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THE TWO WELL PROBLEM WITH SURFACE ENERGY

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Abstract. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, let H be a 2×2 diagonal matrix with $\det(H)=1$. Let $\epsilon>0$ and consider the functional

$$I_\epsilon(u) := \int_\Omega \mathrm{dist}\left(Du(z), SO(2)\cup SO(2)H\right) + \epsilon\left|D^2u(z)\right| dL^2 z$$

over $\mathcal{A}_F \cap W^{2,1}(\Omega)$ where $\mathcal{A}_F$ is the class of functions from Ω satisfying affine boundary condition F. It can be shown by convex integration that there exists $F\notin SO(2)\cup SO(2)H$ and $u\in\mathcal{A}_F$ with $I_0(u)=0$. Let $0<\zeta_1<1<\zeta_2<\infty$,

$$\mathcal{B}_F := \left\{u\in\mathcal{A}_F : u \text{ is } C^1,\ \text{bilipschitz with } \mathrm{Lip}(u)<\zeta_2,\ \mathrm{Lip}\left(u^{-1}\right)<\zeta_1^{-1}\right\}.$$

In this paper we begin the study of the asymptotics of $m_\epsilon := \inf_{\mathcal{B}_F\cap W^{2,1}} I_\epsilon$ for such F. This is one of the simplest minimisation problems involving surface energy for which we can hope to see the effects of convex integration solutions. The only known lower bounds are $\liminf_{\epsilon\to 0}\frac{m_\epsilon}{\epsilon}=\infty$.

We link the behavior of m_ϵ to the minimum of I_0 over a suitable class of piecewise affine functions. Let $\{\tau_i\}$ be a triangulation of Ω by triangles of diameter less than h and let A_F^h denote the class of continuous functions that are piecewise affine on a triangulation $\{\tau_i\}$. For function $u\in\mathcal{B}_F$ let $\tilde{u}\in A_F^h$ be the interpolant, i.e. the function we obtain by defining $\tilde{u}_{\lfloor\tau_i}$ to be the affine interpolation of u on the corners of τ_i. We show that if for some small $\omega>0$ there exists $u\in\mathcal{B}_F\cap W^{2,1}$ with

$$\frac{I_\epsilon(u)}{\epsilon}\leq\epsilon^{-\omega}$$

then for $h=\epsilon^{\frac{1+6399\omega}{3201}}$ the interpolant $\tilde{u}\in A_F^h$ satisfies $I_0(\tilde{u})\leq h^{1-c\omega}$.

Note that it is trivial that $\inf_{v\in A_F^h} I_0(v)\geq c_0 h$ so we reduce the problem of non-trivial (scaling) lower bounds on $\frac{m_\epsilon}{\epsilon}$ to the problem of non-trivial lower bounds on $\inf_{v\in A_F^h} I_0(v)$.

Contents

1. Introduction

In the 1980's from the work of Ball, James [1], [2] and Chipot, Kinderlehrer [4] a well known model for solid-solid phase transformations arose. In the model, microstructures observed in phase mixtures were explained in terms of energy minimisation of deformations of the material.

Let $u:\Omega\to\mathbb{R}^3$ be a deformation of the material which occupies a reference configuration Ω, the total free energy of this deformation is given by

$$I(u) = \int_\Omega \phi\left(Du(z),\theta\right) dL^3 z \tag{1}$$

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where $\phi(\cdot,\theta)$ is the free energy per unit volume in Ω at temperature θ. We fix θ and we normalize ϕ such that $\inf_F \phi(F,\theta)=0$.

Formation of microstructure was shown to be closely related to the behavior of minimising sequences of I. Many features of minimising sequences can be understood from the set $\{F:\phi(F)=0\}$. This set is known as the energy *wells* of the functional I.

Certain natural assumptions on the behavior of ϕ, in particular frame indifference, imply that K has to be of the form

$$K=\{SO(3)A_i : i=1,2,\dots m\} \tag{2}$$

where the A_i are symmetry related and depend on the action of the phase transition.

Given $F\in M^{n\times n}$ let $\mathcal{A}_F$ denote the set of functions $u:\Omega\to\mathbb{R}^n$ satisfying $u(z)=F(z)$ for all $z\in\partial\Omega$. The set of F for which $\inf_{u\in\mathcal{A}_F} I(u)=0$ turns out to agree with the quasiconvex hull K^{qc} (see [22] for the relevant notions). For any $F\in\text{int}(K^{qc})$ it is possible to lower the energy of functional I with a relatively simple function $u\in\mathcal{A}_F$ that is built up from a simple (finite) layering of regions on which Du is made to be affine, these functions are known as *laminates* .

Mathematically speaking, the first real surprise in this theory is the existence of exact minimisers of functional I for certain sets K of the form (2). Formally; given $F\in K^{qc}$ there exists a function $u\in\mathcal{A}_F$ such that

$$Du(z)\in K \text{ for } a.e.\ z\in\Omega. \tag{3}$$

Even though the functional I is not quasiconvex (by the very existence of such exact solutions) and therefore not lower semicontinuous with respect to weak convergence, absolute minimisers exist and can be constructed.

Following the work of Dacorogna and Marcellini [7], Müller and Šverák [20], [21], and later by Sychev [24] and Kirchheim [13] there now exist a wide variety of methods to prove the existence of such solutions. However all these methods start with a delicate construction of an approximating sequence of set $K_n\to K$. The methods of [20] and [24] are in some sense more constructive and related to the approach developed by Gromov [11], which is known as convex integration.

Exact minimisers of functional I are only possible due to the fact that I takes no account of the "cost" of oscillations. This is physically unrealistic. The oscillation term $\int_\Omega \left|D^2u(z)\right| dL^2z$ is known as the *surface energy* . The *bulk energy* is the $\int_\Omega \phi(Du(z))\,dL^2z$ part of the functional.

Functional I was designed to model situations for which the *surface energy* is small. From the mathematical perspective the most natural adaption of the functional that takes account of surface energy is:

$$I_\epsilon(u)=\int_\Omega \phi(Du(z))+\epsilon\left|D^2u(z)\right| dL^2z. \tag{4}$$

This functional is minimised over functions $u\in W^{2,1}(\Omega)\cap\mathcal{A}_F$.

1.1. **The question: How does I_ϵ scale ?** The question we are interested in is whether the existence of exact solutions to inclusion (3) having affine boundary condition has any effect on the scaling of $\inf_{W^{2,1}\cap\mathcal{A}_F} I_\epsilon$ as $\epsilon\to 0$. In some sense this could be expected, in words; as $\epsilon\to 0$ surface energy becomes arbitrarily cheap, we can concern ourselves less and less with oscillations and just concentrate on minimising the bulk part of the functional. It may there for be reasonable to expect that minimisers for sufficiently small ϵ are something like slightly smoothed out solutions of (3).

Let $K=SO(2)\cup SO(2)H$, $F\in\text{int}(K^{qc})$. The differential inclusion

$$Du\in K \quad a.e. \tag{5}$$

for function $u\in\mathcal{A}_F$ is the simplest convex integration result. And the minimisation problem

$$\inf_{u\in\mathcal{A}_F\cap W^{2,1}} I_\epsilon(u) \tag{6}$$

is the simplest "physical" situation where we could hope to see the effect of the existence of solutions to differential inclusion (3). The only known lower bounds on (6) are $\inf_{u\in\mathcal{A}_F\cap W^{2,1}} \frac{I_\epsilon(u)}{\epsilon}\to$

∞ which follows from the result of Dolzmann, Müller [8] (also see Kirchheim [12]) that if u satisfies (5) and Du is BV then u is a laminate. For the special case of a functional whose wells are given by two rank-1 connected matrices a complete understanding of the scaling has been achieved in [15], [6].

Our main tool for studying this question is a two well Liouville Theorem proved in [17] (see Theorem 1.1). In order to use it we will have to minimise over a subset of $\mathcal{A}_F$. Let $0 < \zeta_1 < 1 < \zeta_2 < \infty$ and let

$$\mathcal{B}_F := \left\{ u \in \mathcal{A}_F : u \text{ is } C^1, \text{ bilipschitz with } \mathrm{Lip}\,(u) < \zeta_2, \ \mathrm{Lip}\left(u^{-1}\right) < \zeta_1^{-1} \right\}. \tag{7}$$

From [21] it is clear we can find a sequence $u_k \in \mathcal{B}_F$ with $u_k \overset{W^{1,1}}{\longrightarrow} u$ where u solves (5). So it is valid to study the scaling of I_ϵ over this subset.

Let

$$m_\epsilon := \inf_{u \in \mathcal{B}_F \cap W^{2,1}} I_\epsilon (u).$$

As a consequence of Šverák's characterization of the wells K, [23] (namely that the quasiconvex hull is in the second laminate convex hull) it is not hard (see figure 1) to obtain the upper bound

$$\frac{m_\epsilon}{\epsilon} < c\epsilon^{-\frac{2}{3}}.$$

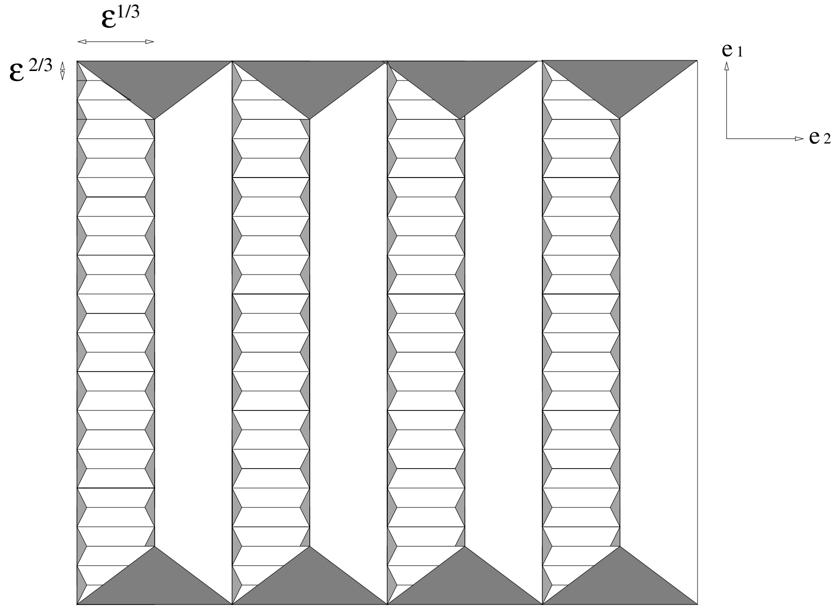

FIGURE 1

If something like exact solutions to differential inclusion (3) start having an effect on our functional for sufficiently small ϵ then we can expect to be able to "beat" the scaling $c\epsilon^{-\frac{2}{3}}$. Conversely if it could be shown that $\frac{m_\epsilon}{\epsilon} \geq c'\epsilon^{-\frac{2}{3}}$ this would say that these solutions do not affect functional I_ϵ. The ultimate goal of the research is to prove optimal (scaling) lower bounds on $\frac{m_\epsilon}{\epsilon}$. We conjecture these lower bounds are given by $c'\epsilon^{-\frac{2}{3}}$.

Now we state the theorem that will be our main tool for studying this question, [17].

Theorem 1.1. *Let $0 < \zeta_1 < 1 < \zeta_2 < \infty$. Let $K := SO(2) \cup SO(2)H$ where $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$. Let $u \in W^{2,1}(Q_1(0))$ be a C^1 bilipschitz function with $\mathrm{Lip}\,(u) < \zeta_1$, $\mathrm{Lip}\left(u^{-1}\right) < \zeta_2^{-1}$. There exists positive constants $c_1, c_3, c_4 < 1$ and $c_2, c_5 > 1$ depending only on σ, ζ_1, ζ_2 such that if*

$\kappa \in (0, c_1]$ and u satisfies the following inequalities

$$\int_{Q_1(0)} d\left(Du\left(z\right), K\right) dL^2 z \leq \kappa \tag{8}$$

$$\int_{Q_1(0)} \left|D^2 u\left(z\right)\right| dL^2 z \leq c_3, \tag{9}$$

then there exists $J \in \{Id, H\}$ and $R \in SO(2)$ such that

$$\int_{Q_{c_4}(0)} \left|Du\left(z\right) - RJ\right| dL^2 z \leq c_5 \kappa^{\frac{1}{800}}.$$

By applying Theorem 1.1 we reduce the problem of non-trivial (scaling) lower bounds on I_ϵ to the problem of non-trivial lower bounds on the finite element approximation to I_0. As we will explain, this is a genuine reduction, the later problem is a minimisation problem involving competition between surface and bulk energies *without* an ϵ weighting on the surface energy. The only parameter in the finite element approximation to I_0 is the grid size h. Before going into details, we need some preliminaries.

1.1.1. *Finite element approximations.* As is standard in finite element approximations, we will say a *triangulation* (denoted $\triangle_h$) of Ω of size h is a collection of pairwise disjoint triangles $\{\tau_i\}$ all of diameter h such that

$$\Omega \subset \bigcup_{\tau_i \in \triangle_h} \tau_i.$$

Given a function u, we can approximate u uniformly by a function $\tilde{u}$ that is piecewise affine on the triangles of $\triangle_h$ by letting $\tilde{u}_{\lfloor \tau_i}$ be the affine map we obtain from interpolating u on the corners of τ_i. We will call $\tilde{u}$ the *interpolant* of u. Given a minimisation problem for functional J over a function class with certain boundary data, if we replace the function class by functions that are piecewise affine on $\{\tau_i\}$ and have the same boundary data, this is known as the finite element approximation to J.

Finite element approximations of functionals such as I have received much interest, for example see [19],[5], [3]. As stated our interest in these approximations comes mainly from the fact that they provide a convenient intermediary step for the study of surface energy problems: Given a triangulation for which the edges of the triangles are not parallel to the rank-1 connections of the wells K, every time the interpolant of a function jumps from one well to another, there must be at least one triangle which is nowhere near the wells. In this way, F.E. approximations reflect a competition between "surface energy" as given by the error contributed from jumps in the derivative, and bulk energy.

F.E. approximations of a three well functional $\tilde{I}$ of the form I_0, over a function class having affine boundary condition in the second laminate convex hull of the wells have been studied by Chipot [3] and the author [16]. If $\triangle_h$ denotes a triangulation of size h and A_F^h denotes the set of functions that are piecewise affine on $\triangle_h$ satisfying the affine boundary condition F. It has been shown $\inf_{u \in A_F^h} \tilde{I}(u) \sim h^{\frac{1}{3}}$. From Šverák's characterization [23] we know the exact arrangement of rank-1 connections between the wells $SO(2) \cup SO(2)H$ and a matrix in the interior of the quasiconvex hull. The finite well functional studied in [16] precisely mimics these rank-1 connections. We conjecture.

Conjecture 1.2. *Let $K = SO(2) \cup SO(2)H$, $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$. Let $\triangle_h$ be a triangulation of Ω of grid size h with the directions of the edges of the triangles some uniform distance away from the set of rank-1 directions of K.*

Let A_F^h denote the set of functions with affine boundary condition $F \in \mathrm{int}(K^{qc})$ that are piecewise affine on the triangulation $\triangle_h$. Let $\mathcal{I}(u) := \int_\Omega d(Du(z), K) dL^2 z$. Then we have

$$\inf_{u \in A_F^h} \mathcal{I}(u) \geq ch^{\frac{1}{3}}.$$

It is relatively elementary to see [2] that there exists some small constant $c>0$ such that

$$\inf_{u\in A_F^h}\mathcal{I}(u)\geq ch.$$

The following theorem reduces the problem of non trivial (scaling) lower bounds on the scaling of $\frac{m_\epsilon}{\epsilon}$ to the problem of non trivial lower bounds on $\inf_{u\in A_F^h}\mathcal{I}(u)$.

Theorem 1.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^2$. Let $K=SO(2)\cup SO(2)H$ where $H=\begin{pmatrix}\sigma & 0\\ 0 & \sigma^{-1}\end{pmatrix}$. Let $F\in \mathrm{int}(K^{qc})$. Let $\mathcal{B}_F$ be defined by (7). Suppose $u\in W^{2,1}(\Omega)\cap\mathcal{B}_F$ satisfies*

$$\frac{I_\epsilon(u)}{\epsilon}\leq\epsilon^{-\omega}$$

for some small ω.

Let $h=\epsilon^{\frac{1+6399\omega}{3201}}$ and $\beta=\frac{3201\omega}{1+6399\omega}$. Given a triangulation $\{\tau_i\}$ of Ω with triangle size h we let $\tilde{u}$ be the interpolation of u on $\{\tau_i\}$, then we have

$$\mathcal{I}(\tilde{u})\leq ch^{1-2\beta}$$

where c depends only on σ, ζ_1, ζ_2.

So informally speaking, we replace the question of scaling with respect to parameter ϵ in the minimisation problem $\inf_{u\in\mathcal{B}_F\cap W^{1,2}(\Omega)}I_\epsilon(u)$ with parameter h in minimisation problem $\inf_{u\in A_F^h}\mathcal{I}(u)$. Note that in the first problem, ϵ is a factor *only* of the surface energy, so the surface energy becomes arbitrarily cheap for small ϵ. In the second problem, for very small h it does not become advantageous to concentrate on minimising of the bulk energy.

The reduction achieved by Theorem 1.3 is far from optimal, this is partly due to the suboptimality of Theorem 1.1. After this paper was submitted, an optimal version of Theorem 1.1 has been achieved by Conti and Schweizer [9], using this theorem a (scaling) optimal version of Theorem 1.3 has been proved, [18]. In addition [18] contains a version of the (optimised) Theorem 1.3 for functionals with L^q norm on the second derivative, which is established using an L^q version of Theorem 1.1.

2. Proof of Theorem 1.3

The proof can easily be seen to work for any Lipschitz domain Ω but to simplify technical details we let $\Omega=Q_1(0)$.

Suppose we have triangulation $T:=\left\{\tau_i: i=1,2,\dots\left[\frac{2}{h^2}\right]\right\}$ of $Q_1(0)$ with triangles of side length h.

Let $\kappa:=h^{3200-6400\beta}$, $\alpha:=3201-6400\beta$. Suppose we have inequalities

$$\int_{Q_1(0)}d(Du(z),K)\,dL^2z\leq h^{\alpha} \tag{10}$$

$$\int_{Q_1(0)}\left|D^2u(z)\right|dL^2z\leq h^{-\beta}. \tag{11}$$

Step 1. We will show that there exists a subcollection of triangles $G\subset T$ with the following properties. Let $\gamma:=\frac{2}{c_4}$.

- For each $\tau_i\in G$ if o_i denotes the center of the triangle, then $Q_{\gamma h}(o_i)\subset Q_1(0)$.
- $$\int_{Q_{\gamma h}(o_i)}d(Du(z),K)\,dL^2z\leq\kappa(\gamma h)^2. \tag{12}$$

[2]Given a triangulation $\{\tau_i\}$ of Ω, suppose we have a function $u\in A_F^h$ such that $\mathcal{I}(u)\leq ch$ for some small c. Then letting $B:=\left\{\tau_i:\int_{\tau_i}d(Du(z),K)\,dL^2z\geq\sqrt{c}h^2\right\}$ its immediate from the fact that $\mathcal{I}(u)\leq ch$ that there must exist a complete column of triangles $\{\tau_{k_1},\tau_{k_2},\dots\tau_{k_m}\}$ running through Ω such that $\{\tau_{k_1},\tau_{k_2},\dots\tau_{k_m}\}\subset\{\tau_i\}\backslash B$. Hence the derivative of the function u must remain close to either $SO(2)$ or $SO(2)H$ and thus by integration along the column we will have two points b_1,b_2 at the top and bottom of the column for which $u(b_1)-u(b_1)\approx RJ(b_1-b_2)$ with $R\in SO(2)$, $J\in\{H,Id\}$ and this is incompatible with the boundary conditions. Contradiction

- $$\int_{Q_{\gamma h}(o_i)} \left|D^2 u(z)\right| dL^2 z \leq c_3 \gamma h. \tag{13}$$

- $$\text{Card}(T \backslash G) \leq \kappa^{-1}\gamma^{-2}h^{\alpha-2} + \frac{h^{-1-\beta}}{c_3\gamma} + 8\gamma h^{-1}. \tag{14}$$

Proof of Step 1.

Let $G_1 := \{\tau_i \in T : Q_{\gamma h}(o_i) \subset Q_1(0)\}$. It is easy to see that

$$\text{Card}(T \backslash G_1) \leq 8\gamma h^{-1}. \tag{15}$$

Let

$$B_1 := \left\{\tau_i \in G_1 : \int_{Q_{h\gamma}(o_i)} d(Du(z), K)\, dL^2 z \geq \kappa (\gamma h)^2\right\}.$$

Let

$$B_2 := \left\{\tau_i \in G_1 : \int_{Q_{h\gamma}(o_i)} \left|D^2 u(z)\right| dL^2 z \geq c_3 \gamma h\right\}.$$

So $\text{Card}(B_1)\,\kappa(\gamma h)^2 \leq h^{\alpha}$ which implies

$$\text{Card}(B_1) \leq h^{\alpha-2}\gamma^{-2}\kappa^{-1}. \tag{16}$$

Similarly $\text{Card}(B_2)\, c_3\gamma h \leq h^{-\beta}$ which implies

$$\text{Card}(B_2) \leq \frac{h^{-1-\beta}}{c_3\gamma}. \tag{17}$$

Let $G := G_1 \backslash (B_1 \cup B_2)$. By (16),(17), (15) G satisfies (14) and by definition of B_1, B_2 any $\tau \in G$ satisfies (12), (13), this completes the proof of Step 1.

Step 2. We will show there exists a positive constant c_6 (depending on σ, ζ_1, ζ_2) such that for any $\tau_i \in G$ we have

$$d(D\tilde{u}(o_i), K) \leq 4c_6\gamma\kappa^{\frac{1}{3200}}. \tag{18}$$

Proof of Step 2. Let $v : Q_1(0) \to \mathbb{R}^2$ be defined by

$$v(z) := \frac{u(\gamma h z + o_i)}{\gamma h}. \tag{19}$$

By scaling of inequality (12)

$$\int_{Q_1(0)} d(Dv(z), K)\, dL^2 z \leq \kappa.$$

Similarly, by scaling of (13) we have

$$\int_{Q_1(0)} \left|D^2 v(z)\right| dL^2 z \leq c_3.$$

So by Theorem 1.1 there exists $R \in SO(2)$, $J \in \{Id, H\}$ such that $\int_{Q_{c_4}(0)} |Dv(z) - RJ|\, dL^2 z \leq c_5\kappa^{\frac{1}{800}}$. As v is ζ_2-Lipschitz we have

$$\begin{aligned}\int_{Q_{c_4}(0)} |Dv(z) - RJ|^4\, dL^2 z &\leq 8\zeta_2^3 \int_{Q_{c_4}(0)} |Dv(z) - RJ|\, dL^2 z \\ &\leq 8c_5\zeta_2^3\kappa^{\frac{1}{800}}.\end{aligned}$$

Let $\psi(z) := v(0) + RJz$. Using Morrey's inequality, (see Theorem 3, Section 4.5.3 [10]) we see there exists some constant c_6 (depending on σ, ζ_1, ζ_2) such that

$$\|v - \psi\|_{L^\infty(Q_{c_4}(0))} \leq c_6\kappa^{\frac{1}{3200}}. \tag{20}$$

So from (19) we have $u(z) = \gamma h v\left(\frac{z-o_i}{\gamma h}\right)$. Let $\tilde{\psi} : Q_{\gamma h}(o_i) \to \mathbb{R}^2$ be defined by $\tilde{\psi}(z) := \gamma h \psi\left(\frac{z-o_i}{\gamma h}\right)$. So $D\tilde{\psi}(o_i) = D\psi(0)$. Now

$$\begin{aligned}
\|u-\tilde{\psi}\|_{L^\infty\left(Q_{c_4\gamma h}(o_i)\right)} &= \sup\left\{\left|u(z)-\tilde{\psi}(z)\right| : z \in Q_{c_4\gamma h}(o_i)\right\} \\
&= \sup\left\{\gamma h\left|v\left(\frac{z-o_i}{\gamma h}\right)-\psi\left(\frac{z-o_i}{\gamma h}\right)\right| : z \in Q_{c_4\gamma h}(o_i)\right\} \\
&= \gamma h\|v-\psi\|_{L^\infty\left(Q_{c_4}(0)\right)} \\
&\overset{(20)}{\leq} c_6\gamma h\kappa^{\frac{1}{3200}}. \qquad (21)
\end{aligned}$$

Now note $\tau_i \subset Q_{c_4\gamma h}(o_i)$. Let t_1, t_2, t_3 denote the corners of the triangle τ_i with t_2 being the point at the right angle corner of the triangle. The function $\tilde{u}$ on τ_i is equal to the affine map given by the interpolation of $\{u(t_1), u(t_2), u(t_3)\}$ and so $D\tilde{u}$ on τ_i is the linear part of this affine map. By choice of triangulation, $e_1 = \pm\frac{t_1-t_2}{|t_1-t_2|}$ and $e_2 = \pm\frac{t_3-t_2}{|t_3-t_2|}$. Assume without loss of generality $e_1 = \frac{t_1-t_2}{|t_1-t_2|}$ and $e_2 = \frac{t_3-t_2}{|t_3-t_2|}$.

Now $u(t_1) - u(t_2) = |t_1-t_2|\, D\tilde{u}(o_i)\, e_1$ so from (21) we have

$$\begin{aligned}
h\left|\left(D\tilde{u}(o_i) - RJ\right)e_1\right| &= \left|\left|t_1-t_2\right| D\tilde{u}(o_i)\, e_1 - |t_1-t_2|\, RJe_1\right| \\
&= \left|\left(u(t_1)-u(t_2)\right) - \left(\tilde{\psi}(t_1)-\tilde{\psi}(t_2)\right)\right| \\
&\overset{(21)}{\leq} 2c_6\gamma h\kappa^{\frac{1}{3200}}.
\end{aligned}$$

Which implies

$$\left|D\tilde{u}(o_i)\, e_1 - RJe_1\right| \leq 2\gamma c_6\kappa^{\frac{1}{3200}}.$$

In the same way we can see

$$\left|D\tilde{u}(o_i)\, e_2 - RJe_2\right| \leq 2\gamma c_6\kappa^{\frac{1}{3200}}.$$

which implies (18). This completes the proof of Step 2.

Proof of Theorem 1.3 continued.

By Step 1 and Step 2 we know that

$$\begin{aligned}
\int_{Q_1(0)} d\left(D\tilde{u}(z), K\right) dL^2 z &\leq \sum_{\tau_i\in T} L^2(\tau_i)\, d\left(D\tilde{u}(o_i), K\right) + 8\zeta_2 h \\
&\leq \sum_{\tau_i\in G} L^2(\tau_i)\, d\left(D\tilde{u}(o_i), K\right) + \sum_{\tau_i\in T\setminus G} 2\zeta_2 L^2(\tau_i) + 8\zeta_2 h \\
&\overset{(14),(18)}{\leq} 4c_6\gamma\kappa^{\frac{1}{3200}} + 2\zeta_2 h^2\left(\kappa^{-1}\gamma^{-2}h^{\alpha-2} + \frac{h^{-1-\beta}}{c_3\gamma} + 12\gamma h^{-1}\right). \qquad (22)
\end{aligned}$$

Now recall $\kappa = h^{3200-6400\beta}$ and $\alpha = 3201 - 6400\beta$. So note $\kappa^{-1}h^{\alpha} = h$. Note also that $\kappa^{\frac{1}{3200}} = h^{1-2\beta}$. Thus from (22) we have

$$\int_{Q_1(0)} d\left(D\tilde{u}(z), K\right) dL^2 z \leq ch^{1-2\beta} \qquad (23)$$

where constant c depends only on σ, ζ_1, ζ_2.

Now we will rewrite initial conditions in form of (10), (11). Recall, our initial hypotheses on u were $\frac{I_\epsilon}{\epsilon}(u) \leq \epsilon^{-\omega}$ which implies

$$\int_{Q_1(0)} d\left(Du(z), K\right) dL^2 z \leq \epsilon^{1-\omega} \qquad (24)$$

and

$$\int_{Q_1(0)} \left|D^2 u(z)\right| dL^2 z \leq \epsilon^{-\omega}. \tag{25}$$

Now from the statement of Theorem 1.3 we know $\beta = \frac{3201\omega}{1+6399\omega}$ and $h = \epsilon^{\frac{1+6399\omega}{3201}}$. So $\omega = \frac{\beta}{3201-6399\beta}$. Now $\epsilon^{-\omega} = \left(h^{\frac{3201}{1+6399\omega}}\right)^{-\omega}$. So from (25) we have for this value of h we have (11). Now we will use (24) to show (10), note $\epsilon^{1-\omega} = \left(h^{\frac{3201}{1+6399\omega}}\right)^{1-\omega}$. And

$$\begin{aligned}
\frac{3201(1-\omega)}{1+6399\omega} &= \frac{3201\left(1-\frac{\beta}{3201-6399\beta}\right)}{1+\frac{6399\beta}{3201-6399\beta}} \\
&= \frac{3201\left(\frac{3201-6400\beta}{3201-6399\beta}\right)}{\frac{3201}{3201-6399\beta}} \\
&= 3201-6400\beta \\
&= \alpha.
\end{aligned}$$

So from (24) for this value of h we have (10). Recall, the interpolant of u on a triangulation T (whose triangles have side length h) is given by $\tilde{u}$, so we have that $\tilde{u}$ satisfies (22), hence $\widetilde{u}$ also satisfies (23) and this completes the proof. □

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