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THE U.S. PUBLIC DEBT VALUATION PUZZLE

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ABSTRACT

The government budget constraint ties the market value of government debt to the expected risk-adjusted present discounted value of fiscal surpluses. We find evidence that U.S. Treasury investors fail to impose this no-arbitrage restriction in the U.S. Both cyclical and long-run dynamics of tax revenues and government spending make the surplus claim risky. In a realistic asset pricing model, this risk in surpluses creates a large gap between the market value of debt and its fundamental value, the PDV of surpluses, suggesting that U.S. Treasurys may be mispriced.

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1 Introduction

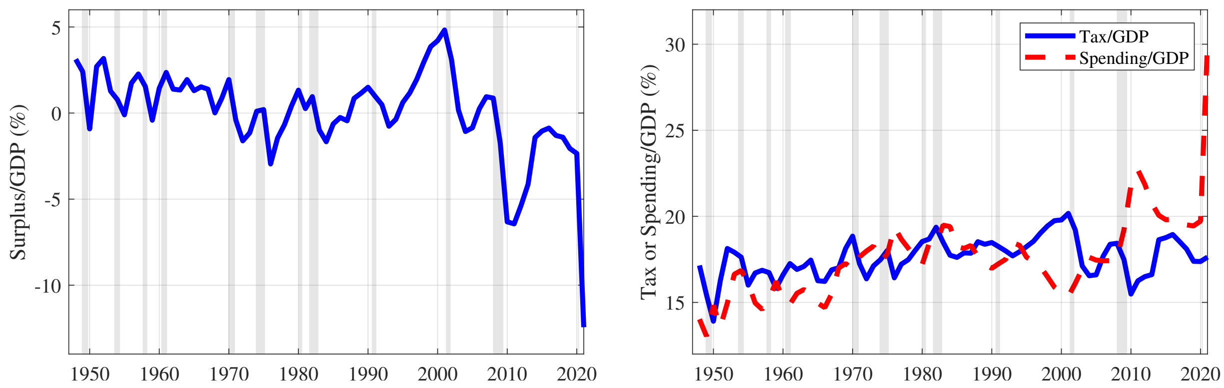
The U.S. Treasury is the largest borrower in the world. At the end of 2019, outstanding federal government debt held by the public was valued at nearly $17 trillion, or around 100% of GDP. It had doubled after the Great Financial Crisis to 78% of U.S. GDP. Before the GFC, there was widespread concern that the U.S. had embarked on an unsustainable fiscal path (see, e.g., Rubin, Orszag, and Sinai, 2004). Yet, recently, some economists have argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because interest rates are below GDP growth rates (Blanchard, 2019; Furman and Summers, 2020). As a case in point, bond investors took the massive spending increase in response to the COVID-19 pandemic, which generated a deficit of 15% of GDP in 2020 and increased the debt to 100% of GDP, in stride. How much fiscal capacity does the U.S. still have?

The central idea in this paper is to price the entire portfolio of outstanding Treasury debt, rather than individual bond securities. In the absence of bubbles, the market value of outstanding debt should equal the present discounted value of current and future primary surpluses. By the same logic, the expected return on the debt portfolio has to reflect the risk profile of primary surpluses. What makes this a valuation equation, not an accounting identity, is that we insist on pricing the surplus claim in a manner that is consistent with the risk pricing in stock and bond markets. We find evidence of mispricing. The value of the bond portfolio exceeds the value of the surplus claim, a gap we label the government debt valuation puzzle. Conversely, yields on the Treasury bond portfolio are lower than the relevant "interest rate" bond investors ought to be earning given the risk in the surplus claim, a gap we label the government debt risk premium puzzle.

To understand why, consider a stock-pricing analogy. The price of a stock is the expected present discount value (PDV) of future dividends. Risk-free interest rates are below expected dividend growth rates, yet the price of the stock is finite. Since the stock's dividend growth is pro-cyclical, its cash flows are low when the investors' marginal utility is high. The relevant "interest rate" for the dividend claim contains a risk premium because of the risk exposure of its cash flows: the equity risk premium. Analogously, a portfolio strategy that buys all new Treasury issues and receives all Treasury coupon and principal payments has as its cash flow the primary surplus of the federal government. Primary surpluses are strongly pro-cyclical just like stock dividends, as shown in Figure 1. Spending by the federal government increases in recessions, while the tax system produces pro-cyclical revenue.[1] In recessions, when marginal utility is high, surpluses are

[1] Non-discretionary spending, including Social Security, Medicare and Medicaid, food stamps, and unemployment benefits, accounts for at least two-thirds of government spending. Many of these transfer payments rise automatically in recessions. In addition, the government often temporarily increases transfer spending in recessions, e.g., the extension of unemployment benefits in 2009 and 2020. On the revenue side, the progressive nature of the tax code generates strongly pro-cyclical variation in tax revenue as a fraction of GDP.

Figure 1: U.S. Government Surplus



(a) Surplus/GDP (b) Tax and Spending/GDP

The figure plots the U.S. federal government primary surplus as a fraction of GDP. The primary surplus is defined as the current revenue minus current spending, excluding net interest spending. The data source is NIPA Table 3.2. The sample period is from 1947 to 2020.

negative and net government bond issuance is high. The Treasury portfolio cash flows have substantial business cycle risk. As explained below, and just like for stock dividends, tax revenue and spending also have substantial long-run risk due to cointegration with GDP. Taken together, the relevant "interest rate" for surpluses contains a substantial risk premium reflecting compensation for both short- and long-run risk exposures.

The expected risk-adjusted PDV of surpluses governs a country's fiscal capacity, i.e., how much debt it can issue. The expected risk-adjusted PDV of primary surpluses divided by output, is obtained as the difference between the expected risk-adjusted PDV of federal tax revenues, given by the tax/output ratio times the price/dividend ratio on a claim to future taxes, $\tau_t \exp(pd_t^\tau)$, and the expected risk-adjusted PDV of federal spending excluding debt service, given by the spending/output ratio times the price/dividend ratio on a claim on future spending, $g_t \exp(pd_t^g)$. In the long run, spending and taxes grow at the same rate as output because of co-integration. The pro-cyclicality of tax revenues makes the tax revenue claim risky and raises its discount rate; the average price/dividend ratio pd_0^τ is low. The counter-cyclicality of government spending makes the spending claim safer; pd_0^g is high. As a result, when the country is running zero average primary surpluses ($\tau_0 = g_0$), as the U.S. has done over the past nine decades, the country's steady-state fiscal capacity, $\tau_0 \exp(pd_0^\tau) - g_0 \exp(pd_0^g)$, cannot be positive, regardless of the difference between the risk-free rate and the growth rate of the economy. This result can only be overturned by rendering the tax claim safer than the spending claim, i.e. by shifting more aggregate risk onto taxpayers, who are short the tax claim, and onto transfer recipients, who are long the spending claim.

Our paper is the first to quantitatively evaluate the magnitude of the value of the surplus claim in a dynamic asset pricing model with priced aggregate risk. We develop a new, straightforward methodology. We first do so by deriving a model-free upper bound on the PDV of surpluses. We then deploy a fully specified dynamic asset pricing model that matches a rich set of asset pricing moments for stocks and bonds. Using either method, we find a large wedge between the model estimate of fiscal capacity and the actual debt/output ratio.

The surplus value can be decomposed as the present value of future surpluses, discounted using the risk-free term structure of interest rates, plus the covariance of future surpluses with the stochastic discount factor. Without aggregate risk, there is no covariance term. In this case, fiscal capacity is unbounded when the average risk-free rate is lower than the average expected growth rate of the economy. Much of the literature, including recent work, has ignored the covariance term. However, in the presence of priced aggregate risk, the covariance term will typically lower the government's fiscal capacity because surpluses move with the business cycle in the short run and are co-integrated with output in the long run. Our work is the first to estimate and quantify the covariance term in a realistic dynamic asset pricing model. When we insist that our model be consistent with key moments of asset prices, we find that fiscal capacity is much lower than conventionally thought, and lower than the market value of outstanding debt.

The above argument relies on a realistic model of quantities and prices of risk. When modeling the quantity of risk in fiscal cash flows, adequately capturing the dynamics of government spending and tax revenue is crucial. We model the growth rates of tax revenues-to-GDP and government spending-to-GDP in a VAR alongside macro-economic and financial variables. This structure allows us to capture the cyclical properties of fiscal cash-flows. A second important feature of fiscal cash flows is that tax revenues and spending are co-integrated with GDP, so that revenues, spending, and GDP adjust when revenue-to-GDP or spending-to-GDP move away from their long-run relationships. This error correction imposes a form of long-run automatic stabilization. With cointegration, GDP innovations permanently alter future surpluses. Both the spending and the revenue claims are exposed to the same long-run risk as GDP.

The only unknowns in our valuation exercise are the risk premia on claims to taxes and spending. We deal with this in two ways. First, to guard against model misspecification, we derive a model-free upper bound on the PDV of surpluses. By setting both the risk premium on the tax and on the spending claim equal to the risk premium on a GDP claim, we obtain an upper bound on the former, a lower bound on the latter, and hence an upper bound on their difference. In the steady-state, the upper bound is given by the product of the surplus and the valuation ratio of a claim to output $(\tau_0 - g_0)\exp(pd_0^y)$. If the output risk premium is 3%, an empirically plausible estimate of the unlevered equity risk premium, the valuation ratio $\exp(pd_0^y)$ is 57. The government's fiscal capacity, the upper bound on debt/GDP, increases by 57% per 1% of surplus/GDP.

The steady-state upper bound is the average surplus-to-GDP, which is 0.05% in post-war U.S. data, times 57 or 3.15% of GDP. This is far below the average debt-to-GDP ratio in U.S. post-war data. The dynamic upper bound additionally reflects time variation in the cash-flow and the discount-rate components of the output valuation ratio. For most of the U.S. sample, the dynamic upper bound is lower than the actual debt-to-GDP ratio. At the end of 2020, the gap between the observed debt-to-GDP ratio and the upper bound exceeds 100% of GDP. This is a lower bound on the wedge with the observed debt-to-GDP ratio. Using bootstrapped confidence intervals for the bound, we reject the null hypothesis that observed debt is below the upper bound after the GFC. The calculations incorporate additional conservatism by assuming that the spending-to-GDP ratio will converge back to its unconditional post-war mean, a strong assumption under current fiscal policies according to the Congressional Budget Office.

The U.S. Treasury earns a convenience yield on the debt it issues, making Treasury yields lower than the risk-free rate. Our paper is the first, to the best of our knowledge, to quantitatively explore the implications of convenience yields for the budget constraint. Convenience yields generate an additional source of revenue, $d_0\lambda_0$, which increase the surplus and steady-state fiscal capacity to $(\tau_0 + d_0\lambda_0)\exp(pd_0^{\tau}) - g_0\exp(pd_0^{g})$. Using the standard convenience yield estimates of Krishnamurthy and Vissing-Jorgensen (2012), the puzzle remains. We estimate seigniorage revenue of 0.20% of U.S. GDP, which raises the steady-state upper bound by only 11% (57 times 0.20%) of U.S. GDP. Even when we use higher convenience yield estimates, we cannot fully close the gap due to an offsetting discount rate effect. The higher the convenience yield, the higher the true risk-free rate for the same observed short-term Treasury yield. The valuation ratios of the spending and tax claims are correspondingly lower. Higher surpluses due to convenience are discounted at a higher rate.

In our second approach, we explicitly model and estimate risk premia. To do so, we posit a stochastic discount factor (SDF) model. Rather than committing to a specific utility function, we use a flexible SDF that accurately prices the nominal and real term structure of Treasury bond yields, matches stock prices, and generates an equity risk premium. The SDF model's rich implications for the term structure of risk allow it to adequately price short- and long-run risk to spending and tax revenue.

Combining features from both quantities and prices of risk, the long-run discount rates on claims to tax revenues, spending, and GDP must all be equal. A claim to GDP is akin to an unlevered equity claim. In any reasonable asset pricing model with a large permanent component in the SDF, the unlevered equity risk premium exceeds the yield on a long-term government bond (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). The discount rate for revenues and spending is high. Because of the dynamic government budget constraint, the relevant "interest rate" on the portfolio of government debt must also be high. In contrast, real-world Treasury investors seem willing to purchase government debt at low yields. The historical return on the U.S. government debt portfolio is only 1.16% in excess of the T-bill rate.

Quantitatively, we find an average surplus value equal to -128.87% of GDP, far below the average market value of outstanding government debt, 39.45% of GDP. The difference between these two, which is 168% on average over the last 75 years, quantifies the government debt valuation puzzle. The gap widens dramatically after the Great Financial Crisis (GFC) as the level of government debt/GDP rises while the valuation of the surplus claim falls. It peaks at 400% of GDP in 2020. The U.S. government has been issuing government debt while simultaneously decreasing the expected surpluses to back up the debt. The puzzle may deepen further as the government continues to incur large deficits in the wake of the COVID-19 pandemic and entitlement programs turn from surplus to deficit. The consideration of aggregate risk lowers fiscal capacity substantially.

The last part of the paper studies several potential resolutions of the government bond valuation and risk premium puzzles. First, the valuation gap can be interpreted as a violation of the transversality condition (TVC) in the Treasury market, due to a rational bubble. In the presence of substantial long-run output risk premia, i.e., in models that resolve the equity risk premium, the TVC is more likely to hold. In addition, rational bubbles in government debt would imply rational bubbles in other long-lived asset, like stocks, but our paper is specifically about the mispricing of U.S. government bonds *relative* to stocks.

Second, we explore the possibility of a future large fiscal correction that is absent from our sample, but present in the minds of investors who value the surplus claim. We back out from the market value of debt what annual probability investor need to assign to such an austerity event to justify the observed market value of debt. Specifically, we back out a 32% average annual probability of a 40% permanent spending cut. The high probability we infer belies the nature of a peso event, and is not consistent with rational expectations. Our results can be interpreted as indirect evidence that U.S. Treasury investors seem to make systematic errors when they forecast future surpluses.[2] We conclude that Treasurys may be mispriced relative to other asset classes.

Related Literature A large literature seeks to relate the riskiness of bonds to macro-economic risks (see Lettau and Wachter, 2007; Baele, Bekaert, and Inghelbrecht, 2010; Gabaix, 2012; David

[2]The future-austerity channel would be more powerful if governments could commit to impose austerity in high marginal-utility states. Our exercise rules this out by assuming no correlation between austerity and the SDF. The empirical evidence from the disaster literature seems consistent with this assumption. In disaster states, sovereign bond investors do not do particularly well. They realize low real returns, and even lower than stock returns in 25% of the disasters (Nakamura, Steinsson, Barro, and Ursúa, 2013).

and Veronesi, 2013; Campbell, Sunderam, and Viceira, 2017; Duffee, 2018; Campbell, Pflueger, and Viceira, 2020; Du, Pflueger, and Schreger, 2020). Our paper contributes to this literature by adding novel no-arbitrage restrictions on the aggregate Treasury portfolio, in addition to the no-arbitrage restrictions on individual bonds. Our contribution is to test a hitherto-untested implication of all no-arbitrage bond pricing models, namely that the value of the government bond portfolio equal the PDV of future surpluses.

Our paper contributes to the literature on the fiscal capacity of the government (see D'Erasmo, Mendoza, and Zhang, 2016, for a recent review). One strand derives time-series restrictions on the government revenue and spending processes that enforce the government's inter-temporal budget constraint, starting with the seminal work of Hansen, Roberds, and Sargent (1991). Recently, the question of U.S. fiscal capacity has received renewed attention (Bassetto and Cui, 2018; Blanchard, 2019; Furman and Summers, 2020; Mehrotra and Sergeyev, 2021; Mian, Straub, and Sufi, 2021; Brunnermeier, Merkel, and Sannikov, 2022; Reis, 2021). Our paper is the first to infer large risk premia on government surpluses when no-arbitrage restrictions on bond and stock markets are imposed, resulting in lower estimates of U.S. fiscal capacity.

Our paper is the first to study fiscal capacity in a world with priced aggregate risk and to estimate the covariance term between the intertemporal marginal rate of substitution and the surplus that first appear in Bohn (1995)'s seminal paper. Our main new qualitative insight is that the overall government bond portfolio is a risky asset since the government must issue debt in high marginal utility states of the world. In other words, the covariance term is negative, reducing fiscal capacity. Our main new quantitative result is that this covariance is large. Fiscal capacity is much smaller due to this covariance term. The presence of a large amount of permanent risk in output, and by virtue of cointegration, in tax revenues, spending, and debt, is crucial for the quantitative result.

There is a parallel literature in asset pricing which tests the present value equation for stocks and other long-lived assets, starting with the seminal work by Shiller (1981); LeRoy and Porter (1981); Campbell and Shiller (1988). That work starts from the definition of a stock return to derive a testable relationship between stock prices and expected discounted dividend growth rates. Similarly, we start from the definition of the government budget constraint and derive a testable relationship between the market value of the government debt portfolio and expected discounted future surpluses. What makes this a testable restriction rather than an accounting identity is that we insist that the discount rates for surpluses be consistent with those for other securities, notably stocks.

Our work connects to the large literature on the convenience yield of U.S. government bonds (Longstaff, 2004; Krishnamurthy and Vissing-Jorgensen, 2012; Nagel, 2016; Binsbergen, Diamond, and Grotteria, 2022). U.S. Treasurys are typically expensive relative to TIPS (Fleckenstein, Longstaff, and Lustig, 2014) , corporate bonds (Bai and Collin-Dufresne, 2019), foreign sovereign bonds (Du, Im, and Schreger, 2018; Jiang, Krishnamurthy, and Lustig, 2021a; Koijen and Yogo, 2019), and duration-matched stocks (Binsbergen, 2020). We find that a portfolio of all Treasuries is expensive relative to the underlying collateral, a claim to surpluses.

Bassetto and Cui (2018); Chien and Wen (2019); Angeletos, Collard, and Dellas (2020); Brunnermeier et al. (2022); Reis (2021) consider heterogeneous-agent models in which government bonds play a key role in allowing agents to smooth idiosyncratic risk, resulting in lower rates and larger convenience yields on Treasurys.

We contribute to a recent literature at the intersection of asset pricing and public finance. Chernov, Schmid, and Schneider (2020); Pallara and Renne (2019) argue that higher CDS premia for U.S. Treasuries since the financial crisis are related to the underlying fiscal fundamentals. Our puzzle holds in the presence of default: the value of defaultable sovereign debt is still backed by future surpluses. Liu, Schmid, and Yaron (2020) argue that increasing safe asset supply can be risky as more government debt increases corporate default risk premia despite providing more convenience. Croce, Nguyen, Raymond, and Schmid (2019) study cross-sectional differences in firms' exposure to government debt. Corhay, Kind, Kung, and Morales (2018) study how quantitative easing affects inflation by changing the maturity structure of government debt.

The rest of the paper is organized as follows. Section 2 describes the data. Section 3 presents theoretical results. Section 4 describes the VAR we use for forecasting spending and tax revenue. Section 5 derives a model-free upper bound on U.S. fiscal capacity. Section 6 estimates the risk premium on taxes and spending in a dynamic asset pricing model and quantifies the puzzle. It also discusses potential resolutions of the puzzle. Section 7 concludes and suggests avenues for further research. The appendix presents proofs of the propositions, and details of model derivation and estimation.

2 Data

We conduct our analysis at annual frequency, arguably a reasonable frequency to study the cash flow risk in fiscal revenues and outlays. We focus on the post-war period from 1947 until 2020. The primary surpluses are constructed using NIPA Table 3.2 from the Bureau of Economic Analysis (BEA). All variables are nominal and seasonally adjusted. Government spending is current expenditures (line 24) minus net interest payments (line 33 minus line 14). Tax revenue is the current government receipts (line 1). The primary surplus is defined as tax revenue minus government spending. Constant-maturity Treasury yields are from Fred.[3] Stock price and dividend data are from CRSP; we use the CRSP value-weighted market portfolio. Dividends are summed across the months in the year. We obtain the time series of GDP from NIPA Table 1.1.5. Inflation is the change in the GDP price index from NIPA Table 1.1.4. Real GDP growth x_t is nominal GDP growth minus inflation.

We construct the market value and the total returns of the marketable government bond portfolio using CUSIP-level data from the CRSP Treasuries Monthly Series. At the end of each period, we multiply the nominal price of each CUSIP by its total amount outstanding (normalized by the face value), and sum across all issuances (CUSIPs). We exclude non-marketable debt which is mostly held in intra-governmental accounts.[4] Marketable debt includes the Treasury holdings of the Federal Reserve Bank. Hence, we choose not to consolidate the Fed and the Treasury, which would add reserves and subtract the Fed's Treasury holdings on the left hand side of (1). Doing so would mainly change the duration of the bond portfolio.

Following Hall and Sargent (2011) and extending their sample to 2020, we construct zero-coupon bond (strip) positions from all coupon-bearing Treasury bonds issued in the past and outstanding in the current period. This is done separately for nominal and real bonds. Since zero-coupon bond prices are also observable, we can construct the left-hand side of eqn. (1) as the market value of outstanding marketable U.S. government debt. The average return in excess of the T-bill rate on the entire Treasury portfolio realized by an investor who buys all of the new issuances and collects all of the coupon and principal payments is 1.16% per year. The portfolio has an average duration of 3.69 years. Given the secular decline in interest rates over the past forty years, the observed average realized return on the bond portfolio is, if anything, provide an over-estimate of investors' expected return.

The government also owns some financial assets, such as outstanding student loans and other credit transaction and cash balances. Based on Congressional Budget Office data, the total value of these assets is 8.8% of the GDP as of 2018, not large enough to affect our conclusions. We ignore them in our analysis.[5]

[3]There are some periods where the 20-year bond was not issued and some periods where the 30 year bond was not issued.

[4]The largest holders of non-marketable debt are the Social Security Administration (SSA) and the federal government's defined benefit pension plan (GDBPP). The revenues and spending from the SSA and GDBPP plan are included in the federal government revenue and spending. As a result, we net out the SSA and GDBPP holdings of Treasuries, since they are an asset of one part of the consolidated government and a liability of the other part.

[5]The government also owns public land and other assets of strategic importance (e.g., defense, ports) but most of them are not for sale and do not generate cash flows. The omission of government assets is counter-balanced by the omission of large liabilities arising from future deficits in the major entitlement programs, which according to the CBO will lead to long-run structural deficits under current fiscal policy. On balance, our approach is conservative.

3 Theoretical Results

We derive two theoretical results which are general in that they rely only on the absence of arbitrage opportunities and two weak assumptions about government cash flows. The first assumption concerns the long-run: tax revenues and government spending are cointegrated with GDP; they share a stochastic trend. The second assumption concerns the short-run: government spending is counter-cyclical spending and tax revenues are pro-cyclical.

3.1 Valuation of Government Debt

Let G_t denote nominal government spending before interest expenses on the debt, T_t denote nominal government tax revenue, and $S_t = T_t - G_t$ denote the nominal primary surplus. Let $P_t^{\$}(h)$ denote the price at time t of a nominal zero-coupon bond that pays \$1 at time $t+h$, where h is the maturity. There exists a multi-period stochastic discount factor (SDF) $M_{t,t+h}^{\$} = \prod_{k=0}^{h} M_{t+k}^{\$}$, which is the product of the adjacent one-period SDFs $M_{t+k}^{\$}$. By no arbitrage, bond prices satisfy $P_t^{\$}(h) = \mathbb{E}_t\left[M_{t,t+h}^{\$}\right] = \mathbb{E}_t\left[M_{t+1}^{\$} P_{t+1}^{\$}(h-1)\right]$. By convention, $P_t^{\$}(0) = M_{t,t}^{\$} = M_t^{\$} = 1$ and $M_{t,t+1}^{\$} = M_{t+1}^{\$}$. The government bond portfolio is stripped into zero-coupon bond positions $Q_t^{\$}(h)$, where $Q_t^{\$}(h)$ denotes the outstanding face value at time t of the government bond payments due at time $t+h$. $Q_{t-1}^{\$}(1)$ is the total amount of debt payments that is due today. The outstanding debt reflects all past bond issuance decisions, i.e., all past primary deficits. Let D_t denote the nominal market value of the outstanding government debt portfolio.

Proposition 1 (**Value Equivalence**)**.** In the absence of arbitrage opportunities and subject to a transversality condition, the market value of the outstanding government debt portfolio equals the expected risk-adjusted present discounted value of current and future primary surpluses:

$$D_t \equiv \sum_{h=0}^{H} P_t^{\$}(h) Q_{t-1}^{\$}(h+1) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}(T_{t+j} - G_{t+j})\right] \equiv P_t^{\tau} - P_t^{g}, \tag{1}$$

where the cum-dividend value of the tax claim and value of the spending claim are defined as:

$$P_t^{\tau} = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} T_{t+j}\right], \quad P_t^{g} = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$} G_{t+j}\right].$$

The proof is given in Appendix A. The proof relies only on the existence of a SDF, i.e., the absence of arbitrage opportunities, not on the uniqueness of the SDF, i.e., complete markets. It imposes a transversality condition (TVC) that rules out a government debt bubble: $\mathbb{E}_t\left[M_{t,t+T}^{\$} D_{t+T}\right] \to 0$ as $T \to \infty$. Imposing the TVC rules out rational bubbles. We return to possible violations of the TVC in Section 6.5.

This valuation equation is not an accounting identity. The bond portfolio can be mispriced, just like a stock can be over- or under-valued. Eqn. (1) requires that the same SDF which prices individual government bonds and stocks also prices a claim to surpluses, i.e., the entire bond portfolio. Even when the SDF correctly prices individual bonds and stocks, this entire bond portfolio could be mis-priced, for example, because agents have misspecified beliefs about future surpluses. This equation is an accounting identity only when we do not impose any restrictions on discount rates.

When the government runs a deficit in a future date and state, it will need to issue new bonds to the investing public. If those dates and states are associated with a high value of the SDF for the representative bond investor, that debt issuance occurs at the "wrong" time. The representative investor who buys all debt issues and participates in all redemptions needs to be induced by low prices (high yields) to absorb that new debt. To see this, we can rewrite the intertemporal budget constraint, with finite horizon T, as:

$$D_t = \sum_{j=0}^{T} P_t^{\$}(j)\mathbb{E}_t\left[S_{t+j}\right] + \sum_{j=0}^{T}\text{Cov}_t\left(M_{t,t+j}^{\$}, T_{t+j}\right) - \sum_{j=0}^{T}\text{Cov}_t\left(M_{t,t+j}^{\$}, G_{t+j}\right) + \mathbb{E}_t\left[M_{t,t+T}D_{t+T}\right] \quad (2)$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the expected PDV for a risk-neutral investor. If the SDF is deterministic, this and the last term are the only terms on the right-hand side. Fiscal capacity is then constrained by the government's ability to generate current and future surpluses. Lower interest rates increase fiscal capacity. The second and third terms encode the riskiness of the government debt portfolio, and arise in the presence of stochastic discount rates. If tax revenues tend to be high when times are good ($M_{t,t+j}$ is low), then the second term is negative. If government spending tends to be high when times are bad ($M_{t,t+j}$ is high), then the third term is positive. If both are true, then the covariance terms lower the government's fiscal capacity. Put differently, the risk-neutral present-value of future surpluses will need to be higher by an amount equal to the absolute value of the covariance terms to support a given, positive amount of government debt D_t. Our paper is the first to quantify these covariance terms, first derived by Bohn (1995) in a simple consumption-CAPM, in a realistic model of risk and return that is not subject to the equity risk premium puzzle. The covariance terms not only have the hypothesized sign, but they are also quantitatively important.

Discounting future surpluses using the term structure of risk-free interest rates, as typically done in the literature, is inappropriate. In fact, as $T \to \infty$, the first term will diverge if the average risk-free rate is lower than the average expected growth rate of the economy. Even when the debt is risk-free, the last term will not converge to zero if we discount at the risk-free rate.

The valuation eqn. (1) holds ex-ante both in nominal and in real terms. Ex-post, the government can erode the real value of outstanding debt by creating surprise inflation. The same valuation equation holds allowing for sovereign default: the valuation of government debt is still backed by the value of future surpluses. Bond prices adjust to reflect the possibility of sovereign default. The proof is given in Appendix A.

3.2 Discount Rates

As tax revenue and government spending have different cyclical properties, their discount rates may differ. We define the holding period returns on the bond portfolio, the tax claim, and the spending claim as:

$$R^d_{t+1} = \frac{\sum_{h=1}^{\infty} P^{\$}_{t+1}(h-1)Q^{\$}_t(h)}{\sum_{h=1}^{\infty} P^{\$}_t(h)Q^{\$}_t(h)}, \quad R^{\tau}_{t+1} = \frac{P^{\tau}_{t+1}}{P^{\tau}_t - T_t}, \quad R^{g}_{t+1} = \frac{P^{g}_{t+1}}{P^{g}_t - G_t}.$$

The expected returns on these three assets are connected as follows:

Proposition 2 (Risk Premium Equivalence). Under the same assumptions of Proposition 1, we have:

$$\mathbb{E}_t \left[R^d_{t+1} \right] = \frac{P^{\tau}_t - T_t}{D_t - S_t} \mathbb{E}_t \left[R^{\tau}_{t+1} \right] - \frac{P^{g}_t - G_t}{D_t - S_t} \mathbb{E}_t \left[R^{g}_{t+1} \right]. \tag{3}$$

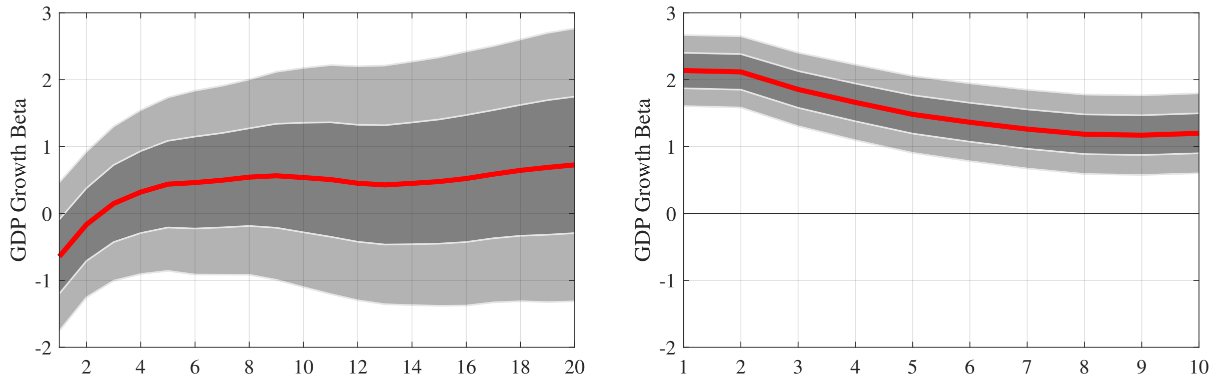
where $D_t - S_t = (P^{\tau}_t - T_t) - (P^{g}_t - G_t)$.

The proof is given in Appendix A. The average discount rate on government debt is equal to the average discount rate on government assets, a claim to primary surpluses. Since the primary surpluses are tax revenues minus government spending, the discount rate on government debt equals the difference between the discount rates of tax revenues and government spending, appropriately weighted. By subtracting the risk-free rate on both sides, we can express the relationship in terms of expected excess returns, or risk premia. If the tax revenue claim is riskier than the spending claim and earns a higher risk premium, then the risk premium on government debt exceeds that on the revenue and the spending claims:

$$\mathbb{E}_t \left[R^d_{t+1} - R^f_t \right] > \mathbb{E}_t \left[R^{\tau}_{t+1} - R^f_t \right] > \mathbb{E}_t \left[R^{g}_{t+1} - R^f_t \right]. \tag{4}$$

We show below that the revenue claim is indeed riskier than the spending claim. The risk premium equivalence then implies that the portfolio of government debt ought to carry a positive risk premium. The right discount rate for government debt, given by (3), cannot be the risk-free rate.

Figure 2: Cash Flow Betas



(a) Spending Beta (b) Tax Beta

The figure plots the GDP growth betas of log government spending and log tax revenue against the horizon (in years), computed by regressing cumulative log spending and tax revenue growth on cumulative log GDP growth. The sample period is from 1947 to 2020. Plotted with 1- and 2-standard error bands. Standard errors generated by bootstrapping 10,000 times from time-series model with cointegration for taxes (spending) and output, as well debt and output. The log of spending/output, the log of taxes/output, and the log of debt/output are AR-processes. Spending growth and tax revenue growth generated by bootstrapping with replacement from joint residuals.

To understand the riskiness of the debt claim, we study the short-run and long-run risk properties of the T- and G-claim. To do so, we study spending and revenue strips. A spending strip is a claim that pays off G_{t+j} at time $t+j$ and nothing at other times. A revenue strip similarly pays off T_{t+j}. Let $R^{g,j}_{t,t+j}$ and $R^{\tau,j}_{t,t+j}$ be the holding-period returns on these strips. At the short end of the maturity spectrum (business cycle frequencies j of 1–3 years), the risk premium on the revenue strip exceeds that on the corresponding-maturity spending strip:

$$\mathbb{E}_t\left[R^{\tau,j}_{t,t+j} - R^f_t\right] > \mathbb{E}_t\left[R^{g,j}_{t,t+j} - R^f_t\right]. \tag{5}$$

The reason is that tax revenue is highly pro-cyclical while government spending is counter-cyclical.

Figure 2 plots the cash-flow betas of U.S. government spending and taxes, defined in section 2, with respect to GDP growth over different horizons (in years). These betas are estimated from regressions of log cumulative spending or tax revenue growth on log cumulative GDP growth. At the 1-year horizon, the tax beta exceeds 2 while the spending beta is negative. At horizons of less than 10 years, the tax revenue betas exceed the spending betas. Since government debt investors have a long position in a riskier claim and a short position in a safer claim, the short end contributes to a positive risk premium on the government debt portfolio.

At the long end of the strip curve, we study the limit of the strip returns as $j \to \infty$. We denote log returns by lowercase letters.

Proposition 3. If the log of government spending/output ratio G/GDP (revenue/output ratio T/GDP) is stationary in levels, then the long-run expected log excess return on long-dated spending (revenue) strips equals that on GDP strips:

$$\lim_{j\to\infty} \mathbb{E}_t \left[r^{g,j}_{t,t+j}\right] = \lim_{j\to\infty} \mathbb{E}_t \left[r^{\tau,j}_{t,t+j}\right] = \lim_{j\to\infty} \mathbb{E}_t \left[r^{Y,j}_{t,t+j}\right] \gg y^{\$}_t(\infty). \tag{6}$$

Government spending and tax revenue have to be cointegrated with GDP; their ratio is stationary in levels. As a result, the cash-flow betas converge to one at long horizons, consistent with the empirical evidence in Figure 2. Under this realistic assumption on cash flows, expected returns on long-dated spending and tax revenue strips tend to the expected return on a long-dated GDP strip. A claim to GDP can be thought of as an unlevered equity claim. In the presence of permanent shocks to marginal utility, the long-run discount rate on GDP (unlevered equity) is much higher than the yield on long-term risk-free bonds (Alvarez and Jermann, 2005). This proposition implies that investors in the government bond portfolio have a net long position in a claim with the same long-run risk as the GDP claim. It follows immediately from this discount rate argument that the value of the long-run spending minus revenue strips will be smaller than what would be obtained when discounting with long-term bond yields.

Combining the properties of short-run and long-run discount rates, the theory predicts that:

$$\mathbb{E}_t \left[R^d_{t+1} - R^f_t\right] > \mathbb{E}_t \left[R^\tau_{t+1} - R^f_t\right] > \mathbb{E}_t \left[R^g_{t+1} - R^f_t\right]. \tag{7}$$

To summarize, any model of government debt that is consistent with asset prices will have to confront two forces that push up the equilibrium return on government debt. First, there is short-run cash-flow risk that pushes the expected return on the revenue claim above the expected return on the spending claim. Second, the long-run discount rates are higher than the yield on a long-maturity bond, because of the long-run cash flow risk in the spending and revenue claims equals that of long-run GDP risk. Government debt investors have a net long position in a claim that is exposed to the same long-run cash flow risk as GDP. The low observed interest rate, or equivalently the high observed value of the government debt portfolio represents a challenge to standard dynamic asset pricing models in light of the fundamental risk of the cash flows backing that debt. Our paper is the first to highlight this tension.

3.3 Convenience Yields

Convenience yields may provide a resolution to this tension. U.S. government bonds occupy a privileged place in the world's financial system. They carry a convenience yield which makes Treasury yields lower than the safe interest rate. The convenience yield produces an additional source of revenue, because the U.S. Treasury can sell its bonds for more than their fundamental value. The question is how far this explanations can go towards accounting for the bond valuation puzzle. We enrich the baseline model to account for convenience.

The convenience yield $\lambda_t(h)$ is the expected returns on government bonds of maturity h that investors are willing to forgo under the risk-neutral measure. Assuming a uniform convenience yield across the maturity spectrum, the Euler equation for a Treasury bond with maturity $h+1$ is:

$$e^{-\lambda_t(h)} \quad = \quad \mathbb{E}_t \left[M^{\$}_{t+1} \frac{P^{\$}_{t+1}(h)}{P^{\$}_t(h+1)} \right].$$

Proposition 4. If the TVC holds, the value of the government debt portfolio equals the value of future surpluses plus the value of future seigniorage revenue:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} \left(T_{t+j} - G_{t+j} \right) \right] + \mathbb{E}_t \left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j} \sum_{h=1}^{H} Q^{\$}_{t+j}(h) P^{\$}_{t+j}(h) (1 - e^{-\lambda_{t+j}(h)}) \right] \quad (8)$$

where $\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h)$ on the left-hand side denotes the value of the government's debt portfolio at the start of period t, and $\sum_{h=1}^{H} Q^{\$}_{t+j}(h)P^{\$}_{t+j}(h)$ on the right-hand side denotes the value of the government's debt portfolio at the end of period $t+j$.

When there is no convenience yield (i.e., $\lambda_t(h) = 0$), we end up back in the case of Proposition 1. If the quantity of current and future outstanding government debt is positive, then a positive convenience yield increases government revenue. This additional income is akin to seigniorage revenue and could potentially turn fiscal deficits into (broadly-defined) surpluses. At the same time, a higher convenience yield λ_t results in a lower discount factor $\mathbb{E}_t[M^{\$}_{t,t+j}]$ for a given observed bond price $P^{\$}_t(h+1) = e^{\lambda_t}\mathbb{E}_t[M^{\$}_{t,t+j}]$. This discount rate effect lowers the value of a given (broadly-defined) surplus stream. Hence, the introduction of convenience yields generates offsetting cash flow and discount rate effects on the valuation of the government bond portfolio. We investigate below which effect dominates.

4 Forecasting Tax Revenue and Government Spending

In order to quantify the value of the claims to tax revenue and government spending in eqn. (1), we need to take a stance on (i) the time-series properties of revenue and spending, and (ii) a stochastic discount factor $M_{t,t+j}$ to discount these cash flows. In this section, we first describe how we model the tax and spending processes.

4.1 Cash Flow Dynamics

State Variables We assume that the $N \times 1$ vector of state variables $\boldsymbol{z}$ follows a Gaussian first-order VAR:

$$\boldsymbol{z}_t = \boldsymbol{\Psi} \boldsymbol{z}_{t-1} + \boldsymbol{u}_t = \Psi \boldsymbol{z}_{t-1} + \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\varepsilon}_t, \tag{9}$$

with $N \times N$ companion matrix Ψ and homoscedastic innovations $\boldsymbol{u}_t \sim i.i.d. \mathcal{N}(0, \Sigma)$. The Cholesky decomposition of the covariance matrix, $\boldsymbol{\Sigma} = \Sigma^{\frac{1}{2}} \left(\Sigma^{\frac{1}{2}}\right)'$, has non-zero elements on and below the diagonal. In this way, shocks to each state variable u_t are linear combinations of its own structural shock $\boldsymbol{\varepsilon}_t$, and the structural shocks to the state variables that precede it in the VAR, with $\boldsymbol{\varepsilon}_t \sim i.i.d. \mathcal{N}(0, I)$. These state variables are defined in Table 1, in order of appearance of the VAR. The vector $\boldsymbol{z}$ contains the state variables demeaned by their respective sample averages.

Our approach takes spending and tax policies as given. By including spending and taxes in the state vector, we assume that the government commits a tax and spending policy that is affine in the state vector. Both policies are allowed to depend on a rich set of state variables with dependencies that are estimated from 74 years of data. The VAR includes $\Delta \log \tau_t$ and $\Delta \log g_t$, the log change in tax revenue-to-GDP ratio and the log change in government spending-to-GDP ratio in its eight and tenth rows. It also includes the log level of revenue/GDP, τ_t, and spending/GDP, g_t, in its ninth and eleventh rows. This fiscal cash flow structure has three important features.

First, our approach allows spending and revenue growth rates to depend not only on their own lags, but also on a rich set of macroeconomic and financial variables. Lagged inflation, GDP growth, interest rates, the slope of the term structure, the stock price-dividend ratio, and dividend growth all predict future revenue and spending growth. In addition, we allow innovations to the fiscal variables to be correlated with contemporaneous innovations in these macro-finance variables.

Second, we include the level variables τ_t and g_t. When there is a positive shock to spending,

Table 1: State Variables

Position	Variable	Mean	Description	Sample Mean
1	π_t	π_0	Log Inflation	3.16%
2	$y_t^{\$}(1)$	$y_0^{\$}(1)$	Log 1-Year Nominal Yield	4.26%
3	$yspr_t^{\$}$	$yspr_0^{\$}$	Log 5-Year Minus Log 1-Year Nominal Yield Spread	0.58%
4	x_t	x_0	Log Real GDP Growth	2.95%
5	Δd_t	μ_d	Log Stock Dividend-to-GDP Growth	−0.18%
6	d_t	$\log d_0$	Log Stock Dividend-to-GDP Level	−1.27
7	pd_t	$\overline{pd}$	Log Stock Price-to-Dividend Ratio	3.54
8	$\Delta \log \tau_t$	μ_τ	Log Tax Revenue-to-GDP Growth	0.02%
9	$\log \tau_t$	$\log \tau_0$	Log Tax Revenue-to-GDP Level	−1.74
10	$\Delta \log g_t$	μ_g	Log Spending-to-GDP Growth	0.65%
11	$\log g_t$	$\log g_0$	Log Spending-to-GDP Level	−1.75

spending tends to revert back to its long-run trend with GDP. Similarly, after a negative shock to tax revenue, future revenues tend to increase back to their long-run level relative to GDP. This mean reversion captures the presence of automatic stabilizers and of corrective fiscal action, as pointed out by Bohn (1998). By having spending/GDP growth $\Delta \log g_t$ (revenue/GDP growth $\Delta \log \tau_t$) depend on the lagged spending/GDP level g_t (lagged revenue/GDP τ_t) with a negative coefficient, the VAR captures this mean reversion. Mean reversion is amplified when spending-to-GDP growth $\Delta \log g_t$ ($\Delta \log \tau_t$) depends on lagged revenue-to-GDP τ_t (g_t) with a positive sign.

Formally, the inclusion of the levels of spending and tax revenue relative to GDP in the VAR is motivated by a cointegration analysis; the system becomes a vector error correction model. We take an a-priori stance that the tax/GDP ratio $\log \tau$ and the spending/GDP ratio $\log g$ are stationary. That is, we assume a cointegration coefficient vector of $(1, -1)$ for both relationships.[6]

In the absence of cointegration, all shocks to spending and tax revenues would be permanent rather than mean-reverting. Importantly, by imposing cointegration, we are being conservative about future fiscal rectitude. Large deficits relative-to-GDP, like the one that occurred in 2020, are assumed to auto-correct in the future. Thus, imposing cointegration raises the expected PDV of future surpluses.

Third, we include both the change and the level of the dividend/GDP ratio d_t. The growth rate loads on the lagged level with a negative coefficient. This specification imposes cointegration of dividends and GDP. As a result, in the long run, claims to taxes, spending, GDP, and aggregate dividends all earn the same risk premium because they are exposed to the same long-run risk.

In-sample Trend in Spending/GDP One empirical issue requires further discussion. The spending-to-GDP ratio trends up in the U.S. data. As shown in Table 1, the sample average of $\Delta \log g_t$ is $\widehat{\mu}^g = 0.65\%$. Tax revenue-to-GDP is approximately stationary: the sample average of $\Delta \log \tau_t$ is $\widehat{\mu}^\tau = 0.02\%$. Because it is theoretically desirable to impose cointegration on the log tax/GDP and the log spending/GDP ratios, the true unconditional growth rates of the tax/GDP and the spending/GDP ratios have to be zero ($\mu_0^\tau = \mu_0^g = 0$).

To avoid biased estimates of the VAR coefficients, we cannot include trending variables in the VAR. Hence, when we estimate the dynamics of the state variables, and only then, we remove the sample averages from the growth rates. We reconstruct the log tax/GDP and log spending/GDP level variables that enter in the VAR as follows:

$$\log \tau_t = \log \tau_1 + \sum_{k=1}^{t} (\Delta \log \tau_k - \widehat{\mu}^\tau), \quad \log g_t = \log g_1 + \sum_{k=1}^{t} (\Delta \log g_k - \widehat{\mu}^g).$$

[6] Appendix B.1 performs Johansen and Phillips-Ouliaris cointegration tests. The results support the existence of cointegration relationships, though the coefficients estimates of the cointegration relationships tend to vary across sample periods.

The initial level $\log g_1$ ($\log \tau_1$) for the log spending/GDP (revenue/GDP) ratios do not affect the results. We set them to their 1947 values.

Importantly, when we price assets and value claims to spending and tax revenues, we always evaluate the state vector at the actual values of τ and g, not the de-trended ones. This approach is conservative starting in 1980, because the actual tax/GDP ratio (spending/GDP) is well below (slightly above) the detrended one. Hence, the model's cash flow forecasts imply larger tax revenue increases (spending declines) in the future than we would obtain if we had used the detrended variables instead.

Estimation We estimate the VAR system in eqn. (9) using OLS. The point estimates of $\mathbf{\Psi}$ are reported in Panel A of Table 2. Lagged macro-finance variables affect fiscal variables, and vice versa. Consistent with the error correction dynamics imposed by cointegration, the response of the dividend/GDP growth rate to the lagged dividend/GDP ratio ($\Psi_{[5,6]}$), the response of the tax/GDP

Table 2: VAR Estimates

		1	2	3	4	5	6	7	8	9	10	11
						Panel (A) $\mathbf{\Psi}$						
		π_{t-1}	$y^{\$}_{t-1}(1)$	$yspr^{\$}_{t-1}$	x_{t-1}	Δd_{t-1}	d_t	pd_{t-1}	$\Delta \log \tau_{t-1}$	$\log \tau_{t-1}$	$\Delta \log g_{t-1}$	$\log g_{t-1}$
1	π_t	**0.48**	**0.20**	*-0.52*	-0.03	0.02	-0.01	0.00	**0.11**	**-0.07**	-0.01	*0.05*
2	$y^{\$}_t(1)$	0.04	**0.86**	-0.06	0.16	*0.07*	-0.01	0.01	-0.03	-0.01	0.01	**0.07**
3	$yspr^{\$}_t$	-0.06	-0.04	**0.40**	**-0.12**	**-0.03**	-0.01	**-0.01**	0.02	0.02	0.00	-0.02
4	x_t	-0.19	**0.38**	*0.97*	0.21	0.09	0.03	*0.02*	-0.07	-0.09	-0.02	0.07
5	Δd_t	-0.13	**-0.80**	*-2.35*	0.35	**0.28**	**-0.12**	-0.00	-0.31	-0.21	-0.18	0.19
6	d_t	-0.13	**-0.80**	*-2.35*	0.35	**0.28**	**0.88**	-0.00	-0.31	-0.21	-0.18	0.19
7	pd_t	*-2.66*	-0.43	-0.31	-1.35	-0.35	-0.11	**0.68**	0.06	0.40	0.34	-0.54
8	$\Delta \log \tau_t$	**-0.71**	**0.76**	-0.97	0.02	0.12	-0.03	*0.04*	**0.35**	**-0.61**	0.08	0.11
9	$\log \tau_t$	**-0.71**	**0.76**	-0.97	0.02	0.12	-0.03	*0.04*	**0.35**	**0.39**	0.08	0.11
10	$\Delta \log g_t$	**1.05**	-0.25	0.15	-0.18	**-0.31**	0.05	-0.04	*0.36*	-0.19	**0.38**	**-0.60**
11	$\log g_t$	**1.05**	-0.25	0.15	-0.18	**-0.31**	0.05	-0.04	*0.36*	-0.19	**0.38**	**0.40**
						Panel (B) $100 \times \mathbf{\Sigma}^{\frac{1}{2}}$						
		ε^{π}_t	$\varepsilon^{y}_t(1)$	ε^{yspr}_t	ε^{x}_t	$\varepsilon^{\Delta d}_t$		ε^{pd}_t	$\varepsilon^{\Delta \log \tau}_t$		$\varepsilon^{\Delta \log g_t}$	
1	π_t	1.06	0	0	0	0	0	0	0	0	0	0
2	$y^{\$}_t(1)$	0.34	1.21	0	0	0	0	0	0	0	0	0
3	$yspr^{\$}_t$	-0.06	-0.32	0.41	0	0	0	0	0	0	0	0
4	x_t	0.18	0.80	-0.14	1.83	0	0	0	0	0	0	0
5	Δd_t	-1.54	0.36	-0.74	-0.65	4.54	0	0	0	0	0	0
6	d_t	-1.54	0.36	-0.74	-0.65	4.54	0	0	0	0	0	0
7	pd_t	-2.49	0.29	0.23	-2.70	-3.94	0	14.14	0	0	0	0
8	$\Delta \log \tau_t$	0.34	0.77	-0.19	1.55	0.65	0	0.35	2.92	0	0	0
9	$\log \tau_t$	0.34	0.77	-0.19	1.55	0.65	0	0.35	2.92	0	0	0
10	$\Delta \log g_t$	0.30	-1.35	0.17	-2.95	-1.22	0	0.18	0.56	0	4.13	0
11	$\log g_t$	0.30	-1.35	0.17	-2.95	-1.22	0	0.18	0.56	0	4.13	0

Panel (a) reports our estimate of the VAR transition matrix $\mathbf{\Psi}$. Numbers in bold have t-statistics in excess of 1.96 in absolute value. Numbers in italics have t-statistics in excess of 1.645 but below 1.96. Panel (b) reports our estimate of the VAR innovation matrix $\Sigma^{\frac{1}{2}}$, multiplied by 100 for readability.

growth rate to the lagged tax/GDP level ($\Psi_{[8,9]}$), and the response of the spending/GDP growth rate to the lagged spending/GDP level ($\Psi_{[10,11]}$) are all negative and statistically significantly different from zero.

The dynamics of $\log d_t$, $\log \tau_t$, and $\log g_t$ in rows 6, 9, and 11 of the VAR are implied by the corresponding dynamics of their first differences Δd_t, $\Delta \log \tau_t$, and $\Delta \log g_t$ in rows 5, 8, and 10, respectively, with the exception of the autoregressive coefficient which is 1 minus the corresponding coefficient. Likewise, there is no independent innovation to these level variables.

Panel B of Table 2 reports the estimate of $\Sigma^{\frac{1}{2}}$, the Cholesky decomposition of the innovation variance-covariance matrix. The innovation in tax revenue/GDP growth rate is positively correlated with the GDP growth rate innovation, while the spending/GDP growth shock is negatively correlated with the GDP growth shock. In other words, tax revenues are strongly pro-cyclical and government spending is strongly counter-cyclical, as anticipated by our earlier discussion.

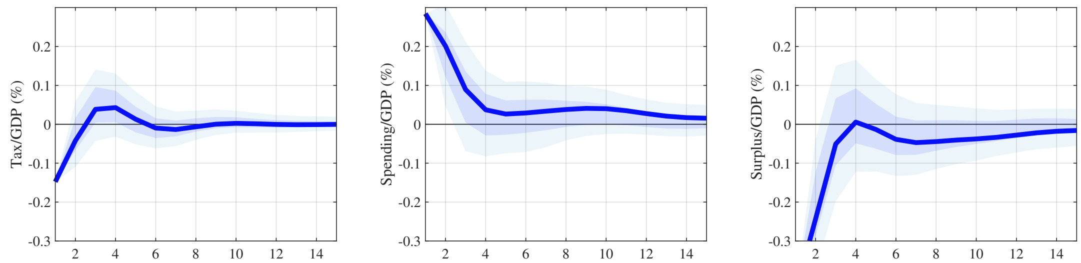
4.2 Implied Revenue and Spending Dynamics

Figure 3 plots the impulse-response functions (IRFs) of the tax revenue-to-GDP ratio (τ_t, left panels), government spending-to-GDP ratio (g_t, middle panels), and surplus-to-GDP ratio (s_t, right panels) to a GDP shock (top row), a revenue shock (middle row), and a spending shock (bottom row). The shocks are calibrated such that the log GDP growth decreases by 1%, the revenue-to-GDP ratio goes down by 1%, and the spending-to-GDP ratio goes up by 1%. The top row shows that the tax revenue-to-GDP ratio declines and the government spending-to-GDP ratio increases in response to a negative GDP shock. The surplus-to-GDP ratio is pro-cyclical. The second and third rows show that mean reversion in spending and revenues brings the responses to their own shocks back to zero within about three years. The instantaneous response of the surplus to all three shocks is negative. There is some evidence of an S-shaped response of the surplus to a spending shock in the bottom right panel as the initial deficits turn into a small surplus after 3 years. However, these surpluses are short-lived and the confidence intervals indicate that even the peak surplus response after 4–5 years is not significantly different from zero. All responses revert back to zero in the long run because of cointegration between spending and GDP and between tax revenues and GDP.

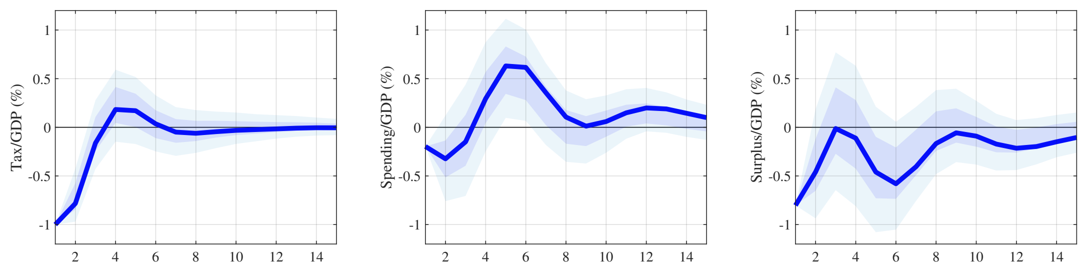
Figure 4 adds further credibility to the cash-flow projections by comparing expected cumulative spending and revenue growth over the next one, five, and ten years to realized future spending and revenue growth over those same horizons. To assess predictive accuracy, we compare the prediction of the VAR (evaluated at actual, un-detrended values of the state variables) to that of the best linear forecaster at that horizon. By design, the VAR prediction is the best linear fore-

Figure 3: Fiscal Impulse Responses

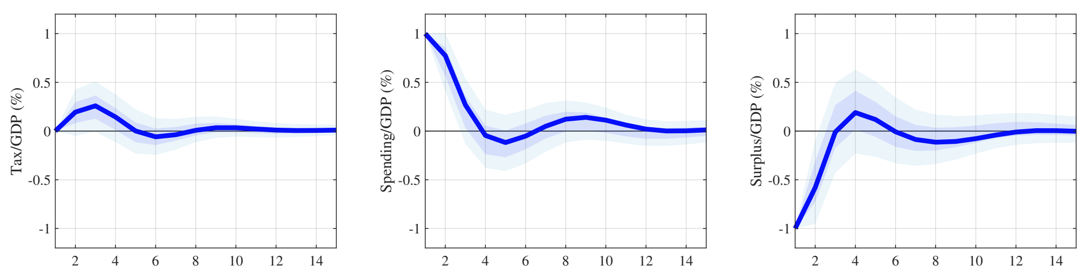
Panel A: −1% Shock to GDP Growth.



Panel B: −1% Shock to Tax-to-GDP.



Panel C: 1% Shock to Spending-to-GDP.



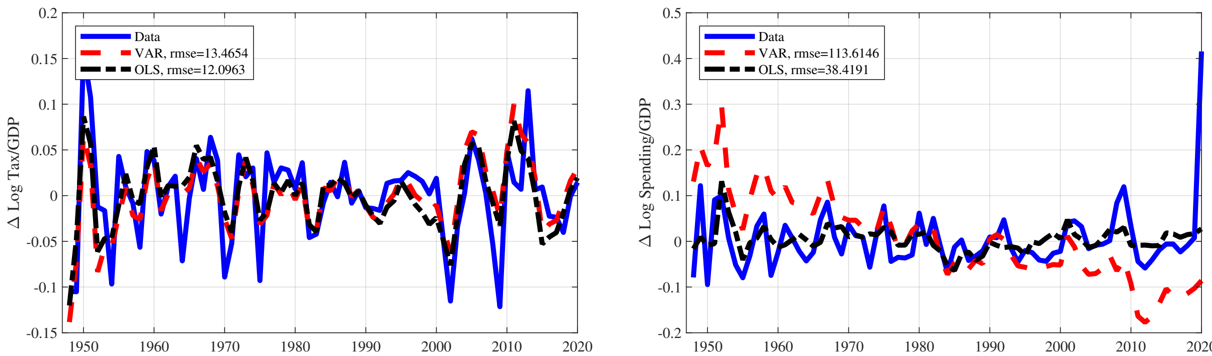
Solid blue line shows the impulse responses for the benchmark VAR. The impulse in the top row is a −1 percentage point shock to GDP growth x_t. The impulse in the middle row is a −1 percentage point shock to tax revenues. The impulse in the bottom row is a +1 percentage point shock to spending growth. We plot the one- and two-standard-deviation confidence intervals based on bootstrapping over 10,000 rounds.

caster at the one-year horizon, but not at the five- and ten-year horizons.[7] Predictive accuracy of the VAR for longer horizons is similar to that of the best linear forecast. The graph shows that the VAR implies reasonable behavior of long-run fiscal cash flows. Note how the long-run spending forecasts from the VAR is low at the end of the sample. This implies that the VAR predicts, if anything, too much mean reversion in the surplus compared to the data. This understatement occurs

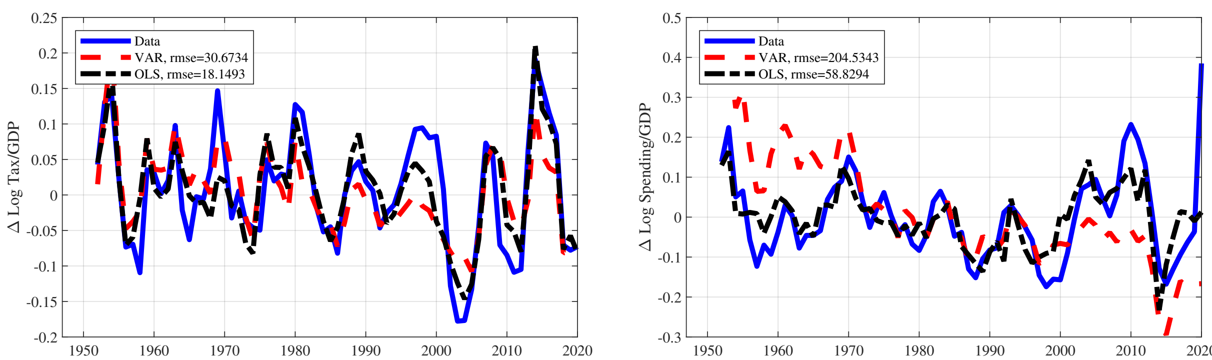
[7] Since we use the actual tax-to-GDP and spending-to-GDP series to compute the VAR predictions but the companion matrix is estimated using the detrended series, the VAR series has a slightly higher RMSE than the OLS prediction at the one-year horizon.

Figure 4: Cash Flow Forecasts

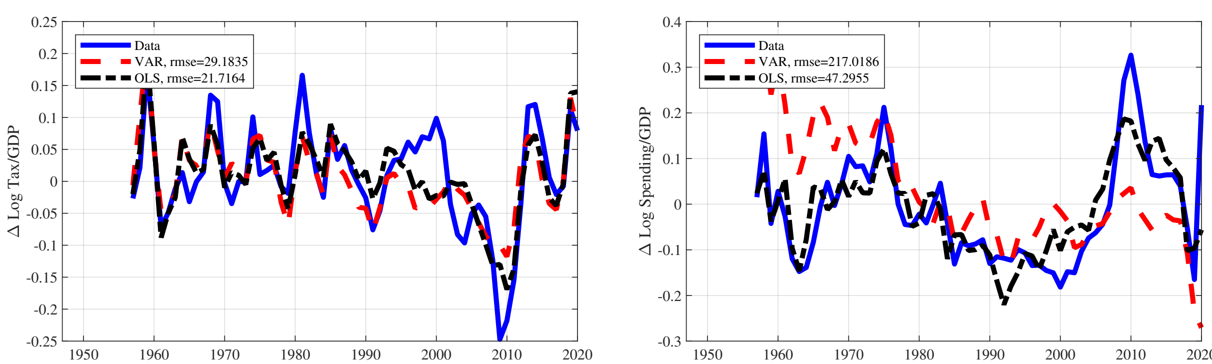
Panel A: Forecast of 1-Year Growth in Log Tax/GDP and Log Spending/GDP.



Panel B: Forecast of 5-Year Growth in Log Tax/GDP and Log Spending/GDP.



Panel C: Forecast of 10-Year Growth in Log Tax/GDP and Log Spending/GDP.



We plot the actual log tax and spending growth rates over 1-year, 5-year and 10-year rolling windows in solid blue lines. The value at each year represents the *k*-year growth rates that end at that year. We also plot these rates as forecasted by our VAR model in dashed red lines and these rates as forecasted by the OLS model in dash-dotted black lines. The value at each year represents the *k*-year growth rates condition on the (un-detrended) state variables *k* years ago.

because we evaluate these forecasts at the actual value of the tax-to-GDP and spending-to-GDP ratios. The former is well below its long-run mean towards the end of the sample, while the latter is well above its mean. The error correction dynamics result in higher future tax revenue-to-GDP and lower future spending-to-GDP forecasts at the end of the sample. This forecast is conservative in that this will result in a higher present value of future surpluses and a smaller bond valuation puzzle.

5 Bounding Fiscal Capacity

We begin by developing an upper bound on fiscal capacity without having to commit to a specific model of risk prices. This approach guards against misspecification of the asset pricing model and should be satisfied in any plausible asset pricing model.

By log-linearizing returns and iterating forward, we obtain the standard Campbell and Shiller (1988) decomposition of the price/dividend ratios on the tax and spending claims:

$$pd_t^\tau = \frac{\kappa_0^\tau}{1-\kappa_1^\tau} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^\tau)^{j-1}\Delta \log T_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^\tau)^{j-1} r_{t+j}^\tau\right],$$

$$pd_t^g = \frac{\kappa_0^g}{1-\kappa_1^g} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^g)^{j-1}\Delta \log G_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^g)^{j-1} r_{t+j}^g\right],$$

where κ_0^k and κ_1^k for $k \in \{\tau, g\}$ are linearization coefficients which depend on the mean of the log price/dividend ratio pd_0^τ. The derivations are detailed in Appendix C. We use rp_t^g and rp_t^τ to denote the log risk premium on the spending and tax claim relative to the yield on a long-term bond. As a result, the expected log return on the tax and spending claims are given by the following expression: $\mathbb{E}_t r_{t+1}^k = y_t^\$(1) + yspr_t^\$ + rp_t^k$ for $k \in \{g, \tau\}$. The risk premia rp_t^τ and rp_t^g are the only "free" parameters that remain to be pinned down.

Restating eqn. (1), the expected risk-adjusted PDV value of surpluses PV_t^s scaled by GDP Y_t is given by:

$$\frac{PV_t^s}{Y_t} = \frac{P_t^\tau}{Y_t} - \frac{P_t^g}{Y_t} = \tau_t \exp(pd_t^\tau) - g_t \exp(pd_t^g). \tag{10}$$

The upper bound on fiscal capacity exploits two insights discussed in Section 3. First, we know from asset pricing theory that the lower bound on the expected return on the tax claim is given by the expected return on the GDP claim. The tax claim is exposed to the same long-run risk as the output claim, because of co-integration with output, but it is also more exposed to business cycle risk. Second, the upper bound on the expected return on the spending claim is given by the expected return on the GDP claim. The spending claim is exposed to the same long-run risk as the output claim, because of co-integration without output, but it also much less exposed to business cycle risk. In summary:

$$rp_t^\tau \geq rp_t^y \geq rp_t^g. \tag{11}$$

Shouldn't the present value of surpluses in eqn. (10) always equal the observed debt/output ratio? We can always back out values for the risk premia rp_t^τ and rp_t^g such that $\frac{PV_t^s}{Y_t} \equiv \frac{D_t}{Y_t}$ holds exactly at all times. But, as we show below, these values will violate (11).

We assume that risk premia are constant at their unconditional levels rp_0^τ and rp_0^g. Using the estimated VAR dynamics in eqn. (10), and denoting by $\boldsymbol{e}_z$ a selector vector that has a one in the position of some state variable z, we obtain the following expressions for the log price/dividend ratio on the tax claim:

$$pd_t^\tau = pd_0^\tau + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_\tau - \boldsymbol{e}_{y1} - \boldsymbol{e}_{yspr})' \boldsymbol{\Psi} (I - \kappa_1^\tau \boldsymbol{\Psi})^{-1} \boldsymbol{z}_t,$$

where $(pd_0^\tau, \kappa_0^\tau, \kappa_1^\tau)$ solve the system of equations:

$$\begin{aligned} pd_0^\tau &= -\frac{(y_0^\$(1) + yspr_0^\$ + rp_0^\tau) - (x_0 + \pi_0)}{(1-\kappa_1^\tau)} + \frac{\kappa_0^\tau}{(1-\kappa_1^\tau)}, \\ \kappa_1^\tau &= \frac{e^{pd_0^\tau}}{e^{pd_0^\tau}+1}, \qquad \kappa_0^\tau = \log(1+\exp(pd_0^\tau)) - \kappa_1^\tau pd_0^\tau. \end{aligned} \tag{12}$$

Because of cointegration, nominal tax revenues grow at the same long-run growth rate as nominal GDP $x_0 + \pi_0$. The expressions for the price/dividend ratios on the spending claim and the GDP claim are analogous. We use the shorthand $\widetilde{CF}_t^\tau = (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_\tau)' \boldsymbol{\Psi} (I - \kappa_1^\tau \boldsymbol{\Psi})^{-1} \boldsymbol{z}_t$ and $\widetilde{DR}_t = (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})' \boldsymbol{\Psi} (I - \kappa_1^\tau \boldsymbol{\Psi})^{-1} \boldsymbol{z}_t$ to denote the time-varying cash-flow and discount-rate components. They are mean-zero because $\boldsymbol{z}_t$ is mean-zero.

Using the valuation ratios of tax revenue and spending claims, we obtain the following expression for the surplus claim, i.e., the model-implied debt-to-GDP ratio:

$$\frac{PV_t^s}{Y_t} = \tau_t \exp(pd_0^\tau + \widetilde{CF}_t^\tau - \widetilde{DR}_t^\tau) - g_t \exp(pd_0^g + \widetilde{CF}_t^g - \widetilde{DR}_t^g). \tag{13}$$

To derive some intuition, we can evaluate the expression at $\boldsymbol{z} = 0$, i.e., when all state variables are at their unconditional mean. Then, the present value of government surplus is given by:

$$\frac{PV_t^s}{Y_t}(\boldsymbol{z} = 0) = \tau_0 \exp(pd_0^\tau) - g_0 \exp(pd_0^g).$$

This expression has important implications for fiscal capacity. A country that runs steady-state deficits ($\tau_0 < g_0$) can only maintain positive debt capacity if the tax revenue process is less risky than the spending process. We need a higher valuation for the tax claim than for the spending claim $pd_0^\tau > pd_0^g$ to obtain positive debt capacity $\frac{PV^s}{Y} > 0$. But given that spending and taxes grow at the same rate as GDP in the long run, we can only satisfy this condition if $rp_0^\tau < rp_0^g$. However, that is inconsistent with eqn. (11), itself motivated by the properties of U.S. tax and spending data.

We derive an upper bound on the value of the surplus claim by maximizing the value of the tax claim and minimizing the value of the spending claim. This is accomplished by equating the expected returns on taxes and spending to the expected return on GDP:

$$rp_0^g = rp_0^y = rp_0^\tau.$$

The upper bound on the present value of government surplus is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)^u = \tau_t \exp(pd_0^y + \widetilde{CF}_t^\tau - \widetilde{DR}_t^\tau) - g_t \exp(pd_0^y + \widetilde{CF}_t^g - \widetilde{DR}_t^g), \tag{14}$$

where the unconditional valuation ratios and the linearization constants $pd_0^y = pd_0^\tau = pd_0^g$, $\kappa_0^y = \kappa_0^\tau = \kappa_0^g$, and $\kappa_1^y = \kappa_1^\tau = \kappa_1^g$ solve the system of equations in (12), and the mean-zero discount rate and expected cash-flow growth terms are given by:

$$\begin{aligned} \widetilde{DR}_t^\tau = \widetilde{DR}_t^g = \widetilde{DR}_t^y &= (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t, \\ \widetilde{CF}_t^k &= (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_k)'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t, \quad \text{for } k \in \{\tau, g\}. \end{aligned}$$

When the state vector is evaluated at its unconditional mean ($\boldsymbol{z} = 0$), the upper bound on the debt/output ratio is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)^u (\boldsymbol{z} = 0) = \exp(pd_0^y)\,(\tau_0 - g_0)\,. \tag{15}$$

First, this steady-state upper bound is positive if and only if steady-state surpluses are positive ($\tau_0 > g_0$). Second, the expression that matters for the (upper bound on) fiscal capacity is not the risk-free rate minus the growth rate: $y_0^\$(1) - (x_0 + \pi_0)$, but the expected return on the GDP claim minus the growth rate: $(y_0^\$(1) + yspr_0^\$ + rp_0^y) - (x_0 + \pi_0)$. On average, the GDP claim earns the unlevered equity risk premium (here defined relative to the average long-term bond yield), i.e. the risk premium on equity of a representative non-financial firm without any debt. When we assume a GDP risk premium of $rp_0^y = 3\%$, the former $y_0^\$(1) - (x_0 + \pi_0)$ is -1.85% while the latter $(y_0^\$(1) + yspr_0^\$ + rp_0^y) - (x_0 + \pi_0)$ is 1.73%. A higher output risk premium lowers the valuation ratio pd_0^y and lowers the upper bound on debt capacity. Third, an increase in pd_0^y only increases the borrowing capacity of the government in the steady-state if the steady-state surplus is positive ($\tau_0 > g_0$). If the government runs a steady-state deficit, a higher pd_0^y lowers fiscal capacity.

In section 5.1, we evaluate the dynamic upper bound in the benchmark 1947—2020 sample. To check the robustness of the results, we explore other specifications. We include debt in the VAR in section 5.2. In section 5.3 we re-estimate the VAR in the longer 1929—2020 sample. We introduce convenience yields in section 5.4.

5.1 Benchmark Results

In the 1947–2020 sample, the realized real growth rate of GDP x_0 is given by 2.95%.[8] With a GDP risk premium of $rp_0^y = 3\%$, the expected real return on the output claim $y_0^\$(1) + yspr_0^\$ + rp_0^y - \pi_0 = 4.68\%$. The linearization coefficient in the Campbell-Shiller decomposition is $\kappa_1^y = 0.98$. The price/dividend ratio for the output claim $\exp(pd_0^y) = 57.39$. Multiplying this steady-state price/dividend ratio with the steady-state surplus-to-GDP ratio of $\tau_0 - g_0 = 0.05\%$, we obtain the steady-state upper bound in eqn. (15) of 3.15% of GDP. The steady-state upper bound on the U.S. debt/output ratio is close to zero. The upper bound increases by 57.39% for each percentage point of surplus-to-GDP. For the steady-state upper bound to accommodate a debt-to-GDP ratio of 100%, the observed value in 2020, the steady-state surplus-to-GDP ratio would need to increase to 1.74%. This would be a dramatic reversal from the deficit in recent years, from the long-run average, and from the March 2021 CBO projections for the next twenty years which indicate an average annual primary deficit of 3.9% of GDP until 2050 under current law.

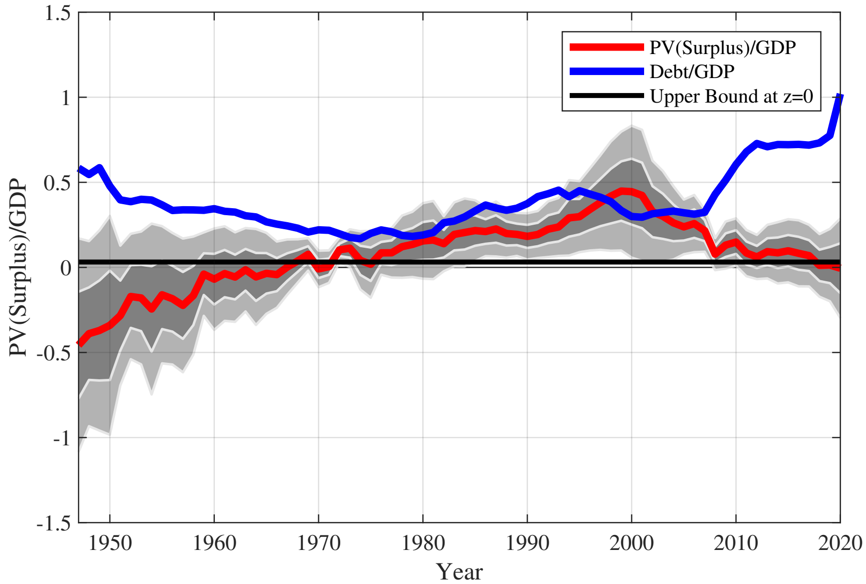
Figure 5 plots the dynamic upper bound in eqn. (14) on the U.S. debt/output ratio (in red) with two-standard error bands, as well as the steady-state upper bound of 3.15% discussed above (in black). The dynamic upper bound incorporates time-varying expected growth rates and discount rates. The latter are driven by long-term interest rates since the risk premium on the GDP claim is assumed to be constant.

The main result is that the dynamic upper bound on fiscal capacity is below the market's valuation of U.S. Treasurys except for a brief period around the year 2000. Before the GFC, the observed market value of debt is often within the 95% confidence interval of the upper bound. However, this changes markedly after the GFC, when the debt/output ratio invariably exceeds the 95% confidence interval until the end of the sample. As of the end of our sample in 2020, the gap exceeds 100% of the GDP.

To generate the standard errors for the upper bound we use a bootstrap. In each bootstrap iteration, we draw with replacement from the VAR residuals $\{\widehat{u}_t\}_{t=1}^T$ and generate a new dataset us-

[8] By using this high number as our estimate of long-run real growth, we end up with a higher pd_0^y and hence a higher upper bound on fiscal capacity. This creates a smaller gap with the observed debt-to-GDP ratio and, hence, is conservative.

Figure 5: Upper Bound on the Value of Surpluses/GDP



The figure plots the upper bound on the present value of government surpluses in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15), and the actual debt/output ratio. We report the benchmark case with a GDP risk premium $rp_0^y = rp_0^g = rp_0^\tau$ of 3%. The sample period is from 1947 to 2020. 2-standard-error confidence intervals generated by bootstrapping 10,000 samples.

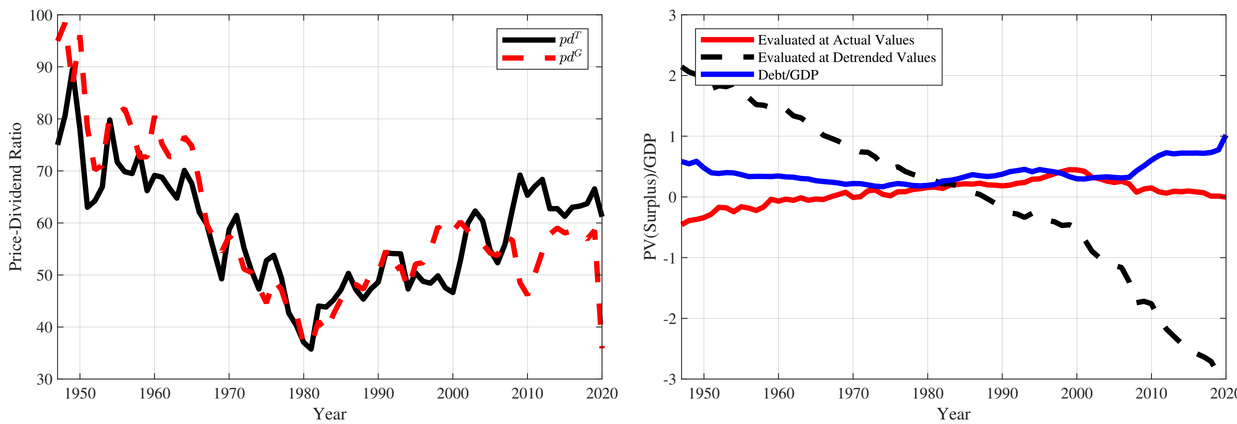
ing the VAR companion matrix. Then, we re-estimate the VAR, and re-compute the upper bound $\{\left(\frac{PV_t^s}{Y_t}\right)^u\}_{t=1}^T$.

Importantly, we evaluate the upper bound $\left(\frac{PV_t^s}{Y_t}\right)^u = \tau_t \exp(pd_t^\tau) - g_t \exp(pd_t^g)$ at the actual values for τ_t and g_t, including when we evaluate the valuation ratios (pd_t^τ, pd_t^g). Because the VAR implies mean-reversion, this implicitly assumes that the spending-to-GDP ratio will revert back to its unconditional mean of 17.40% from its 2020 value of 30.08%. As a result, the expected spending growth rate is extremely low, and therefore so is the 2020 valuation ratio of the spending claim. The left panel of Figure 6 shows the valuation ratio of the spending claim at 35 in 2020. By the same logic, the valuation ratio of the tax claim at 62 is currently very high. If we instead had assumed that the tax-to-GDP and spending/ratios would converge to their trend, then the upper bound in 2020 would decrease from 0 to -3 times GDP, as shown in the right panel of Figure 6. This again illustrates the conservatism in our approach.

To understand this point better, we note that when the average primary surplus is close to zero–as it is in the U.S. data–the cash-flow dynamics become primary determinants of fiscal capacity. A first-order Taylor approximation of the upper bound evaluated at $\tau_t = g_t$ implies that:

$$\left(\frac{PV_t^s}{Y_t}\right)^u \Bigg|_{\tau_t = g_t} \approx \exp(pd_0^y)\tau_t \left(\widetilde{CF}_t^\tau - \widetilde{CF}_t^g\right) = \exp(pd_0^y)\tau_t \left(\boldsymbol{e}_{\Delta\tau} - \boldsymbol{e}_{\Delta g}\right)' \boldsymbol{\Psi}(I - \kappa_1^y \boldsymbol{\Psi})^{-1} \boldsymbol{z}_t.$$

Figure 6: Valuation Ratios and Upper Bounds



The left panel plots the implied valuation ratios for the tax and spending claims in the benchmark case—evaluated at the actual values. We consider the upper bound case when the GDP risk premium $rp_0^y = rp_0^g = rp_0^\tau$ is 3%. The sample period is from 1947 to 2020. The right panel plots the upper bound evaluated at the actual values (Benchmark) and the detrended values.

The dynamics of the short-term interest rate, the slope of the term structure, and the real GDP growth rate are irrelevant since they equally affect the expected cash-flow growth and discount rates of the T and G claims. The last term shows that what matters is the dynamics in tax-to-GDP and spending-to-GDP growth rates. In other words, to generate a positive upper bound, the expected cumulative effect of mean reversion in taxes has to outweigh the expected cumulative effect of mean reversion in spending. The upper bound on fiscal capacity is still around zero in 2020 despite the large primary deficit in that year because of expected future tax increases and spending cuts. As the right panel of Figure 6 illustrates, making a weaker–and possibly more realistic–assumption on the strength of that mean reversion would lead to a much lower upper bound on fiscal capacity (dashed black versus the red line).

As a robustness check, we lower the output risk premium to 2.5%. This increases the steady-state valuation ratio of the output claim to 88. This change only has a small effect on the upper bound, because the average deficits are close zero, as shown in Figure C.1 in the Appendix. While it widens the confidence intervals on the upper bound, it leaves the conclusion that the market value of debt substantially exceeds the upper bound after the GFC intact. In section 6, we estimate an average output risk premium of 2.61%.

To enforce the debt valuation in eqn. (13) as an equality, one would have to reverse-engineer rp_t^τ such that

$$\frac{D_t}{Y_t} = \tau_t \exp(pd_0^\tau - \frac{rp_t^\tau - rp_0^y}{1-\kappa_1^\tau} + \widetilde{CF}_t^T - \widetilde{DR}_t^T) - g_t \exp(pd_0^y + \widetilde{CF}_t^G - \widetilde{DR}_t^G), \quad (16)$$

The right-hand side equals the upper bound in eqn. (14) for $rp_t^\tau = rp_0^y$, $\forall t$. Whenever the debt-to-GDP ratio exceeds the upper bound, one would have a risk premium on the tax claim that is lower than the risk premium on the GDP claim: $rp_t^\tau < rp_0^y$. This violates the risk premium condition in eqn. (11).

Our method is not bound to deliver low values for the upper bound on fiscal capacity. For comparison, the UK primary surplus over the 1947—2020 sample is 1.8%. Applying the U.S. output valuation ratio of 57 to the U.K. average surplus results in a steady-state upper bound of 102%. This bound comfortably accommodates the U.K.'s debt-to-GDP ratio in the last 70 years.

5.2 Debt in the VAR

Based on prior findings that highlight a fiscal response to the level of debt (Bohn (1998); Cochrane (2019a,b)), we also construct a version of the upper bound that includes the log debt-to-GDP ratio as a predictor variable in the state vector, allowing spending and revenue growth to depend on the lagged debt/output ratio.[9] The state variables for this system are shown in Appendix Table C.1.

One empirical challenge is that the log debt/GDP ratio, $\log b_t$, is highly persistent. We estimate an AR(1) model using maximum likelihood and find an AR(1) coefficient estimate of 1.008. In light of this non-stationarity, we test and find strong evidence for a structural break in the debt/output ratio in 2007. Following the approach for stocks in Lettau and Van Nieuwerburgh (2008), we demean $\log b_t$ before 2007 with the pre-2007 sample mean (-1.167) and $\log b_t$ after 2007 with the post-2007 sample mean (-0.379). The structural break introduces a 0.788 log point permanent increase in the debt/output ratio. The AR(1) coefficient of the adjusted series for the log debt-to-GDP ratio is lower at 0.927.[10] We include both the first-difference and the level of the adjusted log debt/GDP ratio in the VAR and impose the same error-correction dynamics as we did for spending-to-GDP and revenues-to-GDP. This way of incorporating debt in the VAR results not only in a better-behaved VAR system, but also in more realistic predictions for future debt and surplus dynamics.[11] Importantly, this approach is conservative in that it results in a stronger response of surpluses to an increase in the debt/GDP ratio.

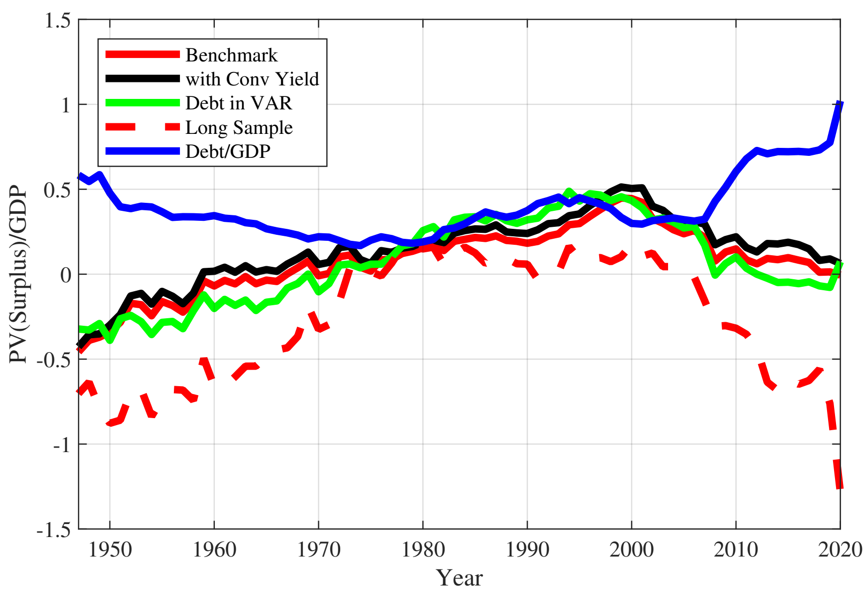
[9] Cochrane (2019a,b) includes debt/GDP in the VAR and argues that this affects the dynamics of the surplus in important ways. In particular, a negative shock to GDP (or a negative shock to the surplus) leads to a deficit on impact. The deficit not only reverts back to zero in subsequent periods, but overshoots into a surplus. The VAR model with debt allows for these dynamics. Separately, an follow-up work to this paper, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2021c) investigate whether the U.S. debt/output ratio forecasts either future surpluses or future bond returns. Contrary to Cochrane (2019a,b), they find no evidence of either predictability. They argue that this finding is consistent with mispricing of Treasurys.

[10] The OLS estimate of the first-order autocorrelation is 0.922 for the raw log debt/GDP ratio, and 0.885 for the adjusted series.

[11] Using the raw debt/output ratio would result in a companion matrix Ψ with a largest eigenvalue in excess of one.

Figure 7 plots the actual debt/output ratio against the dynamic upper bound implied by the model with debt in the VAR (green line). The upper bound is overall quite similar to that in the benchmark model (red line). It is slightly lower before 1975, slightly higher from 1975 to 2000, and slightly lower after the Great Financial Crisis (GFC). The upper bound on fiscal capacity in the extended model with debt cannot accommodate the observed U.S. debt-to-GDP ratio between 1950 and 1970 or after the GFC in 2008, despite several sources of conservatism built into the construction of the upper bound.

Figure 7: Upper Bound on the Value of Surpluses/GDP



The figure plots the upper bound on the present value of government surpluses in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15), and the actual debt/output ratio. The GDP risk premium is 3%. We report the results from 4 different models: the benchmark (red line), the model with debt in the VAR (green line), the long sample (dashed red line), and the model with convenience yields (black line).

5.3 Longer Sample

We study robustness to a longer sample using U.S. data form 1929—2020. In the longer sample, the valuation ratio $\exp(pd^y)$ is 53.90 and the steady state surplus $\tau_0 - g_0$ is given by -0.55%. The U.S. has been running a substantial average primary deficits over the past 92 years. As a result of the lower average surplus, we obtain a lower steady-state upper bound of -29.42%. For the steady-state upper bound to accommodate a debt/GDP ratio of 100%, the steady-state surplus-to-GDP ratio would need to increase by 2.40% to 1.86%.

Figure 7 plots the dynamic upper bound in the longer sample (dashed red line). The observed debt/output ratio now always exceeds the upper bound. Using these estimates, the gap between the upper bound and the current debt/output ratio increases to over 2 times GDP in 2020.

In the longer sample, the spending/output and the tax/output ratio and their growth rates have a different mean. This results in more muted mean-reversion dynamics and hence a larger gap between the surplus value and the observed debt-to-GDP ratio at the end of the sample. Appendix C.2 provides more details on the VAR estimation for the longer sample. Figure C.2 includes confidence intervals and explores different values of the output risk premium. The results are not very sensitive to the choice of output risk premium.

5.4 Convenience Yields

As discussed in Section 3.3, U.S. Treasurys earn a convenience yield, which produces additional revenue for the U.S. government. The question we address here is how far this explanation can go towards accounting for the bond valuation puzzle.

We use the variable k to represent the seigniorage revenue from convenience as a fraction of output. Building on Prop. 4, we can rewrite the intertemporal budget constraint as:

$$\sum_{h=0}^{K} Q_{t-1}^{\$}(h+1)P_t^{\$}(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}(\tau_{t+j}+k_{t+j})Y_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}G_{t+j}\right].$$

The observed nominal Treasury yield $y_t^{\$}(1)$ is given by the following expression: $y_t^{\$}(1) = \rho_t^{\$}(1) - \lambda_t(1)$, where $\rho_t^{\$}(1)$ is the one-period nominal risk-free rate. We include the one-period nominal risk-free rate as the second element of the VAR (replacing the nominal short rate): $z(2,t) = \rho_t^{\$}(1) = y_t^{\$}(1) + \lambda_t(1)$. We continue to assume that $rp_0^y = rp_0^g = rp_0^\tau$ to derive the upper bound. Evaluated at the steady-state, the upper bound on the debt/output ratio with convenience is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)^u\Bigg|_{z=0} = \exp(pd_0^y)\left(\tau_0 + k_0 - g_0\right), \tag{17}$$

where pd_0^y is the valuation ratio of the GDP claim computed above. A country may now be able run steady-state deficits if the seigniorage revenue turns the primary deficit into a surplus.

To proxy for the convenience yield $\lambda_t(1)$, we first construct the spread cy_t between the 3-month Treasury yield and a risk-free benchmark, which is the 3-month CD rate from 1964 and the 3-month banker's acceptance rate before 1964. Panel A of Figure B.1 plots this spread. The average spread cy_0 is 0.60% per year over the period 1947—2020. The real short rate $y_0^{\$}(1) - \pi_0$ increases from 1.10% to 1.66% once convenience is considered.

We assume that Treasury bonds with longer maturities earn lower convenience yields. Specifically, bills with maturities within 1 year earn 100% of cy_t, 1-year bonds earn 90% of cy_t, 2-year bonds earn 80% of cy_t, and so on. Treasury bonds with 10-year maturities and longer earn no convenience yield. We construct the seigniorage revenue in each period based on the amount of bounds outstanding of each maturity. Panel B of Figure B.1 shows the resulting time series. We find an average seigniorage revenue k_0 of 0.20% of U.S. GDP. As a result, the steady-state surplus $(\tau_0 + k_0 - g_0)$ increases from 0.05% to 0.25%.

A narrow measure of convenience yield only affects Treasuries, leaving expected returns on other assets such as stocks or a claim to GDP unchanged. A given expected stock (GDP claim) return–as measured in the data–would then imply a lower equity (GDP) risk premium in the presence of a higher true risk-free rate (observed short rate plus convenience yield). We decrease the output risk premium rp_0^y by 0.60% per year, our estimate of λ_0. The average valuation ratio $\exp(pd_0^y)$ stays roughly constant at 58.66.

When the state vector is evaluated at its long-run mean ($z = 0$), the upper bound on the debt/output ratio is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)^u\bigg|_{z=0} = \exp(pd_0^y)\,(\tau_0 + k_0 - g_0) = 58.66 \times 0.25\% = 14.94\%.$$

Since we decreased the output risk premium by 60 bps, thereby raising $\exp(pd_0^y)$, this calculation is conservative.

We compute the dynamic upper bound, by adding seigniorage revenue as part of tax revenues and including the risk-free rate instead of the nominal short rate in the VAR. The derivation is in Appendix C.2. The black solid line in Figure 7 plots the upper bound with convenience yields. It is modestly above the benchmark upper bound. From the late 1990s until the onset of the GFC, the upper bound with convenience yields is above the actual debt/output ratio. However, since the GFC, the gap has widened dramatically just like in the benchmark model. In conclusion, standard measures of (narrow) convenience yields do not resolve the bond valuation puzzle.

A Broader Measure of Convenience Recently, some have argued that convenience yields may be much larger than previously thought, as high as than 200 bps per annum (Jiang et al., 2021a; Koijen and Yogo, 2019). Higher convenience yields increase government surpluses inclusive of seigniorage revenue. We argue that higher convenience yields are not a panacea for resolving the valuation puzzle. The reason is that the cash-flow effect of higher surpluses is offset by a discount rate effect. Higher convenience yields increase the true risk-free rate. As long as the GDP risk premium falls less than one-for-one with the convenience yield, the expected return on the GDP claim rises. Discounting surpluses at a higher discount rate lowers the expected PDV of surpluses. This less than one-for-one decline of risk premia as convenience yields rise is intuitive. A high enough convenience yield would otherwise turn risk premia negative, which is theoretically undesirable and empirically implausible.

The strength of the discount rate effect depends on the change in the GDP risk premium when the convenience yield increases. The size of the change in the GDP risk premium in turn depends on the source of the convenience yield. Brunnermeier et al. (2022); Reis (2021) entertain models in which governments bonds allow agents to smooth idiosyncratic risk. We label the convenience yield attributable to market incompleteness a "broad convenience yield" in that it affects expected returns on all assets not subject to idiosyncratic return risk.[12] In its starkest form, a broad convenience yield of say 500 bps leaves the risk premium unchanged, raising the true risk-free rate by 500 bps and hence the true expected return on the GDP claim purged of convenience yield effects by the same amount. The increase in the true expected return on the GDP claim lowers the valuation ratio pd_0^y. As a result, the government generates less debt capacity per percentage point of surplus/output.[13]

As one increases the (broad) convenience yield λ_0 by enough, the discount rate effect comes to dominate the cash flow effect; the marginal effect on the upper bound turns negative. To see this, note that:

$$\frac{\partial \left(\frac{PV_t^s}{Y_t}\right)^u \Big|_{z=0}}{\partial \lambda_0} \leq (>)0 \iff \frac{(\tau_0 + k_0 - g_0)}{(1-\kappa_1^y)} \geq (<) d_0.$$

The proof is given in Appendix C.2. The right-hand side is approximately the PDV of steady-state surplus including seigniorage. At low levels of the convenience yield λ_0, seigniorage revenue-to-GDP k_0 is low, $(\tau_0 + k_0 - g_0)/(1-\kappa_1^y) < d_0$, and the derivative of the upper bound is positive. The cash-flow effect dominates. At high enough levels of the convenience yield, k_0 is high enough to produce $(\tau_0 + k_0 - g_0)/(1-\kappa_1^y) > d_0$, and the derivative of the upper bound turns negative. The discount rate effect dominates. In sum, high convenience yields cannot generate high enough upper bounds on fiscal capacity since the latter shrink in the convenience yield for high levels of the convenience yield.

[12]In incomplete market models, imperfect risk sharing affects the equilibrium expected return on all assets. More risk sharing increases the risk-free rate, due to a weaker precautionary savings effect. It leaves risk premia on other assets (stocks or the GDP claim) unchanged if the cross-sectional dispersion in income shares is uncorrelated with the aggregate growth rate of the economy in the time series (Krueger and Lustig, 2010). Brunnermeier et al. (2022) allows for counter-cyclical idiosyncratic risk. In Brunnermeier et al. (2022); Reis (2021), government bonds are the only vehicle for saving not subject to idiosyncratic return risk. For our measurement exercise, many assets, including stocks, allow agents to save without incurring idiosyncratic return risk, and hence would earn the broad convenience yield.

[13]In contrast, a "narrow convenience yield" measure only affects the expected return on Treasurys and not the expected return on other asset classes. An increase in the narrow convenience yield raises the true risk-free rate but not the expected return on other assets. The risk premium on other assets falls one-for-one with the rise in the risk-free rate. For the narrow measure of convenience, there is no offsetting discount rate effect. This was the case we considered in Figure 7, and it was conservative in the sense that it leads to a higher upper bound and a lower gap with the observed market value of debt. As noted, a very large narrow convenience yield is implausible since it results in negative risk premia. If a 500 bps convenience yield only accrued to government bonds, then the implied risk premium on unlevered equity would be -200 bps.

6 Measuring Fiscal Capacity

The upper bound exercise in the previous section assumes equal and constant risk premia rp_0^τ and rp_0^g on the tax and spending claims. In this section, we infer these risk premia from asset prices by using a dynamic asset pricing model. This allows us to quantify the gap between the value of debt and the PDV of surpluses.

6.1 Asset Pricing

We choose a flexible SDF model that only assumes no arbitrage, and use it to price the term structure of interest rates as well as stocks. This approach guarantees that our debt valuation is consistent with observed Treasury bond yields. It also results in an SDF that has enough permanent risk to account for the equity risk premium (Alvarez and Jermann, 2005).[14] The nominal SDF $M_{t+1}^\$ = \exp(m_{t+1}^\$)$ is conditionally log-normal:

$$m_{t+1}^\$ = -y_t^\$(1) - \frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Lambda}_t - \mathbf{\Lambda}_t'\varepsilon_{t+1}.$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^\$ + \pi_{t+1})$, which is also conditionally Gaussian. The priced sources of risk are the structural innovations in the state vector ε_{t+1} from eqn. (9). We use the VAR with the state vector listed in Table 1. These aggregate shocks are associated with a $N \times 1$ market price of risk vector Λ_t of the affine form:

$$\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{z}_t,$$

The $N \times 1$ vector Λ_0 collects the average prices of risk while the $N \times N$ matrix $\mathbf{\Lambda}_1$ governs the time variation in risk premia. This setting allows us to derive analytical solutions for bond yields and stock price-dividend ratios, as detailed in Appendix D. Asset pricing in this model amounts to estimating the market prices of risk in $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$.

Estimation We estimate the model's risk prices by minimizing the distance between several bond and stock price moments in model and data. Appendix E spells out the moments and reports the point estimates for the market price of risk parameters. It shows that the model provides a tight fit for the entire time series of nominal bond yields of the various maturities. It also shows a reasonable fit for real bond yields. The model closely matches the dynamics of the nominal

[14] The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). Using a similar approach, Lustig, Van Nieuwerburgh, and Verdelhan (2013) study the properties of a claim to aggregate consumption, the wealth-consumption ratio, and Gupta and Van Nieuwerburgh (2021) evaluate the performance of private equity funds.

bond risk premium, and generates reasonable behavior on nominal and real yields at very long horizons. Finally, the model produces reasonable equity risk premium levels and dynamics, and provides a close fit to the time-series of the price-dividend ratio. Because it is able to generate an expected equity return that fits the data well, and that is large compared to the long-term real rate, the SDF has a large permanent component. Having formulated and estimated a realistic SDF, we now turn to our main exercise.

6.2 Surplus Pricing

Recall that the model-implied present value of government surplus is given by:

$$\frac{PV_t^s}{Y_t} = \tau_t \exp(pd_t^\tau) - g_t \exp(pd_t^g). \tag{18}$$

When the government runs deficits, we need a larger valuation ratio for the tax claim than the spending claim ($pd_t^\tau > pd_t^g$) to get a positive valuation of the debt. However, our estimates will reveal that the tax claim is riskier than the spending claim. This discount-rate effect needs to be offset by generating higher expected growth of tax revenues in the short run when the government runs deficits. As we show, this cash-flow effect is not strong enough in the data.

We use $r_0^\tau = \mathbb{E}[r_t^\tau]$ to denote the unconditional expected log return on the tax claim. As before, using the log-linearized returns, the log of the valuation ratio on the tax claim is affine in the state vector:

$$pd_t^\tau = pd_0^\tau + (\bar{B}^\tau)' \boldsymbol{z}_t, \tag{19}$$

where the constant valuation ratio is:

$$pd_0^\tau = -\frac{r_0^\tau - (x_0 + \pi_0)}{(1-\kappa_1^\tau)} + \frac{\kappa_0^\tau}{(1-\kappa_1^\tau)},$$

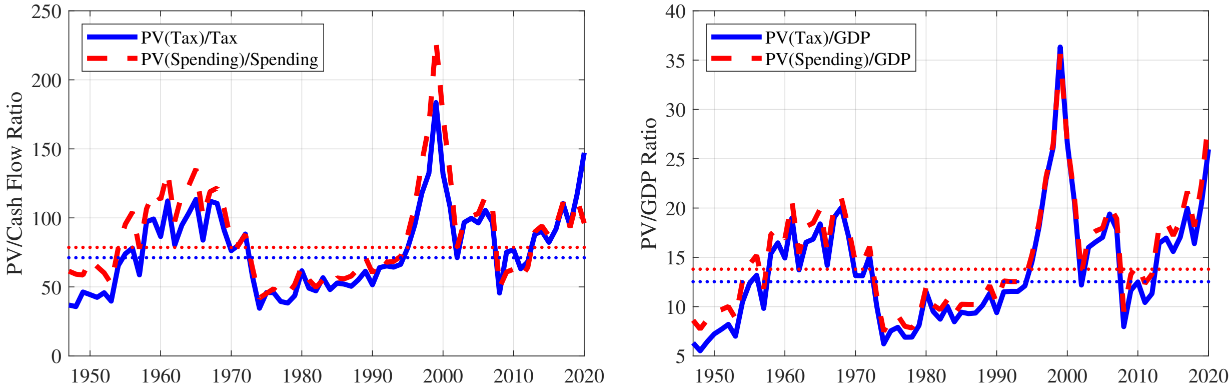
with κ_0^τ and κ_1^τ as defined above. A similar expression applies to the spending claim.

Importantly, differences in the average valuation ratios are driven exclusively by discount rates effects, because the long-run cash flow growth rates of tax revenue and spending claims are identical due to cointegration with GDP. If we equated the expected returns on the output, spending and tax revenue claims ($r_0^g = r_0^\tau = r_0^y$), we recover the upper bound in eqn. (15).

Instead, we now plug the estimated risk prices Λ_0 into the expression for the unconditional risk premium on the tax claim:

$$r_0^\tau - y_0^\$(1) + Jensen = (\boldsymbol{e}_\tau + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^\tau \bar{B}^\tau)' \Sigma^{\frac{1}{2}} \Lambda_0, \tag{20}$$

Figure 8: DAPM Valuations of Taxes and Spending



(a) Normalized by Current Cash Flow (b) Normalized by Current GDP

The figure plots the cum-dividend present values of tax revenues and of government spending. Both time series are scaled by their respective current cash flows in the left panel, and by the current GDP in the right panel. The sample is annual, 1947—2020. The dotted lines represent the values when the state variables are at $z_t = 0$.

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The-right hand side is the unconditional covariance of the log SDF with the log return. The Jensen's term is given by one half of the unconditional variance of the log returns: $\frac{1}{2}\left(\boldsymbol{e}_\tau + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^\tau \bar{B}^\tau\right)' \Sigma \left(\boldsymbol{e}_\tau + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^\tau \bar{B}^\tau\right).$

Similarly, we can calculate the time-varying component of the risk premium on the tax claim:

$$\left(\boldsymbol{e}_\tau + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^\tau \bar{B}^\tau\right)' \Psi - \left(\bar{B}^\tau\right)' - \left(\boldsymbol{e}_{yn}\right)' = \left(\boldsymbol{e}_\tau + \boldsymbol{e}_x + \boldsymbol{e}_\pi + \kappa_1^g \bar{B}^\tau\right)' \Sigma^{\frac{1}{2}} \Lambda_1, \tag{21}$$

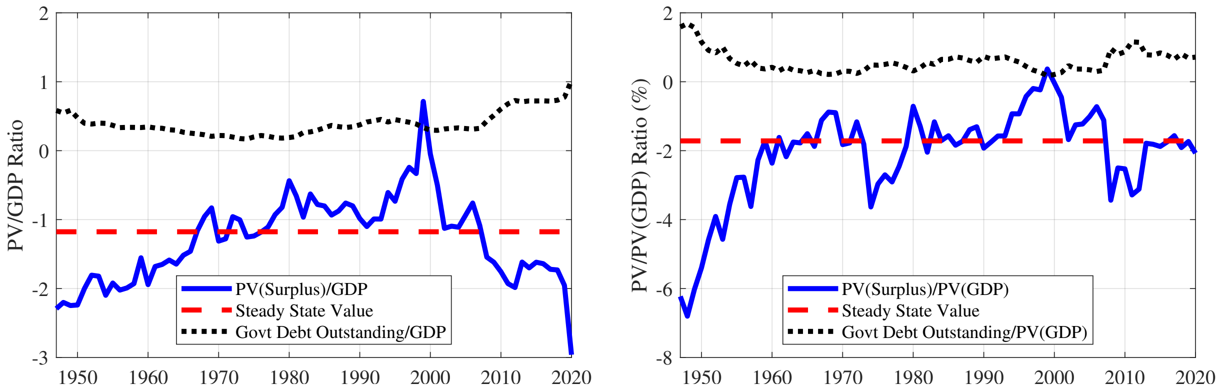
where Λ_1 governs the time-varying component of the market prices of risk. The left-hand side is the time-varying component of the expected excess log return. The right-hand side is the time-varying component of the conditional covariance of the log SDF with the log return on the tax claim. Using this equation, we can recover the conditional expected return and price-dividend ratio for the tax claim. The same procedure applies to the spending claim.[15]

The left panel of Figure 8 plots the present value of tax revenue normalized by tax revenue, P_t^τ / T_t. In steady-state (at $z = 0$), investors are willing to pay 71.18 times annual tax revenue for the tax claim (horizontal dotted line). They are willing to pay larger multiple of 78.62 for the spending claim, because the underlying cash flows are less risky.

The right panel of Figure 8 plots the present value of tax revenue normalized by GDP, P_t^τ / GDP_t.

[15]The implied value of surpluses over output in eqn. (18) is not log-linear in the state vector. When the government adheres to an affine spending and tax rule, the surplus value is not affine. However, we could still include the debt/output ratio as a predictor in the VAR, as in section 5.2, without imposing that the debt/output ratio equal the PDV of surpluses scaled by output. We do not report these results, because they are similar. Alternatively, we could back out the risk-neutral PDV of surpluses from the debt dynamics without assuming a spending and tax policy (see Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2021b).

Figure 9: Present Value of Government Surpluses and Market Value of Government Debt



(a) Normalized by Current GDP (b) Normalized by the Present Value of GDP

The figure plots the cum-dividend present values of the government surplus and the market value of government debt. Both time series are scaled by the current U.S. GDP in the left panel, and by the present value of GDP in the right panel. The sample is 1947 until 2020. In the left (right) panel, the y-axis is denoted in fractions of GDP (percentage points of GDP).

Investors willing to pay 13.67 times the annual GDP on average for the right to receive all current and future tax revenues. The value of the tax claim displays substantial time-variation. With the exception of the late 1990s, a V-shape arises, which is inherited from the inverse V-shape of long-term real interest rate. Real rates are high in the mid-1970s to mid-1980s and low at the beginning and end of the sample. Discounting future tax revenues by a low (high) long-term real rate results in a high (low) valuation ratio. The time-series average of the present value of government spending normalized by GDP, P_t^g/GDP_t, is 14.96. The spending claim is more valuable than the revenue claim, which (in part) reflects the counter-cyclicality of government spending in the short run.

Now we are in a position to value the claim to future government surpluses as the value of the tax claim minus the value of the spending claim, the right-hand side of eqn. (1). Figure 9 plots the present value of government surpluses as the solid blue line. When evaluated at $z = 0$, the steady-state value of the debt/output ratio is given by

$$\left.\frac{PV_t^s}{Y_t}\right|_{z=0} = \tau_0 \exp(pd_0^\tau) - g_0 \exp(pd_0^g) = 17.60\% \times 71.18 - 17.55\% \times 78.62 = -126.77\%.$$

As a result, the model implies a negative steady-state present value of surpluses (horizontal red line in left panel of Figure 9.) The market value of the US government debt is plotted as the dashed black line. The unconditional average present value of the government surplus is -128.87% of GDP, far below the average market value of outstanding government debt, 39.45% of GDP. The gap between the steady-state value of -126.77% and the steady-state upper bound of 3.15% of GDP is entirely due to the higher risk premia for the tax claim than for the spending claim because there are only discount rate effects and no cash flow effects in he steady state.

The valuation gap measures the difference between the market value government debt and the present value of surpluses. It quantifies the government debt valuation puzzle. The gap/GDP ratio is 168% on average. In the time series, the gap widens substantially in the last 20 years of the sample. The level of government debt rises to 55.21% of the GDP while the valuation of the surplus claim decreases to -145% of GDP. In other words, the U.S. government has been issuing government debt while simultaneously decreasing the expected surpluses to back up the debt. The effect is particularly stark in the last few years of the sample. The gap reaches 398% of the GDP in 2020. The bond valuation puzzle will deepen further due to a large fiscal response to the covid pandemic in 2021 and growing deficits from entitlement programs.

We reiterate that imposing cointegration between tax revenues and GDP and spending and GDP is not only imperative to accurately describe fiscal dynamics but also leads to conservative estimates for the gap/GDP ratio. Without the error correction dynamics present in our VAR system, an increase in government spending following a recession is not offset by future reductions in spending or future increases in tax revenues, but rather becomes permanent. In such a world, the spending claim would be much safer and the tax revenue claim much riskier, leading to a much more negative present value of government surpluses and a much larger valuation wedge.

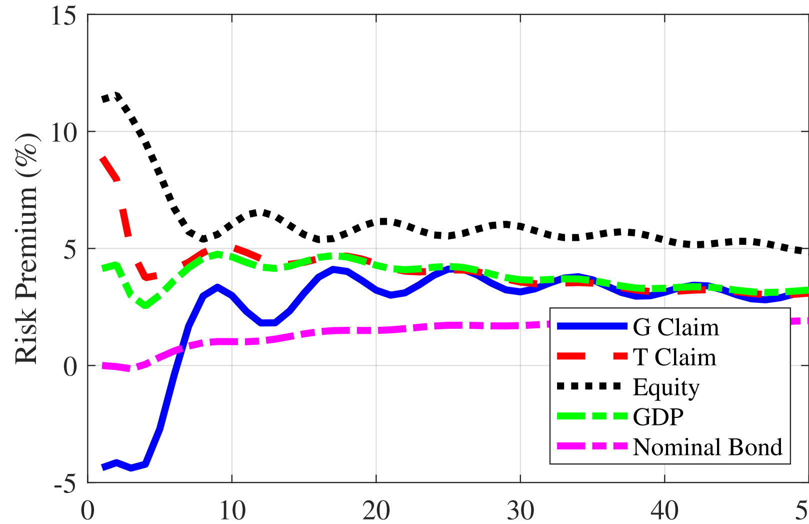
6.3 Risk Premia on Tax and Spending Strips

To further understand the drivers of the negative value of the surplus claim, Figure 10 plots the risk premia on revenue and spending strips for maturities from 1 to 50 years predicted by the asset pricing model. It also plots the risk premia on GDP strips for comparison. At the short end of the maturity spectrum, risk premia on spending strips are very low (-4.37% at the one-year horizon). Because spending is counter-cyclical, these strips are a hedge. In contrast, short-maturity tax revenue strips have high risk premia (8.88% at the one-year horizon) because tax revenues are low in high marginal utility times, making the tax claim a risky asset. Hence, at the short end, the inequality hypothesized in eqn. (5) is satisfied.

As we move to long maturities, risk premia on revenue and spending strips converge towards each other. As noted in eqn. (6), since tax and spending are cointegrated with the GDP, their risk premia also converge towards the risk premium on a GDP strip. Claims to GDP are like unlevered equity claims. They have risk premia well in excess of real bond risk premia but below (levered) equity risk premia. By horizon of 20 years, most of this convergence in risk premia has taken place. In our sample, we estimate an average return on the GDP claim of $r_0^x = 7.45\%$. As a result, the estimated output risk premium relative to the long bond rp_0^y is 2.61%.

We generate these discount rates while maintaining an excellent fit for the term structure of Treasury yields. The claim to surpluses reflects the risk of the government's future debt issuance

Figure 10: Term Structure of Risk Premia on the T-Claim and the G-Claim



This figure plots the cumulative risk premia on the spending strips, the tax strips, equity strips, and the GDP strips in our benchmark model against the holding period. Each point is an annualized holding-period risk premium, as derived in eqn. (D.13) of the Appendix.

strategy.

6.4 Austerity as a Peso Event

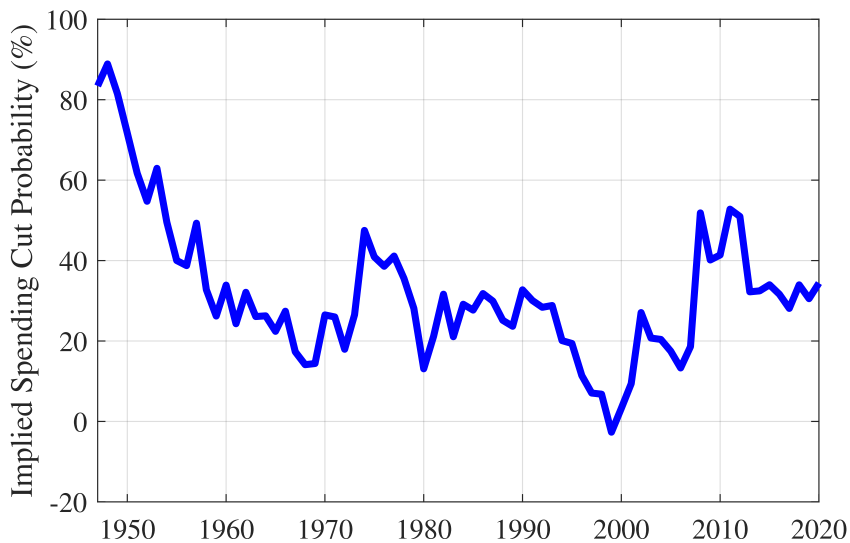
To interpret the magnitude of the valuation puzzle, we consider a simple variant of our model in which bond investors price in the possibility of a major, permanent government spending cut. Such radical austerity never occurs in our 74-year sample but bond investors think it could. How large does the spending cut probability need to be in order to equate the market value of government debt to the present value of surpluses? The possibility of a large future increase in tax revenues is an alternative way to engineer a fiscal correction. We have confirmed that the results are similar.

More precisely, we consider a permanent spending cut that lowers today's and future government spending by 40%. For reference, defense spending accounted for 16% of the federal budget, Social Security for 23%, and Medicare for 15% in 2019. For a typical year with an average spending-to-GDP ratio of 17.55%, the cut lowers it to 10.53%. Given that the spending cut is permanent, we assume the long-run mean of spending-to-GDP, g_0, falls by $\ell = 40\%$. The dynamics of the demeaned state variables, including the demeaned log spending-to-GDP ratio, are still given by the benchmark VAR after the spending cut. As a result, the price of the G-claim scaled by GDP is simply scaled by a factor of $1 - \ell = 60\%$ when the peso event happens. We assume that the peso event itself is not priced; we do not change the market prices of risk Λ_t. Under this simple setting, we ask how likely the spending cut needs to be in each year to exactly match the present value of government surpluses to the market value of debt. We denote the probability of the spending cut that closes the valuation gap by ϕ_t. It satisfies:

$$D_t = T_t \cdot PD_t^{\tau} - G_t \cdot PD_t^{g}(1 - \phi_t \ell), \quad \forall t.$$

Figure 11 reports the resulting time series of ϕ_t. To match the average 168% gap-to-GDP ratio, the probability of the spending cut has to be 32.02% on average. Such a large probability is at odds with the notion of a peso event that never happens in our 74-year sample. We interpret this result as a restatement rather than a resolution of the puzzle.

Figure 11: Probabilities of Spending Cut Implied by Debt-to-GDP Ratio



This figure reports the time series of probabilities of spending cuts implied by the debt-to-GDP ratio, ϕ_t.

Our approach assumes there is no correlation between SDF innovations and the occurrence of a fiscal correction. If instead the fiscal correction took place in high marginal utility states, the implied probability of these fiscal corrections would likely be much smaller.[16] Nakamura et al. (2013) find that real government bond returns suffer in disaster states. They are -3% per annum on average and, in 25% of the disaster cases, the realized real bond returns are as low as those on stocks. This suggests that disaster states are not typically accompanied by large fiscal adjustments

[16] As an example, Elenev, Landvoigt, Shultz, and Van Nieuwerburgh (2021) allow for the possibility of a regime switch in fiscal policy once the debt-to-GDP crosses a boundary. This future regime change is shown to make the tax claim less risky at medium horizons.

that represent good news for government bondholders.

6.5 Bubbles and Limits to Arbitrage

The valuation gap can be interpreted as violation of the transversality condition (TVC) in Treasury markets, consistent with the presence of a rational bubble. The TVC is violated if the value of debt in the far future does not converge to zero:

$$\lim_{T\to\infty} \mathbb{E}_t \left[M_{t,t+T} \frac{D_{t+T}}{Y_{t+T}} Y_{t+T} \right] \neq 0.$$

There is a large literature on rational bubbles in asset markets, starting with the seminal work by Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). However, this interpretation has to clear a few hurdles. First, the key piece of evidence is the plot of GDP risk premia in Figure 10. In the long run, the GDP strip earn a risk premium of more than 2% above the real risk-free rate. Even if the debt is risk-free, as long as the debt is co-integrated with GDP, then we need to discount the claim to future at the risk-free rate plus at least 2%. This follows immediately from Proposition 3. The TVC is unlikely to be violated because the risk-adjusted discount rate on the portfolio of Treasury debt is higher than the growth rate of GDP. To develop intuition for this result, note that when the debt/output ratio is constant ($D_t/GDP_t = b$), the value of the debt in the far future is a multiple of the price of a T-period output strip:

$$\lim_{T\to\infty} \mathbb{E}_t \left[M_{t,t+T} \frac{D_{t+T}}{Y_{t+T}} Y_{t+T} \right] = b \lim_{T\to\infty} \mathbb{E}_t \left[M_{t,t+T} Y_{t+T} \right]$$

While Bohn (1995) recognized this conceptually, we show that the risk premium on debt is actually large enough to make the TVC hold; the economy is dynamically efficient. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2020); Barro (2020) are recent examples showing that the TVC for government debt holds in a plausibly calibrated dynamic asset pricing model with priced aggregate risk.

Second, Brock (1982); Tirole (1982); Milgrom and Stokey (1982); Santos and Woodford (1997) argue that rational bubbles are hard to sustain in the presence of long-lived investors absent other frictions. Third, as Figure 9 shows, the valuation gap is growing faster than GDP, which is inconsistent with rational bubbles. In rational bubble models, the debt/GDP ratio declines over time. Finally, and most importantly, these models give rise to rational bubbles in all long-lived assets, including stocks. Our paper shows that the pricing of Treasurys is off *relative* to the valuation of other long-lived assets such as stocks.

7 Conclusion

Fiscally speaking, the U.S. is an outlier. The U.S. federal government has been running primary deficits for 90 years yet issues positive amounts of debt. In a standard asset pricing framework, this constellation can only be sustained if the U.S. government provides insurance against aggregate risk to its bondholders, regardless of the risk-free rate and the growth rate of the economy. However, the U.S. government tends to run larger deficits in recessions, times when bond investors face high marginal utility. The U.S. government must tap debt markets at inopportune times. Using a state-of-the-art dynamic asset pricing model, we quantify that the riskiness of surpluses dramatically lowers the U.S. government's fiscal capacity. We conclude that the pricing of U.S. Treasury debt seems to violate the no-arbitrage restrictions implied by the government budget constraint, a violation we call the government debt valuation puzzle. Convenience yields cannot explain the puzzle. Perhaps investors expect an unprecedented fiscal correction. If so, we show that they have been expecting a correction for a long time in violation of rational expectations.

We cannot definitively rule out that our estimates of fiscal capacity are off because the spending claim is really riskier than the tax claim. At intermediate frequencies the covariance of spending and tax innovations with the pricing kernel may change in ways that our approach fails to capture. Our paper has already led others to explore related explanations. In one example, Elenev et al. (2021) conjecture that investor anticipate a future fiscal regime change from counter-cyclical to pro-cyclical fiscal policies, not observed in the current sample.[17] Interestingly, the possibility of a future regime change reduces the riskiness of the surplus claim at intermediate frequencies. More work is needed to explore these and other potential explanations of our puzzle.

Our current evidence suggests that U.S. Treasurys may be mispriced when measured against a large class of standard asset pricing models. It is well known in finance that assets can be subject to persistent mispricing when there are limits to arbitrage (Shleifer and Vishny, 1997). More work is needed to investigate the causes and implications of potential mispricing of Treasurys.

[17]In this view, the U.S. government can credibly commit to making taxes less risky and spending more risky in the future by promising to deliver tax increases and transfer cuts in response to future adverse shocks.

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Online Appendix for The U.S. Public Debt Valuation Puzzle

A Proofs of Propositions

Proposition 1

Proof. All objects in this appendix are in nominal terms but we drop the superscript $^{\$}$ for ease of notation. The government faces the following one-period budget constraint:

$$G_t - T_t + Q^{\$}_{t-1}(1) = \sum_{h=1}^{H}(Q^{\$}_t(h) - Q^{\$}_{t-1}(h+1))P^{\$}_t(h),$$

where G_t is total nominal government spending, T_t is total nominal government revenue, $Q^{\$}_t(h)$ is the number of nominal zero-coupon bonds of maturity h outstanding in period t each promising to pay back \$1 at time $t+h$, and $P^{\$}_t(h)$ is today's price for a h-period zero-coupon bond with \$1 face value. A unit of $h+1$-period bonds issued at $t-1$ becomes a unit of h-period bonds in period t. That is, the stock of bonds evolves of each maturity evolves according to $Q^{\$}_t(h) = Q^{\$}_{t-1}(h+1) + \Delta Q^{\$}_t(h)$. Note that this notation can easily handle coupon-bearing bonds. For any bond with deterministic cash-flow sequence, we can write the price (present value) of the bond as the sum of the present values of each of its coupons.

The left-hand side of the budget constraint denotes new financing needs in the current period, due to primary deficit $G - T$ and one-period debt from last period that is now maturing. The right hand side shows that the money is raised by issuing new bonds of various maturities. Alternatively, we can write the budget constraint as total expenses equalling total income:

$$G_t + Q^{\$}_{t-1}(1) + \sum_{h=1}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h),$$

We can now iterate the budget constraint forward. The period t constraint is given by:

$$\begin{aligned} T_t - G_t \quad = \quad & Q^{\$}_{t-1}(1) - Q^{\$}_t(1)P^{\$}_t(1) + Q^{\$}_{t-1}(2)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) + Q^{\$}_{t-1}(3)P^{\$}_t(2) - Q^{\$}_t(3)P^{\$}_t(3) \\ & + \cdots - Q^{\$}_t(H)P^{\$}_t(H) + Q^{\$}_{t-1}(H+1)P^{\$}_t(H). \end{aligned}$$

Consider the period-$t+1$ constraint,

$$\begin{aligned} T_{t+1} - G_{t+1} \quad = \quad & Q^{\$}_t(1) - Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) + Q^{\$}_t(2)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) + Q^{\$}_t(3)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) \\ & + \cdots - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H) + Q^{\$}_t(H+1)P^{\$}_{t+1}(H). \end{aligned}$$

multiply both sides by $M^{\$}_{t+1}$, and take expectations conditional on time t:

$$\begin{aligned} \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)P^{\$}_t(1) - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + Q^{\$}_t(2)P^{\$}_t(2) - \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + Q^{\$}_t(3)P^{\$}_t(3) \\ & - \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)P^{\$}_t(H) \\ & - \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)P^{\$}_t(H+1), \end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t[M^{\$}_{t+1}] = P^{\$}_t(1)$, $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(1)] = P^{\$}_t(2)$, $\cdots$, $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(H-1)] = P^{\$}_t(H)$, and $\mathbb{E}_t[M^{\$}_{t+1}P^{\$}_{t+1}(H)] = P^{\$}_t(H+1)$.

Consider the period $t+2$ constraint, multiplied by $M^{\$}_{t+1}M^{\$}_{t+2}$ and take time-t expectations:

$$\begin{aligned} \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2} - G_{t+2})\right] \quad = \quad & \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] + \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\ & - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] + \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots \\ & + \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)] + \mathbb{E}_t[Q^{\$}_{t+1}(H+1)M^{\$}_{t+1}P^{\$}_{t+1}(H+1)], \end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^{\$}_{t+2}] = P^{\$}_{t+1}(1)$, $\mathbb{E}_{t+1}[M^{\$}_{t+2}P^{\$}_{t+2}(1)] = P^{\$}_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{aligned} & T_t - G_t + \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1} - G_{t+1})\right] + \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2} - G_{t+2})\right] = \textstyle\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) + \\ & -\mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)]. \end{aligned}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{J} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right],$$

where we used the cumulate SDF notation $M^{\$}_{t,t+j} = \prod_{i=0}^{j} M^{\$}_{t+i}$ and by convention $M^{\$}_{t,t} = M^{\$}_t = 1$ and $P^{\$}_t(0) = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \lim_{J\to\infty}\mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j}-G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty}\mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right] = 0.$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right].$$

This is equation (1) in the main text.

Case with Default

Proof. We consider only full default, without loss of generality. Alternatively, we can write the budget constraint that obtains in case of no default at t:

$$G_t + Q^{\$}_{t-1}(1) + \sum_{h=1}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h),$$

and, in case of default at t, the one-period budget constraint is given by:

$$G_t = T_t + \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h).$$

We can now iterate the budget constraint forward. In case of no default, the period t constraint is given by:

$$\begin{aligned} T_t - G_t &= Q^{\$}_{t-1}(1) - Q^{\$}_t(1)P^{\$}_t(1) + Q^{\$}_{t-1}(2)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) + Q^{\$}_{t-1}(3)P^{\$}_t(2) - Q^{\$}_t(3)P^{\$}_t(3) \\ &\quad + \cdots - Q^{\$}_t(H)P^{\$}_t(H) + Q^{\$}_{t-1}(H+1)P^{\$}_t(H). \end{aligned}$$

In case of default, the period t constraint is given by:

$$T_t - G_t = -Q^{\$}_t(1)P^{\$}_t(1) - Q^{\$}_t(2)P^{\$}_t(2) - Q^{\$}_t(3)P^{\$}_t(3) - Q^{\$}_t(H)P^{\$}_t(H).$$

First, consider the period-$t+1$ constraint in case of no default,

$$\begin{aligned} T_{t+1} - G_{t+1} &= Q^{\$}_t(1) - Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) + Q^{\$}_t(2)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) + Q^{\$}_t(3)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) \\ &\quad + \cdots - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H) + Q^{\$}_t(H+1)P^{\$}_{t+1}(H). \end{aligned}$$

Second, consider the period-$t+1$ constraint in case of default,

$$T_{t+1} - G_{t+1} \quad = \quad -Q^{\$}_{t+1}(1)P^{\$}_{t+1}(1) - Q^{\$}_{t+1}(2)P^{\$}_{t+1}(2) - Q^{\$}_{t+1}(3)P^{\$}_{t+1}(3) - Q^{\$}_{t+1}(H)P^{\$}_{t+1}(H).$$

We use χ_t as an indicator variable for default. To simplify, we consider only full default with zero recovery. This is without loss of generality. Next, multiply both sides of the no default constraint by $(1-\chi_{t+1})M^{\$}_{t+1}$, and take expectations conditional on time t:

$$\begin{aligned}\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] - \mathbb{E}_t[Q^{\$}_{t+1}(1)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)] + \mathbb{E}_t[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)]Q^{\$}_t(2) \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(2)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(2)] + \mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)]Q^{\$}_t(3) \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(3)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)\mathbb{E}_t[M^{\$}_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)] \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(H)(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)],\end{aligned}$$

and multiply both sides of the default constraint by $M^{\$}_{t+1}\chi_{t+1}$

$$\begin{aligned}\mathbb{E}_t\left[M^{\$}_{t+1}\chi_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & -\mathbb{E}_t[Q^{\$}_{t+1}(1)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+1}(2)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(3)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+1}(H)\chi_{t+1}M^{\$}_{t+1}P^{\$}_{t+1}(H)].\end{aligned}$$

By adding these 2 constraints, we obtain the following expression:

$$\begin{aligned}\mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + \mathbb{E}_t[(1-\chi_{t+1})M^{\$}_{t+1}P^{\$}_{t+1}(1)]Q^{\$}_t(2) \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + \mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(2)]Q^{\$}_t(3) \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)\mathbb{E}_t[M_{t+1}(1-\chi)P^{\$}_{t+1}(H-1)] \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(H)M_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)].\end{aligned}$$

This can be restated as:

$$\begin{aligned}\mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] \quad = \quad & Q^{\$}_t(1)P^{\$}_t(1) - \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] + Q^{\$}_t(2)P^{\$}_t(2) - \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] + Q^{\$}_t(3)P^{\$}_t(3) \\ & -\mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] + \cdots + Q^{\$}_t(H)P^{\$}_t(H) - \mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] + Q^{\$}_t(H+1)P^{\$}_t(H+1),\end{aligned}$$

where we use the asset pricing equations $\mathbb{E}_t\left[M^{\$}_{t+1}(1-\chi_{t+1})\right] = P^{\$}_t(1)$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(1)] = P^{\$}_t(2)$, $\cdots$, $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H-1)] = P^{\$}_t(H)$, and $\mathbb{E}_t[M^{\$}_{t+1}(1-\chi_{t+1})P^{\$}_{t+1}(H)] = P^{\$}_t(H+1)$.

The rest of the proof is essentially unchanged. Consider the period $t+2$ constraint, multiplied by $M^{\$}_{t+1}M^{\$}_{t+2}(1-\chi_{t+2})$ in the no-default case, and $M^{\$}_{t+1}M^{\$}_{t+2}(\chi_{t+2})$ for the default case, and take time-t expectations (after adding default and no-default states):

$$\begin{aligned}\mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2}-G_{t+2})\right] \quad = \quad & \mathbb{E}_t[Q^{\$}_{t+1}(1)M^{\$}_{t+1}P^{\$}_{t+1}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] + \mathbb{E}_t[Q^{\$}_{t+1}(2)M^{\$}_{t+1}P^{\$}_{t+1}(2)] \\ & -\mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] + \mathbb{E}_t[Q^{\$}_{t+1}(3)M^{\$}_{t+1}P^{\$}_{t+1}(3)] - \cdots \\ & +\mathbb{E}_t[Q^{\$}_{t+1}(H)M^{\$}_{t+1}P^{\$}_{t+1}(H)] - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)] + \mathbb{E}_t[Q^{\$}_{t+1}(H+1)M^{\$}_{t+1}P^{\$}_{t+1}(H+1)],\end{aligned}$$

where we used the law of iterated expectations and $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})] = P^{\$}_{t+1}(1)$, $\mathbb{E}_{t+1}[M^{\$}_{t+2}(1-\chi_{t+2})P^{\$}_{t+2}(1)] = P^{\$}_{t+1}(2)$, etc.

Note how identical terms with opposite signs appear on the right-hand side of the last two equations. Adding up the expected discounted surpluses at t, $t+1$, and $t+2$ we get:

$$\begin{gathered}T_t - G_t + \mathbb{E}_t\left[M^{\$}_{t+1}(T_{t+1}-G_{t+1})\right] + \mathbb{E}_t\left[M^{\$}_{t+1}M^{\$}_{t+2}(T_{t+2}-G_{t+2})\right] = \textstyle\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) + \\ -\mathbb{E}_t[Q^{\$}_{t+2}(1)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(1)] - \mathbb{E}_t[Q^{\$}_{t+2}(2)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(2)] - \cdots - \mathbb{E}_t[Q^{\$}_{t+2}(H)M^{\$}_{t+1}M^{\$}_{t+2}P^{\$}_{t+2}(H)].\end{gathered}$$

Similarly consider the one-period government budget constraints at times $t+3$, $t+4$, etc. Then add up all one-period budget constraints. Again, the identical terms appear with opposite signs in adjacent budget constraints. These terms cancel out upon adding up the budget constraints. Adding up all the one-period budget constraints until horizon $t+J$, we get:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{J} M^{\$}_{t,t+j}(T_{t+j}-G_{t+j})\right] + \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right],$$

where we used the cumulate SDF notation $M^{\$}_{t,t+j} = \prod_{i=0}^{j} M^{\$}_{t+i}$ and by convention $M^{\$}_{t,t} = M^{\$}_t = 1$ and $P^{\$}_t(0) = 1$. The market value of the outstanding government bond portfolio equals the expected present discount value of the surpluses over the next J years plus the present value of the government bond portfolio that will be outstanding at time $t+J$. The latter is the cost the government will face at time $t+J$ to finance its debt, seen from today's vantage point.

We can now take the limit as $J \to \infty$:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] + \lim_{J\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right].$$

We obtain that the market value of the outstanding debt inherited from the previous period equals the expected present-discounted value of the primary surplus stream $\{T_{t+j} - G_{t+j}\}$ plus the discounted market value of the debt outstanding in the infinite future.

Consider the transversality condition:

$$\lim_{J\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+J}\sum_{h=1}^{H} Q^{\$}_{t+J}(h)P^{\$}_{t+J}(h)\right] = 0,$$

which says that while the market value of the outstanding debt may be growing as time goes on, it cannot be growing faster than the stochastic discount factor. Otherwise there is a government debt bubble.

If the transversality condition is satisfied, the outstanding debt today, D_t, reflects the expected present-discounted value of the current and all future primary surpluses:

$$D_t = \sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) \quad = \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right].$$

This is equation (1) in the main text.

□

Proposition 2 From the time-t budget constraint, we get that the primary surplus

$$-S_t = -Q^{\$}_{t-1}(1) + \sum_{h=1}^{H}(Q^{\$}_t(h) - Q^{\$}_{t-1}(h+1))P^{\$}_t(h).$$

It follows that

$$D_t - S_t = \sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - Q^{\$}_{t-1}(1) + \sum_{h=1}^{H}(Q^{\$}_t(h) - Q^{\$}_{t-1}(h+1))P^{\$}_t(h) = \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h).$$

We obtain equation (3) in the main text.

$$\begin{aligned} r^d_{t+1}(D_t - S_t) \quad &= \quad \sum_{h=0}^{\infty} P^{\$}_{t+1}(h)Q^{\$}_t(h+1) = D_{t+1} = P^{\tau}_{t+1} - P^{g}_{t+1} \\ &= \quad (P^{\tau}_t - T_t)r^{\tau}_{t+1} - (P^{g}_{t+1} - G_t)r^{g}_{t+1}. \end{aligned}$$

□

Proposition 3

Proof. We follow the proof in the working paper version of Backus et al. (2018) on page 16 (Example 5). Hansen and Scheinkman (2009) consider the following equation:

$$\mathbb{E}_t[M_{t,t+1}v_{t+1}] = \nu v_t, \tag{A.1}$$

where ν is the dominant eigenvalue and v_t is the eigenfunction. Claims to stationary cash flows earn a return equal to the yield on the long bond. We consider the following decomposition of the pricing kernel:

$$M^1_{t,t+1} \quad = \quad M_{t,t+1}v_{t+1}/\nu v_t, \tag{A.2}$$
$$M^2_{t,t+1} \quad = \quad \nu v_t/v_{t+1}. \tag{A.3}$$

By construction, $\mathbb{E}_t[M^1_{t,t+1}] = 1$. The long yields converge to $-\log\nu$. The long-run bond return converges to $\lim_{n\to\infty} R^n_{t,t+1} = \frac{1}{M^2_{t,t+1}} = v_{t+1}/\nu v_t$. This implies that $\mathbb{E}[\log R^{\infty}_{t,t+1}] = -\log\nu$.

To value claims to uncertain cash flows with one-period growth rate $g_{t,t+1}$, we define $\widehat{p}^n_t$ to denote the price of a strip that pays off $d_{t,t+n}$, n periods from now.

$$\widehat{p}^n_t = \mathbb{E}_t[M_{t,t+1}g_{t,t+1}\widehat{p}^{n-1}_{t+1}] = \mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{p}^{n-1}_{t+1}],$$

where $\widehat{M}_{t,t+1} = M_{t,t+1} g_{t,t+1}$. Consider the problem of finding the dominant eigenvalue:

$$\mathbb{E}_t[\widehat{M}_{t,t+1}\widehat{v}_{t+1}] = \nu \widehat{v}_t. \tag{A.4}$$

If the cash flows are stationary, then the same ν that solves this equation for $M_{t,t+1}$ in eqn. A.1 solves the one for $\widehat{M}_{t,t+1}$. Hence, if (ν, v_t) solves eqn. A.1, then $(\nu, v_t/d_t)$ solves the hat equation eqn. A.4. □

Proposition 4

Proof. Start from the government budget constraint

$$T_t - G_t = Q^{\$}_{t-1}(1) + \sum_{h=1}^{H-1} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h).$$

We present the proof for the general case with default. Consider the period-$(t+1)$ constraint, multiplied by $M^{\$}_{t+1}(1-\chi_{t+1})$ and by $M^{\$}_{t+1}(\chi_{t+1})$ respectively, and take expectations conditional at time t:

$$\begin{aligned}
& \mathbb{E}_t\left[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})(1-\chi_{t+1})\right] \\
= \; & \mathbb{E}_t\left[M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_t(1) + \sum_{h=1}^{H-1} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_t(h+1)P^{\$}_{t+1}(h) - \sum_{h=1}^{H} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\right] \\
= \; & Q^{\$}_t(1)P^{\$}_t(1)e^{-\lambda^1_t} + \sum_{h=1}^{H-1} Q^{\$}_t(h+1)P^{\$}_t(h+1)e^{-\lambda^{h+1}_t} - \mathbb{E}_t\left[\sum_{h=1}^{H} M^{\$}_{t,t+1}(1-\chi_{t+1})Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\right].
\end{aligned}$$

and

$$\mathbb{E}_t\left[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\chi_{t+1}\right] = \mathbb{E}_t\left[-\sum_{h=1}^{H} \chi_{t+1}M^{\$}_{t,t+1}Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\right].$$

So

$$\mathbb{E}_t\left[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\right] = Q^{\$}_t(1)P^1_t e^{-\lambda_t(1)} + \sum_{h=1}^{H-1} Q^{\$}_t(h+1)P^{\$}_t(h+1)e^{-\lambda_t(h+1)} - \mathbb{E}_t\left[\sum_{h=1}^{H} M^{\$}_{t,t+1}Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\right].$$

Combine with the period-t constraint, the sum is

$$\begin{aligned}
& (T_t - G_t) + \mathbb{E}_t\left[M^{\$}_{t,t+1}(T_{t+1} - G_{t+1})\right] \\
= \; & Q^{\$}_{t-1}(1) + \sum_{h=1}^{H-1} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) - \sum_{h=1}^{H} Q^{\$}_t(h)P^{\$}_t(h)(1-e^{-\lambda^h_t}) - \mathbb{E}_t\left[\sum_{h=1}^{H} M^{\$}_{t,t+1}Q^{\$}_{t+1}(h)P^{\$}_{t+1}(h)\right].
\end{aligned}$$

We can iterate this expression to the infinite horizon. If the following transversality condition holds,

$$\lim_{\tau\to\infty} \mathbb{E}_t\left[M^{\$}_{t,t+\tau}\sum_{h=1}^{H} Q^{\$}_{t+\tau}(h)P^{\$}_{t+\tau}(h)\right] = 0,$$

then debt value is the present value of current and future surpluses and seignorage revenues from issuing bonds that earn convenience yields:

$$\sum_{h=0}^{H} Q^{\$}_{t-1}(h+1)P^{\$}_t(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}(T_{t+j} - G_{t+j})\right] + \mathbb{E}_t\left[\sum_{j=0}^{\infty} M^{\$}_{t,t+j}\sum_{h=1}^{H} Q^{\$}_{t+j}(h)P^{\$}_{t+j}(h)(1-e^{-\lambda_{t+j}(h)})\right].$$

□

B VAR Estimation

B.1 Cointegration Tests

We perform a Johansen cointegration test by first estimating the vector error correction model :

$$\Delta w_t \quad = \quad A(B' w_{t-1} + c) + D\Delta w_{t-1} + \varepsilon_t, \quad \text{where } w_t = \begin{pmatrix} \log T_t \\ \log G_t \\ \log GDP_t \end{pmatrix}.$$

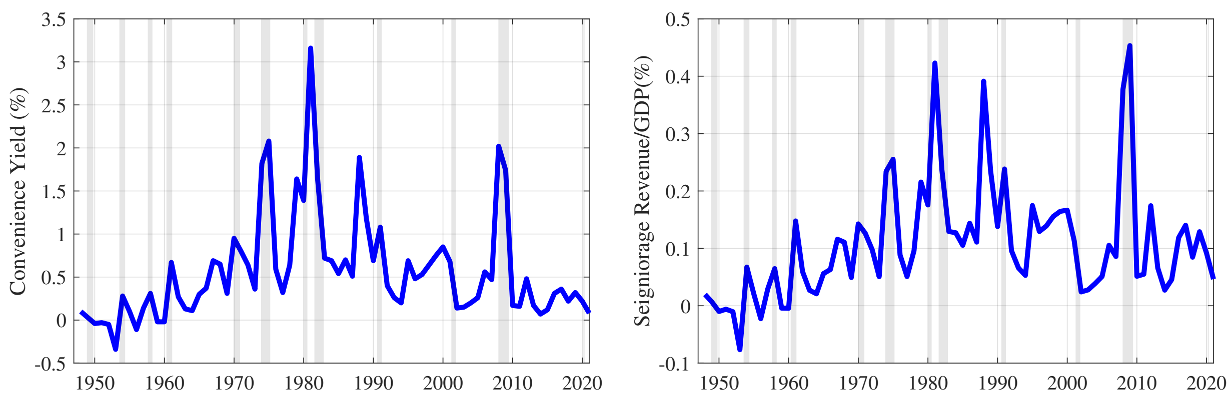
Both the trace test and the max eigenvalue test do not reject the null of cointegration rank 2 (with p-values of 0.16) or rank 1 (with p-values of 0.11), but reject the null of cointegration rank 0 (with p-values lower than 0.01). These results are in favor of cointegration relationships between variables in w_t.

We also conduct the Phillips-Ouliaris cointegration test on the $\{w_t\}$ matrix with a truncation lag parameter of 2, and reject the null hypothesis that w is not cointegrated with a p-value of 0.018.

B.2 Convenience Yield and Seigniorage Revenue

Figure B.1 plots the time series of the convenience yield cy_t in he left-hand side panel and the time series of the seigniorage revenue-to-GDP k_t in the right-hand side panel. Both objects are defined in the main text.

Figure B.1: Time Series of Convenience Yield and Seigniorage Revenue



(a) Convenience Yield (b) Seigniorage Revenue/GDP

The figure plots the convenience yield cy_t and the seigniorage revenue/GDP ratio. The sample is annual, 1947—2020.

C Derivation of the Upper Bound

This section derives the Campbell and Shiller (1988) decomposition for the spending and the tax claim, which is used in the derivation of the upper bound.

C.1 Campbell-Shiller Decomposition of Tax and Spending Claim

Consider the return on a claim to the government's tax revenue:

$$R_{t+1} = \frac{P_{t+1} + T_{t+1}}{P_t} = \frac{\frac{T_{t+1}}{T_t}(1 + pd^{\tau}_{t+1})}{pd^{\tau}_t}.$$

We use pd_t to denote the log price-dividend ratio on the tax revenue claim: $pd^{\tau}_t = p^T_t - \log T_t = \log\left(\frac{P_t}{T_t}\right)$, where price is measured at the end of the period and the dividend flow is over the same period. Also, note that $dp^T_t = -pd^{\tau}_t$. Campbell and Shiller (1988) log-linearize the return equation around the mean log price/dividend ratio to derive the following expression for log returns on the tax claim:

$$r^T_{t+1} = \Delta \log T_{t+1} + \kappa^{\tau}_1 pd^{\tau}_{t+1} + \kappa^{\tau}_0 - pd^{\tau}_t,$$

with a linearization coefficient ρ that depends on the mean of the log price/dividend ratio pd_0^τ:

$$\kappa_1^\tau = \frac{e^{pd_0^\tau}}{e^{pd_0^\tau}+1} < 1, \kappa_0^\tau = \log(1+\exp(pd_0^\tau)) - \kappa_1^\tau pd_0^\tau.$$

By iterating forward on the linearized return equation and imposing a no-bubble condition: $\lim_{j\to\infty}(\kappa_1^\tau)^j pd_{t+j} = 0$, and by taking expectations, we derive the following expression for the log price/dividend ratio of the tax claim:

$$pd_t^\tau = \frac{\kappa^\tau}{1-\kappa_1^\tau} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^\tau)^{j-1}\Delta\log T_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^\tau)^{j-1}r_{t+j}\right].$$

Similarly, the log price/dividend ratio of the spending claim:

$$pd_t^g = \frac{\kappa^G}{1-\kappa_1^g} + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^g)^{j-1}\Delta\log G_{t+j}\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^g)^{j-1}r_{t+j}\right].$$

We use rp_t^j to denote the risk premium on a claim to risky cash flows relative to the long bond:

$$\mathbb{E}_t[r_{t+1}^j] = yspr_t^\$ + y_t^\$(1) + rp_t^j.$$

We assume a constant risk premium on the tax and spending claims, rp_0^τ and rp_0^g. We use $\boldsymbol{e}_\pi$ to denote a column vector of zero with a 1 as the first element. Because the state vector follows VAR(1) dynamics, we can compute expected returns as follows:

$$\mathbb{E}_t\left[r_{t+1}^j\right] = y_0^\$(1) + yspr_0^\$ + rp_0^y + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}^{j+1}\boldsymbol{z}_t, \quad j \in \{\tau, g\}.$$

The DR (discount rate) term is given by the following expression:

$$DR_t^j \quad = \quad \frac{y_0^\$(1) + yspr_0^\$ + rp_0^j}{1-\kappa_1^j} + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^j\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t.$$

The CF (cash flow) term is given by the following expression:

$$CF_t^j = \frac{x_0 + \pi_0}{1-\kappa_1^j} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_j)'\boldsymbol{\Psi}(I - \kappa_1^j\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t,$$

We end up with the following expressions for the price/dividend ratio on the tax and spending claims:

$$pd_t^j \quad = \quad pd_0^j + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_j - \boldsymbol{e}_{y1} - \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^j\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t,$$

where e^j selects the tax-to-GDP growth rate or spending-to-GDP growth rate in the VAR, and $(pd_0^j, \kappa_0^j, \kappa_1^j)$ solve:

$$\begin{aligned} pd_0^j \quad &= \quad \frac{x_0 + \pi_0 - y_0^\$(1) - yspr_0^\$ - rp_0^j}{(1-\kappa_1^j)} + \frac{\kappa_0^j}{(1-\kappa_1^j)}, \\ \kappa_1^j \quad &= \quad \frac{e^{pd_0^j}}{e^{pd_0^j}+1}, \kappa_0^j = \log(1+\exp(pd_0^j)) - \kappa_1^j pd_0^j. \end{aligned}$$

We use $\widetilde{CF}_t^i$ and $\widetilde{DR}_t^i$ to denote the mean-zero time-varying components. The implied debt/GDP ratio is given by:

$$\frac{PV_t^s}{Y_t} = \tau_t \exp(pd_0^\tau + \widetilde{CF}_t^\tau - \widetilde{DR}_t^\tau) - g_t \exp(pd_0^g + \widetilde{CF}_t^g - \widetilde{DR}_t^g).$$

To derive some intuition, we can evaluate the expression at $\boldsymbol{z} = 0$, i.e., when all variables are at their unconditional mean. When the state variables are at their unconditional mean ($\boldsymbol{z} = 0$), the PDV of surpluses/output ratio is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)(\boldsymbol{z} = 0) = \tau_0 \exp(pd_0^\tau) - g_0 \exp(pd_0^g).$$

Upper Bound on Debt Valuation To derive an upper bound, we equate the expected returns on taxes and spending to the expected return on GDP: $rp_0^y = rp_0^g = rp_0^\tau$. This delivers an upper bound on the valuation of future surpluses, because it maximizes the value of the tax claim, and minimizes the value of the spending claim. Given these 2 assumptions, we derive the following expression for the implied log price/dividend ratio on the tax claim and the spending claim:

$$\begin{aligned} pd_t^\tau &= pd_0^y + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}(\Delta \log T_{t+j} - (x_0+\pi_0))\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}(r_{t+j}^Y - (yspr_0^\$ + y_0^\$(1) + erp_0^y)\right], \\ pd_t^g &= pd_0^y + \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}(\Delta \log G_{t+j} - (x_0+\pi_0))\right] - \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}(r_{t+j}^Y - (yspr_0^\$ + y_0^\$(1) + erp_0^y))\right]. \end{aligned}$$

The long-run growth rate of tax and spending equals the long-run growth rate of output: $x_0 + \pi_0$. That follows directly from co-integration. We use a constant GDP risk premium erp_0^y. We can back this number out of the unconditional equity risk premium by unlevering the equity premium. We use $\boldsymbol{e}_\pi$ to denote a column vector of zero with a 1 as the first element. The DR (discount rate) term is defined by:

$$DR_t^T = DR_t^G = DR_t^Y = \frac{y_0^\$(1) + yspr_0^\$ + rp_0^y}{1-\kappa_1^y} + (\boldsymbol{e}_{y1} + \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t.$$

The CF (cash flow) term for the tax claim is defined by:

$$CF_t^T = \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}\Delta \log T_{t+j}\right] = \frac{x_0+\pi_0}{1-\kappa_1^y} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_8)'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t.$$

The CF (cash flow) term for the spending claim is defined by:

$$CF_t^G = \mathbb{E}_t\left[\sum_{j=1}^{\infty}(\kappa_1^y)^{j-1}\Delta \log G_{t+j}\right] = \frac{x_0+\pi_0}{1-\kappa_1^y} + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{10})'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t.$$

We use $\widetilde{CF}_t^i$ and $\widetilde{DR}_t^i$ to denote the time-varying components. Hence, we end up with the following expressions for the price/dividend ratio on the tax and spending claims:

$$\begin{aligned} pd_t^\tau &= pd_0^y + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_8 - \boldsymbol{e}_{y1} - \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t, \\ pd_t^g &= pd_0^y + (\boldsymbol{e}_\pi + \boldsymbol{e}_x + \boldsymbol{e}_{10} - \boldsymbol{e}_{y1} - \boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t. \end{aligned}$$

A first-order Taylor expansion yields the following expression:

$$\left(\frac{PV_t^s}{Y_t}\right)^u \approx (\tau_t - g_t)\exp(pd_0^y) + \tau_t(\widetilde{CF}_t^\tau - \widetilde{DR}_t^\tau)\exp(pd_0^y) - g_t\exp(pd_0^y)(\widetilde{CF}_t^g - \widetilde{DR}_t^g).$$

This expression can be simplified. We obtain the following intuitive expression for an upper bound on the PDV of surpluses:

$$\left(\frac{PV_t^s}{Y_t}\right)^u \approx \exp(pd_0^y)\left((\tau_t - g_t)\left(1 - \widetilde{DR}_t^Y\right) + \tau_t\widetilde{CF}_t^\tau - g_t\widetilde{CF}_t^g\right).$$

Suppose the country currently runs a primary surplus of zero. The the discount rate effects cancel out, again to a first-order approximation. When the country runs a zero primary surplus, the upper bound on the value of debt/GDP is positive only if the expected tax revenue growth exceeds expected spending growth:

$$\left(\frac{PV_t^s}{Y_t}\right)^u \approx \exp(pd_0^y)\tau_t\left(\widetilde{CF}_t^\tau - \widetilde{CF}_t^g\right) > 0 \textit{ iff } \widetilde{CF}_t^\tau > \widetilde{CF}_t^g.$$

This can be further simplified to yield the following expression:

$$\left(\frac{PV_t^s}{Y_t}\right)^u \approx \exp(pd_0^y)\tau_t\,(\boldsymbol{e}_8 - \boldsymbol{e}_{10})'\,\boldsymbol{\Psi}(I - \kappa_1^y\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t.$$

The discount rate dynamics and the dynamics of GDP growth are irrelevant (to a first-order approximation) for the upper bound. What matters is the dynamics in tax/GDP and spending/GDP. In other words, the expected cumulative effect of mean reversion in taxes has to outweigh the expected cumulative effect of mean-reversion in spending.

Upper Bound with Convenience Yields We can rewrite the intertemporal budget constraint as:

$$\sum_{h=0}^{K} Q_{t-1}^{\$}(h+1)P_t^{\$}(h) = \mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}T_{t+j}+K_{t+j}\right]-\mathbb{E}_t\left[\sum_{j=0}^{\infty} M_{t,t+j}^{\$}G_{t+j}\right].$$

We use the variable K to represent the combined tax and seigniorage revenues as a fraction of the current tax revenue:

$$K_{t+j} = (1-e^{-\lambda_{t+j}})\sum_{h=1}^{H} Q_{t+j}^{\$}(h)P_{t+j}^{\$}(h).$$

Next, we include the seigniorage revenue from the convenience yields in the government revenue: $T_t \times K_t$. The observed nominal Treasury yield $y_t^{\$}(1)$ is given by the following expression:

$$y_t^{\$}(1) = \rho_t^{\$}(1) - \lambda_t,$$

where $\rho_t^{\$}(1)$ is the one-period nominal risk-free rate. We include the one-period nominal risk-free rate in the VAR:

$$\boldsymbol{z}(2,t) = \rho_t^{\$}(1) = y_t^{\$}(1) + \lambda_t.$$

The DR (discount rate) term is defined by:

$$DR_t^j = \frac{y_0^{\$}(1)+\lambda_0+yspr_0^{\$}+erp_0^j}{1-\kappa_1^j} + (\boldsymbol{e}_{y1}+\boldsymbol{e}_{yspr})'\boldsymbol{\Psi}(I-\kappa_1^j\boldsymbol{\Psi})^{-1}\boldsymbol{z}_t,$$

where $j \in \{TK, G\}$. The implied upper bound is given by:

$$\left(\frac{PV_t^s}{Y_t}\right)^u = (\tau_t k_t)\exp(pd_0^{\tau}+\widetilde{CF}_t^{TK}-\widetilde{DR}_t^{TK}) - g_t\exp(pd_0^{g}+\widetilde{CF}_t^{g}-\widetilde{DR}_t^{g}).$$

Alternatively, we can evaluate the expression at $\boldsymbol{z}=0$, i.e., when all variables are at their unconditional mean.
When the state variables are at their unconditional mean ($\boldsymbol{z}=0$), the value of the debt/output ratio is given by:

$$PV_t^s(\boldsymbol{z}=0) = (\tau_0+k_0)\exp(pd_0^{T}) - g_0\exp(pd_0^{g}).$$

where $pd_0^j, \rho^j, \kappa_0^j$ solve

$$pd_0^j = -\frac{(y_0^{\$}(1)+\lambda_0+yspr_0^{\$}+erp_0^j)-(x_0+\pi_0)}{(1-\kappa_1^j)} + \frac{\kappa_0^j}{(1-\kappa_1^j)},$$

$$\kappa_1^j = \frac{e^{pd^j}}{e^{pd^j}+1}, \kappa_0^j = \log(1+\exp(pd_0^j)) - \kappa_1^j pd_0^j.$$

Reverse-Engineering Finally, we could also reverse-engineer the discount rate for the tax claim that would imply the value of the debt/output ratio equal to the debt/output ratio. Suppose we fix the risk premium on the spending claim. Then we can determine erp_t^T such that it solves the following equation:

$$e^{d'}\boldsymbol{z}_t = \tau_t\exp(pd_0^{\tau} - \frac{erp_t^{\tau}-erp_)^{T}}{1-\kappa_1^{\tau}} + CF_t^T - DR_t^T) - g_t\exp(pd_0^{y} - \frac{erp_t^{Y}}{1-\kappa_1^{y}} + CF_t^G - DR_t^G),$$

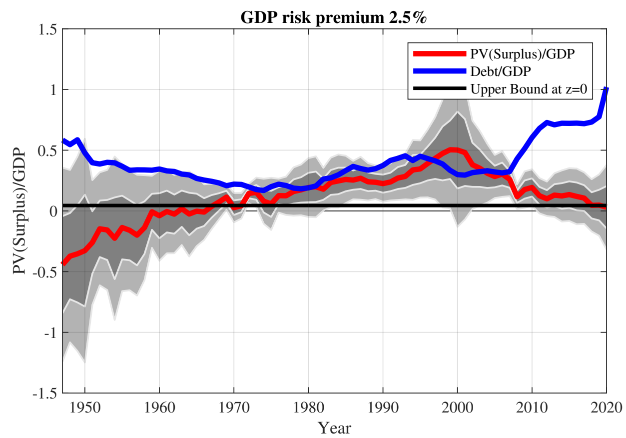
where e^d denotes the element of the VAR that contains the debt/output ratio.

C.2 Additional Upper Bound Results

This section reports additional results for the upper bound calculation.

Risk Premium Figure C.1 plots the upper bound when the GDP risk premium is set to 2.5%. Lowering the risk premium mainly increases the confidence intervals, because occasionally the valuation ratios become much larger. As a result, the upper bound is mostly just outside the 95% confidence interval, until the GFC. Starting with the GFC, the debt/output ratio is comfortably outside of the 95% confidence interval.

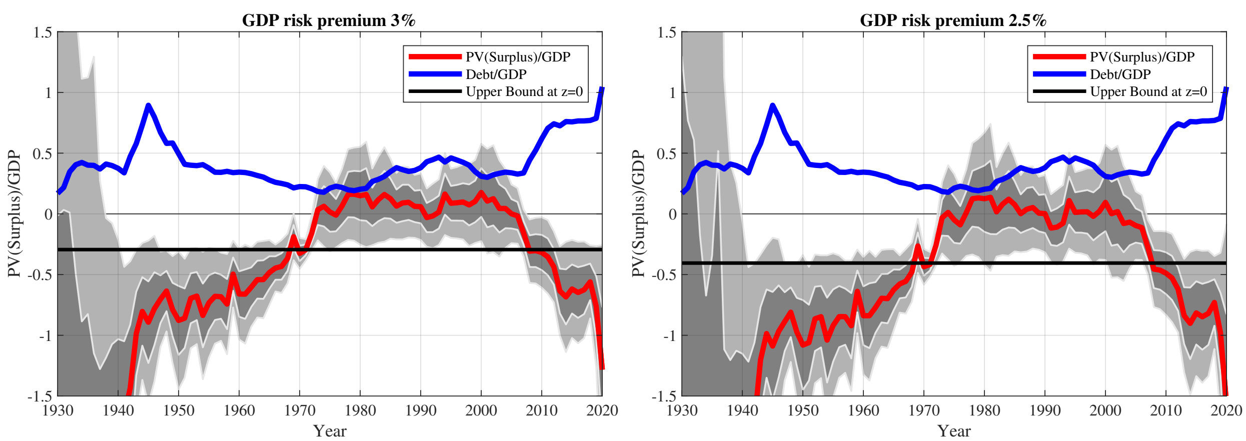
Figure C.1: Upper Bound on the Value of Surpluses/GDP



The figure plots the upper bound on the present value of government surpluses in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15), and the actual debt/output ratio. We report the benchmark case with a GDP risk premium of 2.5%. The sample period is from 1947 to 2020.

Longer Sample Figure C.2 plots the upper bound constructed in the longer sample for the case of $rp_0^y = 3\%$ and the case of $rp_0^y = 2.5\%$. Lowering the risk premium does not have a first-order effect on the upper bound itself, but it widens the size of the confidence interval.

Figure C.2: Upper Bound on the Value of Surpluses/GDP: Longer Sample

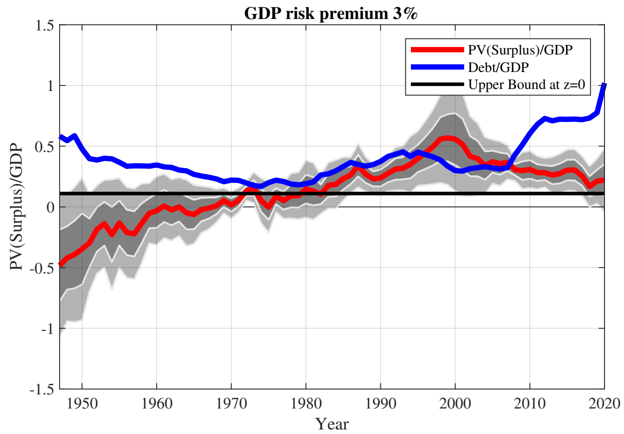


The figure plots the Upper Bound on PDV of Surpluses/GDP in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15) and the actual debt/output ratio. The GDP risk premium is set to 3% or 2.5%. The sample period is from 1947 to 2020.

Convenience Yields Figure C.3 plots upper bound for the case of convenience yields with $rp_0^y = 3\%$. Over the sample period from 1947 to 2020, the average convenience yield λ_0 is 0.60% per year.

Including the Debt/Output Ratio in the State Vector Table C.1 lists all the state variables in the VAR for the case with the debt/output ratio included in the state vector. Figure C.4 plots upper bound for the case with the debt/output ratio in the VAR. In the left (right) panel, we plot the case with $rp_0^y = 3\%$ ($rp_0^y = 2.5\%$). Over the sample period from 1947 to 2020, the average convenience yield λ_0 is 0.60% per year.

Figure C.3: Upper Bound on the Value of Surpluses/GDP with Convenience Yields

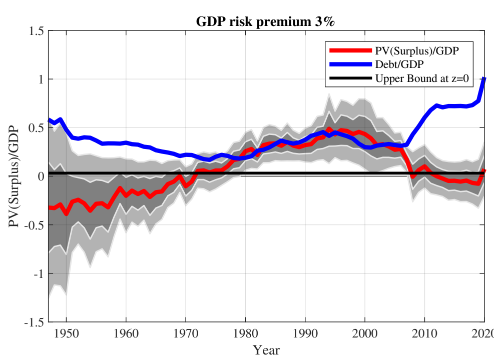


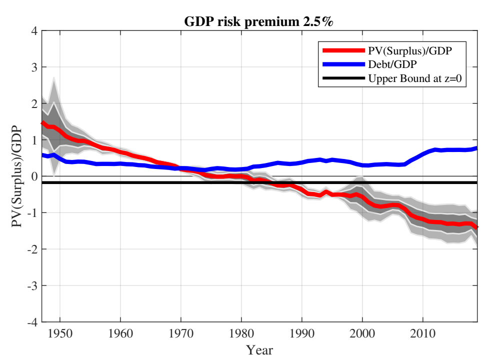
The figure plots the Upper Bound on PDV of Surpluses/GDP in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15) and the actual debt/output ratio. The GDP risk premium is set to 3%. The sample period is from 1947 to 2020.

Table C.1: State Variables

Position	Variable	Mean	Description
1	π_t	π_0	Log Inflation
2	x_t	x_0	Log Real GDP Growth
3	$y_t^{\$}(1)$	$y_0^{\$}(1)$	Log 1-Year Nominal Yield
4	$yspr_t^{\$}$	$\underline{yspr}_0^{\$}$	Log 5-Year Minus Log 1-Year Nominal Yield Spread
5	pd_t	$\overline{pd}$	Log Stock Price-to-Dividend Ratio
6	Δd_t	μ_d	Log Stock Dividend Growth
7	$\Delta \log \tau_t$	μ_τ	Log Tax Revenue-to-GDP Growth
8	$\log \tau_t$	$\log \tau_0$	Log Tax Revenue-to-GDP Level
9	$\Delta \log g_t$	μ_g	Log Spending-to-GDP Growth
10	$\log g_t$	$\log g_0$	Log Spending-to-GDP Level
11	$\Delta \log b_t$	μ_b	Log Debt-to-GDP Growth
12	$\log b_t$	$\log b_0$	Log Debt-to-GDP Level

Figure C.4: Upper Bound on the Value of Surpluses/GDP with Debt in the VAR





The figure plots the upper bound on the present value of government surpluses in eqn. (14), the steady-state upper bound evaluated at $z = 0$ in eqn. (15), and the actual debt/output ratio. We report the benchmark case with a GDP risk premium of 3% and 2.5%. The sample period is from 1947 to 2020.

Proof of Convenience Yield Differentiation

Proof.

$$\frac{\partial \left(\frac{PV_t^s}{Y_t}\right)^u \Big|_{z=0}}{\partial \lambda_0} = \frac{\partial \exp(pd_0^y)}{\partial \lambda_0}(\tau_0 + k_0 - g_0) + \exp(pd_0^y)\frac{\partial k_0}{\partial \lambda_0} \tag{C.1}$$

Note

$$pd_0^y = -\frac{(y_0^{\$}(1) + yspr_0^{\$} + rp_0^y) - (x_0 + \pi_0)}{(1-\kappa_1^y)} + \frac{\kappa_0^y}{(1-\kappa_1^y)}, \tag{C.2}$$

$$\kappa_1^y = \frac{e^{pd_0^y}}{e^{pd_0^y}+1}, \qquad \kappa_0^y = \log(1+\exp(pd_0^y)) - \kappa_1^y pd_0^y \tag{C.3}$$

Substitute in κ_1^y and κ_0^y,

$$pd_0^y = (x_0 + \pi_0) - (y_0^{\$}(1) + yspr_0^{\$} + rp_0^y) + \log(1+\exp(pd_0^y)) \tag{C.4}$$

Given $y_0^{\$}(1)$ moves 1 to 1 w.r.t. λ_0, and rp_0^y is invariant to w.r.t. λ_0, then,

$$\frac{\partial pd_0^y}{\partial \lambda_0} = -1 + \frac{\exp(pd_0^y)}{1+\exp(pd_0^y)}\frac{\partial pd_0^y}{\partial \lambda_0} = -1 + \kappa_1^y \frac{\partial pd_0^y}{\partial \lambda_0} = -\frac{1}{1-\kappa_1^y} \tag{C.5}$$

Since $k_0 \approx d_0 \lambda_0$,

$$\frac{\partial \left(\frac{PV_t^s}{Y_t}\right)^u \Big|_{z=0}}{\partial \lambda_0} = -\exp(pd_0^y)\frac{(\tau_0 + k_0 - g_0)}{(1-\kappa_1^y)} + \exp(pd_0^y) d_0 \tag{C.6}$$

so

$$\frac{\partial \left(\frac{PV_t^s}{Y_t}\right)^u \Big|_{z=0}}{\partial \lambda_0} \geq 0 \text{ iff } \frac{(\tau_0 + k_0 - g_0)}{(1-\kappa_1^y)} \leq d_0 \tag{C.7}$$

□

D Dynamic Asset Pricing Model

This section derives the expression for equilibrium asset prices in the dynamic asset pricing model.

D.1 Risk-free rate

The real short yield $y_t(1)$, or risk-free rate, satisfies $\mathbb{E}_t[\exp\{m_{t+1} + y_t(1)\}] = 1$. Solving out this Euler equation, we get:

$$y_t(1) = y_t^{\$}(1) - \mathbb{E}_t[\pi_{t+1}] - \frac{1}{2}(e_\pi)'\Sigma e_\pi + (e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_t = y_0(1) + \left[(e_{yn})' - (e_\pi)'\Psi + (e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_1\right] z_t. \tag{D.1}$$

$$y_0(1) \equiv y_0^{\$}(1) - \pi_0 - \frac{1}{2}(e_\pi)'\Sigma e_\pi + (e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_0. \tag{D.2}$$

where we used the expression for the real SDF

$$\begin{aligned} m_{t+1} &= m_{t+1}^{\$} + \pi_{t+1} \\ &= -y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + \pi_0 + (e_\pi)'\Psi z_t + (e_\pi)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1} \\ &= -y_t(1) - \frac{1}{2}(e_\pi)'\Sigma e_\pi + (e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_t - \frac{1}{2}\Lambda_t'\Lambda_t - \left(\Lambda_t' - (e_\pi)'\Sigma^{\frac{1}{2}}\right)\varepsilon_{t+1} \end{aligned}$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

D.2 Nominal and real term structure

Proposition 5. Nominal bond yields are affine in the state vector:

$$y_t^{\$}(h) = -\frac{A^{\$}(h)}{h} - \frac{(B^{\$}(h))'}{h} z_t,$$

where the coefficients $A^{\$}(h)$ and $B^{\$}(h)$ satisfy the following recursions:

$$A^{\$}(h+1) = -y_0^{\$}(1) + A^{\$}(h) + \frac{1}{2}(\left(B^{\$}(h)\right))'\Sigma\left(B^{\$}(h)\right) - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \tag{D.3}$$

$$\left(B^{\$}(h+1)\right)' = \left(B^{\$}(h)\right)'\Psi - (e_{yn})' - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1, \tag{D.4}$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

Proof. We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state:

$$\log(P_{t+1}^{\$}(h)) = A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1},$$

and solve for the coefficients $A^{\$}(h+1)$ and $B^{\$}(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned} P_t^{\$}(h+1) &= \mathbb{E}_t[\exp\{m_{t+1}^{\$} + \log\left(P_{t+1}^{\$}(h)\right)\}] \\ &= \mathbb{E}_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A^{\$}(h) + \left(B^{\$}(h)\right)' z_{t+1}\}] \\ &= \exp\{-y_0^{\$}(1) - (e_{yn})'z_t - \frac{1}{2}\Lambda_t'\Lambda_t + A^{\$}(h) + \left(B^{\$}(h)\right)'\Psi z_t\} \times \\ &\quad \mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} P_t^{\$}(h+1) &= \exp\left\{-y_0^{\$}(1) - (e_{yn})'z_t + A^{\$}(h) + \left(B^{\$}(h)\right)'\Psi z_t + \frac{1}{2}\left(B^{\$}(h)\right)'\Sigma\left(B^{\$}(h)\right)\right. \\ &\quad \left. - \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\right\}. \end{aligned}$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(h+1))$:

$$\log\left(P_t^{\$}(h+1)\right) = A^{\$}(h+1) + \left(B^{\$}(h+1)\right)' z_t,$$

where $A^{\$}(h+1)$ satisfies (D.3) and $B^{\$}(h+1)$ satisfies (D.4). The relationship between log bond prices and bond yields is given by $-\log\left(P_t^{\$}(h)\right)/\tau = y_t^{\$}(h)$. □

Define the one-period return on a nominal zero-coupon bond as:

$$r_{t+1}^{b,\$}(h) = \log\left(P_{t+1}^{\$}(h)\right) - \log\left(P_t^{\$}(h+1)\right).$$

The nominal bond risk premium on a bond of maturity τ is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

$$\begin{aligned}\mathbb{E}_t\left[r_{t+1}^{b,\$}(h)\right] - y_t^{\$}(1) + \frac{1}{2}V_t\left[r_{t+1}^{b,\$}(h)\right] &= -Cov_t\left[m_{t+1}^{\$}, r_{t+1}^{b,\$}(h)\right] \\ &= \left(B^{\$}(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_t.\end{aligned}$$

Real bond yields, $y_t(h)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$A(h+1) = -y_0(1) + A(h) + \frac{1}{2}(B(h))'\Sigma(B(h)) - (B(h))'\Sigma^{\frac{1}{2}}\left(\Lambda_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right), \quad \text{(D.5)}$$

$$(B(h+1))' = -(e_{yn})' + (e_\pi + B(h))'\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right). \quad \text{(D.6)}$$

For $\tau = 1$, we recover the expression for the risk-free rate in (D.1)-(D.2).

D.3 Stocks

D.3.1 Aggregate Stock Market

We define the real return on the aggregate stock market as $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$, where P_t^m is the ex-dividend price on the equity market. A log-linearization delivers:

$$r_{t+1}^m = \kappa_0^m + \Delta d_{t+1}^m + \kappa_1^m pd_{t+1}^m - pd_t^m. \quad \text{(D.7)}$$

The unconditional mean log real stock return is $r_0^m = E[r_t^m]$, the unconditional mean real dividend growth rate is $\mu^m = E[\Delta d_{t+1}^m]$, and $\overline{pd^m} = E[pd_t^m]$ is the unconditional average log price-dividend ratio on equity. The linearization constants κ_0^m and κ_1^m are defined as:

$$\kappa_1^m = \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}} + 1} < 1 \text{ and } \kappa_0^m = \log\left(e^{\overline{pd^m}} + 1\right) - \frac{e^{\overline{pd^m}}}{e^{\overline{pd^m}} + 1}\overline{pd^m}. \quad \text{(D.8)}$$

Our state vector z contains the (demeaned) log real dividend growth rate on the stock market, $\Delta d_{t+1}^m - \mu^m$, and the (demeaned) log price-dividend ratio $pd^m - \overline{pd^m}$.

$$\begin{aligned}pd_t^m(h) &= \overline{pd^m} + (e_{pd})'z_t, \\ \Delta d_t^m &= \mu^m + (e_{divm})'z_t,\end{aligned}$$

where $(e_{pd})'$ (e_{divm}) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We *define* the log return on the stock market so that the log return equation holds exactly, given the time series for $\{\Delta d_t^m, pd_t^m\}$. Rewriting (D.7):

$$\begin{aligned}r_{t+1}^m - r_0^m &= \left[(e_{divm} + \kappa_1^m e_{pd})'\Psi - (e_{pd})'\right]z_t + \left(e_{divm} + \kappa_1^m e_{pd}\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}. \\ r_0^m &= \mu^m + \kappa_0^m - \overline{pd^m}(1 - \kappa_1^m).\end{aligned}$$

The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

$$\begin{aligned}
1 &= \mathbb{E}_t\left[M_{t+1}\frac{P^m_{t+1}+D^m_{t+1}}{P^m_t}\right] = \mathbb{E}_t\left[\exp\{m^\$_{t+1}+\pi_{t+1}+r^m_{t+1}\}\right] \\
&= \mathbb{E}_t\left[\exp\left\{-y^\$_{t,1}-\frac{1}{2}\Lambda_t'\Lambda_t-\Lambda_t'\varepsilon_{t+1}+\pi_0+(e_\pi)'z_{t+1}+r^m_0+(e_{divm}+\kappa^m_1 e_{pd})'z_{t+1}-(e_{pd})'z_t\right\}\right] \\
&= \exp\left\{-y^\$_0(1)-\frac{1}{2}\Lambda_t'\Lambda_t+\pi_0+r^m_0+\left[(e_{divm}+\kappa^m_1 e_{pd}+e_\pi)'\Psi-(e_{pd})'-(e_{yn})'\right]z_t\right\} \\
&\quad \times\mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1}+\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right] \\
&= \exp\left\{r^m_0+\pi_0-y^\$_0(1)+\left[(e_{divm}+\kappa^m_1 e_{pd}+e_\pi)'\Psi-(e_{pd})'-(e_{yn})'\right]z_t\right\} \\
&\quad \times\exp\left\{\frac{1}{2}\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)'\Sigma\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)-\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)'\Sigma^{\frac{1}{2}}\Lambda_t\right\}
\end{aligned}$$

Taking logs on both sides delivers:

$$\begin{aligned}
r^m_0+\pi_0-y^\$_0(1)+\left[(e_{divm}+\kappa^m_1 e_{pd}+e_\pi)'\Psi-(e_{pd})'-(e_{yn})'\right]z_t & \\
+\frac{1}{2}\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)'\Sigma\left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right) &= \left(e_{divm}+\kappa^m_1 e_{pd}+e_\pi\right)'\Sigma^{\frac{1}{2}}\Lambda_t \\
\mathbb{E}_t\left[r^{m,\$}_{t+1}\right]-y^\$_{t,1}+\frac{1}{2}V_t\left[r^{m,\$}_{t+1}\right] &= -Cov_t\left[m^\$_{t+1},r^{m,\$}_{t+1}\right]
\end{aligned} \tag{D.9}$$

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

$$\begin{aligned}
\mathbb{E}_t\left[r^m_{t+1}\right]-y_{t,1}+\frac{1}{2}V_t\left[r^m_{t+1}\right] &= -Cov_t\left[m_{t+1},r^m_{t+1}\right] \\
r^m_0-y_0(1) &+ \left[(e_{divm}+\kappa^m_1 e_{pd}+e_\pi)'\Psi-(e_{pd})'-(e_{yn})'-(e_\pi)'\Sigma^{1/2}\Lambda_1\right]z_t \\
+\frac{1}{2}(e_{divm}+\kappa^m_1 e_{pd})'\Sigma(e_{divm}+\kappa^m_1 e_{pd}) &= \left(e_{divm}+\kappa^m_1 e_{pd}\right)'\Sigma^{1/2}(\Lambda_t-\left(\Sigma^{1/2}\right)'e_\pi)
\end{aligned}$$

We combine the terms in Λ_0 and Λ_1 on the right-hand side and plug in for $y_0(1)$ from (D.2) to get:

$$\begin{aligned}
& r^m_0+\pi_0-y^\$_{0,1}+\frac{1}{2}(e_\pi)'\Sigma e_\pi+\frac{1}{2}(e_{divm}+\kappa^m_1 e_{pd})'\Sigma(e_{divm}+\kappa^m_1 e_{pd})+(e_\pi)'\Sigma\left(e_{divm}+\kappa^m_1 e_{pd}\right) \\
& +\left[(e_{divm}+\kappa^m_1 e_{pd}+e_\pi)'\Psi-(e_{pd})'-(e_{yn})'\right]z_t \\
& =\left(e_{divm}+\kappa^m_1 e_{pd}\right)'\Sigma^{1/2}\Lambda_t+(e_\pi)'\Sigma^{\frac{1}{2}}\Lambda_0+(e_\pi)'\Sigma^{1/2}\Lambda_1 z_t
\end{aligned}$$

This recovers eqn. (D.9).

D.3.2 Dividend Strips

Proposition 6. Log price-dividend ratios on dividend strips are affine in the state vector:

$$pd^d_t(h)=A^m(h)+(B^m(h))'z_t,$$

where the coefficients $A^m(h)$ and $B^m(h)$ follow recursions:

$$\begin{aligned}
A^m(h+1) &= A^m(h)+\mu^m-y_0(1)+\frac{1}{2}\left(e_{divm}+B^m(h)\right)'\Sigma\left(e_{divm}+B^m(h)\right) \\
&\quad -\left(e_{divm}+B^m(h)\right)'\Sigma^{\frac{1}{2}}\left(\Lambda_0-\Sigma^{\frac{1}{2}\prime}e_\pi\right), && \text{(D.10)} \\
B^{m\prime}(h+1) &= \left(e_{divm}+e_\pi+B^m(h)\right)'\Psi-(e_{yn})'-\left(e_{divm}+e_\pi+B^m(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1, && \text{(D.11)}
\end{aligned}$$

initialized at $A^m_0=0$ and $B^m_0=0$.

Proof. We conjecture the affine structure and solve for the coefficients $A^m(h+1)$ and $B^m(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
PD_t^d(h+1) &= \mathbb{E}_t\left[M_{t+1}PD_{t+1}^d(h)\frac{D_{t+1}^m}{D_t^m}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^{\$}+\pi_{t+1}+\Delta d_{t+1}^m+pd_{t+1}^d(h)\}\right] \\
&= \mathbb{E}_t\left[\exp\{-y_{t,1}^{\$}-\frac{1}{2}\Lambda_t'\Lambda_t-\Lambda_t'\varepsilon_{t+1}+\pi_0+(e_\pi)'z_{t+1}+\mu^m+(e_{divm})'z_{t+1}+A^m(h)+B(h)^{m\prime}z_{t+1}\}\right] \\
&= \exp\{-y_0^{\$}(1)-(e_{yn})'z_t-\frac{1}{2}\Lambda_t'\Lambda_t+\pi_0+(e_\pi)'\Psi z_t+\mu^m+(e_{divm})'\Psi z_t+A^m(h)+B(h)^{m\prime}\Psi z_t\} \\
&\quad \times\mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1}+(e_{divm}+e_\pi+B^m(h))'\,\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$
\begin{aligned}
pd_t^d(h+1) &= -y_0^{\$}(1)+\pi_0+\mu^m+A^m(h)+\left[(e_{divm}+e_\pi+B^m(h))'\,\Psi-(e_{yn})'\right]z_t \\
&\quad +\frac{1}{2}\,(e_{divm}+e_\pi+B^m(h))'\,\Sigma\,(e_{divm}+e_\pi+B^m(h)) \\
&\quad -(e_{divm}+e_\pi+B^m(h))'\,\Sigma^{\frac{1}{2}}\,(\Lambda_0+\Lambda_1 z_t)
\end{aligned}
$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^d(h+1)$:

$$
pd_t^d(h+1) = A^m(h+1)+B^m(h+1)'z_t,
$$

where:

$$
\begin{aligned}
A^m(h+1) &= A^m(h)+\mu^m-y_0^{\$}(1)+\pi_0+\frac{1}{2}\,(e_{divm}+e_\pi+B^m(h))'\,\Sigma\,(e_{divm}+e_\pi+B^m(h)) \\
&\quad -(e_{divm}+e_\pi+B^m(h))'\,\Sigma^{\frac{1}{2}}\,\Lambda_0, \\
B^m(h+1)' &= (e_{divm}+e_\pi+B^m(h))'\,\Psi-(e_{yn})'-(e_{divm}+e_\pi+B^m(h))'\,\Sigma^{\frac{1}{2}}\,\Lambda_1.
\end{aligned}
$$

We recover the recursions in (D.10) and (D.11) after using equation (D.2). □

We define the dividend strip risk premium as:

$$
\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^{d,\$}(h)\right]-y_{t,1}^{\$}+\frac{1}{2}V_t\left[r_{t+1}^{d,\$}(h)\right] &= -Cov_t\left[m_{t+1}^{\$},r_{t+1}^{d,\$}(h)\right] \\
&= (e_{divm}+e_\pi+B^m(h))'\,\Sigma^{\frac{1}{2}}\,\Lambda_t
\end{aligned}
$$

D.4 Government Spending and Tax Revenue Claims

This appendix computes P_t^T, the value of a claim to future tax revenues, and P_t^G, the value of a claim to future government spending. It contains the proof for Proposition 5.

D.4.1 Spending Claim

Nominal government spending growth equals

$$
\Delta\log G_{t+1} = \Delta\log g_{t+1}+x_{t+1}+\pi_{t+1} = x_0+\pi_0+\mu_0^g+\left(e_{\Delta g}+e_x+e_\pi\right)'z_{t+1}. \tag{D.12}
$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$
pd_t^g(h) = \log\left(pd_t^g(h)\right) = A^g(h)+(B^g(h))'z_t.
$$

We solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
pd_t^g(h+1) &= \mathbb{E}_t\left[M_{t+1}pd_{t+1}^g(h)\frac{G_{t+1}}{G_t}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^{\$}+\Delta\log g_{t+1}+x_{t+1}+\pi_{t+1}+pd_{t+1}^g(h)\}\right] \\
&= \exp\{-y_0^{\$}(1)-(e_{yn})'z_t-\frac{1}{2}\Lambda_t'\Lambda_t+\mu^g+x_0+\pi_0+(e_{\Delta g}+e_x+e_\pi+B^g(h))'\Psi z_t+A^g(h)\} \\
&\quad \times\mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1}+\left(e_{\Delta g}+e_x+e_\pi+B^g(h)\right)'\,\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\right].
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} pd_t^g(h+1) &= \exp\{-y_0^\$(1) + \mu^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - (e_{yn})')z_t + A^g(h) \\ &\quad + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\ &\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned} A^g(h+1) &= -y_0^\$(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\ &\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \\ B^g(h+1)' &= (e_{\Delta g} + e_x + e_\pi + B^g(h))'\Psi - (e_{yn})' - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}\Lambda_1, \end{aligned}$$

and the price-dividend ratio of the cum-dividend spending claim is

$$\sum_{h=0}^{\infty} \exp(A^g(h+1) + B^g(h+1)'z_t)$$

Derivation of Risk Premium We note that the 1-period holding return on a spending strip is

$$\exp(r_{t+1}^g(h)) = \exp\{\Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + pd_{t+1}^g(h) - pd_t^g(h+1)\}$$

so that the Euler equation is $\mathbb{E}_t[\exp(m_{t+1}^\$ + r_{t+1}^g(h))] = 1$.

We can express the expected return as

$$\begin{aligned} \mathbb{E}_t[r_{t+1}^G(h)] &= -\mathbb{E}_t[m_{t+1}^\$] - \frac{1}{2}var_t(m_{t+1}^\$) - \frac{1}{2}var_t(r_{t+1}^g(h)) - cov_t(m_{t+1}^\$, r_{t+1}^g(h)) \\ &= y_t^\$(1) - \frac{1}{2}var_t(r_{t+1}^g(h)) - cov_t(m_{t+1}^\$, r_{t+1}^g(h)) \end{aligned}$$

and the risk premium is

$$\begin{aligned} \mathbb{E}_t[r_{t+1}^g(h)] - y_t^\$(1) &= -\frac{1}{2}var_t(r_{t+1}^g(h)) + cov_t(\Lambda_t'\varepsilon_{t+1}, r_{t+1}^g(h)) \\ &= \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma^{\frac{1}{2}}(\Lambda_0 + \Lambda_1 z_t) - \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \end{aligned}$$

To evaluate the risk premium for the entire duration of the strip, we define the holding-period risk premium as

$$\frac{1}{h}\sum_{k=0}^{h-1} \mathbb{E}_t[r_{t+k+1}^g(h-k) - y_{t+k}^\$(1)]$$

when the state variable is at $z_t = 0$, the expected holding-period risk premium simplifies to

$$\frac{1}{h}\sum_{k=0}^{h-1}\left(e_{\Delta g} + e_x + e_\pi\right)'\Sigma^{\frac{1}{2}}\Lambda_0 - \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h-k)\right)'\Sigma\left(e_{\Delta g} + e_x + e_\pi + B^g(h-k)\right) \tag{D.13}$$

Entire Spending Claim Next, we define the (nominal) return on the claim as $R_{t+1}^g = \frac{P_{t+1}^g}{P_t^g - G_t} = \frac{P_{t+1}^{g,ex} + G_{t+1}}{P_t^{g,ex}}$, where P_t^g is the cum-dividend price on the spending claim and $P_t^{g,ex}$ is the ex-dividend price. We log-linearize the return around $z_t = 0$:

$$r_{t+1}^g = \kappa_0^g + \Delta \log G_{t+1} + \kappa_1^g pd_{t+1}^g - pd_t^g. \tag{D.14}$$

where $pd_t^g \equiv \log\left(\frac{P_t^{g,ex}}{G_t}\right) = \log\left(\frac{P_t^g}{G_t} - 1\right)$. The unconditional mean log return of the G claim is $r_0^g = E[r_t^g]$.

We obtain pd_0^g from the precise valuation formula at $z_t = 0$. We define linearization constants κ_0^g and κ_1^g as:

$$\kappa_1^g = \frac{e^{pd_0^g}}{e^{pd_0^g} + 1} < 1 \text{ and } \kappa_0^g = \log\left(e^{pd_0^g} + 1\right) - \frac{e^{pd_0^g}}{e^{pd_0^g} + 1}pd_0^g. \tag{D.15}$$

Then, under a log-linear approximation, the unconditional expected return is:

$$r_0^g = x_0 + \pi_0 + \kappa_0^g - pd_0^g(1-\kappa_1^g). \tag{D.16}$$

The log ex-dividend price-dividend ratio on the entire spending claim is affine in the state vector and verify the conjecture by solving the Euler equation for the claim.

$$pg_t = pd_0^g + (\bar{B}^g)' z_t \tag{D.17}$$

This allows us to write the return as:

$$r_{t+1}^g = r_0^g + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' z_{t+1} - (\bar{B}^g)' z_t. \tag{D.18}$$

Proof. Starting from the Euler equation:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\exp\{m_{t+1}^\$ + r_{t+1}^g\}\right] \\ &= \exp\{-y_0^\$(1) - (e_{yn})' z_t - \frac{1}{2}\Lambda_t'\Lambda_t + r_0^g + [\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)'] z_t\} \\ &\quad \times \mathbb{E}_t\left[\exp\{-\Lambda_t'\varepsilon_{t+1} + \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right]. \end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned} 1 = &\exp\{r_0^g - y_0^\$(1) + [\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)' - (e_{yn})'] z_t \\ &+ \underbrace{\frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Sigma \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)}_{Jensen} - \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we obtain the following system of equations:

$$r_0^g - y_0^\$(1) + Jensen = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \Lambda_0 \tag{D.19}$$

and

$$\left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Psi - (\bar{B}^g)' - (e_{yn})' = \left(e_{\Delta g} + e_x + e_\pi + \kappa_1^g \bar{B}^g\right)' \Sigma^{\frac{1}{2}} \Lambda_1 \tag{D.20}$$

The left-hand side of this equation is the unconditional expected excess log return with Jensen adjustment. The right hand side is the unconditional covariance of the log SDF with the log return. This equation describes the unconditional risk premium on the claim to government spending. Eqn. (D.20) describes the time-varying component of the government spending risk premium. Given Λ_1, the system of N equations (D.20) can be solved for the vector $\bar{B}^g$:

$$\bar{B}^g = \left(I - \kappa_1^g\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right)'\right)^{-1}\left[\left(\Psi - \Sigma^{\frac{1}{2}}\Lambda_1\right)'\left(e_{\Delta g} + e_x + e_\pi\right) - e_{yn}\right]. \tag{D.21}$$

□

D.4.2 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + (e_{\Delta\tau} + e_x + e_\pi)' z_{t+1}. \tag{D.22}$$

where $\tau_t = T_t/GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero. The remaining proof exactly mirrors the proof for government spending, with

$$pd_t^\tau \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^\tau}{T_t} - 1\right) = pd_0^\tau + (B^\tau)' z_t \tag{D.23}$$

$$r_{t+1}^\tau = r_0^\tau + \left(e_{\Delta\tau} + e_x + e_\pi + \kappa_1^g B^\tau\right)' z_{t+1} - (B^\tau)' z_t, \tag{D.24}$$

and

$$r_0^\tau = x_0 + \pi_0 + \kappa_0^\tau - \overline{p\tau}(1-\kappa_1^\tau).$$

$$r_0^\tau - y_0^\$(1) + Jensen = (e_{\Delta\tau} + e_x + e_\pi + \kappa_1^\tau B^\tau)' \Sigma^{\frac{1}{2}} \Lambda_0. \tag{D.25}$$

E Estimation of Market Prices of Risk

This appendix contains the details on which moments we use to estimate the market prices of risk and what the resulting estimates are.

E.1 Moments Used in Estimation

Bond Moments The first four rows of the VAR contain inflation, the one-year yield, the spread between the five- and the one-year yields, and real per capita GDP growth. Market prices of risk associated with these elements have an outsize impact on the bond pricing moments.

As detailed in the previous appendix, the nominal bond yield of maturity h, $y_t^{\$}(h)$, is affine in the state vector with scalar $A^{\$}(h)$ and loading vector $B^{\$}(h)$ which follow ordinary difference equations repeated here for convenience:

$$A^{\$}_{\tau+1} = -y^{\$}_{0,1} + A^{\$}_{\tau} + \frac{1}{2}\left(B^{\$}_{\tau}\right)'\Sigma\left(B^{\$}_{\tau}\right) - \left(B^{\$}_{\tau}\right)'\Sigma^{\frac{1}{2}}\Lambda_0, \tag{E.1}$$

$$\left(B^{\$}_{\tau+1}\right)' = \left(B^{\$}_{\tau}\right)'\Psi - e'_{yn} - \left(B^{\$}_{\tau}\right)'\Sigma^{\frac{1}{2}}\Lambda_1, \tag{E.2}$$

initialized at $A_0^{\$} = 0$ and $B_0^{\$} = 0$. Note that $A^{\$}(h)$ and $B^{\$}(h)$ depend on the properties of the state vector $(\Psi, \Sigma^{\frac{1}{2}})$ and of the market prices of risk (Λ_0, Λ_1). Similarly, real bond yields are affine in the state vector.

Since both the nominal short rate $(y_t^{\$}(1))$ and the slope of the term structure $(y_t^{\$}(5) - y_t^{\$}(1))$ are included in the VAR, internal consistency requires the SDF model to price these bonds closely. The nominal short rate is matched automatically; it does not identify any market price of risk parameters. Matching the mean and dynamics of the five-year bond yield generates $N+1$ parameter restrictions:

$$y^{\$}_{t,5} = y^{\$}_{0,5} + (e_{yn} + e_{yspr})'z_t = -\frac{A^{\$}_5}{5} - \frac{B^{\$\prime}_5}{5}z_t$$

This restriction identifies one element in the constant Λ_0, specifically

$$y^{\$}_{0,1} + yspr_0 = -\frac{1}{5}A^{\$}_5$$

and N elements in the time-varying market price of risk matrix Λ_1:

$$e'_{y1} + e'_{yspr} = -\frac{1}{5}\left(B^{\$}_5\right)'$$

Define $\widetilde{\Psi} = \Psi - \Sigma^{\frac{1}{2}}\Lambda_1$ to be the risk-neutral companion matrix. Then (E.2) at the five-year maturity can be written as:

$$\frac{-\left(B^{\$}_5\right)'}{5} = \frac{1}{5}e'_{yn}(I - \widetilde{\Psi})^{-1}(I - \widetilde{\Psi}^{\tau+1})$$

The restriction on Λ_1 can be written as:

$$e'_{y1} + e'_{yspr} = \frac{1}{5}e'_{yn}(I - \widetilde{\Psi}^{\tau+1})(I - \widetilde{\Psi})^{-1}$$

The left-hand side is a $N \times 1$ vector with a 1 in elements 2 and 3 and a 0 in the other positions. Hence, the same must be true of the right-hand side. This imposes N restrictions on Λ_1 which affects $\widetilde{\Psi}$, given Ψ. There is one restriction on each of the columns of Λ_1. It is a restriction on a linear combination of the elements in the first three rows of Λ_1 of that column. For example, if the elements in the 10th column and first two rows of Λ_1 are all zero, it is a simple restriction on the element in the third row and 10th column. We choose these N restrictions to pin down the third row (element) of Λ_1 (Λ_0), given the first two rows (elements).

We also allow for a non-zero first element of Λ_0 and two non-zero elements in the first row of Λ_1. We pin down these elements by matching bond yields of maturities 2, 10, 20, and 30 years in each year $t \in 1, \cdots, T$. Since they represent $T \times 4$ moments for only 3 parameters, there are $T \times 4 - 3$ over-identifying restrictions. Since the behavior of long-term interest rates is important for our valuation results, we impose extra weight on matching the 30-year bond yields.

Market prices of risk on the real per capita GDP growth shock are jointly identified by both bond and stock moments. Intuitively, real bond yields depend on shocks to expected growth. We allow for a non-zero mean and a non-zero fourth row of the VAR, associated with GDP growth. We price the yields on real bonds (Treasury inflation-index securities) for maturities 5, 7, 10, 20, and 30 years. They are available over a shorter sample of T_2 years. This adds $T_2 \times 5$ over-identifying restrictions. Again, we overweight matching the 30-year maturity.

With three priced sources of risk, ours is a three-factor model of the term structure. Empirically, three factors explain more than 97% of the variation in bond yields across maturities.

Equity Pricing Rows four, five, and six of the VAR contain real GDP growth, log dividend-to-GDP growth, and the log price-dividend ratio. Market prices of risk associated with these elements have an outsize impact on the equity pricing moments. The time series of dividend growth and the price-dividend ratio imply a time series for stock returns per eqn. (D.7). This formulation imposes cointegration between dividend and GDP. The ratio of dividends to GDP is stationary in the long-run. Aggregate stock market dividend strips may have a higher risk premium than GDP strips over short-to-medium horizons, but in the long-run, the risk premia of dividend and GDP strips converge.

We impose that the expected excess return time series implied by the VAR matches the equity risk premium time series in the model. The latter depends on the covariance of the SDF with stock returns and hence on the market price of risk parameters. The VAR implies an expected excess log stock return including a Jensen adjustment given by the left-hand side of the following equation:

$$r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}e_\pi'\Sigma e_\pi + \frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) + e_\pi'\Sigma\left(e_{divm} + \kappa_1^m e_{pd}\right) \tag{E.3}$$
$$+ \left[(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}'\right] z_t = \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}(\Lambda_0 + \Lambda_1' z_t) + e_\pi'\Sigma^{\frac{1}{2}}\Lambda_0 + e_\pi'\Sigma^{1/2}\Lambda_1 z_t$$

The first term on the second line is the time-varying component of the expected excess stock return. This is just data, i.e. the companion matrix of the VAR Ψ, not asset pricing. The asset pricing is on the right-hand side of the above equation. It reports the equity risk premium, which is negative the conditional covariance of the log stock return and the log SDF. It depends on the market prices of risk. To replicate this time-variation, the seventh row of Λ_1 can be chosen to be such that the time-varying components of the left- and right-hand sides of eqn. (E.3) are equalized:

$$(e_{divm} + \kappa_1^m e_{pd} + e_\pi)'\Psi - e_{pd}' - e_{yn}' = \left(e_{divm} + \kappa_1^m e_{pd}\right)' \Sigma^{1/2}\Lambda_1' + e_\pi'\Sigma^{1/2}\Lambda_1$$

This is a linear system of N equations in N unknowns, which uniquely pins down the N elements in the seventh row of Λ_1, given all rows that precede it (the dividend growth dynamics and the term structure dynamics). Given the structure of Ψ and the structure of the market prices of risk in the first six rows of Λ_1, the seventh row of Λ_1 must have non-zero elements in all columns.

Likewise, the constant in the seventh element of Λ_0 helps to match the unconditional equity risk premium in the model to that in the data, given the first six elements of Λ_0.

$$r_0^m + \pi_0 - y_{0,1}^{\$} + \frac{1}{2}e_\pi'\Sigma e_\pi + \frac{1}{2}(e_{divm} + \kappa_1^m e_{pd})'\Sigma(e_{divm} + \kappa_1^m e_{pd}) + e_\pi'\Sigma\left(e_{divm} + \kappa_1^m e_{pd}\right) = \left(e_{divm} + \kappa_1^m e_{pd} + e_\pi\right)' \Sigma^{1/2}\Lambda_0$$

Let $PD_t^m(h)$ denote the price-dividend ratio of the dividend strip with maturity h (Wachter, 2005; Binsbergen, Brandt, and Koijen, 2012). Then, the aggregate price-to-dividend ratio can be expressed as

$$PD_t^m = \sum_{h=0}^{\infty} PD_t^m(h). \tag{E.4}$$

In this SDF model, log price-dividend ratios on dividend strips are affine in the state vector as shown in the previous appendix:

$$pd_t^m(h) = \log\left(PD_t^m(h)\right) = A^m(h) + (B^m(h))'z_t.$$

Since the log price-dividend ratio on the stock market in part of the state vector, it is affine in the state vector by assumption; see the left-hand side of (E.5):

$$\exp\left(\overline{pd} + (e_{pd})'z_t\right) = \sum_{h=0}^{\infty} \exp\left(A^m(h) + (B^m(h))'z_t\right), \tag{E.5}$$

Equation (E.5) rewrites the present-value relationship (E.4), and articulates that it implies a restriction on the coefficients $A^m(h)$ and $(B^m(h))'$. Matching the time series for the price-dividend ratio in model and data provides $T \times 1$ additional restrictions. We can either thinking of this as over-identifying restrictions or we can think of this condition as helping to pin down the market prices of GDP growth risk. When implementing this restriction, we cut off the sum over dividend strips at 3600 years.

We expect and indeed estimate a positive risk price for innovations to GDP growth $\Lambda_0(4)$, since positive innovations in GDP growth are good news for the economy. High GDP growth innovations therefore also increase aggregate dividend growth and move up stock prices. We also estimate a positive market price of pd shocks. Positive shocks to stock market valuations are good news to the representative investor. A positive innovation to pd is a negative shock to the equity risk premium, since the dp ratio is one of the most prominent drivers of the equity risk premium.

Good Deal Bounds and Regularity Conditions We impose good deal bounds on the standard deviation of the log SDF in the spirit of Cochrane and Saa-Requejo (2000). Specifically, we impose a penalty for annual Sharpe ratios in excess of 1.5.

Second, we impose regularity conditions on (unobserved) nominal and real bond yields of maturities of 100 to 4000 years. Specifically, we impose that yields stabilize and that nominal yields remain above real yields by at least long-run expected inflation. This is tantamount to a weak positivity restriction on the long-run inflation risk premium. We also impose that nominal yields remain above the nominal GDP growth rate, the real yields remain above the real GDP growth rate.

Third, we impose that bond return volatilities on very long-maturity bonds are bounded from below by 20%.

Fourth, we require the maximum eigenvalue of the risk neutral companion matrix $\Psi - \Sigma\Lambda_1$ to be less than 1 to ensure all valuation ratios are well-defined in the model. We also verify that the transversality conditions for the GDP, aggregate dividend, tax revenue, and spending claims are satisfied in the estimation.

E.2 Model Fit

The risk price estimates Λ_0 and Λ_1 are given by:

$$\Lambda_0 = \begin{bmatrix} 0.26 & 0 & -0.04 & 2.09 & 1.99 & 0 & 0.62 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

$$\Lambda_1 = \begin{bmatrix} 3.41 & 0.00 & 0.00 & 19.47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.00 & -3.17 & 0.00 & 19.47 & -0.00 & -3.38 & -1.45 & 1.48 & -0.08 & -3.19 & 8.93 \\ 0.73 & 0.00 & -0.06 & -0.72 & -1.62 & -2.53 & -5.29 & -0.13 & 5.39 & 0.00 & 4.02 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16.93 & -24.55 & -96.72 & -4.84 & 0.23 & -1.97 & -2.81 & -1.49 & 0.63 & 0.68 & -0.76 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure E.1 shows that the model matches the nominal term structure in the data closely. The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. In the estimation of the market prices of risk, we overweigh matching the 5-year bond yield since it is included in the VAR and the 30-year bond yield since the behavior of long-term bond yields is important for the results.

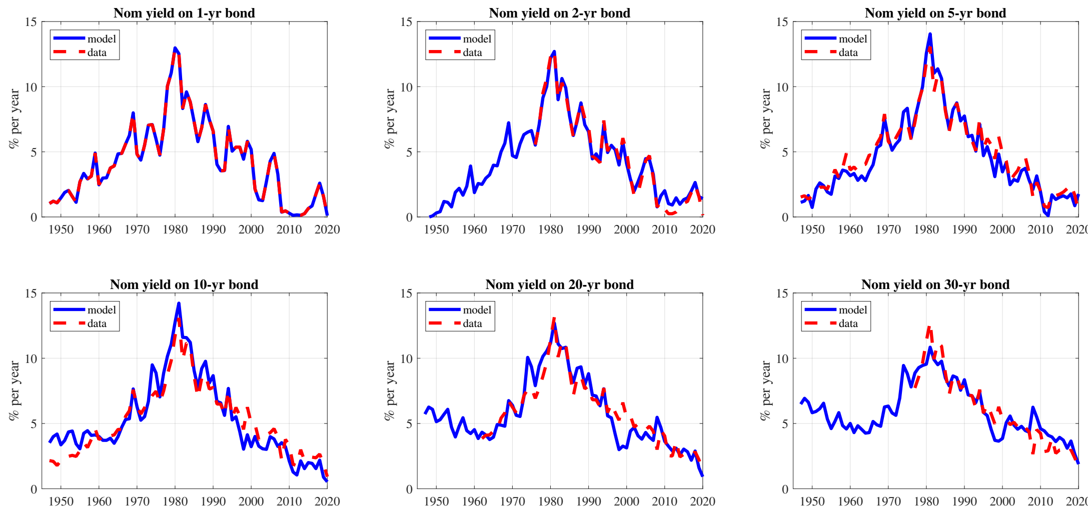
Figure E.2 shows that the model matches the real term structure in the data closely. The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real Treasury bond yields (Treasury Inflation Indexed securities). In the estimation of the market prices of risk, we overweigh matching the 30-year bond yield since the behavior of long-term bond yields is important for the results.

The top panels of Figure E.3 show the model's implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved, with very long-run nominal (real) yields stabilizing at around 6.92% (0.95%) per year. We impose conditions that ensure that the nominal and real term structure are well behaved at very long maturities, for which we have no data. Specifically, we impose that average nominal (real) yields of bonds with maturities of 100, 500, 1000, 2000, 3000, and 4000 years remain above 6.11% (2.95%) per year, which is the long-run nominal (real) GDP growth rate $x_0 + \pi_0$ (x_0) observed in our sample. Second, we impose that nominal yields remain above real yields plus 3.16% expected inflation at those same maturities. This imposes that the inflation risk premium remain positive at very long maturities. Third, we impose that the nominal and real term structures of interest rates flatten out, with an average yield difference between 100- and 50-year yields that must not exceed 2% per year and between 200- and 100-year maturity that must not exceed 1% per year. These restrictions are satisfied at the optimum.

The bottom left panel of Figure E.3 shows that the model matches the dynamics of the nominal bond risk premium, defined as the expected excess return on the five-year nominal bond, quite well. Bond risk premia decline in the latter part of the sample, possibly reflecting the arrival of foreign investors who value U.S. Treasuries highly. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation inflation over the next five years, and the five-year inflation risk premium. On average, the 4.61% nominal bond yield is comprised of a -0.29% real yield, a 3.16% expected inflation rate, and a 1.74% inflation risk premium. The graph shows that the importance of these components fluctuates over time.

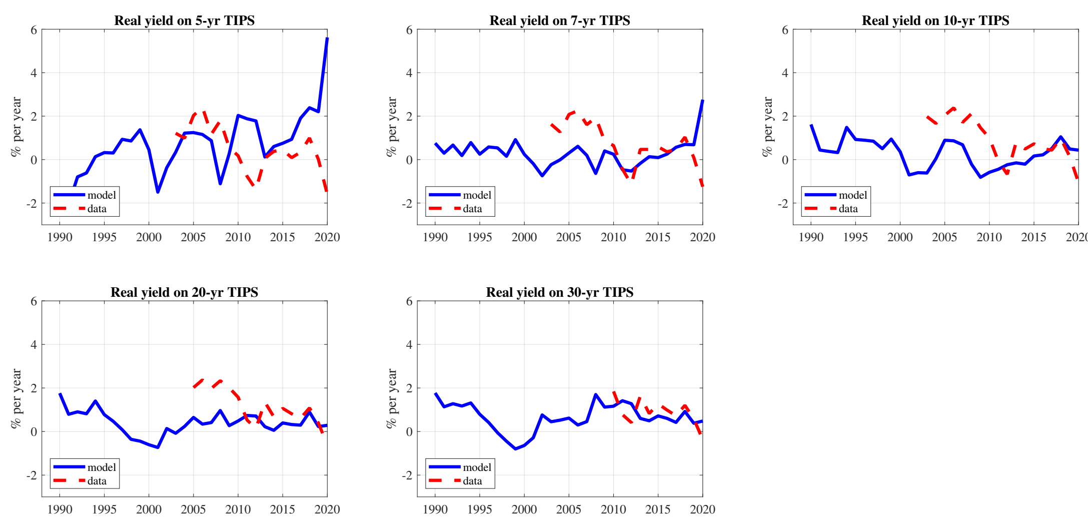
Figure E.4 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panel. The risk premia in the data are the expected equity excess return predicted by the VAR. Their dynamics are sensible, with low risk premia in the dot-com boom of 1999—2000 and very high risk premia in the Great Financial Crisis of 2008—09. The VAR-implied equity risk premium occasionally turns negative. The figure's right panel shows a tight fit for equity price levels. Hence, the model fits both the behavior of expected returns and stock price levels.

Figure E.1: Dynamics of the Nominal Term Structure of Interest Rates



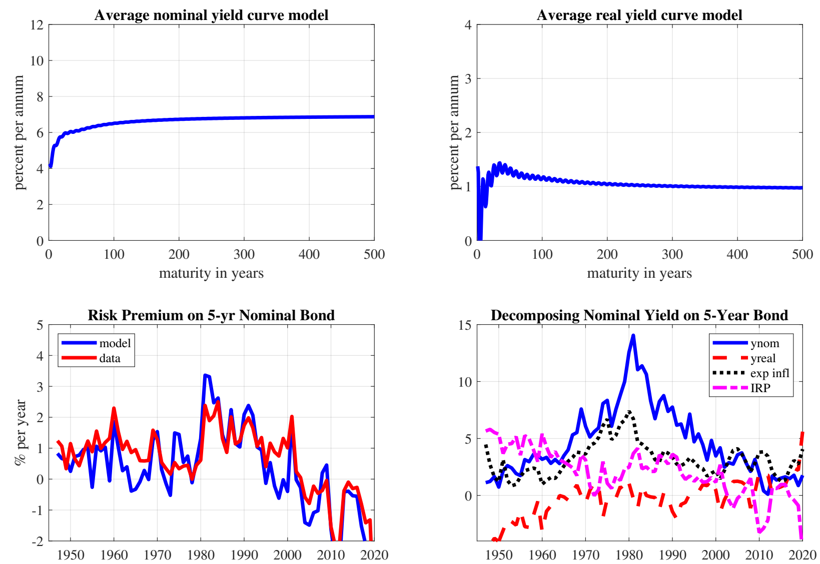
The figure plots the observed and model-implied 1-, 2-, 5-, 10-, 20-, and 30-year nominal Treasury bond yields. Yields are measured at the end of the year. Data are from FRED and FRASER. The sample is annual, 1947—2020.

Figure E.2: Dynamics of the Real Term Structure of Interest Rates



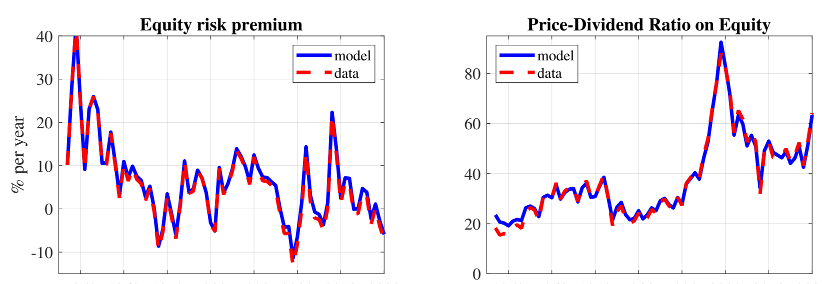
The figure plots the observed and model-implied 5-, 7-, 10-, 20-, and 30-year real bond yields. Data are from FRED and start in 2003. For ease of readability, we start the graph in 1990 but the model of course implies a real yield curve for the entire 1947—2020 period.

Figure E.3: Long-term Yields and Bond Risk Premia



The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 to 500 years. The bottom left panel plots the nominal bond risk premium on the five year bond in model and data. The nominal bond risk premium is measured as the five year bond yield minus the expected one-year bond yield over the next five years. The bottom right panel decomposes the model's five-year nominal bond yield into the five-year real bond yield, the five year expected inflation, and the five-year inflation risk premium.

Figure E.4: Equity Risk Premium and Price-Dividend Ratio



The figure plots the observed and model-implied equity risk premium on the overall stock market in the left panel and the price-dividend ratio in the right panel. The price-dividend ratio is the price divided by the annualized dividend. Data are from 1947—2020. Monthly stock dividends are seasonally adjusted.