pi^4}{240} \quad (\text{F.68})$$

$$G(2, 1, 1, 0; 1) = -\frac{7}{4}\zeta_3 \ln 2 - \frac{1}{12} \ln^4 2 + \frac{1}{12} \pi^2 \ln^2 2 - 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{11\pi^4}{360} \quad (\text{F.69})$$

$$G(2, 1, 1, 1; 1) = \frac{7\pi^4}{720} \quad (\text{F.70})$$

$$G(2, 1, 1, 2; 1) = \frac{3}{4}\zeta_3 \ln 2 - \frac{\pi^4}{288} \quad (\text{F.71})$$

$$G(2, 1, 2, 0; 1) = -\frac{21}{8}\zeta_3 \ln 2 - \frac{1}{6} \ln^4 2 + \frac{1}{8} \pi^2 \ln^2 2 - 4\text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{24} \quad (\text{F.72})$$

$$G(2, 1, 2, 1; 1) = -\frac{7}{2}\zeta_3 \ln 2 - \frac{1}{6} \ln^4 2 + \frac{1}{6} \pi^2 \ln^2 2 - 4\text{Li}_4\left(\frac{1}{2}\right) + \frac{13\pi^4}{288} \quad (\text{F.73})$$

$$G(2, 1, 2, 2; 1) = -\frac{23}{8}\zeta_3 \ln 2 - \frac{1}{8} \ln^4 2 + \frac{1}{6} \pi^2 \ln^2 2 - 3\text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{30} \quad (\text{F.74})$$

$$G(2, 2, 0, 1; 1) = \frac{21}{8}\zeta_3 \ln 2 + \frac{1}{12} \ln^4 2 - \frac{5}{24} \pi^2 \ln^2 2 + 2\text{Li}_4\left(\frac{1}{2}\right) - \frac{7\pi^4}{288} \quad (\text{F.75})$$

$$G(2, 2, 1, 0; 1) = \frac{\pi^4}{480} \quad (\text{F.76})$$

$$G(2, 2, 1, 1; 1) = \frac{7}{4}\zeta_3 \ln 2 + \frac{1}{12} \ln^4 2 - \frac{1}{12} \pi^2 \ln^2 2 + 2\text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{48} \quad (\text{F.77})$$

$$G(2, 2, 1, 2; 1) = \frac{11}{4}\zeta_3 \ln 2 + \frac{1}{8} \ln^4 2 - \frac{1}{8} \pi^2 \ln^2 2 + 3\text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{30} \quad (\text{F.78})$$

$$G(2, 2, 2, 1; 1) = -\frac{7}{8}\zeta_3 \ln 2 - \frac{1}{24} \ln^4 2 + \frac{1}{24} \pi^2 \ln^2 2 - \text{Li}_4\left(\frac{1}{2}\right) + \frac{\pi^4}{90} \quad (\text{F.79})$$

G Goncharov multiple polylogarithm in terms of harmonic polylogarithms

If we consider Goncharov polylogarithms of the form $G(\vec{w}(a); 1)$, it is sometimes possible to reexpress this function in terms of harmonic polylogarithms in a . In the following we present the identities we encountered throughout our computation.

G.1 Polylogarithms of weight one

$$G(a; 1) = -H\left(1; \frac{1}{a}\right) \quad (\text{G.1})$$

G.2 Polylogarithms of weight two

$$G(0, a; 1) = -H\left(0, 1; \frac{1}{a}\right) \quad (\text{G.2})$$

$$G(-1, a; 1) = \ln 2 H\left(-1; \frac{1}{a}\right) + H\left(-1, 1; \frac{1}{a}\right) - H\left(0, 1; \frac{1}{a}\right) \quad (\text{G.3})$$

$$G(a, 0; 1) = H\left(0, 1; \frac{1}{a}\right) \quad (\text{G.4})$$

$$G(-a, -1; 1) = \ln 2 H\left(-1; \frac{1}{a}\right) + \ln 2 H\left(1; \frac{1}{a}\right) - H\left(0, -1; \frac{1}{a}\right) - H\left(1, -1; \frac{1}{a}\right) \quad (\text{G.5})$$

$$G(a, -1; 1) = -\ln 2 H\left(-1; \frac{1}{a}\right) - \ln 2 H\left(1; \frac{1}{a}\right) - H\left(-1, 1; \frac{1}{a}\right) + H\left(0, 1; \frac{1}{a}\right) \quad (\text{G.6})$$

$$G(a, 1; 1) = H\left(0, 1; \frac{1}{a}\right) + H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.7})$$

$$G(a^2, -a; 1) = H\left(-1, -1; \frac{1}{a}\right) - H\left(-1, 1; \frac{1}{a}\right) - H\left(0, -1; \frac{1}{a}\right) + H\left(0, 1; \frac{1}{a}\right) - H\left(1, -1; \frac{1}{a}\right) \quad (\text{G.8})$$

$$G(a^2, a; 1) = -H\left(-1, 1; \frac{1}{a}\right) - H\left(0, -1; \frac{1}{a}\right) + H\left(0, 1; \frac{1}{a}\right) - H\left(1, -1; \frac{1}{a}\right) + H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.9})$$

$$G(-a, a; 1) = -H\left(-1, 1; \frac{1}{a}\right) \quad (\text{G.10})$$

$$G(a, -a; 1) = -H\left(1, -1; \frac{1}{a}\right) \quad (\text{G.11})$$

$$G(a, a; 1) = H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.12})$$

$$G(-a, a^2; 1) = H\left(-1, -1; \frac{1}{a}\right) + H\left(0, -1; \frac{1}{a}\right) - H\left(0, 1; \frac{1}{a}\right) \quad (\text{G.13})$$

$$G(a, a^2; 1) = H\left(0, -1; \frac{1}{a}\right) - H\left(0, 1; \frac{1}{a}\right) + H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.14})$$

G.3 Polylogarithms of weight three

$$G(0, 0, a; 1) = -H\left(0, 0, 1; \frac{1}{a}\right) \quad (\text{G.15})$$

$$G(0, 1, a; 1) = \frac{1}{6}\pi^2 H\left(1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) \quad (\text{G.16})$$

$$G(0, a, 0; 1) = 2H\left(0, 0, 1; \frac{1}{a}\right) \quad (\text{G.17})$$

$$G(0, -a, -1; 1) = \ln 2 H\left(0, -1; \frac{1}{a}\right) + \ln 2 H\left(0, 1; \frac{1}{a}\right) + \frac{1}{12}\pi^2 H\left(1; \frac{1}{a}\right) - 2H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) \quad (\text{G.18})$$

$$G(0, a, -1; 1) = -\ln 2 H\left(0, -1; \frac{1}{a}\right) - \ln 2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{12}\pi^2 H\left(-1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) + 2H\left(0, 0, 1; \frac{1}{a}\right) \quad (\text{G.19})$$

$$G(0, a, 1; 1) = -\frac{1}{6}\pi^2 H\left(1; \frac{1}{a}\right) + 2H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) \quad (\text{G.20})$$

$$+ H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(0, a^2, -a; 1) = H\left(-1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, -1; \frac{1}{a}\right) \quad (\text{G.21})$$

$$- 2H\left(0, -1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 1, -1; \frac{1}{a}\right)$$

$$G(0, a^2, a; 1) = -2H\left(0, -1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1; \frac{1}{a}\right) \quad (\text{G.22})$$

$$- 2H\left(0, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(0, -a, a; 1) = -H\left(0, -1, 1; \frac{1}{a}\right) \quad (\text{G.23})$$

$$G(0, a, -a; 1) = -H\left(0, 1, -1; \frac{1}{a}\right) \quad (\text{G.24})$$

$$G(0, a, a; 1) = H\left(0, 1, 1; \frac{1}{a}\right) \quad (\text{G.25})$$

$$G(0, -a, a^2; 1) = -H\left(-1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) \quad (\text{G.26})$$

$$+ 3H\left(0, 0, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1; \frac{1}{a}\right)$$

$$G(0, a, a^2; 1) = 3H\left(0, 0, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) \quad (\text{G.27})$$

$$+ 2H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(-a, 0, -1; 1) = -\ln 2 H\left(0, -1; \frac{1}{a}\right) - \ln 2 H\left(0, 1; \frac{1}{a}\right) + \frac{1}{12}\pi^2 H\left(-1; \frac{1}{a}\right) \quad (\text{G.28})$$

$$+ H\left(0, 0, -1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right)$$

$$G(a, 0, -1; 1) = \ln 2 H\left(0, -1; \frac{1}{a}\right) + \ln 2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{12}\pi^2 H\left(1; \frac{1}{a}\right) \quad (\text{G.29})$$

$$+ H\left(0, -1, 1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, 1; 1) = \frac{1}{6}\pi^2 H\left(1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) \quad (\text{G.30})$$

$$G(-a, 0, a; 1) = -H\left(-1, 0, 1; \frac{1}{a}\right) \quad (\text{G.31})$$

$$G(a, 0, -a; 1) = -H\left(1, 0, -1; \frac{1}{a}\right) \quad (\text{G.32})$$

$$G(a, 0, a; 1) = H\left(1, 0, 1; \frac{1}{a}\right) \quad (\text{G.33})$$

$$G(-a, 0, a^2; 1) = 2H\left(-1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) \quad (\text{G.34})$$

$$+ H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, a^2; 1) = H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) \tag{G.35}$$

$$- 2H\left(1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 1, 0; 1) = -H\left(0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) \tag{G.36}$$

$$G(a, 1, 1; 1) = -H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right) \tag{G.37}$$

$$G(-a, 1, -a; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) \tag{G.38}$$

$$G(-a, 1, a; 1) = -H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) - H\left(1, -1, 1; \frac{1}{a}\right) \tag{G.39}$$

$$+ H\left(1, 0, -1; \frac{1}{a}\right)$$

$$G(a, 1, -a; 1) = H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) \tag{G.40}$$

$$+ H\left(1, 1, -1; \frac{1}{a}\right)$$

$$G(a, 1, a; 1) = -H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right) \tag{G.41}$$

$$G(-a, 1, a^2; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) \tag{G.42}$$

$$+ H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 1, a^2; 1) = -H\left(-1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) \tag{G.43}$$

$$- H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(a^2, 0, -a; 1) = H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) \tag{G.44}$$

$$+ H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right)$$

$$G(a^2, 0, a; 1) = -H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) \tag{G.45}$$

$$- H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a^2, -a, 0; 1) = -2H\left(-1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) \tag{G.46}$$

$$+ H\left(0, -1, 1; \frac{1}{a}\right) + 3H\left(0, 0, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right)$$

$$+ H\left(1, 0, -1; \frac{1}{a}\right)$$

$$G(a^2, a, 0; 1) = H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) + 3H\left(0, 0, -1; \frac{1}{a}\right) \quad (\text{G.47})$$

$$- 3H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1; \frac{1}{a}\right)$$

$$- 2H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a^2, -a, 1; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) \quad (\text{G.48})$$

$$+ H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) - 2H\left(0, -1, -1; \frac{1}{a}\right) + 2H\left(0, -1, 1; \frac{1}{a}\right)$$

$$+ 3H\left(0, 0, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right)$$

$$- H\left(1, -1, -1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right)$$

$$G(a^2, a, 1; 1) = H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) \quad (\text{G.49})$$

$$+ 2H\left(0, -1, 1; \frac{1}{a}\right) + 3H\left(0, 0, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, -1; \frac{1}{a}\right)$$

$$- 2H\left(0, 1, 1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right)$$

$$+ H\left(1, 1, -1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(a^2, a^2, -a; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) \quad (\text{G.50})$$

$$+ H\left(-1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right)$$

$$- H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right)$$

$$- H\left(1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, -1; \frac{1}{a}\right)$$

$$G(a^2, a^2, a; 1) = -H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) + H\left(-1, 0, 1; \frac{1}{a}\right) \quad (\text{G.51})$$

$$- H\left(-1, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right)$$

$$+ H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right)$$

$$+ H\left(1, 1, -1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(a^2, -a, -a; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) \quad (\text{G.52})$$

$$\begin{aligned}
 &+ H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) \\
 &+ H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, -a, a; 1) &= -H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) + H\left(-1, 0, 1; \frac{1}{a}\right) \quad (G.53) \\
 &- H\left(-1, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) \\
 &- H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, a, -a; 1) &= -H\left(-1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) \quad (G.54) \\
 &+ H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) \\
 &+ H\left(1, -1, 1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, a, a; 1) &= H\left(-1, 1, 1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) \quad (G.55) \\
 &- H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right) \\
 &+ H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, -1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, -a, a^2; 1) &= H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1; \frac{1}{a}\right) \quad (G.56) \\
 &+ H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) \\
 &+ H\left(1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, a, a^2; 1) &= H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) \quad (G.57) \\
 &+ H\left(1, -1, 1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, -1; \frac{1}{a}\right) \\
 &- H\left(1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a, a, 0; 1) = -H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right) \quad (G.58)$$

$$G(-a, -a, 1; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) \quad (G.59)$$

$$\begin{aligned}
 G(-a, a, 1; 1) &= H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) + H\left(0, -1, 1; \frac{1}{a}\right) \quad (G.60) \\
 &+ H\left(1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, 1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a, -a, 1; 1) = -H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 1; \frac{1}{a}\right) \quad (\text{G.61})$$

$$+ H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right)$$

$$G(a, a, 1; 1) = -H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right) \quad (\text{G.62})$$

$$G(-a, a^2, 0; 1) = -H\left(-1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, -1; \frac{1}{a}\right) - 4H\left(0, 0, -1; \frac{1}{a}\right) \quad (\text{G.63})$$

$$+ 4H\left(0, 0, 1; \frac{1}{a}\right)$$

$$G(a, a^2, 0; 1) = -4H\left(0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 1, 1; \frac{1}{a}\right) \quad (\text{G.64})$$

$$- H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(-a, a^2, 1; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) \quad (\text{G.65})$$

$$- H\left(0, -1, -1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1; \frac{1}{a}\right)$$

$$- H\left(0, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, a^2, 1; 1) = H\left(-1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) \quad (\text{G.66})$$

$$- H\left(0, -1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right)$$

$$- H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(-a, a^2, -a; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) \quad (\text{G.67})$$

$$- H\left(0, -1, 1; \frac{1}{a}\right) - 2H\left(0, 0, -1; \frac{1}{a}\right) + 2H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right)$$

$$G(-a, a^2, a; 1) = -H\left(-1, -1, 1; \frac{1}{a}\right) - H\left(0, -1, -1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) \quad (\text{G.68})$$

$$- 2H\left(0, 0, -1; \frac{1}{a}\right) + 2H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right)$$

$$- H\left(1, -1, -1; \frac{1}{a}\right) - H\left(1, -1, 1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, a^2, -a; 1) = H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, -1; \frac{1}{a}\right) \quad (\text{G.69})$$

$$+ H\left(-1, 1, 1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) - 2H\left(0, 0, -1; \frac{1}{a}\right)$$

$$+ 2H\left(0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1; \frac{1}{a}\right)$$

$$G(a, a^2, a; 1) = -H\left(0, -1, 1; \frac{1}{a}\right) - 2H\left(0, 0, -1; \frac{1}{a}\right) + 2H\left(0, 0, 1; \frac{1}{a}\right) \quad (\text{G.70})$$

$$- H\left(0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(-a, a^2, a^2; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) \quad (\text{G.71})$$

$$+ H\left(0, -1, -1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right)$$

$$G(a, a^2, a^2; 1) = H\left(0, -1, -1; \frac{1}{a}\right) - H\left(0, -1, 1; \frac{1}{a}\right) - H\left(0, 1, -1; \frac{1}{a}\right) \quad (\text{G.72})$$

$$+ H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right)$$

$$G(-a, -a, a; 1) = -H\left(-1, -1, 1; \frac{1}{a}\right) \quad (\text{G.73})$$

$$G(-a, a, -a; 1) = -H\left(-1, 1, -1; \frac{1}{a}\right) \quad (\text{G.74})$$

$$G(-a, a, a; 1) = H\left(-1, 1, 1; \frac{1}{a}\right) \quad (\text{G.75})$$

$$G(a, -a, -a; 1) = -H\left(1, -1, -1; \frac{1}{a}\right) \quad (\text{G.76})$$

$$G(a, -a, a; 1) = H\left(1, -1, 1; \frac{1}{a}\right) \quad (\text{G.77})$$

$$G(a, a, -a; 1) = H\left(1, 1, -1; \frac{1}{a}\right) \quad (\text{G.78})$$

$$G(a, a, a; 1) = -H\left(1, 1, 1; \frac{1}{a}\right) \quad (\text{G.79})$$

$$G(-a, -a, a^2; 1) = H\left(-1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) \quad (\text{G.80})$$

$$- H\left(0, 0, 1; \frac{1}{a}\right)$$

$$G(-a, a, a^2; 1) = H\left(-1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1; \frac{1}{a}\right) \quad (\text{G.81})$$

$$+ H\left(0, -1, -1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(1, -1, -1; \frac{1}{a}\right)$$

$$+ H\left(1, -1, 1; \frac{1}{a}\right) + H\left(1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, -a, a^2; 1) = -H\left(-1, 0, -1; \frac{1}{a}\right) + H\left(-1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, -1; \frac{1}{a}\right) \quad (\text{G.82})$$

$$- H\left(-1, 1, 1; \frac{1}{a}\right) + H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right)$$

$$- H\left(1, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 1; \frac{1}{a}\right)$$

$$G(a, a, a^2; 1) = H\left(0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1; \frac{1}{a}\right) - H\left(1, 1, 1; \frac{1}{a}\right) \quad (\text{G.83})$$

G.4 Polylogarithms of weight four

$$G(0, 0, 0, a; 1) = -H\left(0, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.84})$$

$$G(0, 0, 1, a; 1) = \zeta_3 H\left(1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.85})$$

$$G(0, 0, -a, 0; 1) = -3H\left(0, 0, 0, -1; \frac{1}{a}\right) \quad (\text{G.86})$$

$$G(0, 0, a, 0; 1) = 3H\left(0, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.87})$$

$$G(0, 0, -a, -1; 1) = \ln 2H\left(0, 0, -1; \frac{1}{a}\right) + \ln 2H\left(0, 0, 1; \frac{1}{a}\right) + \frac{3}{4}\zeta_3 H\left(1; \frac{1}{a}\right) + \frac{1}{12}\pi^2 H\left(0, 1; \frac{1}{a}\right) - 3H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right) \quad (\text{G.88})$$

$$G(0, 0, a, -1; 1) = -\ln 2H\left(0, 0, -1; \frac{1}{a}\right) - \ln 2H\left(0, 0, 1; \frac{1}{a}\right) - \frac{3}{4}\zeta_3 H\left(-1; \frac{1}{a}\right) - \frac{1}{12}\pi^2 H\left(0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) + 3H\left(0, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.89})$$

$$G(0, 0, a, 1; 1) = -\zeta_3 H\left(1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) + 3H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.90})$$

$$G(0, 0, a^2, -a; 1) = 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, -1, -1; \frac{1}{a}\right) - 4H\left(0, 0, -1, 1; \frac{1}{a}\right) - 12H\left(0, 0, 0, -1; \frac{1}{a}\right) + 12H\left(0, 0, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, 1, -1; \frac{1}{a}\right) \quad (\text{G.91})$$

$$G(0, 0, a^2, a; 1) = -4H\left(0, 0, -1, 1; \frac{1}{a}\right) - 12H\left(0, 0, 0, -1; \frac{1}{a}\right) + 12H\left(0, 0, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, 1, -1; \frac{1}{a}\right) + 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - 4H\left(0, 1, 0, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - 4H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.92})$$

$$G(0, 0, -a, a; 1) = -H\left(0, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.93})$$

$$G(0, 0, a, -a; 1) = -H\left(0, 0, 1, -1; \frac{1}{a}\right) \quad (\text{G.94})$$

$$G(0, 0, a, a; 1) = H\left(0, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.95})$$

$$\begin{aligned} G(0, 0, -a, a^2; 1) &= -3H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\ &\quad - H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) \\ &\quad + 7H\left(0, 0, 0, -1; \frac{1}{a}\right) - 7H\left(0, 0, 0, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.96})$$

$$\begin{aligned} G(0, 0, a, a^2; 1) &= 7H\left(0, 0, 0, -1; \frac{1}{a}\right) - 7H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) \\ &\quad + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.97})$$

$$G(0, 1, 0, a; 1) = \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.98})$$

$$\begin{aligned} G(0, 1, 1, a; 1) &= -\zeta_3 H\left(1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) \\ &\quad - H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.99})$$

$$\begin{aligned} G(0, 1, a, 1; 1) &= -\frac{1}{3}\pi^2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) + 3H\left(0, 0, 0, 1; \frac{1}{a}\right) \\ &\quad + H\left(0, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\ &\quad + 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.100})$$

$$\begin{aligned} G(0, 1, a, a; 1) &= -\frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \\ &\quad + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.101})$$

$$\begin{aligned} G(0, -a, 0, -1; 1) &= -2\ln 2H\left(0, 0, -1; \frac{1}{a}\right) - 2\ln 2H\left(0, 0, 1; \frac{1}{a}\right) \\ &\quad + \frac{1}{12}\pi^2 H\left(0, -1; \frac{1}{a}\right) - \frac{1}{12}\pi^2 H\left(0, 1; \frac{1}{a}\right) + 3H\left(0, 0, 0, -1; \frac{1}{a}\right) + 2H\left(0, 0, 1, -1; \frac{1}{a}\right) \\ &\quad + H\left(0, 1, 0, -1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.102})$$

$$\begin{aligned} G(0, a, 0, -1; 1) &= 2\ln 2H\left(0, 0, -1; \frac{1}{a}\right) + 2\ln 2H\left(0, 0, 1; \frac{1}{a}\right) \\ &\quad + \frac{1}{12}\pi^2 H\left(0, -1; \frac{1}{a}\right) - \frac{1}{12}\pi^2 H\left(0, 1; \frac{1}{a}\right) + H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 0, -1, 1; \frac{1}{a}\right) \end{aligned} \quad (\text{G.103})$$

$$\begin{aligned}
 & -3H\left(0,0,0,1;\frac{1}{a}\right) \\
 G(0,a,0,1;1) &= \frac{1}{3}\pi^2 H\left(0,1;\frac{1}{a}\right) - 3H\left(0,0,0,1;\frac{1}{a}\right) - 2H\left(0,0,1,1;\frac{1}{a}\right)
 \end{aligned} \tag{G.104}$$

$$\begin{aligned}
 & -H\left(0,1,0,1;\frac{1}{a}\right) \\
 G(0,-a,0,a;1) &= -H\left(0,-1,0,1;\frac{1}{a}\right)
 \end{aligned} \tag{G.105}$$

$$G(0,a,0,-a;1) = -H\left(0,1,0,-1;\frac{1}{a}\right) \tag{G.106}$$

$$G(0,a,0,a;1) = H\left(0,1,0,1;\frac{1}{a}\right) \tag{G.107}$$

$$G(0,-a,0,a^2;1) = H\left(0,-1,0,-1;\frac{1}{a}\right) + 2H\left(0,0,-1,-1;\frac{1}{a}\right) \tag{G.108}$$

$$+ 4H\left(0,0,0,-1;\frac{1}{a}\right) - 4H\left(0,0,0,1;\frac{1}{a}\right)$$

$$G(0,a,0,a^2;1) = 4H\left(0,0,0,-1;\frac{1}{a}\right) - 4H\left(0,0,0,1;\frac{1}{a}\right) + 2H\left(0,0,1,1;\frac{1}{a}\right) \tag{G.109}$$

$$+ H\left(0,1,0,1;\frac{1}{a}\right)$$

$$G(0,a,1,0;1) = 2\zeta_3 H\left(1;\frac{1}{a}\right) - 3H\left(0,0,0,1;\frac{1}{a}\right) - H\left(0,1,0,1;\frac{1}{a}\right) \tag{G.110}$$

$$- 2H\left(1,0,0,1;\frac{1}{a}\right)$$

$$G(0,a,1,1;1) = \zeta_3 H\left(1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0,1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1,1;\frac{1}{a}\right) \tag{G.111}$$

$$- 3H\left(0,0,0,1;\frac{1}{a}\right) - 2H\left(0,0,1,1;\frac{1}{a}\right) - 2H\left(0,1,0,1;\frac{1}{a}\right) - H\left(0,1,1,1;\frac{1}{a}\right)$$

$$- 2H\left(1,0,0,1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) - H\left(1,1,0,1;\frac{1}{a}\right)$$

$$G(0,a,1,a;1) = \frac{1}{3}\pi^2 H\left(1,1;\frac{1}{a}\right) - H\left(0,1,0,1;\frac{1}{a}\right) - H\left(0,1,1,1;\frac{1}{a}\right) \tag{G.112}$$

$$- 3H\left(1,0,0,1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) - 2H\left(1,1,0,1;\frac{1}{a}\right)$$

$$G(0,-a,1,a^2;1) = \frac{1}{3}\pi^2 H\left(-1,-1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(-1,1;\frac{1}{a}\right) \tag{G.113}$$

$$- \frac{1}{6}\pi^2 H\left(1,-1;\frac{1}{a}\right) + 3H\left(-1,-1,0,-1;\frac{1}{a}\right) - 2H\left(-1,-1,0,1;\frac{1}{a}\right)$$

$$+ H\left(-1,0,-1,-1;\frac{1}{a}\right) - 6H\left(-1,0,0,-1;\frac{1}{a}\right) + 4H\left(-1,0,0,1;\frac{1}{a}\right)$$

$$- 2H\left(-1,1,0,-1;\frac{1}{a}\right) + 2H\left(-1,1,0,1;\frac{1}{a}\right) + H\left(0,-1,-1,-1;\frac{1}{a}\right)$$

$$\begin{aligned}
& -2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 0, -1; \frac{1}{a}\right) - 4H\left(0, 0, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) + 5H\left(1, 0, 0, -1; \frac{1}{a}\right) - 3H\left(1, 0, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a, 1, a^2; 1) = -\frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) + \frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.114})$$

$$\begin{aligned}
& -3H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
& + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 0, -1; \frac{1}{a}\right) - 4H\left(0, 0, 0, 1; \frac{1}{a}\right) + 2H\left(0, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 6H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, -1; \frac{1}{a}\right) \\
& - 3H\left(1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, 0, -a; 1) = H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.115})$$

$$\begin{aligned}
& -4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 6H\left(0, 0, 0, -1; \frac{1}{a}\right) - 6H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, 0, a; 1) = -2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.116})$$

$$\begin{aligned}
& + 6H\left(0, 0, 0, -1; \frac{1}{a}\right) - 6H\left(0, 0, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, -a, 0; 1) = -6H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 8H\left(-1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.117})$$

$$\begin{aligned}
& -5H\left(0, -1, 0, -1; \frac{1}{a}\right) + 6H\left(0, -1, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 18H\left(0, 0, 0, -1; \frac{1}{a}\right) - 18H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, a, 0; 1) = 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.118})$$

$$\begin{aligned}
 & + 18H\left(0, 0, 0, -1; \frac{1}{a}\right) - 18H\left(0, 0, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
 & + 6H\left(0, 1, 0, -1; \frac{1}{a}\right) - 5H\left(0, 1, 0, 1; \frac{1}{a}\right) + 8H\left(1, 0, 0, -1; \frac{1}{a}\right) - 6H\left(1, 0, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(0, a^2, -a, 1; 1) = -\frac{1}{6}\pi^2 H\left(-1, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) \quad (\text{G.119})$$

$$\begin{aligned}
 & + \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, -1, -1; \frac{1}{a}\right) \\
 & - 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 2H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\
 & - 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(0, -1, -1, -1; \frac{1}{a}\right) - 2H\left(0, -1, -1, 1; \frac{1}{a}\right) \\
 & - 5H\left(0, -1, 0, -1; \frac{1}{a}\right) + 6H\left(0, -1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 1, -1; \frac{1}{a}\right) \\
 & - 8H\left(0, 0, -1, -1; \frac{1}{a}\right) + 8H\left(0, 0, -1, 1; \frac{1}{a}\right) + 18H\left(0, 0, 0, -1; \frac{1}{a}\right) \\
 & - 18H\left(0, 0, 0, 1; \frac{1}{a}\right) + 8H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
 & - 2H\left(0, 1, -1, -1; \frac{1}{a}\right) + 4H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(0, a^2, a, 1; 1) = -\frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.120})$$

$$\begin{aligned}
 & - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 4H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
 & - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 8H\left(0, 0, -1, 1; \frac{1}{a}\right) + 18H\left(0, 0, 0, -1; \frac{1}{a}\right) \\
 & - 18H\left(0, 0, 0, 1; \frac{1}{a}\right) + 8H\left(0, 0, 1, -1; \frac{1}{a}\right) - 8H\left(0, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
 & + 6H\left(0, 1, 0, -1; \frac{1}{a}\right) - 5H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 1, -1; \frac{1}{a}\right) - 2H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
 & + 2H\left(1, 0, -1, 1; \frac{1}{a}\right) + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 2H\left(1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(0, a^2, a^2, -a; 1) = -H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.121})$$

$$\begin{aligned}
 & + 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & + 2H\left(0, -1, -1, -1; \frac{1}{a}\right) - 2H\left(0, -1, -1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 1, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, -1, 1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - 2H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, -1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 1, -1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, a^2, a; 1) = -2H\left(0, -1, -1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \tag{G.122}$$

$$\begin{aligned}
& + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 1, -1; \frac{1}{a}\right) + 2H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
& - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
& - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - 2H\left(0, 1, -1, -1; \frac{1}{a}\right) + 2H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, 1, -1; \frac{1}{a}\right) - 2H\left(0, 1, 1, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
& - 4H\left(1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, -a, a^2; 1) = 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.123}$$

$$\begin{aligned}
& + H\left(-1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 7H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
& + 7H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
& - 2H\left(0, -1, -1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(0, -1, 1, -1; \frac{1}{a}\right) - 2H\left(0, 1, -1, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a^2, a, a^2; 1) = 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \tag{G.124}$$

$$\begin{aligned}
& + 2H\left(0, -1, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, -1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 1, -1; \frac{1}{a}\right) - 2H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
& + 2H\left(1, 0, -1, 1; \frac{1}{a}\right) + 7H\left(1, 0, 0, -1; \frac{1}{a}\right) - 7H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& + 2H\left(1, 0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right) + 4H\left(1, 1, 0, -1; \frac{1}{a}\right) \\
& - 2H\left(1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, a, a, 0; 1) = -2H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) \tag{G.125}$$

$$G(0, a, a, 1; 1) = -\frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) - 2H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) \quad (\text{G.126})$$

$$+ 2H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(0, -a, a^2, 0; 1) = 6H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 8H\left(-1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.127})$$

$$+ H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right)$$

$$- 18H\left(0, 0, 0, -1; \frac{1}{a}\right) + 18H\left(0, 0, 0, 1; \frac{1}{a}\right)$$

$$G(0, a, a^2, 0; 1) = -18H\left(0, 0, 0, -1; \frac{1}{a}\right) + 18H\left(0, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.128})$$

$$- 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - 4H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right)$$

$$- 8H\left(1, 0, 0, -1; \frac{1}{a}\right) + 6H\left(1, 0, 0, 1; \frac{1}{a}\right)$$

$$G(0, -a, a^2, 1; 1) = -\frac{1}{6}\pi^2 H\left(-1, -1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) \quad (\text{G.129})$$

$$+ \frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right)$$

$$- 3H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right)$$

$$+ 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 11H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 11H\left(-1, 0, 0, 1; \frac{1}{a}\right)$$

$$+ 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right)$$

$$- 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right) + 5H\left(0, -1, 0, -1; \frac{1}{a}\right)$$

$$- 7H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right)$$

$$- 18H\left(0, 0, 0, -1; \frac{1}{a}\right) + 18H\left(0, 0, 0, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right)$$

$$+ 3H\left(0, 0, 1, 1; \frac{1}{a}\right) - 4H\left(0, 1, 0, -1; \frac{1}{a}\right) + 3H\left(0, 1, 0, 1; \frac{1}{a}\right)$$

$$+ 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) - 5H\left(1, 0, 0, -1; \frac{1}{a}\right)$$

$$+ 3H\left(1, 0, 0, 1; \frac{1}{a}\right)$$

$$G(0, a, a^2, 1; 1) = \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) \quad (\text{G.130})$$

$$\begin{aligned}
& -\frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) \\
& + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
& - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 18H\left(0, 0, 0, -1; \frac{1}{a}\right) \\
& + 18H\left(0, 0, 0, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) - H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& - 7H\left(0, 1, 0, -1; \frac{1}{a}\right) + 5H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
& + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) - 11H\left(1, 0, 0, -1; \frac{1}{a}\right) + 11H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) - 4H\left(1, 1, 0, -1; \frac{1}{a}\right) \\
& + 3H\left(1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(0, -a, a^2, a^2; 1) &= -H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.131} \\
& - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
& - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& + H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, 1, 1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(0, a, a^2, a^2; 1) &= 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) \tag{G.132} \\
& - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) + 3H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(0, -a, -a, a; 1) = -H\left(0, -1, -1, 1; \frac{1}{a}\right) \quad (\text{G.133})$$

$$G(0, -a, a, -a; 1) = -H\left(0, -1, 1, -1; \frac{1}{a}\right) \quad (\text{G.134})$$

$$G(0, -a, a, a; 1) = H\left(0, -1, 1, 1; \frac{1}{a}\right) \quad (\text{G.135})$$

$$G(0, a, -a, -a; 1) = -H\left(0, 1, -1, -1; \frac{1}{a}\right) \quad (\text{G.136})$$

$$G(0, a, -a, a; 1) = H\left(0, 1, -1, 1; \frac{1}{a}\right) \quad (\text{G.137})$$

$$G(0, a, a, -a; 1) = H\left(0, 1, 1, -1; \frac{1}{a}\right) \quad (\text{G.138})$$

$$G(0, a, a, a; 1) = -H\left(0, 1, 1, 1; \frac{1}{a}\right) \quad (\text{G.139})$$

$$G(-a, 0, 0, -1; 1) = \ln 2H\left(0, 0, -1; \frac{1}{a}\right) + \ln 2H\left(0, 0, 1; \frac{1}{a}\right) + \frac{3}{4}\zeta_3 H\left(-1; \frac{1}{a}\right) \quad (\text{G.140})$$

$$- \frac{1}{12}\pi^2 H\left(0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right)$$

$$G(a, 0, 0, -1; 1) = -\ln 2H\left(0, 0, -1; \frac{1}{a}\right) - \ln 2H\left(0, 0, 1; \frac{1}{a}\right) - \frac{3}{4}\zeta_3 H\left(1; \frac{1}{a}\right) \quad (\text{G.141})$$

$$+ \frac{1}{12}\pi^2 H\left(0, 1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, 0, 1; 1) = \zeta_3 H\left(1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.142})$$

$$+ H\left(0, 0, 1, 1; \frac{1}{a}\right)$$

$$G(-a, 0, 0, a; 1) = -H\left(-1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.143})$$

$$G(a, 0, 0, -a; 1) = -H\left(1, 0, 0, -1; \frac{1}{a}\right) \quad (\text{G.144})$$

$$G(a, 0, 0, a; 1) = H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.145})$$

$$G(-a, 0, 0, a^2; 1) = 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.146})$$

$$+ 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right)$$

$$+ H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, 1, 0; 1) = -2\zeta_3 H\left(1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.147})$$

$$G(a, 0, 1, 1; 1) = -\zeta_3 H\left(1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.148})$$

$$+ H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right)$$

$$G(a, 0, 1, a; 1) = -\frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) \quad (\text{G.149})$$

$$+ 2H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(-a, 0, 1, a^2; 1) = -\frac{1}{3}\pi^2 H\left(-1, -1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) \quad (\text{G.150})$$

$$+ \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) - 4H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, 1; \frac{1}{a}\right)$$

$$- 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right)$$

$$+ 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right)$$

$$+ 5H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right)$$

$$+ H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right)$$

$$+ H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right)$$

$$+ 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, 1, a^2; 1) = \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) - \frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) \quad (\text{G.151})$$

$$- H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right)$$

$$- 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, 1; \frac{1}{a}\right)$$

$$+ H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right)$$

$$- 4H\left(0, 1, 0, -1; \frac{1}{a}\right) + 5H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right)$$

$$+ 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right)$$

$$+ 4H\left(1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) - 4H\left(1, 1, 0, -1; \frac{1}{a}\right)$$

$$+ 4H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 0, a, 0; 1) = -H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.152})$$

$$G(a, 0, a, 1; 1) = \frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.153})$$

$$- 2H\left(1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(-a, 0, a^2, 1; 1) = \frac{1}{3}\pi^2 H\left(-1, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) \tag{G.154}$$

$$\begin{aligned} & - \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \\ & + 4H\left(-1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 8H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\ & + 7H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\ & - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right) \\ & - 3H\left(0, -1, 0, -1; \frac{1}{a}\right) + 3H\left(0, -1, 0, 1; \frac{1}{a}\right) - 3H\left(0, 0, -1, -1; \frac{1}{a}\right) \\ & - H\left(0, 0, -1, 1; \frac{1}{a}\right) - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) \\ & - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\ & + H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) \\ & - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \end{aligned}$$

$$G(a, 0, a^2, 1; 1) = -\frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) + \frac{1}{3}\pi^2 H\left(1, 1; \frac{1}{a}\right) \tag{G.155}$$

$$\begin{aligned} & + H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\ & + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\ & + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) \\ & + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) - 3H\left(0, 0, 1, 1; \frac{1}{a}\right) \\ & + 3H\left(0, 1, 0, -1; \frac{1}{a}\right) - 3H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) \\ & - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\ & + 2H\left(1, 0, -1, 1; \frac{1}{a}\right) + 7H\left(1, 0, 0, -1; \frac{1}{a}\right) - 8H\left(1, 0, 0, 1; \frac{1}{a}\right) \\ & + 2H\left(1, 0, 1, -1; \frac{1}{a}\right) - 4H\left(1, 0, 1, 1; \frac{1}{a}\right) + 4H\left(1, 1, 0, -1; \frac{1}{a}\right) \\ & - 4H\left(1, 1, 0, 1; \frac{1}{a}\right) \end{aligned}$$

$$G(-a, 0, -a, a; 1) = -H\left(-1, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.156})$$

$$G(-a, 0, a, -a; 1) = -H\left(-1, 0, 1, -1; \frac{1}{a}\right) \quad (\text{G.157})$$

$$G(-a, 0, a, a; 1) = H\left(-1, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.158})$$

$$G(a, 0, -a, -a; 1) = -H\left(1, 0, -1, -1; \frac{1}{a}\right) \quad (\text{G.159})$$

$$G(a, 0, -a, a; 1) = H\left(1, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.160})$$

$$G(a, 0, a, -a; 1) = H\left(1, 0, 1, -1; \frac{1}{a}\right) \quad (\text{G.161})$$

$$G(a, 0, a, a; 1) = -H\left(1, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.162})$$

$$G(a, 1, 0, 0; 1) = H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.163})$$

$$G(a, 1, 0, 1; 1) = -\frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) \\ + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.164})$$

$$G(a, 1, 0, a; 1) = -2H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.165})$$

$$G(-a, 1, 0, a^2; 1) = 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \\ + H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\ + 2H\left(0, -1, -1, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\ + H\left(0, 0, -1, -1; \frac{1}{a}\right) + H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) \\ + 4H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.166})$$

$$G(a, 1, 0, a^2; 1) = -2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 4H\left(0, -1, 0, 1; \frac{1}{a}\right) \\ + H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\ - 4H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \\ - H\left(1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.167})$$

$$G(a, 1, 1, 0; 1) = H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.168})$$

$$+ H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 1, 1, 1; 1) = H\left(0, 0, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.169})$$

$$+ H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \\ + H\left(1, 1, 1, 1; \frac{1}{a}\right)$$

$$G(a, 1, 1, a; 1) = H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.170})$$

$$+ H\left(1, 1, 1, 1; \frac{1}{a}\right)$$

$$G(-a, 1, 1, a^2; 1) = H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.171})$$

$$- H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right)$$

$$- H\left(0, -1, 0, -1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) + H\left(0, 0, 0, -1; \frac{1}{a}\right)$$

$$- H\left(0, 0, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right)$$

$$+ 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, -1; \frac{1}{a}\right)$$

$$- H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 1, 1, a^2; 1) = H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.172})$$

$$- H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(0, -1, 0, -1; \frac{1}{a}\right)$$

$$+ 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, 0, -1; \frac{1}{a}\right) - H\left(0, 0, 0, 1; \frac{1}{a}\right)$$

$$+ H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right)$$

$$+ H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$+ H\left(1, 1, 1, 1; \frac{1}{a}\right)$$

$$G(a, 1, a, 0; 1) = H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.173})$$

$$+ H\left(1, 1, 0, 1; \frac{1}{a}\right)$$

$$G(a, 1, a, 1; 1) = H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.174})$$

$$+ H\left(1, 1, 1, 1; \frac{1}{a}\right)$$

$$\begin{aligned}
 G(-a, 1, a^2, 0; 1) = & -H\left(-1, -1, 0, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) \quad (\text{G.175}) \\
 & - 2H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
 & - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) - 4H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - 5H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
 & + 4H\left(1, 0, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a, 1, a^2, 0; 1) = & 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.176}) \\
 & + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
 & - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
 & + H\left(1, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(-a, 1, a^2, 1; 1) = & H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.177}) \\
 & + H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\
 & + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) - 3H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, 0, -1, 1; \frac{1}{a}\right) - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) - 6H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
 & + 3H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) - 5H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
 & + 5H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
 & - 4H\left(1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a, 1, a^2, 1; 1) = & -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) & (G.178) \\
 & - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 5H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
 & - 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & - 6H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
 & - 6H\left(0, 0, 0, -1; \frac{1}{a}\right) + 6H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
 & - 3H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) + 3H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
 & + H\left(1, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(-a, 1, a^2, a^2; 1) = & H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, 0, -1; \frac{1}{a}\right) & (G.179) \\
 & + H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\
 & + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, 0, -1, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
 & + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
 & + H\left(1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a, 1, a^2, a^2; 1) = & -H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) & (G.180) \\
 & - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
 & - H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& -2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
& - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
& - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& - H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a, 1, a, a; 1) = H\left(1, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \quad (\text{G.181})$$

$$G(a^2, 0, 0, -a; 1) = H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.182})$$

$$\begin{aligned}
& + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) - H\left(0, 0, 0, -1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a^2, 0, 0, a; 1) = -H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.183})$$

$$\begin{aligned}
& - H\left(0, 0, 0, -1; \frac{1}{a}\right) + H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a^2, 0, -a, 0; 1) = -2H\left(-1, 0, 0, -1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.184})$$

$$\begin{aligned}
& - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 0, -1, -1; \frac{1}{a}\right) - 2H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
& - 4H\left(0, 0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 0, 1, -1; \frac{1}{a}\right) + 2H\left(1, 0, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a^2, 0, a, 0; 1) = 2H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 0, -1, 1; \frac{1}{a}\right) \quad (\text{G.185})$$

$$\begin{aligned}
& - 4H\left(0, 0, 0, -1; \frac{1}{a}\right) + 4H\left(0, 0, 0, 1; \frac{1}{a}\right) - 2H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
& + 2H\left(0, 0, 1, 1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a^2, 0, -a, 1; 1) = \frac{1}{3}\pi^2 H\left(-1, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(-1, 1; \frac{1}{a}\right) \quad (\text{G.186})$$

$$\begin{aligned}
& - \frac{1}{6}\pi^2 H\left(1, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) \\
& - H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 5H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
& -H\left(-1,0,1,-1;\frac{1}{a}\right) + 2H\left(-1,0,1,1;\frac{1}{a}\right) - H\left(-1,1,0,-1;\frac{1}{a}\right) \\
& -H\left(0,-1,-1,-1;\frac{1}{a}\right) + H\left(0,-1,-1,1;\frac{1}{a}\right) - H\left(0,-1,0,1;\frac{1}{a}\right) \\
& + H\left(0,-1,1,-1;\frac{1}{a}\right) + 3H\left(0,0,-1,-1;\frac{1}{a}\right) - 3H\left(0,0,-1,1;\frac{1}{a}\right) \\
& - 4H\left(0,0,0,-1;\frac{1}{a}\right) + 4H\left(0,0,0,1;\frac{1}{a}\right) - 3H\left(0,0,1,-1;\frac{1}{a}\right) + H\left(0,0,1,1;\frac{1}{a}\right) \\
& + H\left(0,1,-1,-1;\frac{1}{a}\right) - H\left(1,-1,0,-1;\frac{1}{a}\right) - H\left(1,0,-1,-1;\frac{1}{a}\right) + 2H\left(1,0,0,-1;\frac{1}{a}\right) \\
G(a^2,0,a,1;1) = & -\frac{1}{6}\pi^2 H\left(-1,1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1,-1;\frac{1}{a}\right) + \frac{1}{3}\pi^2 H\left(1,1;\frac{1}{a}\right) \quad (G.187)
\end{aligned}$$

$$\begin{aligned}
& + 2H\left(-1,0,0,1;\frac{1}{a}\right) + H\left(-1,0,1,1;\frac{1}{a}\right) + H\left(-1,1,0,1;\frac{1}{a}\right) \\
& - H\left(0,-1,1,1;\frac{1}{a}\right) + H\left(0,0,-1,-1;\frac{1}{a}\right) - 3H\left(0,0,-1,1;\frac{1}{a}\right) \\
& - 4H\left(0,0,0,-1;\frac{1}{a}\right) + 4H\left(0,0,0,1;\frac{1}{a}\right) - 3H\left(0,0,1,-1;\frac{1}{a}\right) \\
& + 3H\left(0,0,1,1;\frac{1}{a}\right) - H\left(0,1,-1,1;\frac{1}{a}\right) - H\left(0,1,0,-1;\frac{1}{a}\right) \\
& - H\left(0,1,1,-1;\frac{1}{a}\right) + H\left(0,1,1,1;\frac{1}{a}\right) + H\left(1,-1,0,1;\frac{1}{a}\right) \\
& - 2H\left(1,0,-1,-1;\frac{1}{a}\right) + H\left(1,0,-1,1;\frac{1}{a}\right) + 3H\left(1,0,0,-1;\frac{1}{a}\right) \\
& - 5H\left(1,0,0,1;\frac{1}{a}\right) + H\left(1,0,1,-1;\frac{1}{a}\right) - 2H\left(1,0,1,1;\frac{1}{a}\right) - 2H\left(1,1,0,1;\frac{1}{a}\right) \\
G(a^2,0,a^2,-a;1) = & 2H\left(-1,-1,0,-1;\frac{1}{a}\right) - 4H\left(-1,-1,0,1;\frac{1}{a}\right) \quad (G.188)
\end{aligned}$$

$$\begin{aligned}
& + 3H\left(-1,0,-1,-1;\frac{1}{a}\right) - 3H\left(-1,0,-1,1;\frac{1}{a}\right) - 5H\left(-1,0,0,-1;\frac{1}{a}\right) \\
& + 5H\left(-1,0,0,1;\frac{1}{a}\right) - 3H\left(-1,0,1,-1;\frac{1}{a}\right) - H\left(-1,1,0,-1;\frac{1}{a}\right) \\
& + 2H\left(-1,1,0,1;\frac{1}{a}\right) - 2H\left(0,-1,0,-1;\frac{1}{a}\right) + 2H\left(0,-1,0,1;\frac{1}{a}\right) \\
& + 2H\left(0,1,0,-1;\frac{1}{a}\right) - 2H\left(0,1,0,1;\frac{1}{a}\right) - H\left(1,-1,0,-1;\frac{1}{a}\right) \\
& + 2H\left(1,-1,0,1;\frac{1}{a}\right) - 2H\left(1,0,-1,-1;\frac{1}{a}\right) + 2H\left(1,0,-1,1;\frac{1}{a}\right) \\
& + 4H\left(1,0,0,-1;\frac{1}{a}\right) - 4H\left(1,0,0,1;\frac{1}{a}\right) + 2H\left(1,0,1,-1;\frac{1}{a}\right) \\
G(a^2,0,a^2,a;1) = & -2H\left(-1,0,-1,1;\frac{1}{a}\right) - 4H\left(-1,0,0,-1;\frac{1}{a}\right) \quad (G.189)
\end{aligned}$$

$$\begin{aligned}
 & + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
 & - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + H\left(1, -1, 0, 1; \frac{1}{a}\right) + 3H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & + 5H\left(1, 0, 0, -1; \frac{1}{a}\right) - 5H\left(1, 0, 0, 1; \frac{1}{a}\right) + 3H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
 & - 3H\left(1, 0, 1, 1; \frac{1}{a}\right) + 4H\left(1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
 G(a^2, 0, -a, a^2; 1) = & -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.190}
 \end{aligned}$$

$$\begin{aligned}
 & + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & - 3H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, -1, -1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
 G(a^2, 0, a, a^2; 1) = & 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) \tag{G.191}
 \end{aligned}$$

$$\begin{aligned}
 & + H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, 1, -1, 1; \frac{1}{a}\right) - 3H\left(0, 1, 0, -1; \frac{1}{a}\right) + 3H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 1, 1; \frac{1}{a}\right) + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - 4H\left(1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) - 4H\left(1, 1, 0, -1; \frac{1}{a}\right) \\
 G(a^2, -a, 0, 0; 1) = & 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \tag{G.192} \\
 & + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& -H\left(0,0,-1,1;\frac{1}{a}\right) - 7H\left(0,0,0,-1;\frac{1}{a}\right) + 7H\left(0,0,0,1;\frac{1}{a}\right) \\
& - H\left(0,0,1,-1;\frac{1}{a}\right) - H\left(0,1,0,-1;\frac{1}{a}\right) - H\left(1,0,0,-1;\frac{1}{a}\right) \\
G(a^2, a, 0, 0; 1) = & -H\left(-1,0,0,1;\frac{1}{a}\right) - H\left(0,-1,0,1;\frac{1}{a}\right) - H\left(0,0,-1,1;\frac{1}{a}\right) \quad (\text{G.193}) \\
& - 7H\left(0,0,0,-1;\frac{1}{a}\right) + 7H\left(0,0,0,1;\frac{1}{a}\right) - H\left(0,0,1,-1;\frac{1}{a}\right) \\
& + H\left(0,0,1,1;\frac{1}{a}\right) - 2H\left(0,1,0,-1;\frac{1}{a}\right) + 2H\left(0,1,0,1;\frac{1}{a}\right) \\
& - 4H\left(1,0,0,-1;\frac{1}{a}\right) + 4H\left(1,0,0,1;\frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, -a, 0, 1; 1) = & -\frac{1}{6}\pi^2 H\left(-1,-1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(-1,1;\frac{1}{a}\right) \quad (\text{G.194}) \\
& + \frac{1}{6}\pi^2 H\left(0,-1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0,1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(1,-1;\frac{1}{a}\right) - 2H\left(-1,0,-1,-1;\frac{1}{a}\right) \\
& + 2H\left(-1,0,-1,1;\frac{1}{a}\right) + 4H\left(-1,0,0,-1;\frac{1}{a}\right) - 4H\left(-1,0,0,1;\frac{1}{a}\right) \\
& + 2H\left(-1,0,1,-1;\frac{1}{a}\right) - 2H\left(-1,0,1,1;\frac{1}{a}\right) - H\left(0,-1,-1,-1;\frac{1}{a}\right) \\
& + H\left(0,-1,-1,1;\frac{1}{a}\right) + H\left(0,-1,0,-1;\frac{1}{a}\right) - H\left(0,-1,0,1;\frac{1}{a}\right) \\
& + H\left(0,-1,1,-1;\frac{1}{a}\right) + 4H\left(0,0,-1,-1;\frac{1}{a}\right) - 4H\left(0,0,-1,1;\frac{1}{a}\right) \\
& - 7H\left(0,0,0,-1;\frac{1}{a}\right) + 7H\left(0,0,0,1;\frac{1}{a}\right) - 4H\left(0,0,1,-1;\frac{1}{a}\right) \\
& + 3H\left(0,0,1,1;\frac{1}{a}\right) + H\left(0,1,-1,-1;\frac{1}{a}\right) - H\left(0,1,0,-1;\frac{1}{a}\right) \\
& + H\left(1,0,-1,-1;\frac{1}{a}\right) - H\left(1,0,0,-1;\frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, a, 0, 1; 1) = & \frac{1}{6}\pi^2 H\left(-1,1;\frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0,-1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0,1;\frac{1}{a}\right) \quad (\text{G.195}) \\
& + \frac{1}{6}\pi^2 H\left(1,-1;\frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1,1;\frac{1}{a}\right) - H\left(-1,0,0,1;\frac{1}{a}\right) - H\left(-1,0,1,1;\frac{1}{a}\right) \\
& - H\left(0,-1,0,1;\frac{1}{a}\right) - H\left(0,-1,1,1;\frac{1}{a}\right) + 3H\left(0,0,-1,-1;\frac{1}{a}\right) - 4H\left(0,0,-1,1;\frac{1}{a}\right) \\
& - 7H\left(0,0,0,-1;\frac{1}{a}\right) + 7H\left(0,0,0,1;\frac{1}{a}\right) - 4H\left(0,0,1,-1;\frac{1}{a}\right) + 4H\left(0,0,1,1;\frac{1}{a}\right) \\
& - H\left(0,1,-1,1;\frac{1}{a}\right) - H\left(0,1,0,-1;\frac{1}{a}\right) + H\left(0,1,0,1;\frac{1}{a}\right) - H\left(0,1,1,-1;\frac{1}{a}\right) \\
& + H\left(0,1,1,1;\frac{1}{a}\right) + 2H\left(1,0,-1,-1;\frac{1}{a}\right) - 2H\left(1,0,-1,1;\frac{1}{a}\right) - 4H\left(1,0,0,-1;\frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
 & + 4H\left(1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
 G(a^2, -a, 0, a^2; 1) = & 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.196}
 \end{aligned}$$

$$\begin{aligned}
 & - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, -1, -1, 1; \frac{1}{a}\right) - 3H\left(0, -1, 0, -1; \frac{1}{a}\right) + 3H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, -1; \frac{1}{a}\right) + H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
 & - H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a^2, a, 0, a^2; 1) = H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) \tag{G.197}$$

$$\begin{aligned}
 & + H\left(-1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, -1, 0, -1; \frac{1}{a}\right) + H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
 & + 3H\left(0, 1, 0, -1; \frac{1}{a}\right) - 3H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
 & - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) + 2H\left(1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a^2, -a, 1, 0; 1) = -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.198}$$

$$\begin{aligned}
 & - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
 & - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
 & + 4H\left(0, -1, 0, -1; \frac{1}{a}\right) - 4H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, 0, -1, 1; \frac{1}{a}\right) - 7H\left(0, 0, 0, -1; \frac{1}{a}\right) + 7H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
 & - 3H\left(0, 1, 0, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a^2, a, 1, 0; 1) = -H\left(-1, 0, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) \tag{G.199}$$

$$\begin{aligned}
 & + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) - 3H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
 & - 7H\left(0, 0, 0, -1; \frac{1}{a}\right) + 7H\left(0, 0, 0, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& + H\left(0, 0, 1, 1; \frac{1}{a}\right) - 4H\left(0, 1, 0, -1; \frac{1}{a}\right) + 4H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
& - H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
& + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
& - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(a^2, -a, 1, a^2; 1) = & H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.200} \\
& - H\left(-1, -1, 0, -1; \frac{1}{a}\right) + H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, -1, 1, -1; \frac{1}{a}\right) \\
& - 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
& - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
& - H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
& - 3H\left(0, 1, 0, -1; \frac{1}{a}\right) + 3H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
& + H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
& + H\left(1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) - 4H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
& + 4H\left(1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
& - 4H\left(1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(a^2, a, 1, a^2; 1) = & -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.201} \\
& - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
& - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
& + H\left(0, -1, -1, 1; \frac{1}{a}\right) + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) - 3H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
 & + H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) \\
 & + H\left(1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
 & + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 1, -1, 1; \frac{1}{a}\right) - H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \\
 G(a^2, a^2, 0, -a; 1) & = H\left(-1, -1, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right) \tag{G.202}
 \end{aligned}$$

$$\begin{aligned}
 & + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 0, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(0, -1, -1, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
 & + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
 & + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
 & - H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
 & - H\left(1, -1, 0, -1; \frac{1}{a}\right) + H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 0, -1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 0, -1; \frac{1}{a}\right) \\
 G(a^2, a^2, 0, a; 1) & = -H\left(-1, -1, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) \tag{G.203}
 \end{aligned}$$

$$\begin{aligned}
 & + H\left(-1, 0, 0, -1; \frac{1}{a}\right) - H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
 & - H\left(-1, 0, 1, 1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) \\
 & + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) \\
 & - H\left(0, -1, 1, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
 & - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& + H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) - H\left(1, 0, 0, -1; \frac{1}{a}\right) \\
& + H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(a^2, a^2, 1, -a; 1) = & H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.204} \\
& - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) - 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) \\
& + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 1, -1; \frac{1}{a}\right) \\
& - H\left(-1, 1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) \\
& + H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
& - H\left(0, 0, -1, 1; \frac{1}{a}\right) - H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) \\
& + H\left(1, -1, -1, 1; \frac{1}{a}\right) + H\left(1, -1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) \\
& + H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
& + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) - H\left(1, 1, -1, 1; \frac{1}{a}\right) \\
& - H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, a^2, 1, a; 1) = & -H\left(-1, -1, -1, 1; \frac{1}{a}\right) - H\left(-1, -1, 0, -1; \frac{1}{a}\right) \tag{G.205} \\
& + H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
& - H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) \\
& + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) \\
& + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
& -H\left(0, -1, 1, 1; \frac{1}{a}\right) + H\left(0, 0, -1, -1; \frac{1}{a}\right) - H\left(0, 0, -1, 1; \frac{1}{a}\right) \\
& -H\left(0, 0, 1, -1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
& -H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
& -H\left(1, -1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, -1, 1; \frac{1}{a}\right) + H\left(1, -1, 1, -1; \frac{1}{a}\right) \\
& -H\left(1, -1, 1, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
& -2H\left(1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) \\
& -H\left(1, 1, -1, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \\
G(a^2, a^2, -a, 0; 1) = & -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.206} \\
& -2H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
& -2H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
& -H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
& -2H\left(0, -1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
& -3H\left(0, 1, 0, -1; \frac{1}{a}\right) + 3H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
& + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
& -H\left(1, 0, -1, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& -H\left(1, 0, 1, -1; \frac{1}{a}\right) - H\left(1, 1, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, a^2, a, 0; 1) = & H\left(-1, -1, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) \tag{G.207} \\
& + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
& -H\left(-1, 0, 1, 1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, 1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
& + H\left(0, -1, -1, 1; \frac{1}{a}\right) + 3H\left(0, -1, 0, -1; \frac{1}{a}\right) - 3H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) \\
& - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) + 3H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
& + 2H\left(1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(1, -1, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(a^2, a^2, -a, 1; 1) & = H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.208}
\end{aligned}$$

$$\begin{aligned}
& - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right) \\
& + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) - H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
& - H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - 2H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
& + 2H\left(0, -1, -1, 1; \frac{1}{a}\right) + 2H\left(0, -1, 1, -1; \frac{1}{a}\right) - 2H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
& + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) + 3H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, -1, -1; \frac{1}{a}\right) - 2H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 0, -1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(0, 1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, -1, 1; \frac{1}{a}\right) \\
& + H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) + H\left(1, -1, 1, -1; \frac{1}{a}\right) \\
& + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - 2H\left(1, 0, -1, 1; \frac{1}{a}\right) - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) - H\left(1, 1, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, a^2, a, 1; 1) & = H\left(-1, -1, 0, 1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) \tag{G.209} \\
& - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + 2H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
 & -3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1, -1; \frac{1}{a}\right) \\
 & - H\left(-1, 1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) + 2H\left(0, -1, -1, 1; \frac{1}{a}\right) \\
 & + H\left(0, -1, 0, -1; \frac{1}{a}\right) - H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 1, -1; \frac{1}{a}\right) - 2H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
 & + 3H\left(0, 0, -1, -1; \frac{1}{a}\right) - 3H\left(0, 0, -1, 1; \frac{1}{a}\right) - 3H\left(0, 0, 1, -1; \frac{1}{a}\right) + 3H\left(0, 0, 1, 1; \frac{1}{a}\right) \\
 & + 2H\left(0, 1, -1, -1; \frac{1}{a}\right) - 2H\left(0, 1, -1, 1; \frac{1}{a}\right) - 2H\left(0, 1, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
 & + H\left(1, -1, -1, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(1, -1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) - H\left(1, 1, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a^2, a^2, a^2, -a; 1) = H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \quad (\text{G.210})$$

$$\begin{aligned}
 & - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) - H\left(-1, 1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) \\
 & - H\left(0, -1, -1, -1; \frac{1}{a}\right) + H\left(0, -1, -1, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) \\
 & - H\left(0, -1, 1, 1; \frac{1}{a}\right) + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, -1, 1; \frac{1}{a}\right) \\
 & + H\left(1, -1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, -1, -1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) - H\left(1, 1, -1, 1; \frac{1}{a}\right) \\
 & - H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$G(a^2, a^2, a^2, a; 1) = -H\left(-1, -1, -1, 1; \frac{1}{a}\right) - H\left(-1, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.211})$$

$$+ H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right)$$

$$\begin{aligned}
 & -H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
 & -H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(0, -1, -1, 1; \frac{1}{a}\right) + H\left(0, -1, 1, -1; \frac{1}{a}\right) - H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
 & + H\left(0, 1, -1, -1; \frac{1}{a}\right) - H\left(0, 1, -1, 1; \frac{1}{a}\right) - H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) + H\left(1, -1, -1, 1; \frac{1}{a}\right) \\
 & + H\left(1, -1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(1, 1, -1, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \\
 G(a^2, a^2, -a, a^2; 1) & = H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.212} \\
 & - H\left(-1, -1, 1, -1; \frac{1}{a}\right) + H\left(-1, -1, 1, 1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right) \\
 & + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) - H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & - H\left(-1, 1, -1, -1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
 & - H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(1, -1, -1, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(1, -1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) + H\left(1, 1, 0, -1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
 G(a^2, a^2, a, a^2; 1) & = H\left(-1, -1, 0, -1; \frac{1}{a}\right) - H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.213} \\
 & + H\left(-1, -1, 1, 1; \frac{1}{a}\right) + H\left(-1, 1, -1, 1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(-1, 1, 1, -1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) + H\left(1, -1, -1, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
 & - H\left(1, -1, 0, 1; \frac{1}{a}\right) + H\left(1, -1, 1, -1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
 & - H\left(1, 0, -1, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(1, 1, -1, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, -a, a^2, 0; 1) = & -2H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.214} \\
 & - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, -1, 1; \frac{1}{a}\right) + 3H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
 & - 3H\left(-1, 0, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
 & - 2H\left(0, -1, -1, -1; \frac{1}{a}\right) + 2H\left(0, -1, -1, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
 & + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(1, 0, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, a, a^2, 0; 1) = & -4H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) \tag{G.215} \\
 & - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(-1, 1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 1, 1; \frac{1}{a}\right) - 2H\left(0, 1, -1, 1; \frac{1}{a}\right) \\
 & - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - 2H\left(0, 1, 1, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & - 3H\left(1, 0, 0, -1; \frac{1}{a}\right) + 3H\left(1, 0, 0, 1; \frac{1}{a}\right) - H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
 & + H\left(1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(1, 1, 0, 1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
 G(a^2, -a, a^2, 1; 1) = & H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.216} \\
 & - H\left(-1, -1, 0, -1; \frac{1}{a}\right) + H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, -1, 1, -1; \frac{1}{a}\right) \\
 & - H\left(-1, 1, -1, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
 & - 2H\left(0, -1, -1, -1; \frac{1}{a}\right) + 2H\left(0, -1, -1, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
 & - 2H\left(0, -1, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
 & + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) - H\left(1, -1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(1, -1, 0, -1; \frac{1}{a}\right) + H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, -1; \frac{1}{a}\right)
 \end{aligned}$$

$$\begin{aligned}
& + 3H\left(1, 0, -1, 1; \frac{1}{a}\right) + 7H\left(1, 0, 0, -1; \frac{1}{a}\right) - 7H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
& + 3H\left(1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(1, 0, 1, 1; \frac{1}{a}\right) + 4H\left(1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(a^2, a, a^2, 1; 1) & = 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.217}
\end{aligned}$$

$$\begin{aligned}
& + 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) - 3H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 7H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
& + 7H\left(-1, 0, 0, 1; \frac{1}{a}\right) - 3H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
& - H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) \\
& - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) - 2H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
& - 2H\left(0, 1, -1, 1; \frac{1}{a}\right) - 2H\left(0, 1, 0, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
& - 2H\left(0, 1, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
& + H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) - H\left(1, 1, -1, 1; \frac{1}{a}\right) \\
& - H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \\
G(a^2, -a, a^2, a^2; 1) & = H\left(-1, -1, -1, -1; \frac{1}{a}\right) - H\left(-1, -1, -1, 1; \frac{1}{a}\right) \tag{G.218}
\end{aligned}$$

$$\begin{aligned}
& - H\left(-1, -1, 0, -1; \frac{1}{a}\right) + H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, -1, 1, -1; \frac{1}{a}\right) \\
& - H\left(-1, 1, -1, -1; \frac{1}{a}\right) - H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
& - H\left(1, -1, -1, -1; \frac{1}{a}\right) - H\left(1, -1, 0, -1; \frac{1}{a}\right) + H\left(1, -1, 0, 1; \frac{1}{a}\right) \\
& - H\left(1, 0, -1, -1; \frac{1}{a}\right) + H\left(1, 0, -1, 1; \frac{1}{a}\right) + H\left(1, 0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(a^2, a, a^2, a^2; 1) & = H\left(-1, 0, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1, 1; \frac{1}{a}\right) \tag{G.219} \\
& - H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(-1, 1, 0, -1; \frac{1}{a}\right) \\
& + H\left(-1, 1, 0, 1; \frac{1}{a}\right) - H\left(-1, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
& + H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, -1, 1, 1; \frac{1}{a}\right) - H\left(1, 1, -1, 1; \frac{1}{a}\right) \\
& - H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) - H\left(1, 1, 1, -1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right)
\end{aligned}$$

$$G(a, a, 0, 0; 1) = H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 0, 1; \frac{1}{a}\right) \quad (\text{G.220})$$

$$G(a, a, 0, 1; 1) = -\frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) + H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) \\ + H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \quad (\text{G.221})$$

$$G(-a, -a, 0, a; 1) = -H\left(-1, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.222})$$

$$G(-a, a, 0, -a; 1) = -H\left(-1, 1, 0, -1; \frac{1}{a}\right) \quad (\text{G.223})$$

$$G(-a, a, 0, a; 1) = H\left(-1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.224})$$

$$G(a, -a, 0, -a; 1) = -H\left(1, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.225})$$

$$G(a, -a, 0, a; 1) = H\left(1, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.226})$$

$$G(a, a, 0, -a; 1) = H\left(1, 1, 0, -1; \frac{1}{a}\right) \quad (\text{G.227})$$

$$G(a, a, 0, a; 1) = -H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.228})$$

$$G(a, a, 1, 0; 1) = H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\ + H\left(1, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.229})$$

$$G(a, a, 1, 1; 1) = H\left(0, 0, 1, 1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\ + H\left(1, 1, 1, 1; \frac{1}{a}\right) \quad (\text{G.230})$$

$$G(-a, a^2, 0, 1; 1) = -\frac{1}{6}\pi^2 H\left(-1, -1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) \\ + \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right) + H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\ - 2H\left(0, -1, -1, -1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\ + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 12H\left(0, 0, 0, -1; \frac{1}{a}\right) - 12H\left(0, 0, 0, 1; \frac{1}{a}\right) \\ + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) + 4H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \quad (\text{G.231})$$

$$G(a, a^2, 0, 1; 1) = -\frac{1}{6}\pi^2 H\left(0, -1; \frac{1}{a}\right) + \frac{1}{6}\pi^2 H\left(0, 1; \frac{1}{a}\right) - \frac{1}{6}\pi^2 H\left(1, 1; \frac{1}{a}\right) \\ - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 4H\left(0, -1, 0, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) \quad (\text{G.232})$$

$$\begin{aligned}
& + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 12H\left(0, 0, 0, -1; \frac{1}{a}\right) - 12H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
& + H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) \\
G(-a, a^2, 1, 0; 1) & = -H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 2H\left(-1, 0, -1, -1; \frac{1}{a}\right) \tag{G.233}
\end{aligned}$$

$$\begin{aligned}
& - 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 12H\left(0, 0, 0, -1; \frac{1}{a}\right) - 12H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
& + 5H\left(1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
G(a, a^2, 1, 0; 1) & = -4H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) \tag{G.234}
\end{aligned}$$

$$\begin{aligned}
& - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
& + 12H\left(0, 0, 0, -1; \frac{1}{a}\right) - 12H\left(0, 0, 0, 1; \frac{1}{a}\right) + 4H\left(0, 0, 1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
& + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
G(-a, a^2, 1, 1; 1) & = H\left(-1, -1, -1, -1; \frac{1}{a}\right) + H\left(-1, -1, 0, -1; \frac{1}{a}\right) \tag{G.235}
\end{aligned}$$

$$\begin{aligned}
& - H\left(-1, -1, 0, 1; \frac{1}{a}\right) - H\left(-1, 0, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1, 1; \frac{1}{a}\right) \\
& - 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) + 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) - H\left(-1, 0, 1, -1; \frac{1}{a}\right) \\
& + H\left(-1, 0, 1, 1; \frac{1}{a}\right) - 2H\left(-1, 1, 0, -1; \frac{1}{a}\right) + H\left(-1, 1, 0, 1; \frac{1}{a}\right) \\
& - H\left(0, -1, -1, -1; \frac{1}{a}\right) - H\left(0, -1, -1, 1; \frac{1}{a}\right) - 4H\left(0, -1, 0, -1; \frac{1}{a}\right) \\
& + 4H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 12H\left(0, 0, 0, -1; \frac{1}{a}\right) - 12H\left(0, 0, 0, 1; \frac{1}{a}\right) \\
& + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
& + H\left(0, 1, -1, 1; \frac{1}{a}\right) + 6H\left(0, 1, 0, -1; \frac{1}{a}\right) - 4H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, -1; \frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
& -H\left(0,1,1,1;\frac{1}{a}\right) - 2H\left(1,-1,0,-1;\frac{1}{a}\right) + H\left(1,-1,0,1;\frac{1}{a}\right) + H\left(1,0,-1,1;\frac{1}{a}\right) \\
& + 5H\left(1,0,0,-1;\frac{1}{a}\right) - 4H\left(1,0,0,1;\frac{1}{a}\right) + H\left(1,0,1,-1;\frac{1}{a}\right) - H\left(1,0,1,1;\frac{1}{a}\right) \\
& + 2H\left(1,1,0,-1;\frac{1}{a}\right) - H\left(1,1,0,1;\frac{1}{a}\right) \\
G(a,a^2,1,1;1) & = H\left(-1,-1,0,-1;\frac{1}{a}\right) - 2H\left(-1,-1,0,1;\frac{1}{a}\right) \tag{G.236}
\end{aligned}$$

$$\begin{aligned}
& + H\left(-1,0,-1,-1;\frac{1}{a}\right) - H\left(-1,0,-1,1;\frac{1}{a}\right) - 4H\left(-1,0,0,-1;\frac{1}{a}\right) \\
& + 5H\left(-1,0,0,1;\frac{1}{a}\right) - H\left(-1,0,1,-1;\frac{1}{a}\right) - H\left(-1,1,0,-1;\frac{1}{a}\right) \\
& + 2H\left(-1,1,0,1;\frac{1}{a}\right) + H\left(0,-1,-1,-1;\frac{1}{a}\right) - H\left(0,-1,-1,1;\frac{1}{a}\right) \\
& - 4H\left(0,-1,0,-1;\frac{1}{a}\right) + 6H\left(0,-1,0,1;\frac{1}{a}\right) - H\left(0,-1,1,-1;\frac{1}{a}\right) \\
& + H\left(0,-1,1,1;\frac{1}{a}\right) - 4H\left(0,0,-1,-1;\frac{1}{a}\right) + 4H\left(0,0,-1,1;\frac{1}{a}\right) \\
& + 12H\left(0,0,0,-1;\frac{1}{a}\right) - 12H\left(0,0,0,1;\frac{1}{a}\right) + 4H\left(0,0,1,-1;\frac{1}{a}\right) \\
& - H\left(0,1,-1,-1;\frac{1}{a}\right) + H\left(0,1,-1,1;\frac{1}{a}\right) + 4H\left(0,1,0,-1;\frac{1}{a}\right) \\
& - 4H\left(0,1,0,1;\frac{1}{a}\right) + H\left(0,1,1,-1;\frac{1}{a}\right) + H\left(0,1,1,1;\frac{1}{a}\right) \\
& - H\left(1,-1,0,-1;\frac{1}{a}\right) + 2H\left(1,-1,0,1;\frac{1}{a}\right) - H\left(1,0,-1,-1;\frac{1}{a}\right) \\
& + H\left(1,0,-1,1;\frac{1}{a}\right) + 4H\left(1,0,0,-1;\frac{1}{a}\right) - 4H\left(1,0,0,1;\frac{1}{a}\right) \\
& + H\left(1,0,1,-1;\frac{1}{a}\right) + H\left(1,0,1,1;\frac{1}{a}\right) + H\left(1,1,0,-1;\frac{1}{a}\right) \\
& - H\left(1,1,0,1;\frac{1}{a}\right) + H\left(1,1,1,1;\frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
G(-a,a^2,1,a^2;1) & = H\left(-1,-1,-1,-1;\frac{1}{a}\right) + H\left(-1,-1,0,-1;\frac{1}{a}\right) \tag{G.237} \\
& - H\left(-1,-1,0,1;\frac{1}{a}\right) - H\left(-1,0,-1,-1;\frac{1}{a}\right) - H\left(-1,0,-1,1;\frac{1}{a}\right) \\
& - 4H\left(-1,0,0,-1;\frac{1}{a}\right) + 4H\left(-1,0,0,1;\frac{1}{a}\right) - H\left(-1,0,1,-1;\frac{1}{a}\right) \\
& + H\left(-1,0,1,1;\frac{1}{a}\right) - 2H\left(-1,1,0,-1;\frac{1}{a}\right) + H\left(-1,1,0,1;\frac{1}{a}\right) \\
& + H\left(0,-1,-1,-1;\frac{1}{a}\right) - H\left(0,-1,-1,1;\frac{1}{a}\right) - 2H\left(0,-1,0,-1;\frac{1}{a}\right)
\end{aligned}$$

$$\begin{aligned}
 & + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) - H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, -1, 1, 1; \frac{1}{a}\right) \\
 & - H\left(0, 1, -1, -1; \frac{1}{a}\right) + H\left(0, 1, -1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) \\
 & - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) + H\left(0, 1, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) \\
 & - 2H\left(1, -1, 0, -1; \frac{1}{a}\right) + H\left(1, -1, 0, 1; \frac{1}{a}\right) + 2H\left(1, 0, -1, -1; \frac{1}{a}\right) \\
 & + H\left(1, 0, -1, 1; \frac{1}{a}\right) + 5H\left(1, 0, 0, -1; \frac{1}{a}\right) - 5H\left(1, 0, 0, 1; \frac{1}{a}\right) \\
 & + H\left(1, 0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right) + 4H\left(1, 1, 0, -1; \frac{1}{a}\right) - 2H\left(1, 1, 0, 1; \frac{1}{a}\right) \\
 G(a, a^2, 1, a^2; 1) = & 2H\left(-1, -1, 0, -1; \frac{1}{a}\right) - 4H\left(-1, -1, 0, 1; \frac{1}{a}\right) \tag{G.238} \\
 & + H\left(-1, 0, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1, 1; \frac{1}{a}\right) - 5H\left(-1, 0, 0, -1; \frac{1}{a}\right) \\
 & + 5H\left(-1, 0, 0, 1; \frac{1}{a}\right) - H\left(-1, 0, 1, -1; \frac{1}{a}\right) - 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\
 & - H\left(-1, 1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, 1, 0, 1; \frac{1}{a}\right) + H\left(0, -1, -1, -1; \frac{1}{a}\right) \\
 & - H\left(0, -1, -1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\
 & - H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, -1, 1, 1; \frac{1}{a}\right) - H\left(0, 1, -1, -1; \frac{1}{a}\right) \\
 & + H\left(0, 1, -1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \\
 & + H\left(0, 1, 1, -1; \frac{1}{a}\right) - H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, -1, 0, -1; \frac{1}{a}\right) \\
 & + 2H\left(1, -1, 0, 1; \frac{1}{a}\right) - H\left(1, 0, -1, -1; \frac{1}{a}\right) + H\left(1, 0, -1, 1; \frac{1}{a}\right) \\
 & + 4H\left(1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(1, 0, 0, 1; \frac{1}{a}\right) + H\left(1, 0, 1, -1; \frac{1}{a}\right) \\
 & + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, -1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \\
 G(a, a, a, 0; 1) = & H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 0, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \tag{G.239} \\
 G(a, a, a, 1; 1) = & H\left(0, 1, 1, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \tag{G.240} \\
 G(a, a, a, a; 1) = & H\left(1, 1, 1, 1; \frac{1}{a}\right) \tag{G.241}
 \end{aligned}$$

$$G(-a, a^2, a^2, 1; 1) = H\left(-1, -1, -1, -1; \frac{1}{a}\right) + H\left(-1, -1, 0, -1; \frac{1}{a}\right) \quad (\text{G.242})$$

$$\begin{aligned} & - H\left(-1, -1, 0, 1; \frac{1}{a}\right) + H\left(-1, 0, -1, -1; \frac{1}{a}\right) - H\left(-1, 0, -1, 1; \frac{1}{a}\right) \\ & - H\left(-1, 0, 1, -1; \frac{1}{a}\right) + H\left(-1, 0, 1, 1; \frac{1}{a}\right) - H\left(0, -1, -1, -1; \frac{1}{a}\right) \\ & - H\left(0, -1, -1, 1; \frac{1}{a}\right) - 2H\left(0, -1, 0, -1; \frac{1}{a}\right) + 2H\left(0, -1, 0, 1; \frac{1}{a}\right) \\ & - H\left(0, -1, 1, -1; \frac{1}{a}\right) + H\left(0, -1, 1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) \\ & + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) \\ & - H\left(0, 1, -1, -1; \frac{1}{a}\right) + H\left(0, 1, -1, 1; \frac{1}{a}\right) + H\left(0, 1, 1, -1; \frac{1}{a}\right) \\ & - H\left(0, 1, 1, 1; \frac{1}{a}\right) - 2H\left(1, 0, -1, -1; \frac{1}{a}\right) - 4H\left(1, 0, 0, -1; \frac{1}{a}\right) \\ & + 4H\left(1, 0, 0, 1; \frac{1}{a}\right) - 2H\left(1, 1, 0, -1; \frac{1}{a}\right) + H\left(1, 1, 0, 1; \frac{1}{a}\right) \end{aligned}$$

$$G(a, a^2, a^2, 1; 1) = -H\left(-1, -1, 0, -1; \frac{1}{a}\right) + 2H\left(-1, -1, 0, 1; \frac{1}{a}\right) \quad (\text{G.243})$$

$$\begin{aligned} & + 4H\left(-1, 0, 0, -1; \frac{1}{a}\right) - 4H\left(-1, 0, 0, 1; \frac{1}{a}\right) + 2H\left(-1, 0, 1, 1; \frac{1}{a}\right) \\ & + H\left(0, -1, -1, -1; \frac{1}{a}\right) - H\left(0, -1, -1, 1; \frac{1}{a}\right) - H\left(0, -1, 1, -1; \frac{1}{a}\right) \\ & + H\left(0, -1, 1, 1; \frac{1}{a}\right) - 4H\left(0, 0, -1, -1; \frac{1}{a}\right) + 4H\left(0, 0, -1, 1; \frac{1}{a}\right) \\ & + 4H\left(0, 0, 1, -1; \frac{1}{a}\right) - 4H\left(0, 0, 1, 1; \frac{1}{a}\right) - H\left(0, 1, -1, -1; \frac{1}{a}\right) \\ & + H\left(0, 1, -1, 1; \frac{1}{a}\right) + 2H\left(0, 1, 0, -1; \frac{1}{a}\right) - 2H\left(0, 1, 0, 1; \frac{1}{a}\right) \\ & + H\left(0, 1, 1, -1; \frac{1}{a}\right) + H\left(0, 1, 1, 1; \frac{1}{a}\right) - H\left(1, 0, -1, -1; \frac{1}{a}\right) \\ & + H\left(1, 0, -1, 1; \frac{1}{a}\right) + H\left(1, 0, 1, -1; \frac{1}{a}\right) - H\left(1, 0, 1, 1; \frac{1}{a}\right) \\ & + H\left(1, 1, 0, -1; \frac{1}{a}\right) - H\left(1, 1, 0, 1; \frac{1}{a}\right) + H\left(1, 1, 1, 1; \frac{1}{a}\right) \end{aligned}$$

H The analytic expression of the remainder function

In this appendix we present the full analytic expression of the remainder function. The result is also available in electronic form from www.arXiv.org. Using the notation introduced in eqs. (3.23) and (5.7), the full expression reads,

$$\begin{aligned}
 R_{6,WL}^{(2)}(u_1, u_2, u_3) = & \quad (H.1) \\
 & \frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) + \\
 & \frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) + \\
 & \frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{24}\pi^2 G\left(\frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) + \\
 & \frac{3}{2}G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) + \\
 & \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_2}; 1\right) + G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_3}; 1\right) + \\
 & G\left(0, \frac{1}{u_1}, 0, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_1}, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) + \\
 & G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_3}; 1\right) + G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_2+u_3}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_2}, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, 0, \frac{1}{1-u_1}; 1\right) + \\
 & \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 0; 1\right) - \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 1; 1\right) + \\
 & \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_1}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_2}; 1\right) + G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_1+u_3}; 1\right) + \\
 & G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) + \\
 & \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, 0, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 0; 1\right) - \\
 & \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) - \\
 & \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, 0, \frac{1}{1-u_2}; 1\right) + \\
 & \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 1; 1\right) + \\
 & \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) - \\
 & \frac{1}{4}G\left(\frac{1}{1-u_1}, 1, \frac{1}{u_3}, 0; 1\right) + \frac{1}{2}G\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, 1, \frac{1}{1-u_1}; 1\right) + \\
 & \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 0, 1; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 0, \frac{1}{1-u_1}; 1\right) +
 \end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 1, 0; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 0; 1\right) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 1; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) - G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_2}; 1\right) + \\
& \frac{1}{2}G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_3}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1+u_3}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_2}, 1, \frac{1}{u_1}, 0; 1\right) + \\
& \frac{1}{2}G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 0, 1; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 0, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 1, 0; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 0; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 1; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, 1; 1\right) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) - G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1}; 1\right) + \\
& \frac{1}{2}G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_3}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_2+u_3}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{1-u_3}, 1, \frac{1}{u_2}, 0; 1\right) + \\
& \frac{1}{2}G\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, 1, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 0, 1; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 0, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 1, 0; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 0; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 1; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{u_1-1}{u_1+u_3-1}, 1; 1\right) - \frac{79\pi^4}{360} + \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) - G\left(\frac{1}{u_3}, 0, 0, \frac{1}{u_1}; 1\right) - \\
& G\left(\frac{1}{u_3}, 0, 0, \frac{1}{u_2}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_3}, 0, 0, \frac{1}{u_1+u_3}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_3}, 0, 0, \frac{1}{u_2+u_3}; 1\right) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) - \\
& \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}; 1\right) + \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right) + \\
& \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}; 1\right) - \frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}; 1\right) + \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) + \\
& \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}; 1\right) - \frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}; 1\right) + \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}; 1\right) + \\
& \frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_1}, v_{123}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_1}, v_{132}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_2}, v_{213}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_2}, v_{231}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_3}, v_{312}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, 0, \frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, v_{123}, \frac{1}{1-u_1}; 1\right) + \mathcal{G}\left(0, 0, v_{132}, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, 0, v_{132}, \frac{1}{1-u_1}; 1\right) + \mathcal{G}\left(0, 0, v_{213}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, v_{213}, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, 0, v_{231}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, 0, v_{312}, \frac{1}{1-u_3}; 1\right) + \mathcal{G}\left(0, 0, v_{321}, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, 0, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, 0, v_{123}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, 0, v_{132}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{123}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{132}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, 0, v_{213}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, 0, v_{231}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{231}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, 0, v_{312}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, 0, v_{321}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{312}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{321}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{123}, 0, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{u_3}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, 0, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{u_1}, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_2}, 0; 1\right) + \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_2}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{u_3-1}{u_2+u_3-1}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{312}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{u_2}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{1-u_3}, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{1-u_3}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{u_1-1}{u_1+u_3-1}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(0, v_{123}, 0, \frac{1}{1-u_1}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, v_{123}, 1, \frac{1}{1-u_1}; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{132}, 0, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, v_{132}, \frac{1}{1-u_1}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{132}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{213}, 0, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{231}, 0, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(0, v_{231}, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{312}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{2}\mathcal{G}\left(0, v_{312}, 1, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(0, v_{321}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{321}, \frac{1}{1-u_3}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(0, v_{321}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, 0, v_{123}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, 0, v_{132}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, 0, \frac{1}{1-u_1}, v_{123}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, 0, \frac{1}{1-u_1}, v_{132}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{123}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{123}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{132}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{132}, \frac{1}{1-u_1}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, 0, v_{123}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, 0, v_{132}; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}; 1\right) - \\
& \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 1, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{u_3}, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 0, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 0, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 1, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 1, \frac{1}{1-u_1}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 0, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 0, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 1, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 1, \frac{1}{1-u_1}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, 0, v_{213}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, 0, v_{231}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, v_{213}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, v_{231}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}, \frac{1}{1-u_2}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, v_{213}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 0, v_{231}; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) - \\
& \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, 0, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, 0, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, 1, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{u_1}, 0; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{1-u_2}, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{u_3-1}{u_2+u_3-1}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1, \frac{1}{1-u_2}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, 1; 1\right) +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 1, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, 0, v_{312}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, 0, v_{321}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, 0, \frac{1}{1-u_3}, v_{312}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, 0, \frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, 0, v_{312}; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, 0, v_{321}; 1\right) - \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) - \\
& \frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 0, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 1, 0; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{u_2}, 0; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{u_1-1}{u_1+u_3-1}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, 0; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0, \frac{1}{1-u_3}; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 1, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 1, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, 0; 1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, 1; 1\right) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, 0; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 1, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 1, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}, 0; 1\right) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\mathcal{G}\left(v_{123}, 0, 1, \frac{1}{1-u_1}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, 0, \frac{1}{1-u_1}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, 1, 0, \frac{1}{1-u_1}; 1\right) - \\
& \frac{5}{4}\mathcal{G}\left(v_{123}, 1, 1, \frac{1}{1-u_1}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}, 0; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}, 1; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}, \frac{1}{1-u_1}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 0, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1, 0; 1\right) - \\
& \frac{5}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1, \frac{1}{1-u_1}; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{132}, \frac{1}{1-u_1}, \frac{1}{1-u_1}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{132}, 1, 1, \frac{1}{1-u_1}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-u_1}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(v_{132}, \frac{1}{1-u_1}, 1, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{213}, 1, 1, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 0, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 0, \frac{1}{1-u_2}, 1; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{231}, 1, 0, \frac{1}{1-u_2}; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{231}, 1, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}, 0; 1\right) - \\
& \frac{5}{4}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 0, 1; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1, 0; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, 0, 1, \frac{1}{1-u_3}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, 0, \frac{1}{1-u_3}, 1; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{312}, 1, 0, \frac{1}{1-u_3}; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{312}, 1, 1, \frac{1}{1-u_3}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}, 0; 1\right) - \\
& \frac{5}{4}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}, \frac{1}{1-u_3}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 0, 1; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1, 0; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1, \frac{1}{1-u_3}; 1\right) + \\
& \frac{1}{2}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, 1, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}, 1; 1\right) - \\
& \frac{1}{4}\mathcal{G}\left(v_{321}, \frac{1}{1-u_3}, 1, 1; 1\right) - \frac{3}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) - \\
& \frac{3}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) - \frac{1}{4}G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) - \\
& \frac{1}{4}G\left(0, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) - \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) H(0; u_1) + \\
& \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) H(0; u_1) - \frac{3}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) - \\
& \frac{3}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) + \frac{1}{2}G\left(\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) + \\
& \frac{1}{2}G\left(\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) + \\
& \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) - \frac{1}{4}G\left(\frac{1}{1-u_2}, 1, \frac{1}{u_1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 1; 1\right) H(0; u_1) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \frac{1}{2}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) H(0; u_1) - \\
& \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 0; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) H(0; u_1) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{u_1-1}{u_1+u_3-1}; 1\right) H(0; u_1) + \frac{1}{2}G\left(\frac{1}{u_3}, 0, \frac{1}{u_1}; 1\right) H(0; u_1) - \\
& \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{u_1}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{u_1-1}{u_1+u_3-1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(0, v_{132}, \frac{1}{1-u_1}; 1\right) H(0; u_1) - \\
& \frac{1}{2}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \frac{1}{2}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{123}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{132}; 1\right) H(0; u_1) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_1) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 1; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 1; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}; 1\right) H(0; u_1) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_1) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, 1; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{u_1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0; 1\right) H(0; u_1) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0; 1\right) H(0; u_1) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}; 1\right) H(0; u_1) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_1) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 0; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{u_1-1}{u_1+u_3-1}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0; 1\right) H(0; u_1) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0; 1\right) H(0; u_1) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(v_{132}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_1) - \\
& \frac{3}{4}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_1) - \frac{3}{4}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_1) + \\
& \frac{3}{4}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_1) + \frac{3}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_1) - \\
& \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(v_{321}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_1) - \\
& \frac{1}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) - \frac{3}{4}G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) - \\
& \frac{3}{4}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) - \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(0, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) + \frac{1}{4}G\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 0; 1\right) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}; 1\right) H(0; u_2) + \frac{1}{2}G\left(\frac{1}{u_1}, 0, \frac{1}{u_2}; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) - \\
& \frac{3}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) - \frac{3}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) + \frac{1}{2}G\left(\frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_2) + \\
& \frac{1}{2}G\left(\frac{1}{u_2}, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, 1, \frac{1}{u_2}; 1\right) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, 1; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \frac{1}{2}G\left(\frac{1}{u_3}, 0, \frac{1}{u_2}; 1\right) H(0; u_2) - \\
& \frac{1}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_2) +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{u_2-1}{u_1+u_2-1}; 1\right) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{u_2}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(0, u_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{2}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(0, v_{213}, \frac{1}{1-u_2}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_2) - \frac{1}{2}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{123}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{132}; 1\right) H(0; u_2) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_2) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 0; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{u_2-1}{u_1+u_2-1}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 0; 1\right) H(0; u_2) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 0; 1\right) H(0; u_2) - \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}; 1\right) H(0; u_2) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_2) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}; 1\right) H(0; u_2) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_2) + \\
& \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{u_2}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, u_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 0; 1\right) H(0; u_2) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 0; 1\right) H(0; u_2) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \\
& \frac{3}{4}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_2) + \frac{3}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(v_{132}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_2) +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_2) - \\
& \frac{3}{4}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_2) - \frac{3}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_2) + \frac{1}{4}\mathcal{G}\left(v_{321}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_2) + \\
& \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) H(0; u_2) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) H(0; u_1) H(0; u_2) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}; 1\right) H(0; u_1) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_1) H(0; u_2) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_1) H(0; u_2) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_1) H(0; u_2) + \frac{5}{24}\pi^2 H(0; u_1) H(0; u_2) - \\
& \frac{1}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) - \frac{1}{4}G\left(0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) + \\
& \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) H(0; u_3) - \frac{3}{4}G\left(0, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) - \\
& \frac{3}{4}G\left(0, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) H(0; u_3) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, 1, \frac{1}{u_3}; 1\right) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) H(0; u_3) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}; 1\right) H(0; u_3) + \frac{1}{2}G\left(\frac{1}{u_1}, 0, \frac{1}{u_3}; 1\right) H(0; u_3) - \\
& \frac{1}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, 0; 1\right) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right) H(0; u_3) - \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}; 1\right) H(0; u_3) + \frac{1}{2}G\left(\frac{1}{u_2}, 0, \frac{1}{u_3}; 1\right) H(0; u_3) - \\
& \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) - \\
& \frac{3}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) - \frac{3}{4}G\left(\frac{1}{u_3}, 0, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) + \\
& \frac{1}{2}G\left(\frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_3) + \frac{1}{2}G\left(\frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_3) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_3) - \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_3) + \\
& \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_3) -
\end{aligned}$$

$$\begin{aligned}
 & \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(0, u_{123}, \frac{1}{u_3}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{u_3-1}{u_2+u_3-1}; 1\right) H(0; u_3) - \\
 & \frac{1}{2}\mathcal{G}\left(0, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_3) + \frac{1}{2}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(0, v_{321}, \frac{1}{1-u_3}; 1\right) H(0; u_3) - \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{123}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, 0, v_{132}; 1\right) H(0; u_3) - \\
 & \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_3) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, \frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_3) - \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, 1; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}, \frac{1}{u_3}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, 0; 1\right) H(0; u_3) - \\
 & \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}, \frac{1}{1-u_1}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 0; 1\right) H(0; u_3) + \\
 & \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}; 1\right) H(0; u_3) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_3) + \\
 & \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, 0; 1\right) H(0; u_3) - \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}, \frac{u_3-1}{u_2+u_3-1}; 1\right) H(0; u_3) - \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, 0; 1\right) H(0; u_3) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}, \frac{1}{1-u_2}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, 0; 1\right) H(0; u_3) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}, \frac{1}{1-u_2}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{312}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, 0, v_{321}; 1\right) H(0; u_3) + \\
 & \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{312}; 1\right) H(0; u_3) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3}, \frac{1}{1-u_3}, v_{321}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, 1; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}, \frac{1}{1-u_3}; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, 1; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}, \frac{1}{1-u_3}; 1\right) H(0; u_3) - \\
 & \frac{3}{4}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_3) - \frac{3}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_3) + \\
 & \frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-u_1}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(v_{132}, \frac{1}{1-u_1}, 1; 1\right) H(0; u_3) - \\
 & \frac{1}{4}\mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_3) + \\
 & \frac{3}{4}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}; 1\right) H(0; u_3) + \frac{3}{4}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 1; 1\right) H(0; u_3) +
 \end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}\mathcal{G}\left(v_{312}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_3) + \\
& \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1\right) H(0; u_3) + \frac{1}{4}\mathcal{G}\left(v_{321}, \frac{1}{1-u_3}, 1; 1\right) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}; 1\right) H(0; u_1) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_1+u_3}; 1\right) H(0; u_1) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, u_{231}; 1\right) H(0; u_1) H(0; u_3) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) H(0; u_1) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}; 1\right) H(0; u_1) H(0; u_3) + \\
& \frac{5}{24}\pi^2 H(0; u_1) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}; 1\right) H(0; u_2) H(0; u_3) + \\
& \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) H(0; u_3) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, u_{123}; 1\right) H(0; u_2) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_2) H(0; u_3) - \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}; 1\right) H(0; u_2) H(0; u_3) + \frac{5}{24}\pi^2 H(0; u_2) H(0; u_3) + \\
& 3H(0; u_2) H(0, 0; u_1) H(0; u_3) + 3H(0; u_1) H(0, 0; u_2) H(0; u_3) + \\
& \frac{1}{4}H(0; u_2) H\left(0, 1; \frac{u_1+u_2-1}{u_2-1}\right) H(0; u_3) + \frac{1}{2}H(0; u_1) H(0, 1; (u_1+u_3)) H(0; u_3) + \\
& \frac{1}{4}H(0; u_1) H\left(0, 1; \frac{u_2+u_3-1}{u_3-1}\right) H(0; u_3) + \frac{1}{2}H(0; u_2) H(0, 1; (u_2+u_3)) H(0; u_3) + \\
& \frac{3}{4}H(0; u_2) H(1, 0; u_1) H(0; u_3) + \frac{3}{4}H(0; u_1) H(1, 0; u_2) H(0; u_3) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) H(0, 0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}; 1\right) H(0, 0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}; 1\right) H(0, 0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}; 1\right) H(0, 0; u_1) - \frac{23}{24}\pi^2 H(0, 0; u_1) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right) H(0, 0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}; 1\right) H(0, 0; u_2) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{312}; 1\right) H(0, 0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3}, v_{321}; 1\right) H(0, 0; u_2) - \\
& \frac{25}{4}H(0, 0; u_1) H(0, 0; u_2) - \frac{23}{24}\pi^2 H(0, 0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right) H(0, 0; u_3) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}; 1\right) H(0, 0; u_3) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{213}; 1\right) H(0, 0; u_3) + \\
& \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{231}; 1\right) H(0, 0; u_3) + 3H(0; u_1) H(0; u_2) H(0, 0; u_3) - \\
& \frac{25}{4}H(0, 0; u_1) H(0, 0; u_3) - \frac{25}{4}H(0, 0; u_2) H(0, 0; u_3) - \frac{23}{24}\pi^2 H(0, 0; u_3) + \frac{1}{12}\pi^2 H(0, 1; u_1) + \\
& \frac{1}{12}\pi^2 H(0, 1; u_2) - \frac{1}{24}\pi^2 H\left(0, 1; \frac{u_1+u_2-1}{u_2-1}\right) + \frac{1}{2}H(0; u_1) H(0; u_2) H(0, 1; (u_1+u_2)) + \\
& \frac{1}{12}\pi^2 H(0, 1; (u_1+u_2)) + \frac{1}{12}\pi^2 H(0, 1; u_3) + \frac{1}{4}H(0; u_1) H(0; u_2) H\left(0, 1; \frac{u_1+u_3-1}{u_1-1}\right) -
\end{aligned}$$

$$\begin{aligned}
 & \frac{1}{24}\pi^2 H\left(0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) + \frac{1}{12}\pi^2 H(0, 1; (u_1 + u_3)) - \frac{1}{24}\pi^2 H\left(0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + \\
 & \frac{1}{12}\pi^2 H(0, 1; (u_2 + u_3)) - \frac{1}{2}G\left(0, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_1) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_1) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_1) + \\
 & \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_1) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_1) + \\
 & \frac{1}{4}G\left(\frac{1}{1 - u_3}, \frac{u_1 - 1}{u_1 + u_3 - 1}; 1\right) H(1, 0; u_1) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_1) - \\
 & \frac{1}{4}G\left(\frac{1}{1 - u_3}, u_{312}; 1\right) H(1, 0; u_1) - \frac{3}{4}H(0, 0; u_2) H(1, 0; u_1) - \frac{3}{4}H(0, 0; u_3) H(1, 0; u_1) + \\
 & \frac{1}{4}H\left(0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) H(1, 0; u_1) - \frac{1}{3}\pi^2 H(1, 0; u_1) - \frac{1}{2}G\left(0, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_2) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right) H(1, 0; u_2) + \\
 & \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1 + u_2}; 1\right) H(1, 0; u_2) + \\
 & \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_2) - \\
 & \frac{1}{4}G\left(\frac{1}{1 - u_1}, u_{123}; 1\right) H(1, 0; u_2) - \frac{3}{4}H(0, 0; u_1) H(1, 0; u_2) - \frac{3}{4}H(0, 0; u_3) H(1, 0; u_2) + \\
 & \frac{1}{4}H\left(0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) H(1, 0; u_2) - \frac{1}{4}H(1, 0; u_1) H(1, 0; u_2) - \frac{1}{3}\pi^2 H(1, 0; u_2) - \\
 & \frac{1}{2}G\left(0, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_3) - \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_3) + \\
 & \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_3) + \frac{1}{4}G\left(\frac{1}{1 - u_2}, \frac{u_3 - 1}{u_2 + u_3 - 1}; 1\right) H(1, 0; u_3) + \\
 & \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_3) - \frac{1}{3}\pi^2 H(1, 0; u_3) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_3) + \\
 & \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_3) - \frac{1}{4}G\left(\frac{1}{1 - u_2}, u_{231}; 1\right) H(1, 0; u_3) + \\
 & \frac{3}{4}H(0; u_1) H(0; u_2) H(1, 0; u_3) - \frac{3}{4}H(0, 0; u_1) H(1, 0; u_3) - \frac{3}{4}H(0, 0; u_2) H(1, 0; u_3) + \\
 & \frac{1}{4}H\left(0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) H(1, 0; u_3) - \frac{1}{4}H(1, 0; u_1) H(1, 0; u_3) - \frac{1}{4}H(1, 0; u_2) H(1, 0; u_3) + \\
 & \frac{1}{24}\pi^2 H(1, 1; u_1) + \frac{1}{24}\pi^2 H(1, 1; u_2) + \frac{1}{24}\pi^2 H(1, 1; u_3) + \frac{1}{2}H(0; u_2) H(0, 0, 0; u_1) + \\
 & \frac{1}{2}H(0; u_3) H(0, 0, 0; u_2) + \frac{1}{2}H(0; u_1) H(0, 0, 0; u_3) - \frac{1}{2}H(0; u_2) H\left(0, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) - \\
 & \frac{1}{2}H(0; u_3) H\left(0, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) - H(0; u_1) H(0, 0, 1; (u_1 + u_2)) - \\
 & H(0; u_2) H(0, 0, 1; (u_1 + u_2)) - \frac{1}{2}H(0; u_1) H\left(0, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - \\
 & \frac{1}{2}H(0; u_2) H\left(0, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - H(0; u_1) H(0, 0, 1; (u_1 + u_3)) -
 \end{aligned}$$

$$\begin{aligned}
& H(0; u_3) H(0, 0, 1; (u_1 + u_3)) - \frac{1}{2} H(0; u_1) H\left(0, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) - \\
& \frac{1}{2} H(0; u_3) H\left(0, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) - H(0; u_2) H(0, 0, 1; (u_2 + u_3)) - \\
& H(0; u_3) H(0, 0, 1; (u_2 + u_3)) - \frac{1}{2} H(0; u_2) H(0, 1, 0; u_1) - \frac{1}{2} H(0; u_3) H(0, 1, 0; u_2) - \\
& \frac{1}{2} H(0; u_1) H(0, 1, 0; u_3) + \frac{1}{4} H(0; u_2) H\left(0, 1, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) - \\
& \frac{1}{4} H(0; u_3) H\left(0, 1, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) + \frac{1}{4} H(0; u_1) H\left(0, 1, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - \\
& \frac{1}{4} H(0; u_2) H\left(0, 1, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - \frac{1}{4} H(0; u_1) H\left(0, 1, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + \\
& \frac{1}{4} H(0; u_3) H\left(0, 1, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + \frac{1}{2} H(0; u_2) H(1, 0, 0; u_1) - \frac{1}{2} H(0; u_3) H(1, 0, 0; u_1) - \\
& \frac{1}{2} H(0; u_1) H(1, 0, 0; u_2) + \frac{1}{2} H(0; u_3) H(1, 0, 0; u_2) + \frac{1}{2} H(0; u_1) H(1, 0, 0; u_3) - \\
& \frac{1}{2} H(0; u_2) H(1, 0, 0; u_3) - \frac{1}{4} H(0; u_3) H\left(1, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) - \\
& \frac{1}{4} H(0; u_2) H\left(1, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - \frac{1}{4} H(0; u_1) H\left(1, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) - \\
& 7H(0, 0, 0, 0; u_1) - 7H(0, 0, 0, 0; u_2) - 7H(0, 0, 0, 0; u_3) + \frac{3}{2} H\left(0, 0, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) + \\
& 3H(0, 0, 0, 1; (u_1 + u_2)) + \frac{3}{2} H\left(0, 0, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) + 3H(0, 0, 0, 1; (u_1 + u_3)) + \\
& \frac{3}{2} H\left(0, 0, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + 3H(0, 0, 0, 1; (u_2 + u_3)) + \frac{9}{4} H(0, 0, 1, 0; u_1) + \\
& \frac{9}{4} H(0, 0, 1, 0; u_2) + \frac{9}{4} H(0, 0, 1, 0; u_3) - \frac{1}{2} H(0, 1, 0, 0; u_1) - \frac{1}{2} H(0, 1, 0, 0; u_2) - \\
& \frac{1}{2} H(0, 1, 0, 0; u_3) + \frac{1}{2} H\left(0, 1, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) + \frac{1}{2} H\left(0, 1, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) + \\
& \frac{1}{2} H\left(0, 1, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + H(0, 1, 1, 0; u_1) + H(0, 1, 1, 0; u_2) + H(0, 1, 1, 0; u_3) - \\
& \frac{1}{4} H\left(0, 1, 1, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) - \frac{1}{4} H\left(0, 1, 1, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) - \\
& \frac{1}{4} H\left(0, 1, 1, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + H\left(1, 0, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) + H\left(1, 0, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) + \\
& H\left(1, 0, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + 2H(1, 0, 1, 0; u_1) + 2H(1, 0, 1, 0; u_2) + 2H(1, 0, 1, 0; u_3) + \\
& \frac{1}{4} H\left(1, 1, 0, 1; \frac{u_1 + u_2 - 1}{u_2 - 1}\right) + \frac{1}{4} H\left(1, 1, 0, 1; \frac{u_1 + u_3 - 1}{u_1 - 1}\right) + \\
& \frac{1}{4} H\left(1, 1, 0, 1; \frac{u_2 + u_3 - 1}{u_3 - 1}\right) + \frac{1}{2} H(1, 1, 1, 0; u_1) + \frac{1}{2} H(1, 1, 1, 0; u_2) + \frac{1}{2} H(1, 1, 1, 0; u_3) - \\
& \frac{1}{24} \pi^2 H(0; u_3) \mathcal{H}\left(1; \frac{1}{u_{123}}\right) - \frac{1}{24} \pi^2 H(0; u_1) \mathcal{H}\left(1; \frac{1}{u_{231}}\right) - \frac{1}{24} \pi^2 H(0; u_2) \mathcal{H}\left(1; \frac{1}{u_{312}}\right) + \\
& \frac{1}{8} \pi^2 H(0; u_2) \mathcal{H}\left(1; \frac{1}{v_{123}}\right) - \frac{1}{8} \pi^2 H(0; u_3) \mathcal{H}\left(1; \frac{1}{v_{123}}\right) + \frac{1}{24} \pi^2 H(0; u_2) \mathcal{H}\left(1; \frac{1}{v_{132}}\right) - \\
& \frac{1}{24} \pi^2 H(0; u_3) \mathcal{H}\left(1; \frac{1}{v_{132}}\right) - \frac{1}{24} \pi^2 H(0; u_1) \mathcal{H}\left(1; \frac{1}{v_{213}}\right) + \frac{1}{24} \pi^2 H(0; u_3) \mathcal{H}\left(1; \frac{1}{v_{213}}\right) -
\end{aligned}$$

$$\begin{aligned}
 & \frac{1}{8}\pi^2 H(0; u_1) \mathcal{H}\left(1; \frac{1}{v_{231}}\right) + \frac{1}{8}\pi^2 H(0; u_3) \mathcal{H}\left(1; \frac{1}{v_{231}}\right) + \frac{1}{8}\pi^2 H(0; u_1) \mathcal{H}\left(1; \frac{1}{v_{312}}\right) - \\
 & \frac{1}{8}\pi^2 H(0; u_2) \mathcal{H}\left(1; \frac{1}{v_{312}}\right) + \frac{1}{24}\pi^2 H(0; u_1) \mathcal{H}\left(1; \frac{1}{v_{321}}\right) - \frac{1}{24}\pi^2 H(0; u_2) \mathcal{H}\left(1; \frac{1}{v_{321}}\right) - \\
 & \frac{1}{4}H(0; u_2) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{u_{123}}\right) - \frac{1}{4}H(1, 0; u_2) \mathcal{H}\left(0, 1; \frac{1}{u_{123}}\right) + \frac{1}{24}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{u_{123}}\right) + \\
 & \frac{1}{24}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{u_{231}}\right) - \frac{1}{4}H(0; u_1) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{u_{231}}\right) - \frac{1}{4}H(1, 0; u_3) \mathcal{H}\left(0, 1; \frac{1}{u_{231}}\right) - \\
 & \frac{1}{4}H(0; u_1) H(0; u_2) \mathcal{H}\left(0, 1; \frac{1}{u_{312}}\right) - \frac{1}{4}H(1, 0; u_1) \mathcal{H}\left(0, 1; \frac{1}{u_{312}}\right) + \frac{1}{24}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{u_{312}}\right) - \\
 & \frac{1}{4}H(0; u_2) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{123}}\right) + \frac{1}{4}H(0, 0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{123}}\right) + \\
 & \frac{1}{4}H(0, 0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{123}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{123}}\right) - \frac{1}{4}H(0; u_2) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{132}}\right) + \\
 & \frac{1}{4}H(0, 0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{132}}\right) + \frac{1}{4}H(0, 0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{132}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{132}}\right) - \\
 & \frac{1}{4}H(0; u_1) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{213}}\right) + \frac{1}{4}H(0, 0; u_1) \mathcal{H}\left(0, 1; \frac{1}{v_{213}}\right) + \\
 & \frac{1}{4}H(0, 0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{213}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{213}}\right) - \frac{1}{4}H(0; u_1) H(0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{231}}\right) + \\
 & \frac{1}{4}H(0, 0; u_1) \mathcal{H}\left(0, 1; \frac{1}{v_{231}}\right) + \frac{1}{4}H(0, 0; u_3) \mathcal{H}\left(0, 1; \frac{1}{v_{231}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{231}}\right) - \\
 & \frac{1}{4}H(0; u_1) H(0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{312}}\right) + \frac{1}{4}H(0, 0; u_1) \mathcal{H}\left(0, 1; \frac{1}{v_{312}}\right) + \\
 & \frac{1}{4}H(0, 0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{312}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{312}}\right) - \frac{1}{4}H(0; u_1) H(0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{321}}\right) + \\
 & \frac{1}{4}H(0, 0; u_1) \mathcal{H}\left(0, 1; \frac{1}{v_{321}}\right) + \frac{1}{4}H(0, 0; u_2) \mathcal{H}\left(0, 1; \frac{1}{v_{321}}\right) + \frac{1}{6}\pi^2 \mathcal{H}\left(0, 1; \frac{1}{v_{321}}\right) - \\
 & \frac{1}{2}H(0; u_2) H(0; u_3) \mathcal{H}\left(1, 1; \frac{1}{v_{123}}\right) + \frac{1}{2}H(0, 0; u_2) \mathcal{H}\left(1, 1; \frac{1}{v_{123}}\right) + \\
 & \frac{1}{2}H(0, 0; u_3) \mathcal{H}\left(1, 1; \frac{1}{v_{123}}\right) + \frac{11}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{123}}\right) - \frac{1}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{132}}\right) - \\
 & \frac{1}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{213}}\right) - \frac{1}{2}H(0; u_1) H(0; u_3) \mathcal{H}\left(1, 1; \frac{1}{v_{231}}\right) + \frac{1}{2}H(0, 0; u_1) \mathcal{H}\left(1, 1; \frac{1}{v_{231}}\right) + \\
 & \frac{1}{2}H(0, 0; u_3) \mathcal{H}\left(1, 1; \frac{1}{v_{231}}\right) + \frac{11}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{231}}\right) - \frac{1}{2}H(0; u_1) H(0; u_2) \mathcal{H}\left(1, 1; \frac{1}{v_{312}}\right) + \\
 & \frac{1}{2}H(0, 0; u_1) \mathcal{H}\left(1, 1; \frac{1}{v_{312}}\right) + \frac{1}{2}H(0, 0; u_2) \mathcal{H}\left(1, 1; \frac{1}{v_{312}}\right) + \frac{11}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{312}}\right) - \\
 & \frac{1}{24}\pi^2 \mathcal{H}\left(1, 1; \frac{1}{v_{321}}\right) + \frac{1}{2}H(0; u_2) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{123}}\right) + \frac{1}{2}H(0; u_3) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{123}}\right) + \\
 & \frac{1}{2}H(0; u_1) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{231}}\right) + \frac{1}{2}H(0; u_3) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{231}}\right) + \frac{1}{2}H(0; u_1) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{312}}\right) + \\
 & \frac{1}{2}H(0; u_2) \mathcal{H}\left(0, 0, 1; \frac{1}{u_{312}}\right) + \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{123}}\right) + \frac{1}{4}H(0; u_1) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{231}}\right) + \\
 & \frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{u_{312}}\right) + \frac{1}{4}H(0; u_2) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{123}}\right) - \frac{1}{4}H(0; u_3) \mathcal{H}\left(0, 1, 1; \frac{1}{v_{123}}\right) -
 \end{aligned}$$

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