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The Two-Parameter Deformation of GL(2), its Differential Calculus, and Lie Algebra

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Abstract:
The Yang-Baxter equation is solved in two dimensions giving rise to a two-parameter deformation of $GL(2)$. The transformation properties of quantum planes are briefly discussed. Non-central determinant and inverse are constructed. A right-invariant differential calculus is presented and the role of the different deformation parameters investigated. While the corresponding Lie algebra relations are simply deformed, the comultiplication exhibits both quantization parameters.

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1. Introduction

The main understanding of the concept of quantum groups goes back to the seminal papers of Faddeev *et. al.* [1], Drinfeld [2], and Manin[3]. Quantum groups have now found wide interest among theoretical physicists whose interest spans from axiomatic quantum field theory to two-dimensional solid-state systems. Quantum groups are a generalization of the concept of groups. More precisely, a quantum group is a deformation of a group that, for particular values of the deformation parameter, coincides with the group. Knowing the role that symmetries play in physics, it is a natural question to ask to what extent the deformed concept of symmetries might be used in physics as well. This is a particular tempting question because the deformation formalism is so to speak "smooth" in the deformation parameter and, therefore, if quantum groups can be applied to physics, the predictions might be arbitrarily close to the predictions of regular symmetries. To pursue this idea it is certainly necessary to get a better understanding of the basic concepts and the computational techniques of quantum groups.

In this paper we ask for the most general possible deformation of the group $GL(2)$ and, with some technical assumptions, obtain a two-parameter deformation. The first occurrence of a two-parameter deformation is presumably due to Kobyzev in collaboration with Manin [4] (recently also [5]). It was also independently found by one of the authors [6]. Next we develop the formalism of differential geometry on these two-parameter deformed quantum groups.

The application of non-commutative differential geometry to quantum matrix groups by Woronowicz [7] was the starting point to study differential geometry on the basic matrix representations of quantum groups. Here we use the same techniques as they were first applied in the treatment of a one-parameter deformation of $GL(1,1)$ [8] and of the quantum planes [9]. On quantum planes the application of this formalism leads to a deformation of the Heisenberg algebra. The two-parameter deformation of $GL(2)$, as it is found in this paper, gives rise to a two-parameter deformation of the differential calculus on the two-dimensional quantum plane. It leads to a two-parameter deformation of the Heisenberg algebra. The deformed Lie algebra, corresponding to the deformed group $GL(2)$ depends essentially on one parameter only. This is in agreement with Drinfeld's uniqueness theorem. We shall show that the Drinfeld algebra is in the enveloping algebra of our deformed Lie algebra. The relevance of the two-parameter deformation, however, comes to bear in the comultiplication rule which truly depends on both parameters.

2. The Quantum Group

Quantum matrix spaces were approached by Manin [3] using an associative graded algebra $A = \bigoplus_{i=0}^{\infty} A_i \quad A_0 = \mathbf{K}$, a field (normally $\mathbf{R}$ or $\mathbf{C}$), $A_1 = \{t^1{}_1, t^1{}_2, \ldots, t^n{}_n\}$ a set that generates the algebra. This algebra is factorized with an ideal generated by quadratic relations. Following Takthajan [10] the relations are expressible in the form

$$R(T \otimes \mathbf{1})(\mathbf{1} \otimes T) = (\mathbf{1} \otimes T)(T \otimes \mathbf{1})\, R \tag{1}$$

where $T = (t^i{}_k)_{i,k=1\ldots n}$ and $R \in \mathbf{C}^{n^2 \times n^2}$, a matrix. More explicitly, the generators $t^i{}_k$ of the algebra have to obey the relations:

$$R^{ik}{}_{rs}\, t^r{}_v\, t^s{}_w = t^k{}_b\, t^i{}_a\, R^{ab}{}_{vw} \tag{2}$$

We shall refer to these relations as RTT-relations. These are n^4 equations for n^2 variables. There is always a trivial solution $t^i{}_k = \delta^i{}_k$ (T the unit matrix). But for suitable R-matrices there are more interesting solutions that correspond to non-trivial deformations of groups. The relations (2) should not generate cubic or higher relations in $t^i{}_k$, they should describe the non-commuting structure completely. This is the case if R satisfies the (quantum) Yang Baxter equation (YBE). In the standard direct product notation this is the equation:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}\ . \tag{3}$$

Altogether, the algebra of quantum matrices can be precisely defined as

$$A_R = \frac{\mathbf{C}[[t^i{}_k]]}{RT_1T_2 - T_2T_1R}\ . \tag{4}$$

The quantum matrices can be understood as transformations of a quantum vector space. The n-dimensional quantum vector space is a space spanned by n variables x^i, subject to the relation:

$$[f(\widehat{R})]^{ij}{}_{kl}\, x^k x^l = 0 \tag{5}$$

where $f(t) \in \mathbf{C}[t]$, but now a polynomial in the matrix $\widehat{R}^{ij}{}_{kl} = R^{ji}{}_{kl}$. In the language of equation (4), the quantum vector space algebra can be characterized as follows:

$$C^n_{f,R} = \frac{\mathbf{C}[[x^1, \cdots, x^n]]}{I^n_{f,R}}, \tag{6}$$

where $I^n_{f,R}$ is the ideal corresponding to relation (5). The quantum matrices act via the coaction $T : C^n_{f,R} \longrightarrow A_R \otimes C^n_{f,R}$ in a combination of matrix and tensor product $T x^i = t^i{}_k \otimes x^k$, which gives $C^n_{f,R}$ an A_R-comodule structure. Note that in general the non-commuting structure of A_R does not completely determine that of $C^n_{f,R}$.

In order to find deformations of $GL(2)$ one has to solve eq. (3) for the case $n = 2$, i.e. $n^6 = 64$ cubic equations in $n^4 = 16$ variables. This problem becomes calculable if one requires some basic assumptions for quantized $GL(2)$. Firstly, the ordered quadratic monomials of the four quantum group parameters $T \equiv \binom{a\,b}{c\,d}$, should be linearly independent also in the quantized cases and the non-commuting structure should allow a lexicographical ordering. If in addition one requires the squares a^2, b^2, c^2, d^2 not to enter in the RTT relations, as is the case for conventional GL quantum groups, it follows that the R-matrix is of block-diagonal type:

$$R = \begin{pmatrix} X & & & \\ & A & B & \\ & C & D & \\ & & & Y \end{pmatrix} \tag{7}$$

The YBE reduces in this case to 14 nontrivial equations giving seven independent relations:

$$\begin{aligned} &C(AD + CX - X^2) = 0, \qquad && C(AD + CY - Y^2) = 0, \\ &B(AD + CX - X^2) = 0, && B(AD + CY - Y^2) = 0, \\ &ABC = 0, \quad BCD = 0, && BC(B - C) = 0\ . \end{aligned} \tag{8}$$

If we demand in addition that the solution has a unit matrix limit, i.e. that there are values for the deformation parameters such that the quantum group parameters become commutative, and that the RTT relation does not reduce the number of group parameters, then the conditions (8) restrict R to be of the form

$$R_{p,q} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-\frac{1}{p} & \frac{q}{p} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{9}$$

(or to the form of the transposed matrix), where p and q are free parameters. Interestingly, the linear independence of all ordered quadratic monomials in combination with the RTT relations also determines the R-matrix uniquely. (Note that the RTT relations can be constructed from the YBE acting on $V_1 \otimes V_2 \otimes V_3$ instead of $V^{\otimes 3}$ and identifying $V_1 = V_2 = V$ and $A = \mathrm{End}(V_3)$ [11]).

$R_{p,q}$ becomes the well-known R_q solution in the limit $p \to q$. For $p \neq q$ two by two matrices can exhibit different commutation structures in rows and columns. The relations are

$$\begin{aligned} ab &= p\,ba\ , & cd &= p\,dc, \\ ac &= q\,ca\ , & bd &= q\,db, \\ bc &= \tfrac{q}{p}\,cb\ , & ad - da &= \left(p - \tfrac{1}{q}\right) bc \end{aligned} \tag{10}$$

One interesting consequence of the existence of a two-parameter deformation is the fact that it implies an infinite number of one-parameter deformations and the conventional choice of $R_q = R_{q,q}$ appears to be a quite special one. This will proove to have interesting consequences for the interplay of the group quantization with the quantization of its Lie algebra in section 5 and 6.

The R-matrix is the sum of two projectors $\widehat{R}_{p,q} = q\mathbf{1} + \mathcal{A}$, $\mathcal{A}^2 \propto \mathcal{A}$. Thus $f(\widehat{R})$ in equation (5) is always of the form $c_1\mathbf{1} + c_2\mathcal{A}$. Only if f is proportional to $\mathcal{A}$ or to $\mathcal{S} = \mathcal{A} + (q - \frac{1}{p})\,\mathbf{1}$, we find x^1x^2 different from zero: For these two cases we get the well-known q plane and $\frac{1}{p}$ exterior plane, respectively:

$$\begin{aligned} &x^1x^2 = q\,x^2x^1\ , \\ &\xi^1\xi^2 = -\frac{1}{p}\xi^2\xi^1\ , \qquad (\xi^1)^2 = (\xi^2)^2 = 0\ . \end{aligned} \tag{11}$$

Consequently, $T \in A_R$ transforms a q plane *and* a $\frac{1}{p}$ exterior plane.

Before we can define the quantum group $GL_{p,q}(2)$ explicitely, we have to take care of the determinant, this will be done in the next section.

3. Determinant and Inverse

In order to define a deformation of $GL(2)$ we need the notion of determinant which provides a criterion of regularity of a matrix and given this, we should be able to write down explicitly the inverse of a quantum matrix. To define the determinant we study the deformation of the simple relation for 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \mathrm{Det} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{12}$$

Making the ansatz

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -\beta b \\ -\gamma c & \alpha a \end{pmatrix} = \mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{13}$$

we find $\alpha = 1, \ \beta = 1/p, \ \gamma = p,$ and

$$\mathcal{D} \ = \ ad - p\,bc \ = \ da - 1/p\,cb . \tag{14}$$

Equivalently, using the quantum exterior plane one can find the determinant by calculating the product of the transformed coordinates

$$\xi^i \to \xi'^i = t^i_k\, \xi^k \ : \quad \xi'^1 \xi'^2 = (a\xi^1 + b\xi^2)(c\xi^1 + d\xi^2) = \mathcal{D}\, \xi^1 \xi^2 . \tag{15}$$

In general $\mathcal{D}$ is not central but obeys the following commutation relations

$$\begin{aligned} a\mathcal{D} &= \mathcal{D}a, \quad & b\mathcal{D} &= \tfrac{q}{p}\mathcal{D}b, \\ c\mathcal{D} &= \tfrac{p}{q}\mathcal{D}c, \quad & d\mathcal{D} &= \mathcal{D}d , \end{aligned} \tag{16}$$

i.e. it becomes central if $p = q$. Nevertheless, it is consistent to extend the algebra by an inverse of $\mathcal{D}$ by introducing the following relations

$$\begin{aligned} a\mathcal{D}^{-1} &= \mathcal{D}^{-1}a, \quad & b\mathcal{D}^{-1} &= \tfrac{p}{q}\mathcal{D}^{-1}b, \\ c\mathcal{D}^{-1} &= \tfrac{q}{p}\mathcal{D}^{-1}c, \quad & d\mathcal{D}^{-1} &= \mathcal{D}^{-1}d, \\ \mathcal{D}\mathcal{D}^{-1} &= \mathcal{D}^{-1}\mathcal{D} & &= 1 . \end{aligned} \tag{17}$$

$\mathcal{D}$ will be called the *quantum determinant* of T since for $p, q \to 1$ it agrees with the ordinary determinant.

After having done so, a consistent definition of the inverse can be given by

$$T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -\frac{1}{p}b \\ -pc & a \end{pmatrix} \mathcal{D}^{-1} = \mathcal{D}^{-1} \begin{pmatrix} d & -\frac{1}{q}b \\ -qc & a \end{pmatrix} \tag{18}$$

since $T\,T^{-1} = T^{-1}T = 1$.

The interesting point of this construction is that despite the fact that the determinant $\mathcal{D}$ is not necessarily central, right and left inverse coincide due to the $q - p$-flip mechanism that changes p into q by pulling through the inverse of the determinant.

Now, we are able to give a formal definition of the two-parameter deformation of $GL(2)$ by means of factorizing a freely generated algebra by the discussed relations:

$$GL_{p,q}(2) \;\equiv\; \frac{\mathbf{C}[[a,b,c,d,\mathcal{D}^{-1}]]}{\left\{\begin{array}{cc} ab - p\,ba\,, \quad ac - q\,ca\,, & cd - p\,dc\,, \quad bd - q\,db\,, \\ bc - \frac{q}{p}\,cb\,, & ad - da - (p - 1/q)bc, \\ (ad - pbc)\mathcal{D}^{-1} - 1\,, & \mathcal{D}^{-1}(ad - pbc) - 1\,, \\ [a,\mathcal{D}^{-1}]\,,\; b\mathcal{D}^{-1} - \frac{p}{q}\mathcal{D}^{-1}b, & c\mathcal{D}^{-1} - \frac{q}{p}\mathcal{D}^{-1}c,\; [d,\mathcal{D}^{-1}] \end{array}\right\}} \tag{19}$$

Strictly speaking, an element of $GL_{p,q}(2)$ can be understood as an 3×3-matrix (*cf.*[3])

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \mathcal{D}^{-1} \end{pmatrix} \tag{20}$$

in order to incorporate all algebraic ingredients.

4. Differential Calculus on $GL_{p,q}(2)$ and its Lie Algebra

Constructing a differential calculus on the quantum group $GL_{p,q}(2)$ exhibits the richness and high degree of consistency of this concept as a generalization of commutative geometry. A comprehensive and pedagogical example of non-commutative differential geometry is the right invariant calculus for $GL_{p,q}(2)$. (As explained by Woronowicz [12], a bicovariant calculus can also be constructed but might involve more Lie algebra generators than in the classical case.) What one has to do is to add an exterior derivative δ (δ instead of d since d is already a group parameter) to the framework developed in the previous two sections. This has to be done in a consistent fashion.

We require δ to be a C-linear operator that is nilpotent and obeys the Leibniz rule:

$$\delta^2 = 0 \tag{21}$$

$$\begin{aligned} \delta(fg) =& (\delta f)g + f\,\delta g\ , \qquad f, g \in GL_{p,q}(2)\,, &(22)\\ \delta(f\omega) =& (\delta f)\omega - f\,\delta\omega\ , \qquad \omega \text{ a one} - \text{form}\,, &(23) \end{aligned}$$

where f and g are functions of the group parameters a, b, c, d.

Instead of working with the differentials $\delta a, \delta b, \delta c, \delta d$ it turns out to be more convenient to use the right invariant one-forms $\Omega = \binom{\omega_1\ \omega_+}{\omega_-\ \omega_2}$.

$$\Omega \equiv \delta T{\cdot}T^{-1} = -T\,\delta(T^{-1}) \qquad \text{or}$$

$$\begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix} = \begin{pmatrix} \omega_1 a + \omega_+ c & \omega_1 b + \omega_+ d \\ \omega_- a + \omega_2 c & \omega_- b + \omega_2 d \end{pmatrix} . \tag{24}$$

The Maurer Cartan formula follows:

$$\delta\Omega = \Omega^2\ . \tag{25}$$

Obviously, one-forms and functions of the group parameters do not commute. The structure of these commutation relations can be assumed to be linear, using real or complex matries A, B, C, D:

$$\begin{aligned} \omega_i\, a =& a\, A_{ij}\omega_j + c\, B_{ij}\omega_j\,, \\ \omega_i\, c =& a\, C_{ij}\omega_j + c\, D_{ij}\omega_j\,. \end{aligned} \tag{26}$$

where $i,j = -,1,2,+$. The remaining relations are obtained by interchanging $(a,c) \to (b,d)$. Commuting an ω_i through the basic relations eq. (10) one finds that the matrices A, B, C, D have to obey the same commutation relations as a, b, c, d do (!). For our purpose here, we make a more restricted ansatz (which does not work for general dimension)

$$\begin{aligned} a\,\omega_\pm =& r^\pm \omega_\pm a\,, \qquad c\,\omega_\pm = s^\pm \omega_\pm c\,, \\ a\,\omega_i =& F_{ij}\omega_j a\,, \qquad c\,\omega_i = G_{ij}\omega_j c\,, \qquad i,j = 1,2\,. \end{aligned} \tag{27}$$

And again $(a,c) \to (b,d)$ gives the remaining relations. For this case we have $FG = GF$, i.e. both matrices can be diagonalized simultaneously.

Applying the extérior derivative on the basic relations (10), commuting through the ω's to one side with help of eq.s (27), and equating he coefficients of the linear independent ω's one finds

$$\begin{aligned} r^+ =& p \quad, \qquad r^- = q\,, \\ s^+ =& 1/q\,, \qquad s^- = 1/p\,, \\ G_{11} =& F_{21} + 1\,, \quad G_{12} = F_{22} - 1\,. \end{aligned} \tag{28}$$

From $FG = GF$ follow two more conditions among the F_{ij}:

$$F_{12}G_{21} = F_{21}G_{12}\,, \qquad (F_{11} - F_{22})G_{12} = F_{12}(G_{11} - G_{22})\,. \tag{29}$$

This allows us to express the matrix elements of G in terms of the elements of the matrix F:

$$G = G(F) = \begin{pmatrix} 1 + F_{21} & F_{22} - 1 \\ \frac{F_{21}}{F_{12}}(F_{22} - 1) & \frac{F_{22}-F_{11}}{F_{12}}(F_{22} - 1) + F_{21} + 1 \end{pmatrix}\,. \tag{30}$$

Applying the exterior derivative to the relations (27), taking care of the minus signs picked up by commuting δ with a one-form, and using the Maurer-Catan formula, we find the following relations, now equating the coefficients of a, b, c, d:

$$
\begin{aligned}
0 &= \omega_+\omega_- + X\,\omega_-\omega_+ \ , \\
0 &= (F_{21}+1)\,\omega_+\omega_1 + F_{11}\,\omega_1\omega_+ + (F_{22}-1)\,\omega_+\omega_2 + F_{12}\,\omega_2\omega_+ \ , \\
0 &= G_{21}\,\omega_+\omega_1 + F_{21}\,\omega_1\omega_+ + G_{22}\,\omega_+\omega_2 + F_{22}\,\omega_2\omega_+ \ , \\
0 &= F_{11}\,\omega_-\omega_1 + (1+F_{21})\,\omega_1\omega_- + F_{12}\,\omega_-\omega_2 + (F_{22}-1)\,\omega_2\omega_- \ , \\
0 &= F_{21}\,\omega_-\omega_1 + G_{21}\,\omega_1\omega_- + F_{22}\,\omega_-\omega_2 + G_{22}\,\omega_2\omega_- \ ,
\end{aligned}
\tag{31}
$$

$$
0 = \ \omega_1^2 \ = \omega_2^2 \ , \tag{32}
$$

$$
\begin{aligned}
\omega_1\omega_2 + \omega_2\omega_1 \ &= \frac{X^2 - XF_{11} + F_{12}}{F_{12}(F_{11}+1)}\,\omega_-\omega_+ - \frac{F_{11}}{F_{12}}\,\omega_1^2 - \frac{F_{11}+1}{F_{12}-1}\,\omega_2^2 \ , \\
\omega_1\omega_2 + \omega_2\omega_1 \ &= \left(\frac{X}{F_{22}} - \frac{1}{F_{21}+1}\right)\,\omega_-\omega_+ - \frac{F_{21}}{F_{22}}\,\omega_1^2 - \frac{F_{22}-1}{F_{21}+1}\,\omega_2^2 \ , \\
\omega_1\omega_2 + \omega_2\omega_1 \ &= \frac{X^2G_{21} - XG_{22} + 1}{XG_{21}(G_{22}+1)}\,\omega_-\omega_+ - \frac{G_{21}-1}{G_{22}+1}\,\omega_1^2 - \frac{G_{22}}{G_{21}}\,\omega_2^2 \ ,
\end{aligned}
\tag{33}
$$

where $X \equiv pq$. The three eq.s (33) give additional constraints for the matrix F. In order to make the derivation of the calculus not unreasonably complicate and to stay close to the classical situation we require that all squares of one-forms vanish and we require also that $\omega_-\omega_+$ should not enter in the eq.s (33), hence its coefficients are equated to zero, giving two relations among the entries of F. As we will show this leads to a consistent solution.

As a consequence, the matrix F and therefore G depend on two parameters F_{11} and F_{22}

$$
F = \begin{pmatrix} F_{11} & X(F_{11}-X) \\ \frac{F_{22}}{X} - 1 & F_{22} \end{pmatrix} , \tag{34}
$$

$$
G = \begin{pmatrix} F_{22}/X & F_{22}-1 \\ \frac{(F_{22}-X)(F_{22}-1)}{X^2(F_{11}-X)} & \frac{F_{22}(F_{22}-X-1)+F_{11}}{X(F_{11}-X)} \end{pmatrix} . \tag{35}
$$

F has the eigenvalues 1 and $F_{11}+F_{22}-X$ while those of G are $1/X$ and $\frac{F_{22}(F_{22}+F_{11}-2X-1)+X}{X(F_{11}-X)} \equiv \alpha$. The common eigenvectors are

$$v_1 = \begin{pmatrix} -X \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} X \\ \frac{X-F_{22}}{X-F_{11}} \end{pmatrix} \tag{36}$$

Transforming $\omega_i \to \tilde{\omega}_i = S^{-1}_{ij}\omega_j \quad (\tilde{\omega}_\pm = \omega_\pm)$, where

$$\begin{aligned} S &= \begin{pmatrix} -X & \lambda X \\ 1 & \lambda\frac{F_{22}-X}{F_{11}-X} \end{pmatrix}, \qquad \lambda \neq 0\,, \\ S^{-1} &= \begin{pmatrix} -\lambda\frac{F_{22}-X}{F_{11}-X} & \lambda X \\ 1 & X \end{pmatrix} \frac{1}{\lambda X\left(1+\frac{F_{22}-X}{F_{11}-X}\right)}\,, \end{aligned} \tag{37}$$

with λ an arbitrary parameter to be fixed later, we obtain:

$$\begin{aligned} \tilde{F} &= S^{-1}FS = \begin{pmatrix} X & \\ & F_{11}+F_{22}-X \end{pmatrix}, \\ \tilde{G} &= S^{-1}GS = \begin{pmatrix} 1/X & \\ & \alpha \end{pmatrix}. \end{aligned} \tag{38}$$

To summarize, the commutation relations of the quantum group parameters and the right-invariant one-forms in a convenient basis, still depending on F_{11} and F_{22}, are:

$$\begin{aligned} a\,\tilde{\omega}_+ &= p\tilde{\omega}_+ a\,, & a\,\tilde{\omega}_- &= q\tilde{\omega}_- a\,, \\ c\,\tilde{\omega}_+ &= \frac{1}{q}\tilde{\omega}_+ c\,, & c\,\tilde{\omega}_- &= \frac{1}{p}\tilde{\omega}_- c\,, \\ a\,\tilde{\omega}_1 &= X\tilde{\omega}_1 a\,, & a\,\tilde{\omega}_2 &= (F_{22}+F_{11}-X)\tilde{\omega}_1 a\,, \\ c\,\tilde{\omega}_1 &= \frac{1}{X}\tilde{\omega}_1 c\,, & c\,\tilde{\omega}_2 &= \alpha\,\tilde{\omega}_1 c\,. \end{aligned} \tag{39}$$

The commutation relations for the one-forms $\tilde{\omega}_i$ are obtained by differentiating relations (39), using the appropriate Maurer Cartan relations and equating the coefficients of linearly independent group parameters a, b, c, d. We find six linearly independent two-forms:

$$
\begin{aligned}
&\tilde\omega_1\tilde\omega_2+\tilde\omega_2\tilde\omega_1=0\,,\\
&\tilde\omega_+\tilde\omega_-+X\tilde\omega_-\tilde\omega_+=0\,,\\
&\tilde\omega_\pm\tilde\omega_1+X^{\pm 2}\tilde\omega_1\tilde\omega_\pm=0\,,\\
&\tilde\omega_\pm\tilde\omega_2+\gamma^{\pm 1}\tilde\omega_2\tilde\omega_\pm=0\,,\qquad \gamma=\frac{F_{11}+F_{22}-X}{\alpha}\;.
\end{aligned}
\tag{40}
$$

Thus the ordinary wedge product is also deformed, but the antisymmetry property is kept save a factor. These relations allow to calculate the Lie algebra of $GL_{p,q}(2)$ via the vector fields related to the one forms. Writing δ as $\delta=\omega_i\nabla_i=\tilde\omega_i\tilde\nabla_i$, we can infer from $\delta^2=0$:

$$
\begin{aligned}
0=\delta^2 =&\delta(\tilde\omega_i\tilde\nabla_i)=\delta\tilde\omega_i\,\tilde\nabla_i-\tilde\omega_i\tilde\omega_j\tilde\nabla_j\tilde\nabla_i\\
=&-\tilde\omega_-\tilde\omega_+(\tilde\nabla_+\tilde\nabla_--X\tilde\nabla_-\tilde\nabla_+-(S^{-1}_{12}-XS^{-1}_{12})\tilde\nabla_1-(S^{-1}_{22}-XS^{-1}_{21})\tilde\nabla_2)\\
&-\tilde\omega_2\tilde\omega_1(\tilde\nabla_1\tilde\nabla_2-\tilde\nabla_2\tilde\nabla_1)\\
&-\tilde\omega_1\tilde\omega_+(\tilde\nabla_+\tilde\nabla_1-X^2\tilde\nabla_1\tilde\nabla_+-(S_{11}-X^2S_{21})\tilde\nabla_+)\\
&-\tilde\omega_1\tilde\omega_-(\tilde\nabla_-\tilde\nabla_1-X^{-2}\tilde\nabla_1\tilde\nabla_--(S_{21}-X^{-2}S_{11})\tilde\nabla_-)\\
&-\tilde\omega_2\tilde\omega_+(\tilde\nabla_+\tilde\nabla_2-\gamma\tilde\nabla_2\tilde\nabla_+-(S_{12}-\gamma S_{22})\tilde\nabla_+)\\
&-\tilde\omega_2\tilde\omega_-(\tilde\nabla_-\tilde\nabla_2-\gamma^{-1}\tilde\nabla_2\tilde\nabla_--(S_{22}-\gamma^{-1}S_{12})\tilde\nabla_-)\;.
\end{aligned}
\tag{41}
$$

Due to the linear independence of the two-forms, its coefficients give quadratic relations for the quantum Lie algebra generators $\tilde\nabla_i$. Using the explicit form for S and S^{-1} we find:

$$
\begin{aligned}
&\tilde\nabla_+\tilde\nabla_--X\tilde\nabla_-\tilde\nabla_+=\tilde\nabla_1\;,\\
&\tilde\nabla_1\tilde\nabla_2=\tilde\nabla_2\tilde\nabla_1\;,\\
&\tilde\nabla_+\tilde\nabla_1-X^2\tilde\nabla_1\tilde\nabla_= +X(X+1)\tilde\nabla_+\;,\\
&X^2\tilde\nabla_-\tilde\nabla_1-\tilde\nabla_1\tilde\nabla_= -X(X+1)\tilde\nabla_-\;,\\
&\tilde\nabla_+\tilde\nabla_2-\gamma\tilde\nabla_2\tilde\nabla_+=+(S_{12}-\gamma S_{22})\tilde\nabla_+\;,\\
&\gamma\tilde\nabla_-\tilde\nabla_2-\tilde\nabla_2\tilde\nabla_-=-(S_{12}-\gamma S_{22})\tilde\nabla_-\;.
\end{aligned}
\tag{42}
$$

The commutation properties of the vector fields with the group parameters can be extracted from the Leibniz rule:

$$
\delta(af)=\delta a\;f+a\,\delta f\qquad\Longrightarrow\qquad \tilde\omega_i\tilde\nabla_i a=\delta a+a\tilde\omega_i\tilde\nabla_i\qquad\text{etc.}
$$

It yields:

$$
\begin{aligned}
\tilde{\nabla}_1 a &= X a \tilde{\nabla}_1 - X a \ , & \tilde{\nabla}_1 c &= \frac{1}{X} c \tilde{\nabla}_1 + c \ , \\
\tilde{\nabla}_2 a &= (F_{11} + F_{22} - X) a \tilde{\nabla}_2 + \lambda X a \ , & \tilde{\nabla}_2 c &= \alpha c \tilde{\nabla}_2 + \lambda \frac{X - F_{22}}{X - F_{11}} c \ , \\
\tilde{\nabla}_+ a &= p a \tilde{\nabla}_+ + c \ , & \tilde{\nabla}_+ c &= \frac{1}{q} c \tilde{\nabla}_+ \ , \\
\tilde{\nabla}_- a &= q a \tilde{\nabla}_- \ , & \tilde{\nabla}_- c &= \frac{1}{p} a \tilde{\nabla}_- + a \ .
\end{aligned}
\tag{43}
$$

And again $(a, c) \to (b, d)$ gives the remaining relations.

The linear relations (43) allow us to pull a quantum group parameter a, b, c, or d through the quadratic relations (41). The calculus for $\tilde{\nabla}_1, \tilde{\nabla}_+, \tilde{\nabla}_-$ is already consistent with this requirement. The relation $[\tilde{\nabla}_1, \tilde{\nabla}_2] = 0$ also agrees due to the homogeneity of $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ in the group parameters. The only additional constraint results from the last two relations of (42). Considering, e.g.,

$$
\begin{aligned}
0 &= \left(\tilde{\nabla}_+ \tilde{\nabla}_2 - \gamma \tilde{\nabla}_2 \tilde{\nabla}_+ - (S_{12} - \gamma S_{22}) \tilde{\nabla}_+ \right) a \\
&= a\, p\, (F_{11} + F_{22} - X) \left(\tilde{\nabla}_+ \tilde{\nabla}_2 - \gamma \tilde{\nabla}_2 \tilde{\nabla}_+ - \frac{1}{\alpha} (S_{12} - S_{22}) \tilde{\nabla}_+ \right) \ .
\end{aligned}
\tag{44}
$$

For consistency, we have to require

$$
S_{12} - \gamma S_{22} = \frac{1}{\alpha} (S_{12} - S_{22}) \ . \tag{45}
$$

This is an equation for F_{11} and F_{22} that can be solved for F_{11}. As the equation is quadratic, we obtain two solutions, $F_{11} = -F_{22} + X + 1$ and $F_{11} = -F_{22}$. We discard the second solution because it has no suitable classical limit ($p \to 1$, $q \to 1$ and therefore $X \to 1$). This can be seen from eq.s (27).

The first solution implies $\alpha = \gamma = 1$. With eq. (45) all other relations are consistent. From eq.s (10), (39) and (40) it can be seen that the remaining free parameter F_{22} does not enter these relations. It is only in the diagonalizing matrix, therefore in the appropriate Maurer Cartan formula for $\tilde{\omega}$, in eq. (43), and in the last two eq.s (42) where F_{22} enters. It is present in the combination

$$
C_X = \frac{1}{X} \frac{F_{22} - X}{1 - F_{22}} \quad : \tag{46}
$$

$$S = \begin{pmatrix} -X & \lambda X \\ 1 & \lambda X \, C_X \end{pmatrix} , \tag{47}$$

$$[\tilde{\nabla}_\pm, \tilde{\nabla}_2] = \pm (1 - C_X) \lambda X \, \tilde{\nabla}_\pm , \tag{48}$$

$$\tilde{\nabla}_2 c = c \tilde{\nabla}_2 - \lambda X \, C_X \, c . \tag{49}$$

All these relations have a suitable classical limit if $C_X \to 1$ with $X \to 1$. It is natural to choose $C_X = 1$ for all values of X. In this case $\tilde{\nabla}_2$ commutes with all other vecor fields and the algebra (42) decomposes into the sl_2 subalgebra and an abelian part. The parameter λ is merely scaling $\tilde{\omega}_2$, we choose $\lambda = 1/X$. With this choice of C_X , which corresponds to

$$F_{22} = \frac{2X}{X+1} \tag{50}$$

we finally obtain:

$$F = \begin{pmatrix} \frac{X^2+1}{X+1} & \frac{X(1-X)}{X+1} \\ \frac{1-X}{X+1} & \frac{2X}{X+1} \end{pmatrix} , \qquad G = F^{-1} , \tag{51}$$

$$S = \begin{pmatrix} -X & 1 \\ 1 & 1 \end{pmatrix} , \tag{52}$$

$$\tilde{F} = S^{-1} F S = \begin{pmatrix} X & \\ & 1 \end{pmatrix} , \qquad \tilde{G} = S^{-1} G S = \begin{pmatrix} 1/X & \\ & 1 \end{pmatrix} , \tag{53}$$

and therefore

$$\begin{aligned} a \, \tilde{\omega}_+ &= p \, \tilde{\omega}_+ a , & a \, \tilde{\omega}_- &= q \, \tilde{\omega}_- a , \\ c \, \tilde{\omega}_+ &= \frac{1}{q} \, \tilde{\omega}_+ c , & c \, \tilde{\omega}_- &= \frac{1}{p} \, \tilde{\omega}_- c , \\ a \, \tilde{\omega}_1 &= X \, \tilde{\omega}_1 a , & a \, \tilde{\omega}_2 &= \tilde{\omega}_1 a , \\ c \, \tilde{\omega}_1 &= \frac{1}{X} \tilde{\omega}_1 c , & c \, \tilde{\omega}_2 &= \tilde{\omega}_1 c . \end{aligned} \tag{54}$$

The vector field action on the group parameters is now fixed:

$$
\begin{aligned}
\tilde{\nabla}_1 a &= X a \tilde{\nabla}_1 - X a\,, & \tilde{\nabla}_1 c &= \frac{1}{X} c \tilde{\nabla}_1 + c\,,\\
\tilde{\nabla}_2 a &= a \tilde{\nabla}_2 + a\,, & \tilde{\nabla}_2 c &= c \tilde{\nabla}_+ + c\,,\\
\tilde{\nabla}_+ a &= p a \tilde{\nabla}_+ + c\,, & \tilde{\nabla}_+ c &= \frac{1}{q} c \tilde{\nabla}_+\,,\\
\tilde{\nabla}_- a &= q a \tilde{\nabla}_-\,, & \tilde{\nabla}_- c &= \frac{1}{p} c \tilde{\nabla}_- + a\,.
\end{aligned}
\tag{55}
$$

For the determinant and its inverse we find from relations (54)

$$
\begin{aligned}
\mathcal{D}\tilde{\omega}_+ &= \frac{p}{q}\,\tilde{\omega}_+\mathcal{D}\,, & \mathcal{D}^{-1}\tilde{\omega}_+ &= \frac{q}{p}\,\tilde{\omega}_+\mathcal{D}^{-1}\,,\\
\mathcal{D}\tilde{\omega}_- &= \frac{q}{p}\,\tilde{\omega}_-\mathcal{D}\,, & \mathcal{D}^{-1}\tilde{\omega}_- &= \frac{p}{q}\,\tilde{\omega}_-\mathcal{D}^{-1}\,,\\
[\mathcal{D},\tilde{\omega}_1] &= [\mathcal{D},\tilde{\omega}_2] = 0\,, & [\mathcal{D}^{-1},\tilde{\omega}_1] &= [\mathcal{D}^{-1},\tilde{\omega}_2] = 0\,.
\end{aligned}
\tag{56}
$$

The only non-vanishing actions of a vector field on the determinant and its inverse are:

$$
\tilde{\nabla}_2\mathcal{D} = \mathcal{D}\left(2+\tilde{\nabla}_2\right)\,, \qquad \tilde{\nabla}_2\mathcal{D}^{-1} = \mathcal{D}^{-1}\left(-2+\tilde{\nabla}_2\right)\,. \tag{57}
$$

In particular the derivative of the determinant is

$$
\delta\mathcal{D} = 2\,\tilde{\omega}_2\,\mathcal{D} = \frac{2}{X+1}(\omega_1 + X\omega_2)\,\mathcal{D}\,, \tag{58}
$$

i.e. for constant $\mathcal{D}$ one one-form vanishes or, equivalently, two one-forms become linearly dependent.

The algebra of generators is now completely determined:

$$
\begin{aligned}
\tilde{\nabla}_+\tilde{\nabla}_- - X\tilde{\nabla}_-\tilde{\nabla}_+ &= \tilde{\nabla}_1\,,\\
X^2\tilde{\nabla}_1\tilde{\nabla}_+ + \tilde{\nabla}_+\tilde{\nabla}_1 &= +X(X+1)\tilde{\nabla}_+\,,\\
\tilde{\nabla}_1\tilde{\nabla}_- - X^2\tilde{\nabla}_-\tilde{\nabla}_1 &= -X(X+1)\tilde{\nabla}_-\,,\\
[\tilde{\nabla}_1,\tilde{\nabla}_2] &= 0\,,\\
[\tilde{\nabla}_\pm,\tilde{\nabla}_2] &= 0\,.
\end{aligned}
\tag{59}
$$

After this journey through lots of commutation relations we arrived at a neat result: The Lie algebra of the two-parameter deformation of $GL(2)$ is essentially a one-parameter deformation of the classical Lie algebra $gl(2)$. The center of the Lie algebra of $GL_{p,q}(2)$ remains undeformed ($\tilde{\omega}_2$ commutes with all group parameters and one-forms). For $p, q \to 1$ the ordinary differential calculus on the group is recovered. While on the one-form level p and q can still be distinguished, only its product enters in the deformed Lie algebra. This is in accordance with Drinfelds uniqueness theorem for semi-simple quantized Lie algebras [2]. Moreover $GL_{p,q}(2)$ is a Hopf algebra with the antipode $S(T) = T^{-1}$ and the counit $\epsilon(T) = \mathbf{1}$. We are left to calculate the comultiplication rules of the algebra. It will turn out that they will depend on both parameters p and q.

5. The Comultiplication

To work out the commutations of arbitrary functions of the group parameters with the right-invariant one-forms, we start with the monomial $a^N c^M$ and obtain from (54):

$$\begin{aligned} a^N c^M \tilde{\omega}_+ &= p^N q^{-M} \, \tilde{\omega}_+ \, a^N c^M \, , \\ a^N c^M \tilde{\omega}_- &= q^N p^{-M} \, \tilde{\omega}_- \, a^N c^M \, , \\ a^N c^M \tilde{\omega}_1 &= X^{N-M} \, \tilde{\omega}_1 \, a^N c^M \, , \\ a^N c^M \tilde{\omega}_2 &= \tilde{\omega}_2 \, a^N c^M \, . \end{aligned} \tag{60}$$

To generalize this for arbitrary functions, we introduce M and N as counting operators, N counts the number of factors a or b, M does this for c and d. The Leibniz rule for the vector fields, that follow from eq.s (60), are:

$$\begin{aligned} \tilde{\nabla}_1(fg) &= (\tilde{\nabla}_1 f)g + X^{(N-M)} f(\tilde{\nabla}_1 g) \, , \\ \tilde{\nabla}_2(fg) &= (\tilde{\nabla}_2 f)g + f(\tilde{\nabla}_2 g) \, , \\ \tilde{\nabla}_+(fg) &= (\tilde{\nabla}_+ f)g + q^{-M} \, p^N f(\tilde{\nabla}_+ g) \, , \\ \tilde{\nabla}_-(fg) &= (\tilde{\nabla}_- f)g + p^{-M} \, q^N f(\tilde{\nabla}_- g) \, . \end{aligned} \tag{61}$$

The comultiplication can now simply be read off eq. (61) and turns out to be two-parameter deformed

$$
\begin{aligned}
\Delta(\tilde{\nabla}_1) &= \tilde{\nabla}_1 \otimes \mathbf{1} + X^{N-M} \otimes \tilde{\nabla}_1 \,, \\
\Delta(\tilde{\nabla}_2) &= \tilde{\nabla}_2 \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{\nabla}_2 \,, \\
\Delta(\tilde{\nabla}_+) &= \tilde{\nabla}_+ \otimes \mathbf{1} + p^N q^{-M} \otimes \tilde{\nabla}_+ \,, \\
\Delta(\tilde{\nabla}_-) &= \tilde{\nabla}_- \otimes \mathbf{1} + p^{-M} q^N \otimes \tilde{\nabla}_- \,.
\end{aligned}
\tag{62}
$$

To complete this section, let us demonstrate the equivalence of the algebra relations eq.s (59) with the Drinfeld algebra and we shall see in what sense quantization is unique. We introduce operators H and K $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$, $H \equiv N - M$, $K \equiv N + M$, we express $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ in terms of H and K using its action on monomials:

$$
\begin{aligned}
\tilde{\nabla}_1 a^N c^M &= \frac{1 - X^{N-M}}{1 - \frac{1}{X}} a^N c^M \,, \\
\tilde{\nabla}_2 a^N c^M &= (N + M) a^N c^M \,,
\end{aligned}
\tag{63}
$$

i.e.

$$
\tilde{\nabla}_1 = \frac{1 - X^{N-M}}{1 - \frac{1}{X}} \,, \qquad \tilde{\nabla}_2 = N + M \,. \tag{64}
$$

Rewriting the relations (59) in terms of H and K we find:

$$
\begin{aligned}
X^{\pm 2} X^H \tilde{\nabla}_\pm = \tilde{\nabla}_\pm X^H \qquad &\Leftrightarrow \qquad [H, \tilde{\nabla}_\pm] = \pm 2 \tilde{\nabla}_\pm \,, \\
X^K \tilde{\nabla}_\pm = \tilde{\nabla}_\pm X^K \qquad &\Leftrightarrow \qquad [K, \tilde{\nabla}_\pm] = 0 \,,
\end{aligned}
\tag{65}
$$

$$
\tilde{\nabla}_+ \tilde{\nabla}_- - X \tilde{\nabla}_- \tilde{\nabla}_+ = \frac{1 - X^H}{1 - \frac{1}{X}} \,, \qquad [K, H] = 0 \,. \tag{66}
$$

We redefine the operators, $T_+ = p^{-\frac{N}{2}} q^{\frac{M}{2}} \tilde{\nabla}_+$, $T_- = q^{-\frac{N}{2}} p^{\frac{M}{2}} \tilde{\nabla}_-$, and find that the algebra relations agree with Drinfeld's algebra.

$$
\begin{aligned}
&[T_+, T_-] = \frac{X^{\frac{H}{2}} - X^{-\frac{H}{2}}}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}} \,, \\
&[H, T_\pm] = \pm 2 T_\pm \,, \qquad [K, T_\pm] = [H, K] = 0 \,.
\end{aligned}
\tag{67}
$$

Drinfeld stated that for semi-simple groups the Lie algebra relations should be unique up to substitution $H \to H + \sum_{i=1}^{\infty} c_i H^i$. The algebra relations, however, are only one part of the definition of a deformed Lie algebra, the other part is the comultiplication and here in fact we find an interesting novelty. The comultiplication is genuinely two-parameter deformed, i.e. for the new generators

$$\begin{aligned}
\Delta(K) &= K \otimes \mathbf{1} + \mathbf{1} \otimes K \,, \\
\Delta(H) &= H \otimes \mathbf{1} + \mathbf{1} \otimes H \,, \\
\Delta(T_+) &= T_+ \otimes p^{-\frac{N}{2}} q^{\frac{M}{2}} + p^{\frac{N}{2}} q^{-\frac{M}{2}} \otimes T_+ \,, \qquad M = (K-H)/2 \,, \\
\Delta(T_-) &= T_- \otimes q^{-\frac{N}{2}} p^{\frac{M}{2}} + q^{\frac{N}{2}} p^{-\frac{M}{2}} \otimes T_- \,, \qquad N = (K+H)/2 \,.
\end{aligned} \tag{68}$$

It is easy to verify that this comultiplication is an algebra homomorphism for the deformed Lie algebra relations eq.s (67).

In order to get a better understanding of the non-commutative structure let us turn to some examples.

(1) Setting $p = q$ $(X = q^2)$, we recover the conventional one-parameter deformation of $GL(2)$. The comultiplication for $T_\pm$ reads

$$\Delta(T_\pm) = T_\pm \otimes q^{-\frac{H}{2}} + q^{\frac{H}{2}} \otimes T_\pm \,. \tag{69}$$

The determinant is central and allows a factorization with the ideal generated by $\mathcal{D} = 1$ giving $SL_q(2)$.

(2) Interestingly, there exists a one-parameter deformation of $GL(2)$ transforming a *commutative* quantum plane $(q = 1)$ despite of the fact that the underlying Lie algebra is quantized. Alternatively the same can be obtained for the superplane $(p = 1)$ but not for both simultaneously.

(3) It is also possible to construct a one-parameter deformation of $GL(2)$ exhibiting a non-commutative structure on the group corresponding to *undeformed* Lie algebra relations. This is the case for $p = 1/q$, here comultiplication is as in eq. (69) with K replacing H

$$\Delta(T_\pm) = T_\pm \otimes q^{-\frac{K}{2}} + q^{\frac{K}{2}} \otimes T_\pm \,. \tag{70}$$

This concludes our discussion of the right-invariant differential calculus for $GL_{p,q}(2)$. In the last section we will give some hints how these results are connected with the basic ingredient we started from, the R-matrix.

6. The Structure of the R-Matrix and its Characteristic Equation

Up to this point we have treated p and q on an equal footing but in order to emphasize the two different quantization degrees of freedom let us now normalize the R-matrix differently:

$$R_{X;q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 1-\frac{1}{X} & \frac{q}{X} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{71}$$

This displays the algebra deformation by the parameter X and the plane deformation by the parameter q which indicates the commutation structure of a quantum plane transformed by $GL_{X;q}(2)$. Diagonal R-matrices, for example, correspond to undeformed Lie algebra relations while off-diagonal entries, depending on X only, would contain information on its deformation. In this parametrization $\widehat{R}$ has q-independent eigenvalues $+1$ and $-\frac{1}{X}$, i.e. it obeys the characteristic equation

$$\left(\widehat{R}-\mathbf{1}\right)\left(\widehat{R}+\frac{1}{X}\mathbf{1}\right) = 0 \tag{72}$$

These properties of R-matrices generalize for arbitrary dimension. Also the commutation relations of the quantum group parameters and the basic algebra as the determinant and the construction of the inverse can be generalized [13,14].

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