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Abstract

Let p and q be distinct odd primes. We analyse a semigroup crossed product C * (G p,q) α 2 similar to the semigroup crossed product which models the Hecke C * -algebra of Bost and Connes. We describe a composition series of ideals in C * (G p,q) α 2 , and show that the structure of one of the subquotients reflects interesting number-theoretic information about the multiplicative orders of q in the rings Z/p^l Z.

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The Two-prime Analogue of the Hecke C^*-algebra of Bost and Connes

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ABSTRACT. Let p and q be distinct odd primes. We analyse a semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ similar to the semigroup crossed product which models the Hecke C^*-algebra of Bost and Connes. We describe a composition series of ideals in $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, and show that the structure of one of the subquotients reflects interesting number-theoretic information about the multiplicative orders of q in the rings $\mathbb{Z}/p^\ell\mathbb{Z}$.

In [3], Bost and Connes introduced and studied a Hecke C^*-algebra $C_\mathbb{Q}$ which has many fascinating connections with number theory. It was shown in [11] that $C_\mathbb{Q}$ can be realised as a crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an endomorphic action α of the multiplicative semigroup $\mathbb{N}^*$ of positive integers, and this realisation gives a great deal of insight into the Bost-Connes analysis (see [9]). Here we fix two odd primes p and q, and analyse the semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ associated to the subgroup $G_{p,q} := \{n/p^k q^\ell \mid n \in \mathbb{Z}\}/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$ and the restriction of α to the subsemigroup $\{p^k q^\ell\} \subset \mathbb{N}^*$, which is isomorphic to the additive semigroup $\mathbb{N}^2$. This crossed product still exhibits rich connections with number theory, though of a somewhat different nature: it has a subquotient, for example, whose ideal structure encodes the multiplicative orders of q in the rings $\mathbb{Z}/p^\ell\mathbb{Z}$.

We begin our analysis by passing to the Fourier transform of our dynamical system, which involves the algebras of continuous functions on the spaces of p-adic and q-adic integers. We describe our dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ and its Fourier transform in Section 1. Next we construct a composition series for $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ using general results about invariant ideals and tensor products of semigroup crossed products which have been worked out in [13]. Our main structure theorem is Theorem 2.2, which is proved in Section 2 and Section 3. Theorem 3.1, which gives a detailed description of an ordinary crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ arising in our analysis, is interesting in its own right: it shows, for

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example, that $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ is simple if and only if q is a primitive root modulo p^ℓ for all ℓ, which happens if and only if it is primitive modulo p^ℓ for any single $\ell > 1$ (see Remark 3.8). In the last section, we describe the topology on the primitive ideal space of $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, thus completely determining its ideal structure.

1. THE DYNAMICAL SYSTEM AND ITS FOURIER TRANSFORM

Let p and q be distinct odd primes. We consider the additive group

$$\mathbb{Z}[p^{-1}, q^{-1}] = \{rp^{-k}q^{-\ell} \mid r, k, \ell \in \mathbb{Z}\}$$

and its quotient $G_{p,q} := \mathbb{Z}[p^{-1}, q^{-1}]/\mathbb{Z}$. We write α for the action of $\mathbb{N}^2$ by endomorphisms of the group C^*-algebra $C^*(G_{p,q})$ which is characterised on the canonical generating unitaries $\{\delta_r \mid r \in G_{p,q}\}$ by

$$\alpha_{m,n}(\delta_r) = \frac{1}{p^m q^n} \sum_{\{s \in G_{p,q} \mid p^m q^n s = r\}} \delta_s; \tag{1.1}$$

we can see that there is such an action either by modifying [11, Proposition 2.1] or by applying the general method of [14, Section 1] to the action of $\mathbb{N}^2$ on $\mathbb{Z}$ defined by $\eta_{m,n}(k) = p^m q^n k$ (see [14, Example 1.2]). As in [10, Proposition 2.1], the action satisfies

$$\alpha_{k,\ell}(1)\alpha_{m,n}(1) = \alpha_{k \vee m, \ell \vee n}(1). \tag{1.2}$$

A covariant representation of the dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ consists of a nondegenerate representation π of $C^*(G_{p,q})$ and a representation V of $\mathbb{N}^2$ by isometries on the same space such that

$$\pi(\alpha_{m,n}(a)) = V_{m,n}\pi(a)V_{m,n}^*, \quad \text{for } a \in C^*(G_{p,q}) \text{ and } (m,n) \in \mathbb{N}^2; \tag{1.3}$$

the relation (1.2) then implies that the isometric representation V is Nica covariant, in the sense that $V_{k,\ell}V_{k,\ell}^* V_{m,n}V_{m,n}^* = V_{k\vee m, \ell \vee n}V_{k \vee m, \ell \vee n}^*$. One can see that the system has nontrivial covariant representations by modifying the constructions in [11], or by applying [14, Lemma 1.7]. Thus there is a crossed product $(C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2, i_A, i_S)$, which is a universal C^*-algebra for covariant representations of the system (see [10, Proposition 2.1]). (To avoid complicated notation, we always write i_A and i_S for the algebra and semigroup components of the universal covariant representation.) This crossed product carries a dual action $\hat{\alpha}$ of $\mathbb{T}^2$ which leaves $i_A(C^*(G_{p,q}))$ invariant and satisfies $\hat{\alpha}_{w,z}(i_S(m,n)) = w^m z^n i_S(m,n)$.

To compute the Fourier transform of the system, we need a description of the dual group $\hat{G}_{p,q}$. Note that with $G_p := \mathbb{Z}[p^{-1}]/\mathbb{Z}$, the map $(r,s) \mapsto r + s$ is an isomorphism of $G_p \times G_q$ onto $G_{p,q}$, and, dually, we have $\hat{G}_{p,q} \cong \hat{G}_p \times \hat{G}_q$. To describe $\hat{G}_p$, note that $\mathbb{Z}[p^{-1}] = \bigcup_{\ell=1}^{\infty} p^{-\ell}\mathbb{Z}$, so $G_p = (\bigcup p^{-\ell}\mathbb{Z})/\mathbb{Z}$ has a natural description as a direct limit $\varinjlim p^{-\ell}\mathbb{Z}/\mathbb{Z}$, and $\hat{G}_p$ is an inverse limit $\varprojlim (p^{-\ell}\mathbb{Z}/\mathbb{Z})^{\wedge}$ of finite groups. The usual pairing $\langle t, n\rangle = \exp 2\pi i t n$ of $\mathbb{Z}$ with $\mathbb{R}/\mathbb{Z}$ induces an isomorphism of $\mathbb{Z}/p^{\ell}\mathbb{Z}$ onto $(p^{-\ell}\mathbb{Z}/\mathbb{Z})^{\wedge}$, and it is easy to check that the dual of the inclusion $p^{-\ell}\mathbb{Z}/\mathbb{Z} \hookrightarrow p^{-(\ell+1)}\mathbb{Z}/\mathbb{Z}$ is the map of $\mathbb{Z}/p^{\ell+1}\mathbb{Z}$ onto $\mathbb{Z}/p^{\ell}\mathbb{Z}$ given by reduction mod p^{ℓ}. Thus $\hat{G}_p$ is naturally identified as a compact group with the inverse limit $\varprojlim \mathbb{Z}/p^{\ell}\mathbb{Z}$.

Each $\mathbb{Z}/p^{\ell}\mathbb{Z}$ is a ring, and the reduction maps are ring homomorphisms, so $\varprojlim \mathbb{Z}/p^{\ell}\mathbb{Z}$ is a compact topological ring $\mathbb{Z}_p$, which is called the ring of p-adic integers; in the previous paragraph, we identified $\hat{G}_p$ with the additive group of $\mathbb{Z}_p$. However, the multiplicative structure of $\mathbb{Z}_p$ plays a crucial role in our analysis, for two reasons. First, we can use it to describe the action α: the reduction maps $\mathbb{Z} \to \mathbb{Z}/p^{\ell}\mathbb{Z}$ induce an embedding of $\mathbb{Z}$ in $\mathbb{Z}_p$, and $\alpha_{m,n}$ is, loosely speaking, division by $p^m q^n$ (see Lemma 1.1 below). Second, the group $\mathcal{U}(\mathbb{Z}_p)$ of units in $\mathbb{Z}_p$ (the multiplicatively invertible elements) appears in our theorems. We need to know that there is a natural identification of $\mathcal{U}(\mathbb{Z}_p)$ with $\varprojlim \mathcal{U}(\mathbb{Z}/p^{\ell}\mathbb{Z})$, and that an integer m is a unit in $\mathbb{Z}_p$ precisely when m is coprime to p. For these and other properties of $\mathbb{Z}_p$, we refer to [16, Chapter II].

We are now ready to describe the Fourier-transform system. The dual of $G_{p,q}$ is $\mathbb{Z}_p \times \mathbb{Z}_q$; if π_ℓ denotes the canonical map of $\mathbb{Z}_p$ onto $\mathbb{Z}/p^{\ell}\mathbb{Z}$, then the pairing is given by

$$(1.4)\quad \langle r + s, (x, y)\rangle = \exp 2\pi i(r\pi_\ell(x) + s\pi_\ell(y)) \quad \text{for } r \in \mathbb{Z}[p^{-1}],\ s \in \mathbb{Z}[q^{-1}], \text{ and } \ell \text{ large.}$$

Lemma 1.1. *The Fourier transform $C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q)$ carries the action defined by* (1.1) *into the action given by*

$$(1.5)\quad \alpha_{m,n}(f)(x, y) = \begin{cases} f(p^{-m}q^{-n}x, p^{-m}q^{-n}y) & \text{if } x \in p^m q^n \mathbb{Z}_p \text{ and } y \in p^m q^n \mathbb{Z}_q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We aim to apply [13, Proposition 4.5]. To do this, note that $\alpha_{m,n}$ is defined by averaging over the solutions s of $\beta_{m,n}(s) = r$, where $\beta_{m,n}$ is the endomorphism of $G_{p,q}$ defined by $\beta_{m,n}(s) = p^m q^n s$. From the pairing (1.4), we see that the endomorphism $\hat{\beta}_{m,n}$ of $\mathbb{Z}_p \times \mathbb{Z}_q$ is given in terms of the ring structure by $\hat{\beta}_{m,n}(x, y) = (p^m q^n x, p^m q^n y)$. Thus the Lemma follows directly from [13, Proposition 4.5]. □

2. THE STRUCTURE THEOREM

Our main theorem describes the structure of $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$—or, equivalently, of the crossed product $C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes_\alpha \mathbb{N}^2$ of the Fourier-transform system described in Lemma 1.1. To state it, we need a number-theoretic lemma. If k and m are coprime integers, so that m is a unit in $\mathbb{Z}/k\mathbb{Z}$, we write $o_k(m)$ for the order of m in $\mathcal{U}(\mathbb{Z}/k\mathbb{Z})$.

Lemma 2.1. *Let p and q be distinct odd primes. Then there is a positive integer $L = L_p(q)$ such that*

$$o_{p^\ell}(q) = \begin{cases} o_p(q) & \text{if } 1 \le \ell \le L, \\ p^{\ell-L} o_p(q) & \text{if } \ell > L. \end{cases} \tag{2.1}$$

This lemma is presumably well-known; certainly some of its immediate consequences are (see Remark 3.8). We are not going to prove it now, because we shall prove a slightly more general result in Theorem 3.1. However, we want to use the integers $L_p(q)$ from this lemma in the statement of our main theorem.

Theorem 2.2. *Let p and q be distinct odd primes. Then there are $\hat{\alpha}$-invariant ideals I_1 and I_2 in $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ such that $I_1 \subset I_2$,*

$$I_1 \cong \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)), \tag{2.2}$$

$$I_2/I_1 \cong (\mathcal{K}(\ell^2(\mathbb{N})) \otimes C) \oplus (\mathcal{K}(\ell^2(\mathbb{N})) \otimes D), \quad \text{and} \tag{2.3}$$

$$(C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2)/I_2 \cong C(\mathbb{T}^2), \tag{2.4}$$

where C is the direct sum of $(p-1)p^{L_p(q)-1}/o_p(q)$ Bunce-Deddens algebras with supernatural number $o_p(q)p^\infty$ and D is the direct sum of $(q-1)q^{L_q(p)-1}/o_q(p)$ Bunce-Deddens algebras with supernatural number $o_q(p)q^\infty$.

The algebra $C^*(G_{p,q}) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q)$ decomposes as a tensor product $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$, and the action α given by (1.5) decomposes as a tensor product of two actions of $\mathbb{N}^2$. At this point, we cannot separate the actions of the two copies of $\mathbb{N}$ (as Bost and Connes say, the two primes interact), but there is a large invariant ideal $C_0(\mathbb{Z}_p \setminus \{0\})$ in $C(\mathbb{Z}_p)$ where the action does split as a tensor product of two actions of $\mathbb{N}$. The ideals I_1 and I_2 will be crossed products of different invariant ideals in $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$ built from $C_0(\mathbb{Z}_p \setminus \{0\})$ and its twin.

For ordinary crossed products $A \rtimes G$ by group actions, invariant ideals in A give rise to short exact sequences

$$0 \longrightarrow I \rtimes G \longrightarrow A \rtimes G \longrightarrow (A/I) \rtimes G \longrightarrow 0.$$

For semigroup crossed products $A \rtimes_\alpha S$, one has to know that the ideal I is *extendibly invariant*, in the sense that each endomorphism α_s extends to endomorphisms of $M(I)$ and $M(A)$ in such a way that $\alpha_s(1_{M(I)}) = \alpha_s(1_{M(A)})$ as multipliers of I (see [1, 13]). Since the endomorphism $x \mapsto p^m q^n x$ of $\mathbb{Z}_p$ leaves both $\mathbb{Z}_p \setminus \{0\}$ and $\{0\}$ invariant, it follows from Lemma 1.1 and [13, Theorem 4.3] that $I := C_0(\mathbb{Z}_p \setminus \{0\})$ and $J := C_0(\mathbb{Z}_q \setminus \{0\})$ are extendibly invariant ideals in $A := C(\mathbb{Z}_p)$ and $B := C(\mathbb{Z}_q)$. We can therefore apply [13, Theorem 3.1] to deduce that the ideals $I_1 := (I \otimes J) \rtimes \mathbb{N}^2$ and $I_2 := (I \otimes B + A \otimes J) \rtimes \mathbb{N}^2$ form a composition series in which

$$I_1 \cong (I \otimes J) \rtimes_\alpha \mathbb{N}^2, \tag{2.5}$$

$$I_2/I_1 \cong ((A/I) \otimes J) \rtimes \mathbb{N}^2 \oplus (I \otimes (B/J)) \rtimes \mathbb{N}^2, \quad \text{and} \tag{2.6}$$

$$(A \otimes B) \rtimes_\alpha \mathbb{N}^2/I_2 \cong ((A/I) \otimes (B/J)) \rtimes \mathbb{N}^2. \tag{2.7}$$

Notice that because the ideals are crossed products, they are $\hat{\alpha}$-invariant. To prove Theorem 2.2, therefore, we have to identify the subquotients.

We begin by noting that the maps $f \mapsto f(0)$ induce isomorphisms $A/I \cong \mathbb{C}$ and $B/J \cong \mathbb{C}$, so $(A/I) \otimes (B/J) \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Thus (2.7) is $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$. When the action is unital, as the identity action id certainly is, the covariance relation (1.3) implies that the isometries are all unitary; thus $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$ is the universal C^*-algebra generated by a unitary representation of $\mathbb{Z}^2$. In other words, $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2 = C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, and we have proved (2.4).

For the other two parts, we need the promised decomposition of the action of $\mathbb{N}^2$ on $I = C_0(\mathbb{Z}_p \setminus \{0\})$.

Lemma 2.3. *The map $(n, x) \mapsto p^n x$ is a homeomorphism of $\mathbb{N} \times \mathcal{U}(\mathbb{Z}_p)$ onto $\mathbb{Z}_p \setminus \{0\}$.*

Proof. Since every nonzero p-adic number can be uniquely written as a power of p times a unit (by Proposition 2 of [16, Chapter II], for example), the map is a bijection. It is a homeomorphism because it carries the basic open sets $\{n\} \times V$ for the topology on $\mathbb{N} \times \mathcal{U}(\mathbb{Z}_p)$ into the basic open sets $p^n V$ for the topology on $\mathbb{Z}_p \setminus \{0\}$. □

The lemma implies that $I = C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p))$. To describe what happens to the action α under this isomorphism, we need some notation. We let τ denote the action of $\mathbb{N}$ on $c_0(\mathbb{N})$ by forward shifts; if we think of elements of $c_0(\mathbb{N})$ as functions on $\mathbb{N}$, then

$$\tau_m(f)(k) = \begin{cases} f(k - m) & \text{if } k \geq m \\ 0 & \text{if } k < m. \end{cases}$$

Since $(q, p) = 1$, q is a unit in $\mathbb{Z}_p$, and division by powers of q defines an action $\sigma = \sigma^{p,q}$ of $\mathbb{Z}$ by automorphisms of $C(\mathcal{U}(\mathbb{Z}_p))$: $\sigma_n(f)(x) = f(q^{-n}x)$. We now have the following immediate corollary of Lemma 2.3:

Corollary 2.4. *The isomorphism $C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p))$ induced by the homeomorphism of Lemma 2.3 carries α into the tensor product action $\tau \otimes \sigma$: $(m, n) \mapsto \tau_m \otimes \sigma_n$.*

Lemma 2.5. *There is an isomorphism*

$$(2.8)\quad I_2/I_1 \cong (\mathcal{K}(\ell^2(\mathbb{N})) \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{Z})) \oplus (\mathcal{K}(\ell^2(\mathbb{N})) \otimes (C(\mathcal{U}(\mathbb{Z}_q)) \rtimes_{\sigma^{q,p}} \mathbb{Z})).$$

Proof. First, recall that $A/I \cong \mathbb{C}$ and $B/J \cong \mathbb{C}$, so from (2.6) we have

$$(2.9)\qquad I_2/I_1 \cong (I \rtimes_\alpha \mathbb{N}^2) \oplus (J \rtimes_\alpha \mathbb{N}^2).$$

Next, we use the decomposition of Corollary 2.4 and [13, Theorem 2.6] (which applies because our action satisfies (1.2)), to see that

$$(2.10)\qquad I \rtimes_\alpha \mathbb{N}^2 \cong (c_0(\mathbb{N}) \rtimes_\tau \mathbb{N}) \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{N}).$$

Because $\sigma^{p,q}$ consists of automorphisms, the isometries in any covariant representation of $(C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{N}, \sigma)$ are unitary, and $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{N}$ is the usual crossed product $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$.

To handle the other factor in (2.10), recall that $c \rtimes_\tau \mathbb{N} = B_{\mathbb{N}} \rtimes_\tau \mathbb{N}$ is the Toeplitz algebra, and $c_0(\mathbb{N}) \rtimes_\tau \mathbb{N}$ is the ideal of compact operators. More precisely, let M denote the representation of c by multiplication operators on $\ell^2(\mathbb{N})$, and let S be the unilateral shift on $\ell^2(\mathbb{N})$. Then (M, S) is a covariant representation of $(c, \mathbb{N}, \tau)$ such that $M \times S$ is an isomorphism of $c \rtimes_\tau \mathbb{N}$ onto the C^*-algebra generated by S. (This formulation of Coburn's Theorem is described in [2], for example.) It is easy to check that $M \rtimes S$ carries the ideal $c_0 \rtimes_\tau \mathbb{N}$ onto $\mathcal{K}(\ell^2(\mathbb{N}))$. Thus (2.10) implies that $I \rtimes_\alpha \mathbb{N}^2 \cong \mathcal{K} \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z})$. Swapping p and q gives an analogous description of $J \rtimes_\alpha \mathbb{N}^2$, and the Lemma follows from (2.9). □

The description of I_2/I_1 in (2.3) will follow from this lemma and Theorem 3.1.

To describe $I_1 := (I \otimes J) \rtimes_\alpha \mathbb{N}^2$, we use two applications of Corollary 2.4 to get an isomorphism

$$I \otimes J = C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C_0(\mathbb{Z}_q \setminus \{0\}) \cong C_0(\mathbb{N} \times \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$$

which carries the endomorphism $\alpha_{m,n}$ into $\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}$. We now borrow another idea from the theory of ordinary crossed products: recall that $(C_0(G) \otimes A) \rtimes_{\tau\otimes\beta} G \cong (C_0(G) \rtimes_\tau G) \otimes A$ for any action β. Because $q \in \mathcal{U}(\mathbb{Z}_p)$ and $p \in \mathcal{U}(\mathbb{Z}_q)$, the endomorphism φ of $C_0(\mathbb{N}\times\mathbb{N}\times\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q))$ defined by $\varphi(f)(k, \ell, x, y) = f(k, \ell, q^\ell x, p^k y)$ is an automorphism. A quick calculation shows that $\varphi \circ (\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}) = \tau_m \otimes \tau_n \otimes \mathrm{id} \otimes \mathrm{id}$, so φ induces an isomorphism

$$(I \otimes J) \rtimes_\alpha \mathbb{N}^2 \cong (c_0(\mathbb{N} \times \mathbb{N}) \rtimes_{\tau\otimes\tau} (\mathbb{N} \times \mathbb{N})) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)).$$

To finish off the proof of (2.2), either note that

$$\begin{aligned} c_0(\mathbb{N}^2) \rtimes_{T\otimes T} \mathbb{N}^2 &\cong (c_0 \rtimes_T \mathbb{N}) \otimes (c_0 \rtimes_T \mathbb{N}) \\ &\cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(\ell^2(\mathbb{N})) = \mathcal{K}(\ell^2(\mathbb{N}^2)), \end{aligned}$$

or check directly that the natural covariant representation of $B_{\mathbb{N}^2} \rtimes_T \mathbb{N}^2$ on $\ell^2(\mathbb{N}^2)$ restricts to an isomorphism of $c_0(\mathbb{N}^2) \rtimes \mathbb{N}^2$ onto $\mathcal{K}(\ell^2(\mathbb{N}^2))$.

To prove Theorem 2.2, therefore, it remains to prove Lemma 2.1 and to identify $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$ with the appropriate number of Bunce-Deddens algebras. We do this in Theorem 3.1.

3. THE CROSSED PRODUCTS $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$

Our analysis of $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,q}} \mathbb{Z}$ does not require that q is prime, only that it is coprime to p. We therefore fix an odd prime p and an integer m coprime to p, and consider the action $\sigma = \sigma^{p,m}$ of $\mathbb{Z}$ on $C(\mathcal{U}(\mathbb{Z}_p))$ defined by

$$\sigma_n^{p,m}(f)(x) = f(m^{-n}x). \tag{3.1}$$

Theorem 3.1. *Suppose that p is an odd prime and $(m,p) = 1$, and denote by $o_{p^\ell}(m)$ the order of m in $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$. Then there is a positive integer L such that*

$$o_{p^\ell}(m) = \begin{cases} o_p(m) & \text{if } 1 \le \ell \le L \\ p^{\ell-L}o_p(m) & \text{if } \ell > L, \end{cases} \tag{3.2}$$

and $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,m}} \mathbb{Z}$ is the direct sum of $p^{L-1}(p-1)/o_p(m)$ Bunce-Deddens algebras with supernatural number $o_p(m)p^\infty$.

We begin by establishing the number-theoretic statements. Because $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ is cyclic of order $(p-1)p^{\ell-1}$ (see Theorem 2 of [8, Chapter 4], for example), we can apply the following elementary lemma about cyclic groups.

Lemma 3.2. *Suppose that $(n,p) = 1$ and G, H are cyclic groups of orders $p^\ell n$, $p^{\ell-1}n$, respectively. If $\pi : G \to H$ is a surjective homomorphism and g is a generator of G, then the order of $\pi(g^r)$ is given by*

$$o(\pi(g^r)) = \begin{cases} |G|/(r,|G|) & \text{if } p^\ell \text{ divides } r \\ |G|/p(r,|G|) & \text{if } p^\ell \text{ does not divide } r \end{cases}$$

Proof. Since $\pi(g)$ is a generator of H, we have

$$o(\pi(g^r)) = o(\pi(g)^r) = \frac{|H|}{(r,|H|)} = \frac{|G|}{p(r,|H|)}.$$

If p^ℓ divides r, say $r = sp^\ell$, then

$$p(r,|H|) = p(p^\ell s, p^{\ell-1}n) = p^\ell(ps,n) = p^\ell(s,n) = (r,p^\ell n) = (r,|G|),$$

as claimed. If p^ℓ does not divide r, then $(r,|G|) = (r,p^\ell n) = (r,p^{\ell-1}n) = (r,|H|)$. □

Corollary 3.3. *Suppose p is prime and $(p,m) = 1$. Then*

$$o_{p^\ell}(m) = \begin{cases} o_{p^{\ell+1}}(m) & \text{if } p \text{ does not divide } o_{p^{\ell+1}}(m) \\ o_{p^{\ell+1}}(m)/p & \text{if } p \text{ does divide } o_{p^{\ell+1}}(m). \end{cases}$$

Proof. Since a number is coprime to p^ℓ iff it is coprime to $p^{\ell+1}$, the reduction map π is a homomorphism of $\mathcal{U}(\mathbb{Z}/p^{\ell+1}\mathbb{Z})$ onto $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$, and Lemma 3.2 applies. Indeed, there is a generator g such that $m = g^r$ where $r := (p-1)p^\ell/o_{p^{\ell+1}}(m)$. Then

$$o_{p^\ell}(m) = o(\pi(g^r)) = \begin{cases} o_{p^{\ell+1}}(m) & \text{if } p^\ell \text{ divides } (p-1)p^\ell/o_{p^{\ell+1}}(m), \\ o_{p^{\ell+1}}(m)/p & \text{if } p^\ell \text{ does not divide } \\ & (p-1)p^\ell/o_{p^{\ell+1}}(m), \end{cases}$$

which translates into what we want. □

Corollary 3.4. *There is a positive integer L such that* (3.2) *holds.*

Proof. We first note that the sequence $\{o_{p^\ell}(m) \mid \ell \in \mathbb{N}\}$ must be unbounded: for fixed N, m^N is eventually less than p^ℓ, and then $o_{p^\ell}(m) > N$. In particular, $\{o_{p^\ell}(m)\}$ is certainly not constant. Let L be the first integer such that $o_{p^L}(m) < o_{p^{L+1}}(m)$. Then $o_{p^\ell}(m) = o_p(m)$ for $1 \le \ell \le L$, and by Corollary 3.3, we have $o_{p^{L+1}}(m) = po_p(m)$, and p divides $o_{p^{L+1}}(m)$. Since $o_{p^{L+1}}(m)$ divides $o_{p^\ell}(m)$ for all $\ell > L$, it follows that p divides $o_{p^\ell}(m)$ for all $\ell > L$, and $\ell - L$ applications of Corollary 3.3 show that $o_{p^\ell}(m) = p^{\ell-L}o_{p^L}(m) = p^{\ell-L}o_p(m)$. □

Remark 3.5. The referee has pointed out that one can also deduce Corollary 3.4 from the isomorphism of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \times p\mathbb{Z}_p^+$ onto $\mathcal{U}(\mathbb{Z}_p)$ provided by sending elements of $\mathcal{U}(\mathbb{Z}/p\mathbb{Z})$ to their Teichmüller representatives and the exponential isomorphism of the additive group $p\mathbb{Z}_p^+$ onto $1 + p\mathbb{Z}_p$ (see [7, Corollary 4.5.10], for example). This isomorphism is compatible with the inverse limit decompositions of $\mathcal{U}(\mathbb{Z}_p)$ and $p\mathbb{Z}_p^+$, and hence it suffices to prove the analogous properties of additive orders in $p\mathbb{Z}_p^+$.

Let H be the closed subgroup of $\mathcal{U}(\mathbb{Z}_p)$ generated by m. Then H is invariant under multiplication by powers of m, and the formula (3.1) also defines an action σ of $\mathbb{Z}$ on $C(H)$. This is where the Bunce-Deddens algebras come from:

Proposition 3.6. *The crossed product $C(H) \rtimes_\sigma \mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $o_p(m)p^\infty$.*

The *Bunce-Deddens algebras* were originally defined to be the C^*-algebras generated by certain weighted shifts on ℓ^2 [5, Section V.3], but we shall recognize them as crossed products associated to odometer actions. Let $\{n_k\}$ be a sequence of integers each of which is at least 2, and let $X_k = \{0, 1, \ldots, n_k - 1\}$. The *odometer action* τ of $\mathbb{Z}$ on $\prod_{k\geq 1} X_k$ is given by addition with carry over: let $N_1 = 1$, $N_k := \prod_{i<k} n_i$ for $k > 1$, and then

$$\tau_n(\{a_k\}) = \{b_k\}, \quad \text{where } \sum_{k\geq 1}^{\ell} b_k N_k :\equiv n + \sum_{k\geq 1}^{\ell} a_k N_k \pmod{N_{\ell+1}}.$$

The crossed product $C(\prod_{k\geq 1} X_k) \rtimes_\tau \mathbb{Z}$ is then a Bunce Deddens algebra with supernatural number $\prod_{k\geq 1} n_k$ [5, Theorem VIII.4.1]. In general, Bunce-Deddens algebras are simple [5, Theorem V.3.3], and are determined up to isomorphism by their supernatural number [5, Theorem V.3.5].

Proof. Write d for $o_p(m)$, and let

$$\mathcal{O} := \{0, 1, \ldots, d-1\} \times \{0, 1, \ldots, p-1\}^{\mathbb{N}}.$$

For $\ell > L$, we define $h_\ell : \mathcal{O} \to \mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ by

$$h_\ell(\{a_n\}) = m^{a_0 + da_1 + dpa_2 + \cdots + dp^{\ell-L-1}a_{\ell-L}} \pmod{p^\ell\mathbb{Z}};$$

because the order of m in $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ is $dp^{\ell-L}$, the maps h_ℓ satisfy $h_{\ell+1}(\{a_n\}) = h_\ell(\{a_n\}) \pmod{p^\ell\mathbb{Z}}$. Since the h_ℓ are continuous by definition of the product topology, they induce a continuous map $h : \mathcal{O} \to \mathcal{U}(\mathbb{Z}_p) = \varprojlim \mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$, which is an injection because $h_\ell(\{a_n\})$ determines $a_0, \ldots, a_{\ell-L}$ uniquely. The range of h is a compact subgroup, and contains the positive powers of m, which are the images of the sequences in $\mathcal{O}$ which are eventually zero; since such sequences are dense in $\mathcal{O}$, their images generate the range. In other words, h is a continuous injection of $\mathcal{O}$ onto H, and is therefore a homeomorphism. Since $h(\tau\{a_n\}) = mh(\{a_n\})$ for all $\{a_n\}$, we deduce that the Bunce-Deddens algebra $C(\mathcal{O}) \rtimes_\tau \mathbb{Z}$ is isomorphic to $C(H) \rtimes_\sigma \mathbb{Z}$. □

To finish the proof of our theorem, we need to decompose the dynamical system $(C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{Z}, \sigma)$ as a sum of copies of $(C(H), \mathbb{Z}, \sigma)$. This needs a simple group-theoretic lemma.

Lemma 3.7. *Suppose $G = \varprojlim G_n$ is a compact group which is the inverse limit of finite groups G_n, and suppose that the canonical maps $\pi_n : G \to G_n$ are surjective. If H is a closed subgroup of G and there is an integer k such that $|G_n/\pi_n(H)| = k$ for all n, then $|G/H| = k$.*

Proof. Certainly $|G/H| \geq |\pi_n(G)/\pi_n(H)| = k$. Suppose $g_1H, \dots, g_{k+1}H$ are cosets in G/H; we shall prove that two must be the same. The hypothesis implies that for each n, two of $\pi_n(g_iH)$ coincide. Since there are only finitely many possibilities, we can assume by passing to a subsequence that the same two coincide in each $G_n/\pi_n(H)$; say $\pi_n(g_1H) = \pi_n(g_2H)$ for all n. Then $\pi_n(g_1g_2^{-1}) \in \pi_n(H)$; say $\pi_n(g_1g_2^{-1}) = \pi_n(h_n)$. By definition of the topology on the inverse limit, we have $h_n \to g_1g_2^{-1}$ in G, so that $g_1g_2^{-1} \in H$ and $g_1H = g_2H$. □

End of the proof of Theorem 3.1. Since $\pi_\ell(H)$ is the subgroup of $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ generated by m, we have

$$\begin{aligned} |\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})/\pi_\ell(H)| &= (p-1)p^{\ell-1}/o_{p^\ell}(m) \\ &= (p-1)p^{L-1}/o_p(m) \quad \text{for all } \ell \geq L. \end{aligned}$$

We can therefore apply Lemma 3.7 to $\mathcal{U}(\mathbb{Z}_p) = \varprojlim(\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z}), \ell \geq L)$ to deduce that H has index $N := (p-1)p^{L-1}/o_p(m)$ in $\mathcal{U}(\mathbb{Z}_p)$.

Next, note that because H is a closed subgroup of finite index, it is also open: its complement is the finite union of cosets of H, and hence closed. Since H is by definition invariant under multiplication by powers of m, it follows that $\mathcal{U}(\mathbb{Z}_p)$ is the disjoint union of N open and closed invariant sets of the form xH, and $C(\mathcal{U}(\mathbb{Z}_p))$ is the direct sum of σ-invariant ideals of the form $C(xH)$. The dynamical systems $(C(xH), \mathbb{Z}, \sigma)$ are all conjugate to $(C(H), \mathbb{Z}, \sigma)$. Thus the Theorem follows from Proposition 3.6. □

Remark **3.8**. An integer m which generates $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ is called a *primitive root modulo* p^ℓ. If m is a primitive root modulo p^ℓ for one $\ell > 1$, then (3.2) implies that $L_p(m) = 1$ and $o_p(m) = p-1$, and hence that m is a primitive root modulo p^k for all k. (This is known; see [6, Section 17, Exercise VI.4], for example.) Theorem 3.1 gives a curious C^*-algebraic characterisation of primitive roots: m is primitive modulo p^ℓ for all ℓ if and only if $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}$ is simple. More generally, the cardinality of $\operatorname{Prim} C(\mathcal{U}(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ determines the orders $o_{p^\ell}(m)$ of m in $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$.

The relations (3.2) are the only restrictions on the possible values of $o_p(m)$. Indeed, given an odd prime p, a divisor d of $p-1$, and an integer $L \geq 1$, there are infinitely many primes q with $o_p(q) = d$ and $L_p(q) = L$. To see this, choose k such that $o_{p^{L+1}}(k) = pd$. Then every integer q in the arithmetic progression $k + np^{L+1}$ has $o_p(q) = d$ and $o_{p^{L+1}}(q) = pd$, and it follows from (3.2) that $o_{p^\ell}(q) = p^{\ell-L}d$ for all $\ell > L$. Now our assertion follows from Dirichlet's Theorem: every arithmetic progression $k + nr$ with $(k, r) = 1$ contains infinitely many primes [8, Section 16.1].

4. THE PRIMITIVE IDEAL SPACE

Since $\operatorname{Prim} C(X,\mathcal{K})$ is homeomorphic to X [15, Example A.24] and Bunce-Deddens algebras are simple [5, Theorem V.3.3], Theorem 2.2 gives us a setwise description of the primitive ideal space of the algebra $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2$. It consists of a copy $\{I_{x,y}\}$ of $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$ embedded as an open subset, a copy $\{L_{w,z}\}$ of $\mathbb{T}^2$ embedded as a closed subset, and two finite sets $\{J_{xH_p}\}$, $\{K_{yH_q}\}$ parametrised by the quotients $\mathcal{U}(\mathbb{Z}_p)/H_p = \mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}$ and $\mathcal{U}(\mathbb{Z}_q)/H_q = \mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}$ whose cardinalities determine the number of Bunce-Deddens algebras in the subquotients. The topology on $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2)$ is then given by:

Theorem 4.1. *The maps $(x,y)\mapsto I_{x,y}$, $xH_p\mapsto J_{xH_p}$, $yH_q\mapsto J_{yH_q}$ and $(w,z)\mapsto L_{w,z}$ combine to give a bijection of the disjoint union*

$$(\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q))\sqcup\mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}\sqcup\mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}\sqcup\mathbb{T}^2 \tag{4.1}$$

onto $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2)$. *Write π_p for the map $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)\to\mathcal{U}(\mathbb{Z}_p)\to\mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}$. Then the hull-kernel closure of a nonempty subset F of* (4.1) *is*

(a) *the usual closure of F in $\mathbb{T}^2$ if $F\subset\mathbb{T}^2$;*
(b) *$F\cup\mathbb{T}^2$ if $F\subset\mathcal{U}(\mathbb{Z}_p)/\overline{q^{\mathbb{Z}}}\sqcup\mathcal{U}(\mathbb{Z}_q)/\overline{p^{\mathbb{Z}}}$;*
(c) *the usual closure of F in $\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$ together with $\pi_p(F)\cup\pi_q(F)\cup\mathbb{T}^2$ if $F\subset\mathcal{U}(\mathbb{Z}_p)\times\mathcal{U}(\mathbb{Z}_q)$.*

We prove this by finding irreducible representations of $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes_\alpha\mathbb{N}^2$ realising each of these primitive ideals, identifying their kernels as crossed products of invariant ideals in $C(\mathbb{Z}_p\times\mathbb{Z}_q)$ using results from [13], and then reading off the topology from standard properties of the topology on $\mathbb{Z}_p\times\mathbb{Z}_q$.

The ideals $L_{w,z}$ are lifted from the quotient $(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2)/I_2=\mathbb{C}\rtimes_{\mathrm{id}}\mathbb{N}^2$, and are the kernels of the characters $\gamma_{w,z}:(m,n)\mapsto w^m z^n$; more precisely, $L_{w,z}=\ker(\varepsilon_{0,0}\times\gamma_{w,z})$, where $\varepsilon_{0,0}(f):=f(0,0)$. Because $\operatorname{Prim}(\mathbb{C}\rtimes_{\mathrm{id}}\mathbb{N}^2)$ is a closed subset of $\operatorname{Prim}(C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2)$, this also proves part (a) of Theorem 4.1.

The ideals J_{xH_p} are inverse images under the map $(\mathrm{id}\otimes\varepsilon_0)^*$ of $C(\mathbb{Z}_p\times\mathbb{Z}_q)\rtimes\mathbb{N}^2$ onto $C(\mathbb{Z}_p)\rtimes\mathbb{N}^2$ induced by $\mathrm{id}\otimes\varepsilon_0: C(\mathbb{Z}_p\times\mathbb{Z}_q)\to C(\mathbb{Z}_p)$, and are determined by the intersections of their images with the ideal $C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2$. Recall that the homeomorphism $h_p:(k,x)\mapsto p^k x$ induces an isomorphism

$$h_p^*: C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2\cong C(\mathcal{U}(\mathbb{Z}_p),c_0(\mathbb{N})\rtimes_\tau\mathbb{N})\rtimes_{\sigma\otimes\mathrm{id}}\mathbb{Z}. \tag{4.2}$$

Because $M\times T$ is an isomorphism of $c_0(\mathbb{N})\rtimes_\tau\mathbb{N}$ onto $\mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathbb{Z}$ acts freely on $\mathcal{U}(\mathbb{Z}_p)=\operatorname{Prim} C(\mathcal{U}(\mathbb{Z}_p),\mathcal{K})$, the primitive ideals of the right-hand side of (4.2) are induced from the ideals $\ker(M\times T)\circ\varepsilon_x$. In particular, we have

$$J_{xH_p}\cap(C_0(\mathbb{Z}_p\setminus\{0\})\rtimes\mathbb{N}^2)=\ker((\operatorname{Ind}_{\{0\}}^{\mathbb{Z}}(M\times T)\circ\varepsilon_x)\circ h_p^*\circ(\mathrm{id}\otimes\varepsilon_0)^*).$$

We can now use the standard form $\tilde{\pi} \times \lambda$ of the induced representation to see that the ideal J_{xH_p} is the kernel of the representation $\rho_x \times (T \otimes \lambda)$ of $C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes_\alpha \mathbb{N}^2$ on $\ell^2(\mathbb{N} \times \mathbb{Z})$, where $(\rho_x(f)\xi)(k,\ell) := f(p^k q^\ell x, 0)\xi(k,\ell)$. Similarly, with $\sigma_y : C(\mathbb{Z}_p \times \mathbb{Z}_q) \to B(\ell^2(\mathbb{Z} \times \mathbb{N}))$ defined by

$$(\sigma_y(f)\xi)(k,\ell) = f(0, p^k q^\ell y)\xi(k,\ell),$$

we have $\ker(\sigma_y \times (\lambda \otimes T)) = K_{yH_q}$.

The ideals $I_{x,y}$ are determined by their intersection with I_1, and $I_{x,y} \cap I_1$ is pulled back under the isomorphism (2.2) from the kernel of the evaluation map $\varepsilon_{x,y} : C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q), \mathcal{K}) \to \mathcal{K}$. This isomorphism is induced by the homeomorphism $h : (\ell, k, x, y) \mapsto p^k q^\ell (x, y)$ of $\mathbb{N}^2 \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$ onto $(\mathbb{Z}_p \setminus \{0\}) \times (\mathbb{Z}_q \setminus \{0\})$, and the Toeplitz representation $M \times T$ of $c_0(\mathbb{N}^2) \rtimes_\tau \mathbb{N}^2$ onto $\mathcal{K}(\ell^2(\mathbb{N}^2))$. The representation $(\pi_{x,y}(f)\xi)(k,\ell) = f(p^k q^\ell x, p^k q^\ell y)\xi(k,\ell)$ satisfies $\pi_{x,y}|_{C_0((\mathbb{Z}_p \setminus \{0\}) \times (\mathbb{Z}_q \setminus \{0\}))} = M \circ \varepsilon_{x,y} \circ (h^{-1})^*$, and it follows that $(\pi_{x,y}, T)$ is a covariant representation of $(C(\mathbb{Z}_p \times \mathbb{Z}_q), \mathbb{N}^2, \alpha)$ with $I_{x,y} = \ker(\pi_{x,y} \times T)$.

To identify the kernels of these representations, we shall use the following analogue of the standard characterisations of faithful representations.

Lemma 4.2. *Let (η, T) be a covariant representation of a semigroup dynamical system $(A, \mathbb{N}^k, \alpha)$ with extendible endomorphisms. Suppose that* $\ker \eta$ *is an extendibly α-invariant ideal, and and W is a unitary representation of $\mathbb{T}^k$ such that $(\eta \times T, W)$ is a covariant representation of the dual system $(A \rtimes \mathbb{N}^k, \mathbb{T}^k, \hat{\alpha})$. Then*

$$\begin{aligned}\ker(\eta \times T) &= (\ker \eta) \rtimes \mathbb{N}^k \\ &= \overline{\operatorname{span}}\{i_S(m)^* i_A(a) i_S(n) \mid m, n \in \mathbb{N}^k,\ a \in \ker \eta\}.\end{aligned}$$

Proof. We know from [13, Theorem 1.8] that $(\ker \eta) \rtimes \mathbb{N}^k$ is naturally isomorphic to the ideal

$$\overline{\operatorname{span}}\{i_S(m)^* i_A(a) i_S(n) \mid m, n \in \mathbb{N}^k,\ a \in \ker \eta\} \subset A \rtimes \mathbb{N}^k,$$

and that the quotient map $\pi : A \to A/(\ker \eta)$ induces a homomorphism $\pi \times \mathrm{id}$ of $A \rtimes \mathbb{N}^k$ onto $(A/\ker \eta) \rtimes \mathbb{N}^k$ with kernel $(\ker \eta) \rtimes \mathbb{N}^k$. There is a faithful representation ζ of $A/\ker \eta$ such that $\eta = \zeta \circ \pi$, and then (ζ, T) and $(\zeta \times T, W)$ are covariant. It suffices to prove that $\zeta \rtimes T$ is faithful, for then $\eta \times T = (\zeta \times T) \circ (\pi \times \mathrm{id})$, and

$$\ker(\eta \times T) = \ker(\zeta \times T) \circ (\pi \times \mathrm{id}) = \ker(\pi \times \mathrm{id}) = (\ker \eta) \rtimes \mathbb{N}^k.$$

To prove $\zeta \times T$ faithful, we follow the standard procedure of [4, Lemma 2.2]. Write $C = A/\ker \eta$, and let $\theta : C \rtimes \mathbb{N}^k \to C \rtimes \mathbb{N}^k$ be the expectation obtained by averaging over the dual action $\hat{\alpha}$ on $C \rtimes \mathbb{N}^k$, which is faithful on positive elements by [10, Remark 3.6]. Because $S = \mathbb{N}^k$ is abelian, $C \rtimes \mathbb{N}^k$ is spanned by the elements $i_S(m)^* i_C(c) i_S(n)$ [13, Lemma 1.3], and hence $\theta(C \rtimes \mathbb{N}^k)$ is spanned by the elements $i_S(m)^* i_C(c) i_S(m)$; because every finite set of elements in $\mathbb{N}^k$ has an upper bound, we can imitate the proof of [2, Lemma 1.5] to see that $\zeta \times T$ is faithful on $\theta(C \rtimes \mathbb{N}^k)$. Now we can use the covariance of $(\zeta \times T, W)$ to get an estimate

$$\begin{aligned}\|(\zeta \times T)(\theta(f))\| &= \left\| \int_{\mathbb{T}^k} W_z^* (\zeta \times T)(f) W_z \, dz \right\| \\ &\leq \int_{\mathbb{T}^k} \|W_z^* \zeta \times T(f) W_z\| \, dz \\ &= \|\zeta \times T(f)\|,\end{aligned}$$

and follow the argument of [4, Lemma 2.2] to see that $\zeta \times T$ is faithful. □

The ideal $\ker \pi_{x,y}$ consists of the functions which vanish on the closure of the orbit $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$; to check that $\ker \pi_{x,y}$ is extendibly invariant, we need to know exactly what this closure is.

Lemma 4.3. *Let $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q$. Then $q^{\mathbb{N}} x$ has the same closure in $\mathbb{Z}_p$ as $q^{\mathbb{Z}} x$, and the closure of $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$ in $\mathbb{Z}_p \times \mathbb{Z}_q$ is*

$$p^{\mathbb{N}} q^{\mathbb{N}}(x, y) \cup (\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}) \cup (\{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}). \tag{4.3}$$

Proof. Since $q \in \mathcal{U}(\mathbb{Z}_p)$, multiplication by q is a homeomorphism of $\mathcal{U}(\mathbb{Z}_p)$, and defines a free and minimal action of $\mathbb{Z}$ on $\overline{q^{\mathbb{Z}} x}$. The sequence $\{q^k x \mid k \in \mathbb{N}\}$ has a convergent subsequence, $q^{k_n} x \to x_0$, say, and then $\overline{q^{\mathbb{Z}} x_0} = \overline{q^{\mathbb{Z}} x}$ by minimality. Thus every element of $\overline{q^{\mathbb{Z}} x}$ can be approximated first by $q^n x_0$, and then by elements $q^{n+k_n} x$ of $q^{\mathbb{N}} x$. Thus $\overline{q^{\mathbb{N}} x} = \overline{q^{\mathbb{Z}} x}$. This argument also shows that every element of $\overline{q^{\mathbb{Z}} x}$ is the limit of a sequence $q^{m_n} x$ in which $m_n \to \infty$.

Since $(0,0) = \lim_n p^n q^n (x, y)$, it certainly belongs to the orbit closure. Suppose $p^{k_n} q^{\ell_n} x \to s$ and $s \neq 0$. Write $s = p^i s_0$ for $s_0 \in \mathcal{U}(\mathbb{Z}_p)$. Then $p^i \mathcal{U}(\mathbb{Z}_p)$ is an open neighbourhood of s, so $k_n = i$ for large n, and $q^{\ell_n} x \to p^{-i} s$. As observed above, we may as well suppose $\ell_n \to \infty$; but then $q^{\ell_n} y \to 0$, and $p^{k_n} q^{\ell_n}(x, y) \to (s, 0)$. Thus $\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}$ is contained in the orbit closure, and, by symmetry, so is $\{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}$.

For the other inclusion, suppose $(w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ and $p^{k_n} q^{\ell_n}(x, y) \to (w, z)$. It is obvious that (w, z) belongs to (4.3) if one of w or z is 0, so suppose w and z are both nonzero. We can write $(w, z) = (p^i w_0, q^j z_0)$ for units w_0, z_0 and $i, j \in \mathbb{N}$, and then $p^i \mathcal{U}(\mathbb{Z}_p) \times q^j \mathcal{U}(\mathbb{Z}_q)$ is a neighbourhood of (w, z). Thus $(k_n, \ell_n) = (i, j)$ for large n, and $(w, z) = p^i q^j (x, y)$ belongs to $p^{\mathbb{N}} q^{\mathbb{N}}(x, y)$, as required. □

Lemma 4.4. *Let $(x, y) \in \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$. Then*

(a) $J_{xH_p} = \overline{\operatorname{span}}\{i_S(i,j)^* i_A(f) i_S(m,n) \mid f \equiv 0 \text{ on } \overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}\}$;
(b) $K_{yH_q} = \overline{\operatorname{span}}\{i_S(i,j)^* i_A(f) i_S(m,n) \mid f \equiv 0 \text{ on } \{0\} \times \overline{p^{\mathbb{Z}} q^{\mathbb{N}} y}\}$; *and*
(c) $I_{x,y} = \overline{\operatorname{span}}\{i_S(i,j)^* i_A(f) i_S(m,n) \mid f \equiv 0 \text{ on } \overline{p^{\mathbb{N}} q^{\mathbb{N}}(x,y)}\}$.

Proof. For part (a), we want to apply Lemma 4.2 with $\eta = \rho_x$, and we therefore need to know that $\ker \rho_x$ is extendibly invariant. We have $\rho_x(f) = 0$ iff $f \equiv 0$ on $\overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$, which is equivalent by Lemma 4.3 to $f \equiv 0$ on $\overline{p^{\mathbb{N}} q^{\mathbb{Z}} x} \times \{0\}$. Thus it is enough by [13, Theorem 4.3] to prove that $\overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$ and its complement are invariant under multiplication by $p^k q^\ell$. This is trivially true for $\overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$. Suppose $(w,z) \notin \overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$. If $z \neq 0$, then $p^k q^\ell (w,z)$ is certainly not in $\overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$. So we consider the case $z = 0$, and suppose $p^{k_n} q^{\ell_n} x \to p^k q^\ell w$. Since w cannot be 0, we can write $w = p^i w_0$ for $w_0 \in \mathcal{U}(\mathbb{Z}_p)$. Eventually $p^{k_n} q^{\ell_n} x \in p^{k+i} \mathcal{U}(\mathbb{Z}_p)$, so $k_n = k + i$ for large n, and $q^\ell w = \lim p^{k_n - k} q^{\ell_n} x$ belongs to $p^i(\overline{q^{\mathbb{N}} x})$. Since $\overline{q^{\mathbb{N}} x} = \overline{q^{\mathbb{Z}} x}$, this implies that $w \in p^i(\overline{q^{\mathbb{N}} x})$, and hence that $(w,z) \in \overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$, which is a contradiction. So $p^k q^\ell (w,z) \notin \overline{p^{\mathbb{N}} q^{\mathbb{N}} x} \times \{0\}$ for all $k, \ell \in \mathbb{N}$, and we have shown that $\ker \rho_x$ is extendibly invariant.

Next we observe that $W_{w,z}\xi(k,\ell) := w^k z^\ell \xi(k,\ell)$ defines a unitary representation W of $\mathbb{T}^2$ on $\ell^2(\mathbb{N} \times \mathbb{Z})$ such that $(\rho_x \times (T \otimes \lambda), W)$ is covariant for the dual action. Thus we can deduce from Lemma 4.2 that $J_{xH_p} = \ker(\rho_x \times (T \otimes \lambda))$ has the required form. This gives (a), and of course (b) is exactly the same.

For (c), we apply the same argument to

$$\ker \pi_{x,y} = \{f \in C(\mathbb{Z}_p \times \mathbb{Z}_q) \mid f \equiv 0 \text{ on } \overline{p^{\mathbb{N}} q^{\mathbb{N}}(x,y)}\};$$

as above, the crux is to prove that if $p^k q^\ell (w,z)$ is in the closure of $p^{\mathbb{N}} q^{\mathbb{N}}(x,y)$, then so is (w,z). So suppose $(w,z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ and $p^{k_n} q^{\ell_n}(x,y) \to p^k q^\ell (w,z)$. If w or z is 0, we are in the situation covered by the first paragraph. So suppose w and z are both nonzero: say $w = p^i w_0$ and $z = q^j z_0$ for units w_0, z_0. By Lemma 4.3, we must have $p^k q^\ell (w,z) = p^m q^n (x,y)$ for some $m, n \in \mathbb{N}$. Then $p^{k+i} q^\ell w_0 = p^m q^n x$ and $p^k q^{\ell+j} z_0 = p^m q^n y$. The first of these equations implies that $k + i = m$, so $k \leq m$, and the second that $\ell \leq n$. Thus $(w,z) = p^{m-k} q^{n-\ell}(x,y)$ belongs to $p^{\mathbb{N}} q^{\mathbb{N}}(x,y)$. This proves that $\ker \pi_{x,y}$ is extendibly invariant. Part (c) follows from an application of Lemma 4.2 with W given by the same formula as before. □

Proof of Theorem 4.1. We have already observed that (a) is easy. For (b), notice that for any $x \in \mathcal{U}(\mathbb{Z}_p)$, the spanning elements $i_S(i,j)^* i_A(f) i_S(m,n)$ of J_{xH_p} go to $f(0,0) i_S(i,j)^* i_S(m,n)$ in the quotient $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{N}^2$, and hence $J_{xH_p} \subset L_{w,z}$ for all $(w,z) \in \mathbb{T}^2$.

For (c), we observe that $\bar{F} \cap (\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$ is the usual closure because $(x,y) \mapsto I_{x,y}$ is a homeomorphism of $\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)$ onto the open set $\operatorname{Prim} I_1$.

That the closure contains the other points follows from Lemma 4.4: $f \in \ker \pi_{x,y}$ implies $f \in \ker \rho_x$, so all the generators for $I_{x,y}$ described in Lemma 4.4 belong to J_{xH_p}, and $(x, y) \in F$ implies

$$J_{xH_p} \in \bar{F} = \Big\{P \in \operatorname{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{N}^2) \;\Big|\; \bigcap_{Q\in F} Q \subset P\Big\}.$$

To see that $J_{x_0H_p}$ does not belong to $\bar{F}$ when $x_0H_p \notin \pi_p(F)$, let F_1 be the union of the cosets in $\pi_p(F)$. Choose $g \in C(\mathcal{U}(\mathbb{Z}_p))$ such that $g(x_0) = 1$ and $g \equiv 0$ on F_1, extend g to a continuous function on $\mathbb{Z}_p$ by taking it to be zero outside $\mathcal{U}(\mathbb{Z}_p)$, and define $f(x, y) = g(x)$. Now we can see from Lemma 4.3 that f vanishes on the closure of $p^{\mathbb{N}}q^{\mathbb{N}}(x, y)$ for every $(x, y) \in F$, and hence $i_A(f)$ belongs to $\bigcap\{I_{x,y} \mid x, y \in F\}$ but not to $J_{x_0H_p}$. Thus $\bar{F} \cap \mathcal{U}(\mathbb{Z}_p)/H_p$ is precisely $\pi_p(F)$, and part (c) follows from (b). □

Remark 4.5. It is interesting to compare our description of $\operatorname{Prim} C^*(G_{p,q}) \rtimes \mathbb{N}^2$ with that obtained for the Bost-Connes algebra $C_{\mathbb{Q}}$ in [12]. In $\operatorname{Prim} C_{\mathbb{Q}}$, the finite sets coming from $\operatorname{Prim} I_2/I_1$ do not appear; loosely speaking, we believe this happens because $C_{\mathbb{Q}}$ contains all the primes, and some of these will act minimally on any given $\mathcal{U}(\mathbb{Z}_p)$ (see Remark 3.8). So the numbers $o_{p^\ell}(q)$ cannot be recovered from $\operatorname{Prim} C_{\mathbb{Q}}$. Of course this information is still buried somewhere in $C_{\mathbb{Q}}$: it follows from [14, Theorem 2.1] that the inclusion of $G_{p,q}$ in $\mathbb{Q}/\mathbb{Z}$ induces an isomorphism of $C^*(G_{p,q}) \rtimes \mathbb{N}^2$ into $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^* = C_{\mathbb{Q}}$.

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