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THE ULTIMATE RUIN PROBABILITY OF A DEPENDENT DELAYED-CLAIM RISK MODEL PERTURBED BY DIFFUSION WITH CONSTANT FORCE OF INTEREST

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Abstract. Recently, Li [12] gave an asymptotic formula for the ultimate ruin probability in a delayed-claim risk model with constant force of interest and pairwise quasi-asymptotically independent and extended-regularly-varying-tailed claims. This paper extends Li's result to the case in which the risk model is perturbed by diffusion, the claims are consistently-varying-tailed and the main-claim interarrival times are widely lower orthant dependent.

1. Introduction

Consider a delayed-claim risk model perturbed by diffusion with constant force of interest, which involves two kinds of insurance claims, namely the main claims and the by-claims, such that each main claim may cause a by-claim occurring after a period of delay. In this model, the main claims $\{X_i, i \geq 1\}$, by-claims $\{Y_i, i \geq 1\}$, inter-arrival times of main claims $\{\theta_i, i \geq 1\}$ are three sequences of nonnegative and identically distributed, but not necessarily independent, random variables (r.v.s) with common distributions F, G and K, respectively. Denote by $\tau_i = \sum_{k=1}^{i} \theta_k,\ i \geq 1$, the arrival times of successive main claims, which constitute a counting process

$$N(t) = \sup\{i \geq 1 : \tau_i \leq t\}, \quad t \geq 0;$$

and by $\{T_i, i \geq 1\}$ the delay times of by-claims, which are nonnegative (but possibly degenerated at 0), arbitrarily dependent and identically distributed r.v.s with common distribution H.

Assume that the total amount of premiums accumulated before time $t \geq 0$, denoted by $C(t)$, is a nonnegative and nondecreasing stochastic process with $C(0) = 0$ and $C(t) < \infty$ almost surely for every $0 \leq t < \infty$; and that the diffusion process, as a perturbed term, $\{B(t), t \geq 0\}$ is a standard Brownian

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motion with volatility parameter $\sigma \geq 0$ and independent of the other sources of randomness. In practice, the diffusion-perturbed term can be interpreted as an additional uncertainty of the aggregate claims or the premium income of an insurance company. Let $r > 0$ be the constant force of interest and $x \geq 0$ be the insurer's initial reserve. Then the total reserve up to time $t \geq 0$, denoted by $R(t)$, satisfies

$$\begin{aligned} R(t) = xe^{rt} + \int_{0-}^{t} e^{r(t-s)} dC(s) - \sum_{i=1}^{\infty} X_i e^{r(t-\tau_i)} \mathbf{1}_{\{\tau_i \leq t\}} \\ - \sum_{i=1}^{\infty} Y_i e^{r(t-\tau_i-T_i)} \mathbf{1}_{\{\tau_i+T_i \leq t\}} + \sigma \int_{0-}^{t} e^{r(t-s)} dB(s), \end{aligned} \tag{1.1}$$

where $\mathbf{1}_E$ is the indicator function of an event E. Hence the ultimate ruin probability is defined by

$$\psi_r(x) = P\left(R(t) < 0 \text{ for some } t \geq 0\right). \tag{1.2}$$

It is well-known that the delayed-claim risk model was firstly introduced by Waters and Papatriandafylou [22] so that the independence assumption between claim sizes and their inter-arrival times can be relaxed, and since then it has been extensively investigated by many researchers. See, for example, Yuen and Guo [26], Xiao and Guo [23], Li and Wu [13], among others. For the continuous-time counterparts, the readers are referred to Yuen et al. [27], Xie and Zou [24, 25], Meng and Wang [15], Zou and Xie [28], and references therein. We notice that all the references above only discussed the case when the claims are light-tailed and mutually independent. But recently, Li [12] considered the delayed-claim risk model with heavy-tailed and dependent claims, and an asymptotic formula for the ultimate ruin probability was reached. So in the following, we introduce some dependence structures and some classes of heavy-tailed distributions.

Wang et al. [20] introduced the widely lower orthant dependent (WLOD) structure. Say that r.v.s $\{\xi_i, i \geq 1\}$ are WLOD, if there exists a sequence of finite positive numbers $\{g_L(n), n \geq 1\}$ such that for each $n \geq 1$ and all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n} \left\{\xi_i \leq x_i\right\}\right) \leq g_L(n) \prod_{i=1}^{n} P(\xi_i \leq x_i).$$

Clearly, if $\{\xi_i, i \geq 1\}$ are WLOD r.v.s, then for each $n \geq 1$ and any $s > 0$,

$$E \exp\left\{-s \sum_{i=1}^{n} \xi_i\right\} \leq g_L(n) \prod_{i=1}^{n} E e^{-s\xi_i}. \tag{1.3}$$

Also, Chen and Yuen [2] proposed a more general dependence structure below. Say that r.v.s $\{\xi_i, i \geq 1\}$ are pairwise quasi-asymptotically independent

(PQAI), if for any $1 \le i \ne j < \infty$,

$$\lim_{x\to\infty} P\big(|\xi_i| \wedge \xi_j > x \mid \xi_i \vee \xi_j > x\big) = 0,$$

where we denote $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. For further study on the above dependence structures and their analogues, we refer to Geluk and Tang [11], Wang and Cheng [21], Liu et al. [14], Gao and Jin [6], Gao and Liu [8], Gao and Yang [9, 10], Gao et al. [7], Gao and Bao [5], and others.

Henceforth, all limit relationships are for $x \to \infty$ unless mentioned otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(1)b(x)$ if $\limsup a(x)/b(x) = C < \infty$, write $a(x) = o(1)b(x)$ if $C = 0$, write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $C \le 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$, write $a(x) \asymp b(x)$ if $a(x) = O(1)b(x)$ and $b(x) = O(1)a(x)$.

In the paper, we are concerned with some classes of heavy-tailed distributions, which have no finite exponential moments. Say that a r.v. ξ or its distribution V belongs to the ERV class of extended-regularly-varying-tailed distributions if there exist some $0 < \alpha \le \beta < \infty$ such that $y^{-\beta} \le \overline{V}_*(y) \le \overline{V}^*(y) \le y^{-\alpha}$ for all $y \ge 1$, where $\overline{V}_*(y) = \liminf \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) = \limsup \overline{V}(xy)/\overline{V}(x)$; belongs to the consistently-varying-tailed class, denoted by $V \in \mathcal{C}$, if $\lim_{y\searrow 1} \overline{V}_*(y) = 1$ or $\lim_{y\nearrow 1} \overline{V}^*(y) = 1$; belongs to the dominatedly-varying-tailed class, denoted by $V \in \mathcal{D}$, if $\overline{V}^*(y) < \infty$ for all $y > 0$; belongs to the long-tailed class, denoted by $V \in \mathcal{L}$, if

$$\overline{V}(x + y) \sim \overline{V}(x) \quad \text{for all } y > 0. \tag{1.4}$$

In conclusion,

$$ERV \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D}.$$

More details of heavy-tailed distributions and their applications can be found in Bingham et al. [1] and Embrechts et al. [4].

Recently, Li [12] showed that in the delayed-claim risk model (1.1) with $\sigma = 0$ and $C(\cdot)$ a deterministic linear function, if the claim sizes $\{X_i, Y_i, i \ge 1\}$ are PQAI, random pairs $\{(X_i, Y_i), i \ge 1\}$ are identically distributed with marginal distributions $F \in ERV$ and $G \in ERV$, the inter-arrival times of main claims $\{\theta_i, i \ge 1\}$ are independent, and $\{X_i, Y_i, i \ge 1\}$, $\{\theta_i, i \ge 1\}$ and $\{T_i, i \ge 1\}$ are mutually independent, then

$$\psi_r(x) \sim \int_{0-}^{\infty} \overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}(xe^{r(u+t)})dH(u)dEN(t). \tag{1.5}$$

Motivated by Li's result in [12], in the paper we further consider the delayed-claim risk model (1.1) perturbed by diffusion, in which the claim sizes (including main claims and by-claims) are PQAI and consistently-varying-tailed, the inter-arrival times of main claims satisfy WLOD structure, and the premium income follows a general stochastic process. In our main result, we will discuss two cases, one is that the premium process $\{C(t), t \ge 0\}$ is independent of the other sources of randomness, and the other is that $\{C(t), t \ge 0\}$ is not necessarily so.

In the rest part of this paper, we will present the main result in Section 2, and prove it in Section 3 after preparing some lemmas.

2. Main result

In this section, we state our main result. For a distribution V and any $y > 0$, we set

$$J_V^+ = -\lim_{y\to\infty} \log \overline{V}_*(y)/\log y \quad \text{and} \quad J_V^- = -\lim_{y\to\infty} \log \overline{V}^*(y)/\log y.$$

Assume that the total discounted amount of premiums is finite, namely,

$$0 \le \widetilde{C} = \int_{0-}^{\infty} e^{-rs} dC(s) < \infty \quad \text{almost surely.} \tag{2.1}$$

Our main result is given below.

Theorem 2.1. *Consider the delayed-claim risk model* (1.1)*, in which the claim sizes* $\{X_i, Y_i, i \ge 1\}$ *are PQAI r.v.s, random pairs* $\{(X_i, Y_i), i \ge 1\}$ *are identically distributed by marginal distributions* $F \in \mathcal{C}$ *and* $G \in \mathcal{C}$ *with* $J_F^- > 0$ *and* $J_G^- > 0$*, respectively, and* $\{X_i, Y_i, i \ge 1\}$, $\{\theta_i, i \ge 1\}$ *and* $\{T_i, i \ge 1\}$ *are mutually independent. Assume that the inter-arrival times of main claims* $\{\theta_i, i \ge 1\}$ *are WLOD r.v.s such that*

$$\lim_{n\to\infty} g_L(n) e^{-\epsilon n} = 0 \tag{2.2}$$

holds for every $\epsilon > 0$*, depending on* F*,* G*,* K *and* H*. Then relation* (1.5) *holds, if one of the following conditions is true*:

1. *the premium process* $\{C(t), t \ge 0\}$ *is independent of the other sources of randomness*;
2. *the total discounted amount of premiums satisfies*

$$P(\widetilde{C} > x) = o(1)\big(\overline{F}(x) + \overline{G}(x)\big). \tag{2.3}$$

Remark 2.1. As was pointed out by Tang [16], condition (2.3), which does not require the independence between the premium process and the other sources of randomness, allows for a more realistic case that the premium income varies as a deterministic or stochastic function of the insurer's current reserve.

3. Proof of Theorem 2.1

Before proving Theorem 2.1, we prepare the following lemmas. The first one is a direct result of Proposition 2.2.1 of Bingham et al. [1] or Lemma 3.5 of Tang and Tsitsiashvili [17].

Lemma 3.1. *If a distribution* $V \in \mathcal{D}$ *with* $J_V^- > 0$*, then for any* $0 < \hat{p} < J_V^- \le J_V^+ < p < \infty$*, there exist some* $C > 1$ *and* $D > 0$ *such that*

$$C^{-1}\Big(\frac{x}{y}\Big)^{\hat{p}} \le \frac{\overline{V}(y)}{\overline{V}(x)} \le C\Big(\frac{x}{y}\Big)^{p} \quad \textit{for all } x \ge y \ge D. \tag{3.1}$$

The second lemma is from Theorem 3.3(iv) of Cline and Samorodnitsky [3] and Lemma 2.5 of Wang et al. [19].

Lemma 3.2. *Let ξ and η be two independent r.v.s, where ξ is distributed by V, and η is nonnegative, not degenerate at 0 and satisfying $E\eta^p < \infty$ for some $p > J_V^+$.*

(1) *If $V \in \mathcal{D}$, then $P(\xi\eta > x) \asymp \overline{V}(x)$.*

(2) *If $V \in \mathcal{C}$, then the distribution of $\xi\eta$ still belongs to the class $\mathcal{C}$.*

The third lemma is a restatement of Theorem 2.1 of Li [12]. Also, see Lemma 3.2 of Gao and Yang [10]. It should be mentioned that the asymptotic formula in the lemma was first developed by Tang and Tsitsiashvili [18].

Lemma 3.3. *If $\{\xi_i, 1 \le i \le n\}$ are n PQAI and real-valued r.v.s with distributions $V_i \in \mathcal{C}$, $1 \le i \le n$, respectively, then for any fixed $0 < a \le b < \infty$,*

$$P\left(\sum_{i=1}^{n} c_i \xi_i > x\right) \sim \sum_{i=1}^{n} \overline{V}_i\left(\frac{x}{c_i}\right)$$

holds uniformly for all $(c_1, c_2, \ldots, c_n) \in [a, b]^n$.

The lemma below is due to Lemma 3.1 of Chen and Yuen [2] or Theorem 2.2 of Li [12].

Lemma 3.4. *Let $\{\xi_i, 1 \le i \le n\}$ be n PQAI and real-valued r.v.s with distributions $V_i \in \mathcal{D}$, $1 \le i \le n$, respectively, and $\{\eta_i, 1 \le i \le n\}$ be n nonnegative r.v.s, independent of $\{\xi_i, 1 \le i \le n\}$ and satisfying $E\eta_i^p < \infty$ for some $p > \bigvee_{i=1}^n J_{V_i}^+$, $1 \le i \le n$. Then $\{\xi_i\eta_i, 1 \le i \le n\}$ are still PQAI.*

Lemma 3.5. *If $\{\xi_i, \eta_i,\ i \ge 1\}$ are PQAI and real-valued r.v.s with ξ_i and η_i distributed by $V_i \in \mathcal{C}$ and $W_i \in \mathcal{C}$, $i \ge 1$, respectively, then $\{\zeta_i = \xi_i + \eta_i,\ i \ge 1\}$ are still PQAI r.v.s with distributions $U_i \in \mathcal{C}$, $i \ge 1$.*

Proof. By Theorem 2.5(ii) of Li ([12]), we know that $\{\zeta_i, i \ge 1\}$ are PQAI. By Lemma 3.3, it holds that

$$\overline{U}_i(x) \sim \overline{V}_i(x) + \overline{W}_i(x), \quad i \ge 1$$

which, along with $V_i \in \mathcal{C}$ and $W_i \in \mathcal{C}$, $i \ge 1$, leads to $U_i \in \mathcal{C}$, $i \ge 1$. □

The last lemma comes from Lemma 3.3 of Gao et al. [7].

Lemma 3.6. *Consider the main-claim arrival process $\{N(t), t \ge 0\}$ with WLOD inter-arrival times $\{\theta_i, i \ge 1\}$ satisfying (2.2) for every $\epsilon > 0$. Then for any fixed $t > 0$ and any $p > 0$,*

$$E(N(t))^p < \infty.$$

Now we proceed to prove Theorem 2.1.

Proof of Theorem 2.1. By the reserve process (1.1), we get its discounted value as

$$\widetilde{R}(t) = x + \int_{0-}^{t} e^{-rs} dC(s) - \sum_{i=1}^{\infty} X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t\}} - \sum_{i=1}^{\infty} Y_i e^{-r(\tau_i+T_i)} \mathbf{1}_{\{\tau_i+T_i \le t\}} + \sigma \int_{0-}^{t} e^{-rs} dB(s), \quad t \ge 0. \tag{3.2}$$

Then by (1.2), it follows that

$$\psi_r(x) = P\left(\widetilde{R}(t) < 0 \ \text{ for some } t \ge 0\right). \tag{3.3}$$

Let $\widetilde{B} = \sigma \sup_{t\in[0,\infty]} |\int_{0-}^{t} e^{-rs} dB(s)|$. It is well-known that the stochastic integral $\int_{0-}^{t} e^{-rs} dB(s)$, $0 < t \le \infty$, follows a normal distribution with mean 0 and variance $\int_0^t e^{-2rs} ds$, then by many classic martingale inequalities, $\widetilde{B}$ has finite moments of arbitrary orders. So if a distribution $U \in \mathcal{D}$, then

$$P(\widetilde{B} > x) = o(1)\overline{U}(x). \tag{3.4}$$

Clearly, by (3.2) we get

$$\widetilde{R}(t) \ge x - \sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i}\right) e^{-r\tau_i} - \widetilde{B}, \quad t \ge 0,$$

which, along with (3.3), implies that

$$\psi_r(x) \le P\left(\sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i}\right) e^{-r\tau_i} + \widetilde{B} > x\right). \tag{3.5}$$

On the other hand, again by (3.2) we have

$$\widetilde{R}(t) \le x + \widetilde{C} - D_r(t) + \widetilde{B}, \quad t \ge 0,$$

where $D_r(t) = \sum_{i=1}^{\infty} X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \le t\}} + \sum_{i=1}^{\infty} Y_i e^{-r(\tau_i+T_i)} \mathbf{1}_{\{\tau_i+T_i \le t\}}$, and $\widetilde{C}$ is defined in (2.1). Hence by (3.3), we derive that

$$\begin{aligned} \psi_r(x) &\ge P\left(D_r(t) - \widetilde{B} > x + \widetilde{C} \quad \text{for some } t \ge 0\right) \\ &= P\left(\bigcup_{t\ge 0} \left\{D_r(t) - \widetilde{B} > x + \widetilde{C}\right\}\right) \\ &\ge P\left(D_r(\infty) - \widetilde{B} > x + \widetilde{C}\right) \\ &= P\left(\sum_{i=1}^{\infty} \left(X_i + Y_i e^{-rT_i}\right) e^{-r\tau_i} - \widetilde{B} > x + \widetilde{C}\right). \end{aligned} \tag{3.6}$$

Firstly, we deal with the asymptotic upper-bound of $\psi_r(x)$. By Lemmas 3.2(2) and 3.4, one can easily see that the common distribution of $Y_i e^{-rT_i}$, $i \ge 1$, belongs to the class $\mathcal{C}$, and $\{X_i, Y_i e^{-rT_i}, i \ge 1\}$ are PQAI. Then by

Lemma 3.5, we know that $X_i + Y_i e^{-rT_i}$, $i \geq 1$, are also PQAI and identically distributed by $U \in \mathcal{C}$. So, following the proof of Lemma 3.5 of Gao and Liu [8] or going along the similar lines of the proof of Lemma 3.5 of Gao and Jin [6], there exists a positive integer n_0 such that for any $0 < v < 1$,

$$P\left(\sum_{i=n_0+1}^{\infty}\Big(X_i + Y_i e^{-rT_i}\Big)e^{-r\tau_i} > \frac{vx}{2}\right) = o(1)P\Big(\big(X_1 + Y_1 e^{-rT_1}\big)e^{-r\tau_1} > x\Big). \tag{3.7}$$

Note that the common distribution of $X_i + Y_i e^{-rT_i}$, $i \geq 1$, belongs to $\mathcal{C}$, then again by Lemma 3.2(2), the distributions of $(X_i + Y_i e^{-rT_i})e^{-r\tau_i}$, $i \geq 1$, also belong to $\mathcal{C}$. Hence for any given $\varepsilon > 0$, there exists a number v_0, $0 < v_0 < 1$, such that for all large x,

$$\sum_{i=1}^{n_0} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > (1 - v_0)x\Big) \leq (1+\varepsilon)\sum_{i=1}^{n_0} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x\Big). \tag{3.8}$$

Let n_0 and v_0 be fixed as above. By (3.5), we have

$$\begin{aligned}\psi_r(x) &\leq P\left(\sum_{i=1}^{n_0}\Big(X_i + Y_i e^{-rT_i}\Big)e^{-r\tau_i} > (1 - v_0)x\right) \\ &\quad + P\left(\sum_{i=n_0+1}^{\infty}\Big(X_i + Y_i e^{-rT_i}\Big)e^{-r\tau_i} > \frac{v_0 x}{2}\right) + P\left(\widetilde{B} > \frac{v_0 x}{2}\right) \\ &= \sum_{i=1}^{3} H_i.\end{aligned} \tag{3.9}$$

For H_1, by Theorem 3.2 of Chen and Yuen [2] and (3.8), we show that for all large x,

$$\begin{aligned}H_1 &\sim \sum_{i=1}^{n_0} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > (1 - v_0)x\Big) \\ &\leq (1+\varepsilon)\sum_{i=1}^{n_0} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x\Big).\end{aligned}$$

For H_2, by (3.7) with $v = v_0$, we get

$$H_2 = o(1)P\Big(\big(X_1 + Y_1 e^{-rT_1}\big)e^{-r\tau_1} > x\Big).$$

For H_3, by (3.4), $U \in \mathcal{C} \subset \mathcal{D}$ and Lemma 3.2(1), we obtain

$$H_3 = o(1)\Big(\big(X_1 + Y_1 e^{-rT_1}\big)e^{-r\tau_1} > x\Big).$$

Therefore, substituting the derivations for H_i, $i = 1, 2, 3$, into (3.9) and considering the arbitrariness of $\varepsilon > 0$ can imply that

$$\psi_r(x) \lesssim \sum_{i=1}^{\infty} P\Big(\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x\Big)$$

$$\sim \int_{0-}^{\infty} \overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty} \overline{G}(xe^{r(u+t)})dH(u)dEN(t), \tag{3.10}$$

where in the second step we used Lemma 3.3.

Subsequently, we turn to the asymptotic lower-bound of $\psi_r(x)$. As mentioned above, $X_i + Y_i e^{-rT_i}$, $i \geq 1$, are identically distributed by $U \in \mathcal{C} \subset \mathcal{D}$. Thus by (3.1), it holds that for all $x \geq D$ and any $0 < t < \infty$,

$$\frac{\int_t^{\infty} \overline{U}(xe^{rs})dEN(s)}{\int_{0-}^{\infty} \overline{U}(xe^{rs})dEN(s)} = \frac{\int_t^{\infty} \overline{U}(xe^{rs})/\overline{U}(x)dEN(s)}{\int_{0-}^{\infty} \overline{U}(xe^{rs})/\overline{U}(x)dEN(s)} \leq C^2 \frac{\int_t^{\infty} e^{-r\hat{p}s}dEN(s)}{\int_{0-}^{\infty} e^{-rps}dEN(s)}. \tag{3.11}$$

By (1.3), we get

$$\int_{0-}^{\infty} e^{-rps}dEN(s) = \sum_{n=1}^{\infty} E(e^{-rp\tau_n}) \leq \sum_{n=1}^{\infty} g_L(n)(Ee^{-rp\theta_1})^n.$$

Take $\epsilon = -\log(Ee^{-rp\theta_1}) - c$ for some $c > 0$ in (2.2), then there exists a positive integer n_1 such that for all $n \geq n_1$,

$$g_L(n) \leq e^{-cn}\exp\{-n\log(Ee^{-rp\theta_1})\}.$$

Thus, we have

$$\int_{0-}^{\infty} e^{-rps}dEN(s) \leq \sum_{n=1}^{n_1-1} g_L(n)\left(Ee^{-rp\theta_1}\right)^n + \sum_{n=n_1}^{\infty} e^{-cn} < \infty.$$

Similarly,

$$\int_{0-}^{\infty} e^{-r\hat{p}s}dEN(s) < \infty.$$

Hence, the third item of (3.11) tends to 0 as $t \to \infty$, which yields that for any given $\varepsilon > 0$, there exists a number t_0, $0 < t_0 < \infty$, such that for all $x \geq D$,

$$\int_{t_0}^{\infty} \overline{U}(xe^{rt})dEN(t) \leq \varepsilon \int_{0-}^{\infty} \overline{U}(xe^{rt})dEN(t). \tag{3.12}$$

Write $\widetilde{D}_r(t_0) = \sum_{i=1}^{\infty}(X_i + Y_i e^{-rT_i})e^{-r\tau_i}\mathbf{1}_{\{\tau_i \leq t_0\}}$, where t_0 is fixed as above. Now we estimate the asymptotic lower-bound of $P(\widetilde{D}_r(t_0) > x)$, which will be used in the proof below. For an arbitrarily fixed integer m, we derive from Lemma 3.3 that

$$P(\widetilde{D}_r(t_0) > x) \geq \sum_{n=1}^{m} P\left(\sum_{i=1}^{n}\big(X_i + Y_i e^{-rT_i}\big)e^{-r\tau_i} > x, N(t_0) = n\right) \tag{3.13}$$

$$= \sum_{n=1}^{m} \int_{\{0<t_1\le t_2\le\cdots\le t_n\le t_0,\ t_{n+1}>t_0\}} P\left(\sum_{i=1}^{n} \left(X_i + Y_i e^{-rT_i}\right)e^{-rt_i} > x\right) dG(t_1,t_2,\ldots,t_{n+1})$$

$$\sim \left(\sum_{n=1}^{\infty} - \sum_{n=m+1}^{\infty}\right) \sum_{i=1}^{n} P\Big(\left(X_i + Y_i e^{-rT_i}\right)e^{-r\tau_i} > x, N(t_0)=n\Big)$$

$$= \sum_{i=1}^{\infty} P\Big(\left(X_i + Y_i e^{-rT_i}\right)e^{-r\tau_i}\mathbf{1}_{\{\tau_i\le t_0\}} > x\Big) - H_4,$$

where $G(t_1,t_2,\ldots,t_{n+1})$ is the joint distribution of $(\tau_1,\tau_2,\ldots,\tau_{n+1})$, $1\le n\le m$. For H_4, it follows from Lemma 3.6 that

$$H_4 \le \overline{U}(x)\sum_{n=m+1}^{\infty} nP(N(t_0)=n)$$
$$= \overline{U}(x)EN(t_0)\mathbf{1}_{\{N(t_0)>m\}} = o(1)\overline{U}(x) \quad \text{as } m\to\infty,$$

which, along with $U\in\mathcal{C}\subset\mathcal{D}$ and Lemma 3.2(1), yields that

$$H_4 = o(1)P\Big(\left(X_1 + Y_1 e^{-rT_1}\right)e^{-r\tau_1}\mathbf{1}_{\{\tau_1\le t_0\}} > x\Big). \tag{3.14}$$

So, we substitute (3.14) into (3.13) to obtain that

$$P\left(\widetilde{D}_r(t_0) > x\right) \gtrsim \sum_{i=1}^{\infty} P\Big(\left(X_i + Y_i e^{-rT_i}\right)e^{-r\tau_i}\mathbf{1}_{\{\tau_i\le t_0\}} > x\Big)$$
$$= \int_{0-}^{t_0} \overline{U}(xe^{rt})dEN(t). \tag{3.15}$$

Under condition 1 of Theorem 2.1, we show from (3.6) and (3.15) that for all large $x\ge D$,

(3.16) $\psi_r(x)$

$$\ge P\left(\widetilde{D}_r(t_0) > x + \widetilde{C} + \widetilde{B}\right)$$
$$\gtrsim \int_{0-}^{\infty}\int_{0-}^{t_0} \overline{U}((x+y)e^{rt})dEN(t)P\left(\widetilde{C}+\widetilde{B}\in dy\right)$$
$$\sim \int_{0-}^{t_0} \overline{U}(xe^{rt})dEN(t)$$
$$= \left(\int_{0-}^{\infty} - \int_{t_0}^{\infty}\right)\overline{U}(xe^{rt})dEN(t)$$
$$\ge (1-\varepsilon)\left(\int_{0-}^{\infty}\overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty}\overline{G}(xe^{r(u+t)})dH(u)dEN(t)\right),$$

where the third step is due to $U\in\mathcal{C}\subset\mathcal{L}$ and the local uniformity of (1.4), and the last step is due to (3.12) and Lemma 3.3. Therefore, by (3.10), (3.16) and the arbitrariness of $\varepsilon > 0$, we obtain that relation (1.5) holds under condition 1 of this theorem.

Under condition 2 of Theorem 2.1, we know from $U \in \mathcal{C}$ that for the given $\varepsilon > 0$, there exists a number $\delta_0 > 0$ such that for all large x,

$$\overline{U}((1+\delta_0)x) \geq (1-\varepsilon)\overline{U}(x). \tag{3.17}$$

By (3.6), we see that for $\delta_0 > 0$ as above,

$$\psi_r(x) \geq P\left(\widetilde{D}_r(t_0) - \widetilde{B} > (1+\delta_0)x\right) - P\left(\widetilde{C} > \delta_0 x\right) = H_5 - H_6. \tag{3.18}$$

For H_5, arguing as (3.16) and using (3.17) leads to that for all large x,

(3.19)

$$\begin{aligned} H_5 &\gtrsim \int_{0-}^{t_0} \overline{U}((1+\delta_0)xe^{rt})dEN(t) \\ &\geq (1-\varepsilon)\left(\int_{0-}^{\infty} - \int_{t_0}^{\infty}\right)\overline{U}(xe^{rt})dEN(t) \\ &\geq (1-\varepsilon)^2\left(\int_{0-}^{\infty}\overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty}\overline{G}(xe^{r(u+t)})dH(u)dEN(t)\right), \end{aligned}$$

where the last step is due to (3.12) and Lemma 3.3. For H_6, by (2.3), $F \in \mathcal{C} \subset \mathcal{D}$, $G \in \mathcal{C} \subset \mathcal{D}$ and Lemma 3.2(1), we have

(3.20)

$$\begin{aligned} H_6 &= o(1)\left(\overline{F}(x) + \overline{G}(x)\right) \\ &= o(1)\left(P(X_1e^{-r\tau_1} > x) + P(Y_1e^{-r(\tau_1+T_1)} > x)\right) \\ &= o(1)\sum_{i=1}^{\infty}\left(P(X_ie^{-r\tau_i} > x) + P(Y_ie^{-r(\tau_i+T_i)} > x)\right) \\ &= o(1)\left(\int_{0-}^{\infty}\overline{F}(xe^{rt})dEN(t) + \int_{0-}^{\infty}\int_{0-}^{\infty}\overline{G}(xe^{r(u+t)})dH(u)dEN(t)\right). \end{aligned}$$

Consequently, by (3.10), (3.18)-(3.20) and the arbitrariness of $\varepsilon > 0$, we arrive at relation (1.5) under condition 2 of the theorem, and hence the proof is completed. □

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References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[2] Y. Chen and K. C. Yuen, *Sums of pairwise quasi-asymptotically independent random variables with consistent variation*, Stoch. Models **25** (2009), no. 1, 76–89.

[3] D. B. H. Cline and G. Samorodnitsky, *Subexponentiality of the product of independent random variables*, Stochastic Process. Appl. **49** (1994), no. 1, 75–98.

[4] P. Embrechts, C. Klüppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, 1997.

[5] Q. Gao and D. Bao, *Asymptotic ruin probabilities in a generalized jump-diffusion risk model with constant force of interest*, J. Korean Math. Soc. **51** (2014), no. 4, 735–749.

[6] Q. Gao and N. Jin, *Randomly weighted sums of pairwise quasi upper-tail independent increments with application to risk theory*, Comm. Statist. Theory Methods (2015), 10.1080/03610926.2013.851234.

[7] Q. Gao, N. Jin, and H. Shen, *Asymptotic behavior of the finite-time ruin probability with pairwise quasi-asymptotically independent claims and constant interest force*, Rocky Mountain J. Math. **44** (2014), no. 5, 1505–1528.

[8] Q. Gao and X. Liu, *Uniform asymptotics for the finite-time ruin probability with upper tail asymptotically independent claims and constant force of interest*, Statist. Probab. Lett. **83** (2013), no. 6, 1527–1538.

[9] Q. Gao and X. Yang, *Asymptotic ruin probabilities in a generalized bidimensional risk model perturbed by diffusion with constant force of interest*, J. Math. Anal. Appl. **419** (2014), no. 2, 1193–1213.

[10] Q. Gao and Y. Yang, *Uniform asymptotics for the finite-time ruin probability in a general risk model with pairwise quasi-asymptotically independent claims and constant interest force*, Bull. Korean Math. Soc. **50** (2013), no. 2, 611–626.

[11] J. Geluk and Q. Tang, *Asymptotic tail probabilities of sums of dependent subexponential random variables*, J. Theoret. Probab. **22** (2009), no. 4, 871–882.

[12] J. Li, *On pairwise quasi-asymptotically independent random variables and their applications*, Statist. Probab. Lett. **83** (2013), no. 9, 2081–2087.

[13] J. Li and R. Wu, *The Gerber-Shiu discounted penalty function for a compound binomial risk model with by-claims*, Acta Math. Appl. Sin. Engl. Ser. **31** (2015), no. 1, 181–190.

[14] X. Liu, Q. Gao, and Y. Wang, *A note on a dependent risk model with constant interest rate*, Statist. Probab. Lett. **82** (2012), no. 4, 707–712.

[15] H. Meng and G. Wang, *On the expected discounted penalty function in a delayed-claims risk model*, Acta Math. Appl. Sin. Engl. Ser. **28** (2012), no. 2, 215–224.

[16] Q. Tang, *Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation*, Scand. Actuar. J. **2005** (2005), no. 1, 1–5.

[17] Q. Tang and G. Tsitsiashvili, *Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks*, Stochastic Process. Appl. **108** (2003), no. 2, 299–325.

[18] ———, *Randomly weighted sums of subexponential random variables with application to ruin theory*, Extremes **6** (2003), no. 3, 171–188.

[19] D. Wang, C. Su, and C. Zeng, *Uniform estimate for maximum of randomly weighted sums with applications to insurance risk theory*, Sci. China Ser. A **48** (2005), no. 10, 1379–1394.

[20] K. Wang, Y. Wang, and Q. Gao, *Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate*, Methodol. Comput. Appl. Probab. **15** (2013), no. 1, 109–124.

[21] Y. Wang and D. Cheng, *Basic renewal theorems for a random walk with widely dependent increments and their applications*, J. Math. Anal. Appl. **384** (2011), no. 2, 597–606.

[22] H. R. Waters and A. Papatriandafylou, *Ruin probabilities allowing for delay in claims settlement*, Insurance Math. Econom. **4** (1985), no. 2, 113–122.
[23] Y. Xiao and J. Guo, *The compound binomial risk model with time-correlated claims*, Insurance Math. Econom. **41** (2007), no. 1, 124–133.
[24] H. Xie and W. Zou, *Expected present value of total dividends in a delayed claims risk model under stochastic interest rates*, Insurance Math. Econom. **46** (2010), no. 2, 415–422.
[25] ———, *On the expected discounted penalty function for the compound Poisson risk model with delayed claims*, J. Comput. Appl. Math. **235** (2011), no. 8, 2392–2404.
[26] K. C. Yuen and J. Guo, *Ruin probabilities for time-correlated claims in the compound binomial model*, Insurance Math. Econom. **29** (2001), no. 1, 47–57.
[27] K. C. Yuen, J. Guo, and K. W. Ng, *On ultimate ruin in a delayed-claims risk model*, J. Appl. Probab. **42** (2005), no. 1, 163–174.
[28] W. Zou and H. Xie, *On the probability of ruin in a continuous risk model with delayed claims*, J. Korean Math. Soc. **50** (2013), no. 1, 111–125.

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