Imperial/TP/87-88/37
CERN. TH-5136/88

The Ultraviolet Properties of Supersymmetric Field Theories

P.S. Howe

Department of Mathematics
University of London King's College
London WC2, England

and

K.S. Stelle

CERN — Geneva, Switzerland
and
The Blackett Laboratory
Imperial College
London SW7, England

ABSTRACT

We review the structure of ultraviolet divergence cancellations in supersymmetric field theories. We discuss the various non-renormalization theorems of superspace perturbation theory, both for extended and for simple supersymmetry. These theorems and the background field method are applied to super Yang-Mills theories in four and higher dimensions, to supergravity theories and to two-dimensional supersymmetric non-linear σ-models.

I — INTRODUCTION

It has been known since 1930 that radiative corrections to quantum field theories contain ultraviolet divergences [1]. At first, it was not clear whether these were simply an artifact of the perturbative expansion or whether they represented a difficulty inherent in the structure of the theory. It was the discovery in 1947 [2] that there is a slight lifting of the degeneracy between the $2^2P_{1/2}$ and $2^2P_{3/2}$ levels of the hydrogen atom that made it clear that there are finite physical consequences of the radiative corrections despite the presence of divergences. Thereafter, the theory of renormalization was rapidly developed in a series of famous papers [3]. The consequences of renormalization for the behaviour of the theory at different energy scales were later made clear through the development of the renormalization group equations [4].

Renormalizable and asymptotically free quantum field theories have been tremendously successful in explaining the structure of the strong, weak and electromagnetic interactions. Nevertheless, there is an almost universal feeling among physicists that the fundamental theory should be a finite one. Whatever the nature of this ultimate theory, it is to be expected that renormalization-group methods will remain important in extracting the low-energy physics. This effective low-energy renormalizable theory should arise as a leading approximation to the full theory after integrating out the high mass excitations. Obviously, the treatment of gravity in such an effective theory poses special problems, since there is no unitary renormalizable field theory of gravity interacting with a finite number of other fields. One response to these difficulties is to treat processes involving gravity semiclassically. The inclusion of gravity-matter interactions is of particular importance for the spontaneous breaking of supersymmetry [5], where supergravity coupling is essential to produce realistic models. A key motivation for incorporating supersymmetry is to stabilize the hierarchy problem, in which the renormalization-group equations would otherwise require an impossibly precise fine tuning of the renormalized parameters of a model.

The key property of supersymmetric theories that makes them candidates for solving the hierarchy problem is their softer ultraviolet behaviour than conventional theories. The original example of such softening is in the Wess-Zumino model [6], where à priori one might expect there to be six independent renormalization constants. Early calculations [7] showed that supersymmetric invariance of the counterterms reduces this number to three and that there is a further reduction to just one independent renormalization constant due to less obvious implications of supersymmetry. This further reduction eliminates the quadratic mass renormalization and relates the coupling constant and mass renormalizations to the wavefunction renormalization; this was the first example of a supersymmetric non-renormalization theorem [8]. This result was later shown to be a natural consequence of manifestly supersymmetric superspace Feynman rules [9].

Supersymmetric Yang-Mills theories were also found at an early stage to have a softening of their ultraviolet behaviour. There are supersymmetric Yang-Mills theories with $N = 1$ [10], $N = 2$ [11] and $N = 4$ [12] supersymmetries. In all of these cases, supersymmetric invariance of the counterterms permits only a single coupling-constant renormalization for a pure super Yang-Mills theory without supermatter. The situation is more dramatic for the $N = 4$ case, where the coupling-constant renormalization was found by explicit calculation to vanish at 1 [13], 2 [14] and 3 [15] loops. These results led many people to conjecture that the $N = 4$ theory would be finite to all orders. An interesting suggestion [16] was that the $N = 4$ theory might be a self-dual theory in the sense that its monopole sector might be described in some phase by an $N = 4$ massless gauge theory. Due to the Dirac quantization condition for the monopole charge, this would be inconsistent with asymptotic freedom except in the limiting case of a finite theory.

The explanation for the special behaviour of the $N = 4$ theory has progressed through a number of stages. An elegant proposal for the underlying mechanism of these cancellations focused on the anomaly structure of the theory [17]. This argument assumed that the rigid $SU(4)$ symmetry associated with the supersymmetry algebra is preserved in the quantum theory. Supersymmetry, on the other hand, associates to the conformal anomaly an anomaly in some axial $U(1)$ current. The $U(1)$ axial symmetries of the $N = 4$ theory are all contained in the rigid $SU(4)$, however, and hence should remain anomaly-free. Hence, there should be no conformal anomaly, and consequently the β-function of the theory should vanish.

Another way of viewing the cancellations in the $N = 4$ theory is to combine the non-renormalization theorem of $N = 1$ supersymmetry with use of the background field method and an assumption that the rigid symmetry $SO(4) \subset SU(4)$ is preserved in the quantum theory [18]. The non-renormalization theorem of $N = 1$ supersymmetry and use of the background field method give two relations between the à priori four independent renormalization constants. The assumption that $SO(4)$ invariance is preserved gives a

further two relations, with the consequence that all the renormalization constants are unity.

Both the argument based upon the conformal anomaly structure and the $N = 1$ non-renormalization theorem argument make an unproven assumption about the preservation of classical symmetries in the $N = 4$ super Yang-Mills quantum theory. In order to try to understand the quantum structure of the $N = 4$ theory without making such an essential assumption, one needs to use a formalism for quantization that encodes manifestly as many as possible of the classical symmetries. In practice, this has meant learning how to quantize the theory in extended superfields of various types. Unfortunately, it is not possible to manifestly realize all of the extended supersymmetry in a superfield action with a finite number of auxiliary fields. This is the content of the "no-go" theorem for $N = 4$ super Yang-Mills theory [19,20]. This circumstance forces one to make a choice of those symmetries to be linearly and those to be non-linearly realized.

One choice of the linearly-realized symmetries is to require manifest Lorentz invariance, which is compatible in an action with a maximum of $N = 2$ extended supersymmetry if the number of fields is finite. In order to derive the full consequences of the extended superfield Feynman rules, it is essential to use the powerful background field method [21,22], in the form modified for use above the one-loop order [23,24]. Moreover, one has to use a form of the background field method that is adapted to superfield calculations [25]. This use of the background field method is at the heart of the superfield analysis of theories with extended supersymmetry. For example, in pure $N = 2$ super Yang-Mills theory, the methods of ref. [25] imply that the counterterms for graphs with external background lines only must be local functionals constructed from the spinorial part of the superspace Yang-Mills connection $\mathcal{A}^i_\alpha$. The superspace Feynman rules also imply that the counterterms must be written as integrals over the full $N = 2$ superspace. However, these restrictions do not simultaneously apply in the specific case of the one-loop graphs, where complications due to gauge fixing and ghosts weaken the implications of the superspace Feynman rules for gauge multiplets. In the case of $N = 2$ supersymmetry, the full superspace measure $d^4x d^8\theta$ has dimension zero, and since $\mathcal{A}^i_\alpha$ has dimension $\frac{1}{2}$ in units of mass, there are no admissible superspace counterterms for $N = 2$ super Yang-Mills theory above the one-loop order.

In terms of $N = 2$ supermultiplets, the $N = 4$ super Yang-Mills theory consists of the $N = 2$ super Yang-Mills theory coupled to an $N = 2$ massless matter hypermultiplet in the adjoint representation of the gauge group [26]. An important step in understanding the superspace quantization of the $N = 4$ theory was the construction of an off-shell unconstrained $N = 2$ superfield formulation of the hypermultiplet [27]. The procedure for quantizing the $N = 2$ super Yang-Mills theory also had to be developed. As with all superspace gauge theories, the $N = 2$ superspace connection $\mathcal{A}_A = (\mathcal{A}^i_\alpha, \mathcal{A}_{\dot{\alpha}i}, \mathcal{A}_a;\ \ a = 0,\ldots,3\ \ \alpha,\dot{\alpha} = 1,2\ \ i = 1\ldots N(=2))$ satisfies constraints [28] that must be solved in terms of an unconstrained "prepotential" which is to be used as the quantum superfield. The solution to these constraints was found for the Abelian theory in ref. [29] and later extended to the non-Abelian case [30,31]. The special features of the one-loop graphs show up in this formalism as the impossibility of using a completely background-gauge-covariant gauge fixing procedure without introducing an infinite series of "ghosts for ghosts" [32]. This problem can be confined to the one-loop level by a judicious choice of gauge-fixing condition for a ghost field that does not appear in the higher loop graphs [30]. In the end, the $N = 2$ non-renormalization theorem can be shown to apply both to the $N = 2$ super Yang-Mills multiplet and to the hypermultiplet. As a consequence of this non-renormalization theorem, the $N = 4$ theory has no allowable counterterms above the one-loop order, and it can be verified explicitly that there are no divergences at the one-loop order.

An alternative procedure in quantizing the $N = 4$ theory is to sacrifice the manifest covariance under the full Lorentz group and work in a light-cone superspace formalism [33]. The light-cone superfields involve only physical degrees of freedom and are thus more manageable than the $N = 2$ extended superfields. Moreover, the internal $SU(4)$ symmetry remains linearly realized. Using these superfields, it can be shown by detailed analysis of the Feynman graphs that the $N = 4$ theory is finite [34].

A third type of extended superfield formulation of the $N = 4$ theory uses harmonic superspace [35]. In this formulation, the $N = 4$ theory may be treated either as an $N = 2$ or as an $N = 3$ theory. The full Lorentz invariance is manifestly maintained at the cost of an infinite number of auxiliary fields. For quantization in terms of $N = 2$ harmonic superfields, the full details of the Feynman rules have been worked out [36]. In this case it is again found that the counterterms have to be written in terms of integrals over the full superspace. The use of the background field method with harmonic superfields is complicated at the one-loop level by the infinite number of auxiliary fields. However, formulation of the Feynman rules in terms of the $N = 3$ superfields should show the theory to be finite by power counting without use of the background field method.

Of the various methods for quantizing extended supersymmetric theories, the Lorentz-covariant superfields together with use of the background field method provide the clearest perspective on the general structure of the ultraviolet divergences. The methods of refs. [27,30] and of refs. [35,36] both make it clear that all $N = 2$ renormalizable supersymmetric theories will have no divergences beyond the one-loop order. Although the light-cone superspace method of refs. [33,34] is more manageable for the $N = 4$ super Yang-Mills theory, it is not known how to apply it to understand the cancellations of divergences beyond one loop in the $N = 2$ models. If the matter sector of an $N = 2$ model is chosen appropriately, the gauge coupling constant renormalization can be made to vanish by construction at one loop, and the non-renormalization theorem then guarantees finiteness to all orders [37]. Strictly speaking, to apply the non-renormalization theorem to an $N = 2$ model according to the construction of ref. [27], the hypermultiplet should be in a real representation of the gauge group. In fact, there is a "no-go theorem" here, which states that a complex hypermultiplet requires an infinite number of auxiliary fields [38]. Harmonic superspace, however, provides an adequate framework for the quantization of such models.

Lorentz-covariant superfield techniques also give the correct picture of the divergence structures in non-renormalizable theories. For example, super Yang-Mills theories in dimensions $d > 4$ are non-renormalizable. The loop order at which divergences can first occur is correctly given by the extended superspace power counting after taking into account all the other symmetry requirements of counterterms. One finds that the maximal super Yang-Mills theory can diverge at $l = 4, 3\,\&\,2$ loops for $d = 5, 6\,\&\,7$ respectively [39]. Explicit calculations [40] show indeed that the $d = 6$ theory is still finite at the two-loop order, while the $d = 7$ theory diverges for the first time at that order, confirming the completeness of the superspace power counting. Performing a similar analysis for all the extended (including $N = 1$) pure supergravity theories in $d = 4$ shows that they should all prove to be divergent from the three-loop order onward. Supersymmetric counterterms corresponding to the three-loop order are known to exist for all the pure supergravity theories. For the $N = 1$ case, the counterterm was first given in ref. [41]; for the $N = 8$ theory, the counterterm was constructed in terms of $N = 8$ superfields in refs. [42,43]. This $N = 8$ invariant was written as the integral of a suitably constrained integrand over a subspace of the full $N = 8$ superspace. Subsequently, it was shown [30] that the $N = 8$ counterterm could be written as a full $N = 4$ superspace integral and that, in principle, the $N = 8$ theory could be quantized in $N = 4$ superfields [44], thus allowing the three-loop counterterm as the first divergence.

Recognition of the implications of extended supersymmetry for the cancellation of ultraviolet divergences in renormalizable theories in four dimensions led to the investigation of more general finite models. One generalization was to include mass terms with $N = 1$ supersymmetry in models based on the kinetic terms of $N = 4$ super Yang-Mills theory [45], then extended to cover models based on the more general class of ultraviolet-finite $N = 2$ super Yang-Mills theories [46]. By power counting, either using light-cone methods or through the use of "spurion" techniques [47] to induce the soft breaking of the extended supersymmetry, the presence of such mass terms can be shown not to upset the cancellations of the logarithmically-divergent coefficients of the kinetic terms of these models, so the finiteness of such models is preserved, since the mass terms themselves are covered by the non-renormalization theorem of $N = 1$ supersymmetry.

The indications that certain $N = 1$ supersymmetric models could be finite led to a search for more general models with ultraviolet cancellations. One can choose a representation of the Yang-Mills gauge group for the matter fields such that the one-loop gauge coupling constant β-function vanishes. One-loop finiteness can then be arranged by relating the Yukawa couplings to the gauge couplings so that the matter wavefunction renormalization constants are unity. Moreover, the cancellation of one-loop divergences turns out to imply the cancellation of two-loop divergences as well [48]. This procedure of exact loop-by-loop cancellation can be extended to a given loop order by constructing a suitably complicated initial theory [49]. However, for a given theory with kinetic terms that have only $N = 1$ supersymmetry, this exact loop-by-loop cancellation eventually has to break down.

Another approach to the construction of finite models with $N = 1$ supersymmetry is to abandon the requirement of loop-by-loop cancellation and to search for possible cancellations between different loop orders. For the $N = 1$ models that are finite at one loop, this can be done by an iterative procedure of relating the Yukawa couplings to power series (with $\hbar$-dependent coefficients) in the gauge coupling constant [50,51]. The demonstration that this can be carried through to all orders is by induction. Since this quantization is carried out using $N = 1$ superfields, it is also possible in this case to use the more mathematically rigorous BPHZ subtraction scheme, where the infrared problems of $N = 1$ superfield Yang-Mills theories have been overcome [52]. Using these techniques, the cancellations in the β-functions of the one-loop-finite $N = 1$ models can be proved to

be extendable to all orders [53]. This permits a treatment of the finite $N = 2$ and $N = 4$ theories in $N = 1$ superfields, where these theories are distinguished by the fact that the one-loop relations between the Yukawa couplings and the gauge coupling constant are not corrected at higher orders.

The whole class of $N = 1, 2 \& 4$ theories with vanishing β-functions furnishes explicit examples of perturbative fixed points of the renormalization group flows. A set of $N = 1$ theories of this type has been derived in refs. [54].

II — THE BACKGROUND FIELD METHOD

The background field method is an essential computational tool in gauge theories, gravity and non-linear σ-models. The basic idea is rather simple – the (total) field is split up into a classical background and a quantum field over which one integrates in the path integral. This device enables one to maintain manifestly the symmetries associated with the background, such as gauge invariance. Provided that there is an invariant regularization scheme available, the background symmetry greatly simplifies the analysis of the counterterms of the theory. This idea was first applied to Yang-Mills theories and to quantum gravity at the one-loop level [21] and was then modified for use in the non-linear σ-model (also at one loop) [22]. Subsequently, it was shown how to extend the technique to all orders in perturbation theory [23,24].

In Yang-Mills theory, the difference between two connections is a Lie-algebra-valued one-form (which transforms homogeneously), and in gravity, the difference between two metrics is also a symmetric second-rank tensor, so that linear background-quantum splits are convenient to use in these cases. In the non-linear σ-model, on the other hand, the field is a map from spacetime (or some other source space such as the world-sheet of the string) to the target space, $\mathcal{M}$, and so the difference between two such fields is not a nicely defined geometrical quantity. This difficulty is surmounted by using a non-linear split [55], which will be reviewed in §2.3. The outcome is an effective action which depends on the background field and which is invariant under local reparameterizations of the field space (i.e. the space of maps from spacetime to $\mathcal{M}$). Recently, similar ideas have been applied [56] to Yang-Mills theory and to gravity in order to construct effective actions that are reparameterization-invariant and thus fully gauge-independent. We shall not consider these important developments further here, since the standard background field method will be adequate for the treatment of ultraviolet divergence cancellations in the cases that we shall consider.

Manifest covariance in supersymmetric theories requires the use of a superspace formalism and this leads to some new features associated with the fact that there are often constraints present. This is well illustrated by $N = 1$ supersymmetric Yang-Mills theory in four spacetime dimensions, where the spinorial part of the superspace connection is a constrained superfield. There are two possibilities: one can either solve the constraints and then implement a (non-linear) background-quantum split, or one can make a linear split of the total connection and then solve the constraints on the quantum part, leaving the background in constrained form; both of these methods will be reviewed in §2.4. We begin by illustrating the background field technique using a simple example of ϕ^4 scalar field theory.

2.1 ϕ^4 theory in $d = 4$

Consider the action

$$I = \int d^4x \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right). \tag{2.1}$$

The generating functional for connected Green functions $W[J]$ is defined by

$$e^{iW[J]} = \int \mathcal{D}\phi \exp i(I[\phi] + \textstyle\int J[\phi]) \tag{2.2}$$

and is related to the generating functional for 1PI graphs $\Gamma[\hat{\phi}]$ by a Legendre transformation

$$W[J] = \Gamma[\hat{\phi}] + \int J\hat{\phi}. \tag{2.3}$$

In (2.3), the field $\hat{\phi}$ is the vacuum expectation value of the quantum field ϕ in the presence of the source J. Now split the total field ϕ into a background part φ and a quantum fluctuation part π,

$$\phi(x) = \varphi(x) + \pi(x) \tag{2.4}$$

and then define a new functional $\hat{W}[\varphi, J]$ by

$$e^{i\hat{W}[\varphi,J]} = \int \mathcal{D}\pi \exp i\left(I[\varphi+\pi] + \textstyle\int J\pi\right). \tag{2.5}$$

Then clearly

$$\hat{W}[\varphi, J] = W[J] - \int J\varphi. \tag{2.6}$$

If one takes the Legendre transform of $\hat{W}$ with respect to J, one gets a functional $\hat{\Gamma}$ given by

$$\hat{\Gamma}[\varphi, \hat{\pi}] = \hat{W}[\varphi, J] - \int J\hat{\pi}. \tag{2.7}$$

This functional is invariant under the trivial shift symmetry

$$\delta\varphi(x) = \eta(x), \quad \delta\pi(x) = -\eta(x), \tag{2.8}$$

from which one easily finds the Ward identity

$$\frac{\delta\hat{\Gamma}}{\delta\varphi}[\varphi, \hat{\pi}] = \frac{\delta\hat{\Gamma}}{\delta\hat{\pi}}[\varphi, \hat{\pi}], \tag{2.9}$$

i.e.

$$\hat{\Gamma}[\varphi, \hat{\pi}] = \hat{\Gamma}[\varphi + \hat{\pi}], \tag{2.10}$$

from which one obtains

$$\hat{\Gamma}[\varphi, 0] = \Gamma[\varphi]. \tag{2.11}$$

Equation (2.11) is the key equation; it states that the conventional 1PI effective action can be found by computing 1PI graphs with external background lines but no external quantum lines. It is easy to find a functional integral formula for $\Gamma[\varphi]$,

$$e^{i\Gamma[\varphi]} = \int \mathcal{D}\pi \exp i\left(I[\varphi+\pi] - \textstyle\int \pi \frac{\delta\Gamma}{\delta\varphi}[\varphi]\right). \tag{2.12}$$

So far, all our considerations have been formal and we have not taken into account the effects of renormalization. Expanding out the interaction term in the action, one finds

$$\frac{\lambda}{4!}\phi^4 = \frac{\lambda}{4!}(\varphi^4 + 4\varphi^3\pi + 6\varphi^2\pi^2 + 4\varphi\pi^3 + \pi^4) \tag{2.13}$$

and in principle the vertices on the right-hand side could be renormalized differently. Now, of course, this doesn't happen because of the linear-splitting Ward identity (2.9). Furthermore, from (2.9) we also find that the wavefunction renormalizations for the background and quantum fields are equal,

$$Z_\varphi = Z_\pi. \tag{2.14}$$

Thus, the Ward identity (2.9) has the consequence that the counterterms are functionals of the total field ϕ and may be deduced from graphs with no external quantum lines. In particular, the renormalization of the various vertices involving the quantum field is performed by renormalizing λ as deduced from graphs with no external quantum lines and then substituting the corresponding bare λ into the right-hand side of (2.13). The renormalization of the quantum field can similarly be deduced from graphs with only background external lines. However, these multiplicative quantum wavefunction renormalizations cancel out in graphs with only background external lines, as can easily be seen graphically or alternatively from the functional integral.

2.2 Yang-Mills theory

Let $\mathcal{A}_a$ be a gauge field taking its values in the Lie algebra of a gauge group $\mathcal{G}$,

$$\mathcal{A}_a = \mathcal{A}_a^r T_r; \quad (T_r)^\dagger = T_r \tag{2.15}$$

$$[T_r, T_s] = i f_{rst} T_t. \tag{2.16}$$

Under a gauge transformation, we have

$$\mathcal{A}_a \to S\mathcal{A}_a S^{-1} + \frac{i}{g} S\partial_a S^{-1} \tag{2.17}$$

where g is the gauge coupling constant and S is an x-dependent element of the gauge group. Infinitesimally,

$$\delta\mathcal{A}_a^r = D_a(\mathcal{A})u^r \tag{2.18}$$

for $S \cong 1 + igu^r T_r$.

If one splits the total field $\mathcal{A}_a$ into a background part B_a and a quantum part Q_a linearly,

$$\mathcal{A}_a^r = B_a^r + Q_a^r, \tag{2.19}$$

then the gauge transformation (2.18) can be interpreted as a background gauge transformation under which the quantum field transforms homogeneously,

$$\delta B_a^r = D_a(B)u^r; \quad \delta Q_a^r = g f^{rst} u^s Q_a^t \tag{2.20}$$

or as a quantum field transformation under which the background field B_a is invariant,

$$\delta B_a^r = 0; \quad \delta Q_a^r = D_a(\mathcal{A})u^r. \tag{2.21}$$

The covariant derivatives in (2.18), (2.20) and (2.21) are defined in an obvious way,

$$D_a(\mathcal{A}) = \partial_a - ig\mathcal{A}_a^r T_r; \quad D_a(B) = \partial_a - igB_a^r T_r. \tag{2.22}$$

In order to quantize the theory, we must first fix the gauge for the quantum fields. This can be done in such a way as to maintain the background gauge invariance (2.20). A convenient choice of gauge-fixing condition is

$$h^r = D^a(B)Q_a^r. \tag{2.23}$$

The action including gauge-fixing and ghost terms is then

$$I = \int d^4x \left(-\frac{1}{4} F_{ab}^r(\mathcal{A}) F^{r\,ab}(\mathcal{A}) - \frac{1}{2\alpha} h^r h^r + \bar{C}_r D_a(B) D^a(\mathcal{A}) C^r \right) \tag{2.24}$$

where F_{ab} is the usual Yang-Mills field strength tensor and C^r, $\bar{C}_r$ are the ghost and antighost fields respectively. The action (2.24) is manifestly invariant under the background gauge transformations (2.20), with the ghosts transforming homogeneously. The action is also invariant under BRS transformations corresponding to the quantum gauge transformations (2.21),

$$\begin{aligned} SB_a^r &= 0; \quad SQ_a^r = -D_a(\mathcal{A})C^r \\ S\bar{C}_r &= -\frac{1}{\alpha} h_r; \quad SC^r = \frac{1}{2} g f^{rst} C^s C^t. \end{aligned} \tag{2.25}$$

Although the gauge-invariant kinetic term $(-\frac{1}{4}F_{ab}^2)$ is left invariant under a shift transformation analogous to (2.8), $\delta B_a^r = -\delta Q_a^r = \eta_a^r$, the gauge-fixing and ghost terms are not invariant under this transformation. Nevertheless, a vestigial form of this symmetry is present for the modified action [57]

$$I' = I + \int d^4x M_a^r D_a(\mathcal{A}) \bar{C}^r, \tag{2.26}$$

where M_a^r is an anticommuting source. The modified action is invariant under modified BRS transformations where the additional terms in the variations are given by

$$\begin{aligned} (SB_a^r)^{\text{additional}} &= M_a^r; \quad (SQ_a^r)^{\text{additional}} = -M_a^r \\ SM_a^r &= 0. \end{aligned} \tag{2.27}$$

The effective action $\hat{\Gamma}[B, \hat{Q}]$ is now constructed in the usual way. However, it is essential to use the quantum BRS Ward identities to control the renormalization procedure. These identities imply that the quantum fields are multiplicatively renormalized. If an invariant regularization scheme such as dimensional regularization is used, then the counterterms will be background gauge invariant. In particular, this implies that there is only one possible counterterm involving the background field only, i.e. $-1/4F_{\mu\nu}^2(B)$, and hence we have

$$Z_B^{\frac{1}{2}} Z_g = 1. \tag{2.28}$$

This is a useful result, since it means that the coupling constant renormalization can be computed from the renormalization of the propagator. Furthermore, these can be determined at a given loop order ℓ from the divergence in $\hat{\Gamma}[B,0]$ at the ℓ'th loop order (the lower order divergences having been subtracted). There is a technical complication, however, because although the renormalizations of Q, C and $\bar{C}$ cancel out in graphs with no external quantum lines, the renormalization of Q must be taken into account because of the gauge-fixing term. This is unrenormalized, so that

$$Z_\alpha = Z_Q \tag{2.29}$$

(where $\alpha_{(0)} = Z_\alpha \alpha$, $Q_{(0)} = Z_Q^{1/2} Q$) and hence after a change of integration variable from Q to $Q_{(0)}$, one gets the term

$$\frac{-1}{2Z_\alpha \alpha} \int d^4x \, (D^a(B)Q_a^r)^2 \tag{2.30}$$

in the effective action. Since $\hat{\Gamma}[B,0]$ is not α-independent, the renormalization in (2.30) has to be taken into account (except in the gauge $\alpha = 0$). $\hat{\Gamma}[B,0]$ is related to the conventional 1PI Yang-Mills functional $\Gamma[\hat{A}]$ but in an unusual gauge. By manipulating the functional integral, one can show that

$$\hat{\Gamma}[B,0] = \Gamma_B[\hat{A}]\big|_{\hat{A}=B} \tag{2.31}$$

where $\Gamma_B[\hat{A}]$ is the effective action computed using the gauge-fixing term

$$h^r = \partial^a(\mathcal{A}_a^r - B_a^r) + g f^{rst} B^{as}(\mathcal{A}_a^t - B_a^t). \tag{2.32}$$

2.3 The non-linear σ-model

We now turn to the non-linear σ-model, which we shall discuss in two spacetime dimensions for two reasons: 1) these models are renormalizable only in two spacetime dimensions and 2) the two-dimensional theories are relevant for string theory. The non-linear σ-model is the prototype non-linear field theory, and knowing how to quantize

it properly will enable us to handle more general non-linear theories. We consider the simplest case with a set of bosonic scalar fields ϕ^i and the action

$$I = \int d^2x \frac{1}{2} g_{ij} \partial_a \phi^i \partial^a \phi^j. \tag{2.33}$$

Here ϕ is a map from two-dimensional space-time Σ to the σ-model target space $\mathcal{M}$, and g is a Riemannian metric on $\mathcal{M}$. The action (2.33) is invariant under diffeomorphisms f of $\mathcal{M}$ in the following sense:

$$I[f_* g, f \circ \phi] = I[g, \phi] \tag{2.34}$$

where f_* is the induced push-forward map on tensors. Infinitesimally, we have

$$\int \mathcal{L}_v g(\phi) \frac{\delta I}{\delta g(\phi)} - \int v^i(\phi) \frac{\delta I}{\delta \phi^i} = 0, \tag{2.35}$$

where v is the vector field generating the diffeomorphism and $\mathcal{L}_v$ denotes the Lie derivative,

$$(\mathcal{L}_v g)_{ij} = v^k \partial_k g_{ij} = (\partial_i v^k) g_{kj} + (\partial_j v^k) g_{ik}. \tag{2.36}$$

Note that (2.34) is only a symmetry of the usual type (with an associated conserved current) if f is an isometry of g; otherwise, it is a generalized symmetry which involves a variation of the parameters of the model (i.e. g) as well as the fields. Nonetheless, since two models with diffeomorphic metrics differ only by a field redefinition, they are physically equivalent (for example, they have the same S-matrix).

By power counting, the counterterms for the model defined by (2.31) have dimension two and naïvely one would expect them to have the form $\frac{1}{2}\int T_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j$, up to wave-function renormalizations. Each T_{ij} should be a tensorial quantity constructed from the metric, Riemann curvature tensor and the Levi-Civita covariant derivative. The original proof [58] that renormalization can be carried out in this fashion within the space of metrics proceeded by expanding the fields ϕ^i about a fixed *constant* point on the target space $\mathcal{M}$, and then showed that the resulting theory quantized in the tangent space to this point was independent of the original separation between the constant background and the quantum field. For most purposes, however, the most convenient framework in which to compute the counterterms is the background field method [21-24,55,57,59], which makes the covariance properties of the counterterms manifest.

If we split the total field ϕ linearly into a spacetime-dependent background field φ and a quantum field π, $\phi = \varphi + \pi$, then the resulting generating functional $\hat{W}[\varphi, J]$ defined by

$$e^{i\hat{W}[\varphi,J]} = \int \mathcal{D}\pi \exp i \left[I[\varphi + \pi] + \textstyle\int J_i \pi^i \right] \tag{2.37}$$

does not behave nicely under diffeomorphisms because π doesn't. It is therefore advantageous to introduce a non-linear split between the background and the quantum fields.

The idea is to represent a point $\phi \in \mathcal{M}$ by a point $(\varphi, \xi) \in T\mathcal{M}$ (the tangent bundle over $\mathcal{M}$) where ξ is a local fibre coordinate (and we have temporarily suppressed the x-dependence of the fields). In other words, we need a map $\rho : T\mathcal{M} \to \mathcal{M}$ such that

$$\rho^i(\varphi, \xi) = \phi^i. \tag{2.38}$$

Note that ρ is only locally defined; for each $\varphi \in \mathcal{M}$, ρ should be a one-to-one map from a neighborhood of the origin in $T_\varphi\mathcal{M}$ onto a neighborhood of $\varphi \in \mathcal{M}$. If we make an arbitrary variation of φ^i, there will be a corresponding variation of ξ^i such that ϕ^i is invariant:

$$\delta\varphi^i = \eta^i, \quad \delta\xi^i = -\eta^j\gamma_j^i(\varphi, \xi) \Rightarrow \delta\phi^i = 0, \tag{2.39}$$

from which we deduce

$$\partial_j\rho^i - \gamma_j^k\partial'_k\rho^i = 0 \tag{2.40}$$

where

$$\partial_i \equiv \frac{\partial}{\partial\varphi^i}, \quad \partial'_i \equiv \frac{\partial}{\partial\xi^i}. \tag{2.41}$$

The map ρ determines a flat non-linear connection in the tangent bundle as follows: define the horizontal vector fields locally by

$$h_i = \partial_i - \gamma_i^j\partial'_j \tag{2.42}$$

(note that the complementary basis for vertical vector fields is just ∂'_i); then from (2.40) one finds that the corresponding curvature

$$r_{ij}^k = 2\left\{\partial_{[i}\gamma_{j]}^k - \gamma_{[i}^s\partial'_s\gamma_{j]}^k\right\} \tag{2.43}$$

vanishes, where $[h_i, h_j] = -r_{ij}^k\partial'_k$.

Clearly, there are many possible choices of background-quantum field split of this type, for a change in ρ can be compensated by a change in the quantum field ξ; if the latter is vectorial, then the desirable properties of the split are maintained. The simplest choice, and the one that is almost always used in practice, is the one based on geodesics [55]. If we let $\Phi(s)$ be a geodesic connecting $\varphi = \Phi(0)$ with $\phi = \Phi(1)$, with the tangent vector along the geodesic at φ equal to ξ, then $\Phi(s)$ satisfies

$$\frac{d^2}{ds^2}\Phi^i(s) + \Gamma^i_{jk}(\Phi(s))\frac{d\Phi^j}{ds}\frac{d\Phi^k}{ds} = 0 \tag{2.44}$$

$$\Phi^i(0) = \varphi^i, \quad \Phi^i(1) = \phi^i, \quad \frac{d}{ds}\Phi^i(0) = \xi^i \tag{2.45}$$

where Γ^i_{jk} is the Levi-Civita connection. The solution to these equations gives ϕ in terms of φ and ξ as

$$\phi^i = \varphi^i + \xi^i - \sum_{n=2}^{\infty}\frac{1}{n!}\Gamma^i_{j_1\ldots j_n}\xi^{j_1}\ldots\xi^{j_n} \tag{2.46}$$

where

$$\Gamma^i_{j_1\ldots j_n} = \tilde\nabla_{(j_1}\tilde\nabla_{j_2}\ldots\tilde\nabla_{j_{n-2}}\Gamma^i_{j_{n-1}j_n)} \tag{2.47}$$

with $\tilde\nabla$ denoting the Levi-Civita covariant derivative applied to the lower indices only. The corresponding non-linear connection is

$$\gamma_i^j = \delta_i^j + \Gamma^j_{ik}\xi^k - \frac{1}{3}R^j{}_{k,li}\xi^k\xi^l + \mathcal{O}(\xi^3) \tag{2.48}$$

where all the higher terms are tensorial. Any other allowable ρ can be obtained by modifying the right-hand side of (2.44) with terms constructed from this curvature tensor or other covariant expressions.

The above discussion is trivially generalized to the case of fields $\varphi(x)$, $\xi(x)$ and $\phi(x)$, since no spacetime derivatives are involved (i.e. everything in this case is ultralocal). Since $\xi(x) \in T_{\varphi(x)}\mathcal{M}$, it follows that, as a field, ξ is a cross-section of $\varphi^*T\mathcal{M}$, the bundle obtained by pulling back the tangent bundle to Σ using the background field φ. (The fibre over $x \in \Sigma$ is identified with the fibre of $T\mathcal{M}$ over $\varphi(x) \in \mathcal{M}$.) Thus, the quantum field has a clean geometrical interpretation, as desired. We can now define a new generating functional $\hat W[\varphi, J]$ by

$$e^{i\hat W[\varphi,J]} = \int \mathcal{D}\xi \exp i\left[I[\phi] + \textstyle\int J_i\xi^i\right]. \tag{2.49}$$

The corresponding effective action is

$$\hat\Gamma[\varphi, \hat\xi] = \hat W[\varphi, J] - \int J_i\hat\xi^i \tag{2.50}$$

and we have a diffeomorphism Ward identity

$$\int \mathcal{L}_v g(\phi)\frac{\delta\hat\Gamma}{\delta g(\phi)} - \int \partial_k v^i\xi^k(x)\frac{\delta\hat\Gamma}{\delta\hat\xi^i(x)} - \int v^i(\varphi)\frac{\delta\hat\Gamma}{\delta\varphi^i(x)} = 0 \tag{2.51}$$

and a Ward identity for the shift symmetry (2.39) (where η now depends on x)

$$\frac{\delta\hat\Gamma}{\delta\varphi^i(x)} - (\gamma_i^j(x)\cdot\hat\Gamma)\frac{\delta\hat\Gamma}{\delta\hat\xi^j(x)} = 0. \tag{2.52}$$

In (2.52), the notation $\gamma \cdot \hat{\Gamma}$ denotes the set of 1PI graphs with one insertion of the composite operator γ_i^j. The Ward identities (2.51) and (2.52) can both be established in the renormalized theory, but this does not simply depend upon the use of an invariant subtraction scheme such as that using dimensional regularization. Note that the transformation of ξ under diffeomorphisms is linear in ξ, so (2.51) is almost trivial. On the other hand, the shift transformation of ξ is non-linear, so this symmetry must be dealt with using B.R.S. techniques.

Defining the effective action at zero quantum field to be

$$\Gamma[\varphi] = \hat{\Gamma}[\varphi, 0], \tag{2.53}$$

one finds that (2.52) reduces to

$$\frac{\delta \Gamma}{\delta \varphi^i(x)} = C_i^j(x) \frac{\delta \hat{\Gamma}}{\delta \hat{\xi}^j}\Big|_{\hat{\xi}=0} \tag{2.54}$$

where

$$C_i^j \equiv \gamma_i^j \cdot \hat{\Gamma}|_{\hat{\xi}=0} \tag{2.55}$$

(Note that the term involving Γ^i_{jk} explicitly is linear in $\hat{\xi}$ and thus drops out of the right-hand side of (2.55), so that C_i^j is tensorial.) One can write a functional integral for $\Gamma[\varphi]$:

$$e^{i\Gamma[\varphi]} = \int \mathcal{D}\xi \exp i \left[I[\phi] - \xi^i (C^{-1})_i^j \frac{\delta \Gamma}{\delta \xi^j} \right] . \tag{2.56}$$

This is the Vilkovisky-De Witt effective action for the non-linear σ-model; it is manifestly covariant under reparameterizations and is one-particle-irreducible in graphical terms. In the quantization procedure outlined above [59], it can be consistently renormalized while preserving the manifest reparameterization invariance.

Quantum corrections are computed by first expanding out the action in powers of ξ using the geodesic equation. If one defines

$$L(s) = \frac{1}{2} g_{ij}(\Phi(s)) \partial_a \Phi^i(s) \partial^a \Phi^j(s) \tag{2.57}$$

then

$$L = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{ds^n} L(s)\big|_{s=1} = \sum_{n=0}^{\infty} \frac{1}{n!} (\nabla_s)^n L(s)\big|_{s=1} \tag{2.58}$$

where ∇_s is the covariant derivative along the curve $\Phi(s)$. The successive derivatives on the right-hand-side of (2.58) are easily evaluated using the formulas [60]

$$\nabla_s g_{ij} = 0, \qquad \nabla_s \frac{d\Phi^i}{ds} = 0$$
$$[\nabla_s, \nabla_a] X^i = R^i{}_{jkl} X^j \frac{d\Phi^k}{ds} \partial_a \Phi^l, \quad \nabla_a = \partial_a + \partial_a \Phi^i \Gamma^{\bullet}_{i\bullet} \tag{2.59}$$

One then finds

$$L = \frac{1}{2} g_{ij}(\varphi) \partial_a \varphi^i \partial^a \varphi^j + g_{ij}(\varphi) \partial_a \varphi^i \nabla^a \xi^j$$
$$+ \frac{1}{2} g_{ij}(\varphi) \nabla_a \xi^i \nabla^a \xi^j + \frac{1}{2} R_{ijkl}(\varphi) \partial_a \varphi^i \partial^a \varphi^l \xi^j \xi^k + \mathcal{O}(\xi^3). \tag{2.60}$$

The propagator is taken from the third term and the remaining terms give rise to an infinite number of vertices that all involve tensorial functions of the target space metric g. A technical complication is that the quantum field ξ is non-linearly renormalized and, moreover, that these renormalizations do not cancel out (as subgraph contributions) in graphs without external ξ-lines. Thus, the non-linear wavefunction renormalizations of ξ must be taken into account in order to properly renormalize the theory using loop-by-loop counterterms. For further details, we refer the reader to ref. [59].

2.4 Supersymmetric Yang-Mills theory

We illustrate the background field method in supersymmetric theories for the case of $N = 1$ supersymmetric Yang-Mills formulated in flat superspace. We recall that this superspace has coordinates $(x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}; \alpha, \dot{\alpha} = 1, 2)$ and that the covariant derivatives $D_A = (D_a, D_\alpha, \bar{D}_{\dot{\alpha}})$ obey

$$\{D_A, D_B\} = -t_{AB}^C D_C, \quad \begin{cases} t^c_{\alpha\dot{\beta}} = -i(\sigma^c)_{\alpha\dot{\beta}} \\ t_{AB}^C = 0 \qquad \text{otherwise.} \end{cases} \tag{2.61}$$

In the standard coordinates, one has

$$D_a = \partial_a, \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \frac{i}{2} \bar{\theta}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} \partial_a. \tag{2.62}$$

The super Yang-Mills fields are introduced in the usual way via a superspace gauge connection $\mathcal{A}_A = (\mathcal{A}_a, \mathcal{A}_\alpha, \bar{\mathcal{A}}_{\dot{\alpha}}) = \mathcal{A}_A^r T_r$, and the superspace gauge-covariant derivative is

$$\nabla_A = D_A - i\mathcal{A}_A \tag{2.63}$$

where we have set the coupling constant g to 1 for simplicity. We then have

$$[\nabla_A, \nabla_B\} = -t_{AB}^{C}\nabla_C - iF_{AB}. \tag{2.64}$$

(In (2.61) and (2.62), the bracket is "graded", i.e. it denotes an anticommutator when A & B are both odd indices.) The field strength tensor F_{AB} is constrained to satisfy

$$F_{\alpha\beta} = 0, \quad F_{\alpha\dot{\beta}} = 0. \tag{2.65}$$

The second of these conditions allows one to solve algebraically for $\mathcal{A}_a$ in terms of $\mathcal{A}_\alpha$ & $\bar{\mathcal{A}}_{\dot{\alpha}}$, while the first is an integrability condition for the existence of covariantly chiral superfields which describe matter coupled to Yang-Mills fields. The remaining components of F_{AB} can be written in terms of the field strength superfield W_α given by

$$W_\alpha = (\sigma^a)_\alpha^{\dot{\beta}} F_{a\dot{\beta}}. \tag{2.66}$$

The constraints (2.65) imply that W_α is covariantly chiral,

$$\bar{\nabla}_{\dot{\alpha}} W_\alpha = 0. \tag{2.67}$$

As a result, the following action is supersymmetric and locally gauge-invariant in superspace:

$$I = \int d^4x d^2\theta \mathrm{tr}(W_\alpha W^\alpha) + \text{c.c.} \tag{2.68}$$

The first of the constraints (2.65) is solved by the "pure gauge" type solution

$$\mathcal{A}_\alpha = ie^{iU} D_\alpha e^{-iU} \tag{2.69}$$

where U is complex, however. The gauge transformations of the prepotential U are given by

$$e^{-iU} \to e^{i\bar{\Lambda}} e^{-iU} e^{-iK}, \quad \bar{D}_{\dot{\alpha}}\Lambda = 0 \quad K = \bar{K}. \tag{2.70}$$

The chiral "pregauge" transformation with parameter Λ leaves $\mathcal{A}_\alpha$ invariant. The K-gauge transformation reproduces the original supersymmetric Yang-Mills gauge transformation on $\mathcal{A}_\alpha$; since K is real, it can be used to make U pure imaginary ($U = iV/2$). V is the quantum field in terms of which superspace Yang-Mills theory was originally formulated [7].

The background-quantum split can be done either at the level of the potential $\mathcal{A}_\alpha$ [25] or at the level of the prepotential U [9]. The splitting in terms of the potential

simplifies the analysis of the ultraviolet divergence cancellations, as we shall see in the next chapter. We write

$$\mathcal{A}_\alpha = \mathcal{B}_\alpha + Q_\alpha \tag{2.71}$$

and solve the constraints on the quantum field Q_α

$$\nabla_\alpha^B Q_\beta + \nabla_\beta^B Q_\alpha - i\{Q_\alpha, Q_\beta\} = 0 \tag{2.72}$$

to find

$$Q_\alpha = ie^{iU^Q}\nabla_\alpha^B e^{-iU^Q} \tag{2.73}$$

where

$$\nabla_\alpha^B e^{-iU^Q} = D_\alpha e^{-iU^Q} - i[\mathcal{B}_\alpha, e^{-iU^Q}]. \tag{2.74}$$

Under background K-gauge transformations, we have

$$\mathcal{B}_\alpha \to e^{iK}\mathcal{B}_\alpha e^{-iK} + ie^{iK}D_\alpha e^{-iK} \tag{2.75}$$

$$U^Q \to e^{iK}U^Q e^{-iK}. \tag{2.76}$$

Under quantum transformations, $\mathcal{B}_\alpha$ is invariant, but U^Q transforms under both quantum K-gauge and quantum Λ-gauge transformations,

$$e^{-iU^Q} \to e^{i\bar{\Lambda}}e^{-iU^Q}e^{-iK} \tag{2.77}$$

where Λ is now background-covariantly chiral

$$\bar{\nabla}_{\dot{\alpha}}^B \Lambda = 0. \tag{2.78}$$

Alternatively, if we denote the total prepotential by U and the background prepotential by U^B, we can make the equivalent exponential splitting

$$e^{-iU} = e^{-iU^B}e^{-iU^Q}. \tag{2.79}$$

Under gauge transformations, U^Q transforms as above and U^B is invariant under quantum gauge transformations but transforms under background transformations by

$$e^{-iU^B} \to e^{i\bar{\Lambda}}e^{-iU^B}e^{-iK}, \quad \bar{D}_{\dot{\alpha}}\Lambda = 0. \tag{2.80}$$

Note that the parameter of the background Λ-gauge transformations is ordinarily (i.e. not covariantly) chiral.

The shift transformation for super Yang-Mills theory analogous to the transformation (2.39) for the non-linear σ-model can be written in the form

$$\mathcal{B}_\alpha \to e^{iL}\mathcal{B}_\alpha e^{-iL} + ie^{iL}D_\alpha e^{-iL} \tag{2.81}$$

$$e^{-iU^Q} \to e^{iL}e^{-iU^Q}, \tag{2.82}$$

corresponding to the split (2.71). Note that L is a complex superfield. For the split (2.79), (2.81) is replaced by

$$e^{-iU^B} \to e^{-iU^B}e^{-iL}. \tag{2.83}$$

As in the case of ordinary Yang-Mills theory, the gauge-fixing and ghost terms break the shift symmetry, but there is a residual B.R.S. symmetry which combines quantum gauge B.R.S. transformations with quantum shift transformations [57], analogously to (2.26, 2.27).

III — NON-RENORMALIZATION THEOREMS AND THEIR CONSEQUENCES

3.1 The background field method and ultraviolet divergences

The background field method is useful in studying the ultraviolet divergence structure of quantum field theories for two reasons: firstly, it enables one to exploit manifest background symmetries, provided that an invariant regularization scheme is available; secondly, there is often improved ultraviolet power-counting behaviour due to the structure of the theory. This is particularly the case in supersymmetric theories. In this section, we give simple examples of both of these aspects.

As an example of the use of symmetry, we consider the renormalization of the Chern-Simons operator in a Yang-Mills theory with minimally-coupled Dirac fermions [61]. This operator is given by

$$K^a = 4g^2\varepsilon^{abcd}\left[\mathcal{A}^r_a(\partial_b\mathcal{A}^r_c + \frac{g}{3}f^{rst}\mathcal{A}^s_b\mathcal{A}^t_c)\right] \tag{3.1}$$

so that

$$\partial_a K^a = g^2\varepsilon^{abcd}F^r_{ab}F^r_{cd}. \tag{3.2}$$

Expanding K^a out using the background-quantum split $\mathcal{A}_a = \mathcal{B}_a + Q_a$ in powers of Q_a, one finds that all but the first two terms are background-gauge-invariant. Therefore, the vertices involved in computing 1PI graphs with one insertion of this operator are background-gauge-invariant and so the counterterms for such graphs with no external

quantum lines must be dimension-three gauge-invariant objects constructed from $\mathcal{B}_a$. Since there are none of these, we conclude that K^a is a finite operator. The factor of g^2 is present because $g\mathcal{B}_a$ is an unrenormalized quantity in the background field method in the sense that

$$g^{(0)}\mathcal{B}^{(0)}_a = g\mathcal{B}_a \tag{3.3}$$

where $g^{(0)}$ and $\mathcal{B}^{(0)}_a$ are respectively the bare coupling constant and the bare background field. It is easy to see that applying an ordinary derivative to K^a preserves this finiteness and so $g^2\varepsilon^{abcd}F^r_{ab}F^r_{cd}$ is also finite.

In the presence of Dirac fermions, one knows that the anomalous divergence of the gauge-invariant $U(1)$ axial current j^5_a in the theory is proportional to $g^2\varepsilon^{abcd}F^r_{ab}F^r_{cd}$. By rearranging the anomaly equation, one sees that there is a conserved but non-gauge-invariant current

$$j'^5_a = j^5_a - cK_a, \tag{3.4}$$

where c is the anomaly coefficient. Since j'^5_a is conserved, there is a natural subtraction scheme in which it is a finite operator and so

$$\mu\frac{\partial}{\partial\mu}j'^5_a = 0. \tag{3.5}$$

Since j^5_a is gauge-invariant, it cannot mix with K_a, so the anomalous dimension γ_{jK} vanishes. On the other hand, we have seen that the counterterms for K_a must be gauge-invariant, so the only non-vanishing anomalous dimension for K_a is the off-diagonal γ_{Kj}. Since j^5_a and K_a are independent operators, from (3.5) one then learns

$$\gamma_{jj} - c\gamma_{Kj} = 0 \tag{3.6}$$

$$\mu\frac{\partial c}{\partial\mu} = 0, \tag{3.7}$$

where γ_{jj} is the anomalous dimension for j^5_a. We have therefore shown that there is a natural subtraction scheme in which the anomalous divergence of j^5_a is purely a one-loop effect. This example [61] neatly demonstrates the power of the background field method; with its help one can see the origins of ultraviolet divergence cancellations in general. The above argument can be confirmed more rigorously, for example by a detailed study of the Ward identities, or by the original Feynman diagram analysis [62].

The simplest example of improved power counting due to the structure of the theory is afforded by the $N = 1$ supersymmetric Wess-Zumino model, described by a chiral scalar superfield ϕ. The basic renormalizable action for a set of such superfields ϕ^i is

$$I = \int d^4x d^4\theta \bar{\phi}_i \phi^i + (\lambda_{ijk} \int d^4x d^2\theta \phi^i \phi^j \phi^k + \text{c.c.}) \tag{3.8}$$

where the λ's are the coupling constants. This multiplet can also be coupled minimally to $N = 1$ supersymmetric Yang-Mills theory without substantially altering the argument we shall now present.

We split ϕ into background and quantum parts linearly

$$\phi = \varphi + Q \tag{3.9}$$

and solve the chirality constraint on Q by

$$Q = \bar{D}^2 X \qquad \bar{D}^2 \equiv \bar{D}^{\dot\alpha} \bar{D}_{\dot\alpha}, \tag{3.10}$$

where X is an unconstrained complex prepotential and the solution (3.10) introduces a gauge invariance

$$\delta X = \bar{D}^{\dot\alpha} \bar{\Lambda}_{\dot\alpha}. \tag{3.11}$$

Expanding out the action (3.8), one finds that all the terms except for the leading one involving no quantum fields can be written as integrals over the full $(4 + 4)$-dimensional superspace. This must be so by supersymmetry, since X is an unconstrained superfield. By standard arguments that we shall outline shortly, one can then show that all of the counterterms must be integrals of local functionals of the superfields taken over the full superspace. If we restrict our attention to graphs with no external quantum lines, then these counterterms must be full superspace integrals of local functionals of the *constrained* background field φ. Since the interaction term in (3.8) is an integral over the chiral subspace, it is *not* of this form and so it cannot be renormalized. We therefore obtain a relationship between the renormalization constants of the fields ϕ^i and the coupling constants λ_{ijk}.

For this particular model, it is in fact simpler to quantize using chiral superfields, but it is still possible to show from the superspace Feynman rules that the counterterms must be full superspace integrals. In the general case, it is not practical to use the constrained fields as quantum fields and one is obliged to solve the constraints (on the quantum fields) as we have done in the above example.

3.2 Extended superspace non-renormalization theorems

In the extended supersymmetric cases that we shall consider, the general situation is very similar to the $N = 1$ example outlined above. That is, we have a total field ϕ which we split into a background part φ and a quantum part Q. The background field φ is left as a constrained superfield and the constraints on the quantum field Q are solved in terms of a quantum prepotential X. Expanding out the action, one finds that all of the terms involving X must be full superspace integrals in order to be supersymmetric, although the term independent of X is not. The superspace Feynman rules for the quantum effective action will then involve integrations over $d^n\theta$ (where n is the fermionic dimension of the superspace being used) for each vertex in the graph. However, the propagator and the vertices will also involve superspace covariant derivatives D (such as (2.62)) and these can be manipulated first before doing the Feynman integrals over the loop momenta.

As a result of these manipulations, one can show that all of the θ-integrations except for one can be done, so that the counterterms must be full superspace integrals of local functionals constructed from the fields corresponding to the external lines. If we restrict our attention to graphs with no external quantum lines, then these functionals obviously depend only on the background field φ. Since φ has a higher (mass) dimension than the prepotential X, the candidate counterterms may be restricted. That is to say, if Q is related to X by (schematically) p powers of D, the background field method allows one to remove a factor of D^p for each external background line, thus significantly improving the superficial divergence counting in comparison to the case of ordinary field theory, where the external lines would have been the X's themselves. Note that in the case of non-renormalizable theories, the counterterms at each loop order will have a prefactor of a loop-dependent power of the coupling constant. Thus, the dimensions of the counterterms, regarded as functionals of the fields and with the coupling constant contribution factored out, will increase with increasing loop order. The non-renormalization theorems do *not* therefore give a cure for non-renormalizability; in a non-renormalizable case, one can use them only to deduce the lowest loop-order at which divergences are first expected to occur.

The above theorem concerning the nature of the counterterms does not apply at one loop in super Yang-Mills theories or supergravity theories. This is due to the gauge-fixing and ghost terms that must be introduced. In order to maintain background gauge invariance, one would like to use gauge-fixing conditions that involve background-covariant derivatives, as in the case of ordinary Yang-Mills theory. However, in superspace, the gauge-parameter superfields are larger superfields than the prepotentials, as for example in the Wess-Zumino model (see eq. (3.11)), and this implies that the ghost action has a new gauge invariance, which must be fixed in turn. Thus, one gets an infinite sequence of ghosts [32], all of which couple to the background field if background-gauge-invariant gauge-fixing terms are employed. For example, for the chiral scalar multiplet we get the sequence $X \to C_\alpha \to C_{\alpha\beta} \to C_{\alpha\beta\gamma}\ldots$, where each of the ghost fields (C) is totally symmetric in its spinor indices. In order to define the theory, it is therefore necessary to restrict this sequence of ghosts. This can be done in a way that preserves background gauge invariance, but at the cost of introducing the background prepotentials. However, one can arrange things so that the finite number of ghosts that couple to all orders in perturbation theory do so via the constrained background potentials, whereas the ghosts that couple to the background prepotentials contribute only in the one-loop graphs. Thus, at one loop only, one is allowed to construct counterterms out of the background prepotential, e.g. X^B, and so the power-counting improvement is lost. For a fuller discussion, including the Nielsen-Kallosh ghosts which are also required, we refer the reader to ref. [30].

In order to implement the above ideas in detail, it is necessary to introduce a regularization scheme. It is well known that supersymmetric gauge-invariant regularization schemes are difficult to construct, but the method of higher covariant derivatives works for graphs at $\ell \geq 2$ loops. Since the theorem on the structure of counterterms only holds for $\ell \geq 2$ loops, the higher-derivative regularization is sufficient under the reasonable assumption that any one-loop subdivergences may be subtracted using an auxiliary one-loop regularization. We emphasize, however, that it is not the requirement of manifest K-gauge invariance of the counterterms (cf. eqn. (2.70)) which plays the leading role, but rather the structural and power-counting aspects of the theory. The improvement in power counting may be thought of as a manifestation of pregauge invariance (such as the Λ-gauge invariance in eq. (2.70)). A further technical difficulty is the off-shell infrared problem. There is no known general solution to this problem, but it has been rigorously solved for the $N = 1$ super Yang-Mills case [52]. It is very likely that this off-shell infrared problem is just a (solvable) technical difficulty also in many of the other theories that we shall consider, although it may be of more significance in two-dimensional theories.

In order to apply the general non-renormalization theorem to a given supersymmetric theory, it is necessary to find the maximally-extended superfields that are available for the theory under consideration. This is a non-trivial problem, which we will discuss in outline in the next section. For some models, it is not possible to maintain manifestly

the full number of supersymmetries of the equations of motion. In certain models, this difficulty can be circumvented by the use of harmonic superspace [35,36,76], for which the general non-renormalization theorem takes exactly the same form as it does in ordinary superspace. The general situation is that a model with N extended supersymmetries at the level of the equations of motion will only admit $M \leq N$ manifestly realized supersymmetries off-shell. As far as the non-renormalization theorem is concerned, it is M that is the significant supersymmetry extension of the model. However, one also expects that counterterms (or at least the counterterms corresponding to the first non-vanishing divergences) should possess N-extended supersymmetry on-shell. This requirement may impose a further restriction of the allowable counterterms, but it does *not* imply that the counterterms need be capable of being written as integrals over the whole of N-extended superspace.

3.3 $N = 2$ Supersymmetric Yang-Mills theories in $d = 4$

$N = 2$ superspace has coordinates $(x^a,\ \theta^\alpha_i,\ \bar\theta^{\dot\alpha i})$ $i = 1,2$ and the "flat" covariant derivatives are

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} - \frac{i}{2}\bar\theta^{\dot\alpha i}(\sigma^a)_{\alpha\dot\alpha}\partial_a; \qquad D_a = \partial_a. \tag{3.12}$$

They obey

$$\{D^i_\alpha, \bar D_{\dot\beta j}\} = i\delta^i_j(\sigma^a)_{\alpha\dot\beta}\partial_a; \qquad \{D^i_\alpha, D^j_\beta\} = 0. \tag{3.13}$$

To describe N = 2 super Yang-Mills, we introduce the gauge field $\mathcal{A}_A = (\mathcal{A}^i_\alpha,\ \bar{\mathcal{A}}_{\dot\alpha i},\ \mathcal{A}_a)$ and impose the following constraint [28] on the field strength superfield F_{AB},

$$F^{ij}_{\alpha\beta} = \varepsilon_{\alpha\beta}\varepsilon^{ij}\bar W; \qquad F^i_{\alpha\dot\beta j} = 0, \tag{3.14}$$

where the field strength superfield is covariantly chiral

$$\bar\nabla_{\dot\alpha i}W = 0. \tag{3.15}$$

The Yang-Mills action is then

$$I = \int d^4x d^4\theta \mathrm{tr}(W^2) + \text{c.c.} \tag{3.16}$$

Solving the constraints (3.14) is much harder for the $N = 2$ case than it is for $N = 1$, and the solution cannot be written in a neat closed form, but can only be found iteratively. The prepotential is a dimension (-2) $SU(2)$ triplet of fields V_{ij} (where $\mathcal{A}^i_\alpha$ has dimension 1/2) and the corresponding pre-gauge parameter is $\Lambda^\alpha_{ijk} = \Lambda^\alpha_{(ijk)}$, with

$$\delta V_{ij} = D^k_\alpha \Lambda^\alpha_{ijk} + \ldots \tag{3.17}$$

In the matter sector, the real hypermultiplet is described by a set of real constrained superfields L, L_{ij}, L_{ijkl}, all of dimension one. The constraints are [27]

$$\begin{aligned} D_{\alpha(i}L_{jk)} &= D^k_\alpha L_{ijkl}; \qquad D_{\alpha(i}L_{jklm)} = 0 \\ D^i_{(\alpha}D^j_{\beta)}L &= [D^i_\alpha, \bar{D}_{\dot\alpha i}]L = 0. \end{aligned} \tag{3.18}$$

The physical component fields are the leading ($\theta = 0$) components of L and L_{ij} and the spinor $\lambda_{\alpha i} = D^j_\alpha L_{ij}\big|_{\theta=0}$. The constraints (3.18) are solved in terms of prepotentials ρ^i_α (dimension -3/2) and X_{ijkl} (dimension -1),

$$\begin{aligned} L^{ij} &= \frac{3}{4}\bar{D}^{ij}D^{3k}{}_\alpha\rho^\alpha_k + \frac{1}{2}\bar{D}^{(i}_k D^{j)k}D^l_\alpha\rho^\alpha_l + \text{c.c} \\ L^{ijkl} &= \frac{2}{5}D^{(ij}\bar{D}^{kl)}[D^m_\alpha\rho^\alpha_m + \bar{D}^{\dot\alpha}_m\bar{\rho}^m_{\dot\alpha}] \\ L &= D^{ij}\bar{D}^{kl}X_{ijkl}; \qquad X_{ijkl} = X_{(ijkl)} \end{aligned} \tag{3.19}$$

where

$$D^{ij} = \frac{1}{2}\varepsilon^{\alpha\beta}D^i_\alpha D^j_\beta; \qquad D^{3i}{}_\alpha = D_{\alpha j}D^{ij}. \tag{3.20}$$

The free hypermultiplet matter action is then given by

$$I = \int d^4x d^8\theta [(L_{ij}D^i_\alpha\rho^{\alpha j} + \text{c.c.}) + L^{ijkl}X_{ijkl}] \tag{3.21}$$

and leads to the equations of motion

$$L_{ijkl} = 0, \qquad D_{\alpha j}L^{ij} = D^i_\alpha L. \tag{3.22}$$

To couple the hypermultiplet to the super Yang-Mills multiplet, one merely replaces the D^i_α in (3.18) by the gauge-covariant derivatives ∇^i_α (and likewise in the solution (3.19), although additional terms involving W are also required [27]).

We may now apply the non-renormalization theorem to a general $N = 2$ model coupled to $N = 2$ matter in a real representation of the gauge group. The background fields are $\varphi = \{\mathcal{A}^i_\alpha,\ L,\ L^{ij},\ L^{ijkl}\}$ and the quantum prepotentials are $X = \{V_{ij},\ \rho^i_\alpha,\ X_{ijkl}\}$. Since the $N = 2$ full superspace measure $d^4x d^8\theta$ has dimension zero

(recall that Berezinian integration is really differentiation), it is immediately apparent that there are no allowed counterterms at $\ell \geq 2$ loops, since all the φ's have dimension $\geq 1/2$. Therefore, the perturbative ultraviolet divergence structure of these theories is completely determined by their one-loop behaviour.

There are no renormalizable self-interactions of the hypermultiplet (in four dimensions), so the matter-coupled $N = 2$ theories have only a single (gauge) coupling constant g. Therefore, in order to obtain a finite $N = 2$ model, it is sufficient to choose the representation under which the matter multiplets transform in such a way as to ensure the vanishing of the one-loop β-function. For m_i hypermultiplets in representations $\mathcal{R}_i$ of $\mathcal{G}$ this is given by

$$\beta^{1\ \text{loop}}(g) = \frac{2g^3}{16\pi^2}\left(\sum_i m_i T(\mathcal{R}_i) - C_2(\mathcal{G})\right), \tag{3.23}$$

where $C_2(\mathcal{G})$ is the quadratic Casimir eigenvalue for the group and $T(\mathcal{R})$ is the quadratic eigenvalue for the representation $\mathcal{R}$. Thus we obtain finite models if [37]

$$\sum_i m_i T(\mathcal{R}_i) - C_2(\mathcal{G}) = 0. \tag{3.24}$$

There are many solutions to this equation, including $N = 4$ super Yang-Mills, for which $\mathcal{R}$ is the adjoint representation and the multiplicity m is one.

The result that $N = 2$ super Yang-Mills - matter models are finite at $\ell \geq 2$ loops remains true for hypermultiplets in complex gauge-group representations, for which the above superspace formulation is not applicable. This was originally verified at the 2-loop order by an explicit calculation and can be justified to all orders by using the same techniques as above in harmonic superspace.

3.4 Supersymmetric Yang-Mills theories in $d > 4$

There are $N = 1$ supersymmetric Yang-Mills theories in $d = 5, 6, 7, 8, 9\,\&\,10$ dimensions and $N = 2$ theories in $d = 5\,\&\,6$. All of these dimensionally reduce to $N = 4$ in four dimensions except for the $N = 1$ theories in $d = 5\,\&\,6$. In this section, we discuss the ultraviolet counterterm structure of these theories from the point of view of the general non-renormalization theorem [39]. We shall focus on the $d = 6$ theories, briefly mentioning the results for the others.

In $N=1$, $d=6$ superspace, the spinorial coordinates θ^α_i, $\alpha=1,\ldots,4$, $i=1,2$ are right-handed $SU(2)$ Majorana-Weyl spinors obeying

$$\bar{\theta}^{\alpha i}=\varepsilon^{ij}\theta_{\alpha j}. \qquad (3.25)$$

(There is a matrix connecting dotted and undotted indices in six dimensions, multiplication by which is included in the conjugation (3.25)). For $N=2$ supersymmetry, there are two versions of $N=2$ superspace, depending on the handedness of the additional spinoral coordinates (similar to the 2a and 2b superspaces in $d=10$). An $N=2$ supersymmetric Yang-Mills theory can only be formulated in the superspace where the additional spinorial coordinates are left-handed $SU(2)$ Majorana-Weyl spinors $\theta^{i'}_\alpha$, $i'=1,2$. The covariant derivatives obey

$$\{D_A,D_B\}=-t_{AB}{}^C D_C$$
$$D_A=(\partial_a,D^i_\alpha)_{(N=1)} \quad \text{or} \quad (\partial_a,D^i_\alpha,D^\alpha_{i'})_{(N=2)} \qquad (3.26)$$

where

$$t^C_{AB}=\begin{cases}-i(\sigma^c)_{\alpha\beta}\varepsilon_{ij}, \ \text{others}=0 & N=1\\ -i(\sigma^c)_{\alpha\beta}\varepsilon_{ij},(\sigma^c)^{\alpha\beta}\varepsilon_{i'j'}, \ \text{others}=0 & N=2.\end{cases} \qquad (3.27)$$

The constraints on the Yang-Mills field strength tensors in the $N=1$ and $N=2$ cases are

$$F^{ij}_{\alpha\beta}=0 \qquad N=1, \qquad (3.28)$$

$$F^{ij}_{\alpha\beta}=F^{\alpha\beta}_{i'j'}=0$$
$$F^{i\beta}_{\alpha j'}=\delta^\beta_\alpha W^i_{j'} \qquad N=2, \qquad (3.29)$$

where $W^i_{j'}$ is real,

$$\bar{W}^{j'}_i=\varepsilon^{j'l'}W^k_{l'}\varepsilon_{ki}. \qquad (3.30)$$

In the $N=1$ case, the Bianchi identities imply

$$F_{a\beta}{}^j=(\sigma_a)_{\beta\gamma}W^{\gamma j}. \qquad (3.31)$$

$W^{\gamma j}$ and $W^i_{j'}$ are the $d=6$, $N=1$ resp. $N=2$ field strength superfields. In the $N=1$ case, the constraints (3.28) give an off-shell supermultiplet, while the $N=2$ constraints (3.29) imply the super Yang-Mills field equations, thus giving only an on-shell supermultiplet.

A general $d=6$, $N=1$ model will also involve hypermultiplets. Real hypermultiplets are described, as in the $N=2$, $d=4$ case, by a set of scalar superfields similar to those given in equation (3.18).

The generic form of a gauge-invariant ℓ-loop counterterm for the $N=1$, $d=6$ theory is

$$\Delta I^{(\ell)}=g^{2(\ell-1)}\int d^6x d^8\theta\,\mathcal{L}^{(\ell)}, \qquad (3.32)$$

where the integrand has the schematic form

$$\mathcal{L}^{(\ell)}=D^{4\ell-3q}(gW^{\alpha i})^q. \qquad (3.33)$$

At $\ell=2$ loops, the only possibility has the form D^2W^2, but all expressions of this form either vanish or become total divergences when the equations of motion are satisfied, so there are no surviving $\ell=2$ loop counterterms on shell. At $\ell=3$ loops, we have possible candidates of the schematic forms

$$\mathcal{L}^{(3)}=(gW^{\alpha i})^4, \quad D^2(gW^{\alpha i})^3, \quad D^4(gW^{\alpha i})^2. \qquad (3.34)$$

On shell, the first of these $\ell=3$ structures survives (but only this one); thus we conclude that, dropping terms that vanish subject to the classical field equations, $N=1$ super Yang-Mills in $d=6$ is finite at 2 loops but not at 3 loops. At one loop, the theory has been shown to be finite on shell by explicit calculation. If hypermultiplet matter is included, then in general the two-loop finiteness is lost. The general form of a pure hypermultiplet counterterm is

$$\mathcal{L}^{(\ell)}=D^{(4\ell-2q)}(gL)^q, \qquad (3.35)$$

where L represents any of the scalar superfields involved in the hypermultiplet. Although there are no such terms that do not vanish on shell at one loop, non-vanishing two-loop counterterms can be constructed, e.g. $\mathcal{L}^{(2)}=(gL)^4$.

A special case of the $N=1$ Yang-Mills theory coupled to hypermultiplet matter is the $N=2$ theory. In this case, we observe that the theory is on-shell finite at one loop firstly because its hypermultiplet sector is finite as we have just seen, but this is in turn connected to the rest of the theory by N = 2 supersymmetry. At two loops, there at first appear to be possible counterterms written in terms of $N=1$ superfields. However, the $N=2$ supersymmetry of this theory should still be a symmetry of the first non-vanishing counterterms when the classical equations of motion are imposed. The $N=2$

supersymmetry is not linearly realized on the quantum fields (since there is no $N = 2$ off-shell superfield formalism), but the non-linear Ward identities for this symmetry in an $N = 1$ superfield formulation still imply that the first non-vanishing counterterms must be $N = 2$ supersymmetric on-shell. Now, in the case of the $N = 2$, $d = 6$ theory at two loops, there are no surviving counterterms with $N = 2$ supersymmetry on shell. Thus, we conclude that the $d = 6$, $N = 2$ theory is finite at one and at two loops.

At three loops, the $N = 1$ counterterm $(gW^{\alpha i})^4$ occurs as the $N = 1$ gauge multiplet part of the non-vanishing $N = 2$ invariant counterterm

$$\Delta I^3_{N=2} = g^8 \int d^6x d^8\theta d^4\theta^{i'j'k'l'} \mathcal{L}_{i'j'k'l'} + (\text{primed} \leftrightarrow \text{unprimed; left} \leftrightarrow \text{right}), \quad (3.36)$$

where

$$\mathcal{L}_{i'j'k'l'} = \mathrm{tr}\left([W^k_{i'}, W_{kj'}][W^n_{k'}, W_{nl'}]\right) \quad (3.37)$$

and the superspace measure has the indicated symmetries. Note that (3.36) is *not* an integral over the full $N = 2$ superspace but it is nevertheless $N = 2$ supersymmetric because the Lagrangian $\mathcal{L}_{i'j'k'l'}$ satisfies the constraints

$$D^{\alpha}_{(r'}\mathcal{L}_{i'j'k'l')} = 0. \quad (3.38)$$

In five dimensions (and in general for d odd), there are no odd-loop divergences in regularization schemes that have dimensionless parameters, such as dimensional reduction (this scheme should be valid at the low loop orders we are discussing here). There may also be further restrictions due to discrete symmetries. In $d > 6$, there are no off-shell covariant superfields available, but we can use $N = 1$, $d = 6$ superfields at the cost of manifest Lorentz covariance. The net result is that the maximally-extended supersymmetric Yang-Mills theory diverges at the $\ell = 4, 3\,\&\,2$ loop orders for $d = 5, 6\,\&\,7$ respectively. The correctness of this analysis is confirmed by complete agreement with the explicit calculations of ref. [40].

3.5 Supergravity theories

Supergravity theories are naturally formulated in curved superspaces of the appropriate dimensionality. The geometries of these superspaces are not (pseudo-) Riemannian, however, but involve tangent space groups of the form (spin group)×(automorphism group) of the appropriate supersymmetry algebra, i.e. there is no supersymmetry in the

tangent space [63]. The basic variable is the supervielbein $E_M{}^A$, from which one can construct a superconnection $\Omega_{MA}{}^B$ by imposing suitable restrictions on the torsion tensor $T_{AB}{}^C$. However, there are considerably more component fields in $E_M{}^A$ than are required to describe the physical states or the Lagrangian of the theory, so it is necessary to impose further constraints on $T_{AB}{}^C$ (because of the structure of the tangent space group, the superspace curvature tensor is actually a function of $T_{AB}{}^C$ and of covariant derivatives of $T_{AB}{}^C$). The analysis of these constraints is a somewhat lengthy procedure and will not be discussed here (see, eg., [64]). For our purposes, it is sufficient to consider only the linearized on-shell theory. The strategy is then to look for all possible on-shell counterterms and to ask whether they can be written in terms of M-extended superfields as full M-extended superspace integrals, where M is the maximum number of supersymmetries that can be linearly realized off shell. We shall focus here on the four-dimensional supergravity theories.

Flat N-extended superspace has coordinates $(x^a,\ \theta^\alpha_i,\ \bar\theta^{\dot\alpha i})$ $\alpha, \dot\alpha = 1, 2,$ $i = 1, \ldots, N$ with covariant derivatives $D_A = (\partial_a, D^i_\alpha, \bar D_{\dot\alpha i})$ satisfying

$$\{D^i_\alpha, \bar D_{\dot\beta j}\} = i\delta^i_j(\sigma^a)_{\alpha\dot\beta}\partial_a, \qquad \{D^i_\alpha, D^j_\beta\} = 0. \quad (3.39)$$

The on-shell field strength superfield for linearized N-extended supergravity is a scalar, W_{ijkl} $(= W_{[ijkl]})$ for $N \geq 4$ and a superfield with $M = 4 - N$ symmetric spinor indices for $N < 4$. The constraints obeyed by these superfields are [65-67]

$$\left.\begin{aligned} \bar D_{\dot\alpha i} W_{\beta_1\ldots\beta_m} &= 0 \\ D^i_\alpha W_{\beta_1\ldots\beta_m} &= D^i_{(\alpha} W_{\beta_1\ldots\beta_m)} \end{aligned}\right\}\ N < 4,\ m = 4 - N \quad (3.40)$$

and

$$\left.\begin{aligned} D^i_\alpha W_{jklm} &= -\frac{4}{N-3}\,\delta^i_{[j} D^n_\alpha W_{klm]n} \\ \bar D_{\dot\alpha i} W_{jklm} &= \bar D_{\dot\alpha[i} W_{jklm]} \end{aligned}\right\}\ N \geq 4 \quad (3.41)$$

with

$$\bar W^{ijkl} = \frac{1}{4!}\epsilon^{ijklmnpq} W_{mnpq} \quad \text{for } N = 8. \quad (3.42)$$

Each component of these superfields is a field F with $2s$ symmetrized undotted spinor indices, $F_{\alpha_1\ldots\alpha_{2s}}$, (or the complex conjugate with dotted indices) obeying

$$\partial^{\alpha_1}_{\dot\alpha} F_{\alpha_1\ldots\alpha_{2s}} = 0 \quad (3.43)$$

$$\Box F_{\alpha_1\ldots\alpha_{2s}} = 0 \quad (3.44)$$

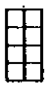
with (3.44) alone being satisfied for $s = 0$. The constraints (3.40), (3.41) and (3.42) ensure that $s \leq 2$ for $N \leq 8$ and that the multiplicity of states corresponding to different values of the helicity s fall into the binomial pattern of on-shell massless maximum-spin-two representations of the N-extended Poincaré supersymmetry algebra. (For $N < 4$, the W's are $SU(N)$ singlets, but they transform under the $U(1)$ factor of the $U(N)$ automorphism group of the supersymmetry algebra; in the $N = 8$ theory, (3.42) explicitly breaks $U(8)$ down to $SU(8)$.)

When $N \geq 4$, there are scalar fields in the theory which are described in the full non-linear theory by a $\mathcal{G}/\mathcal{H}$ non-linear σ-model with $\mathcal{G} = \mathcal{G}_{(N)}$, a non-compact group with the number of non-compact generators equal to the number of scalar fields, and $\mathcal{H}$ $= U(N)$ ($SU(8)$ for the $N = 8$ case), which is the maximal compact subgroup of $\mathcal{G}_{(N)}$. For example, for $N = 8$ we have $\mathcal{G}_{(8)} = E_7$. At the linearized level, the non-compact generators give rise to constant shifts of the superfields W_{ijkl}; we shall take this into account when analysing the allowed counterterms.

Consider first the $N = 8$ theory. An ℓ-loop counterterm has a prefactor of $\kappa^{2\ell-2}$, so the simplest counterterm that one can write down is

$$\Delta I = \kappa^{12} \int d^4x d^{32}\theta (W_{ijkl}\bar{W}^{ijkl})^2, \qquad (3.45)$$

which is a 7 loop counterterm (we take the linearized field W_{ijkl} to have dimension zero) [65]. To get a counterterm for $\ell < 7$, we must integrate over a sub-superspace of $N = 8$ superspace. In addition, in order to take into account the non-renormalization theorems, such a counterterm must be expressible as an integral over the whole of M-extended superspace. For $N = 8$ supergravity, $M = 4$, which implies that $\ell \geq 3$. From the multiplet W_{ijkl}, one can construct the multiplet [43,67] $(W^2)_{\mathbf{1764}}$ where $\mathbf{1764}$ is the irreducible representation of $SU(8)$ into which the lowest-dimension scalar component of the multiplet falls. This representation has the Young tableau



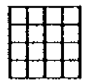
This multiplet has maximum spin 4 and the pure spin-2 contribution to this spin-4 tensor is in fact the Bel-Robinson tensor T_{abcd} where, in 2-component spinor notation,

$$T_{\alpha\dot{\alpha},\beta\dot{\beta},\gamma\dot{\gamma},\delta\dot{\delta}} = C_{\alpha\beta\gamma\delta}C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \qquad (3.42)$$

and $C_{\alpha\beta\gamma\delta}$ is the spinor version of the Weyl tensor. The candidate three-loop counterterm is

$$\Delta I = \kappa^4 \int d^4x (d^{16}\theta)_{232,848} (W^4)_{232,848}, \qquad (3.47)$$

where the 232,848-dimensional representation of SU(8) corresponds to the Young Tableau



Doing the θ-integration in (3.47), one finds that the pure graviton contribution to the x-space counter-Lagrangian is just the square of the Bel-Robinson tensor. The action (3.47) is $N = 8$ supersymmetric, as can be proven using the techniques developed for handling such integrals in ref. [43]; it is clearly SU(8) invariant (and is equal to the non-manifestly $SU(8)$-invariant version first proposed in ref. [42]). It also has linearized E_7 invariance. To see this, it is necessary to integrate out over all the θ's after having made a constant shift of W_{ijkl}. One can then show that at least one derivative falls on the constant shift and so linearised E_7 invariance follows. To show that it can be written as a full $M = 4$ superspace integral, one first decomposes W_{ijkl} into $M = 4$ superfields and then shows that each term in the resultant decomposition of the action (3.47) can indeed be written as an $M = 4$ integral [30]. This is as expected, since the full $M = 4$ superspace measure is precisely of the form $d^{16}\theta$. Thus, the onset of divergences in $N = 8$ supergravity occurs at the three-loop order, precisely the same result as for $N = 1$ [41] (and, clearly, for all $N \leq 8$).

In higher dimensions, the situation deteriorates, and no $d \geq 5$ supergravity theory is finite at $\ell \geq 2$ loops. In particular, $d = 10$ supergravity theories diverge at the one-loop order and the $d = 11$ theory diverges at two loops (it is one-loop finite only because of the odd dimensionality of the spacetime involved).

3.6 Supersymmetric non-linear σ-models in $d = 2$

The $d = 2$ Lorentz group $SO(1,1)$ has one generator. A field transforming under an irreducible representation of $SO(1,1)$ is a one-component object ϕ, characterized by its "Lorentz weight" w: under an infinitesimal Lorentz transformation with parameter ℓ, ϕ transforms as

$$\delta\phi = w\ell\phi. \qquad (3.48)$$

For fermions, w is half-odd-integral, while for bosons w is integral. We use the convention that $+(-)$ denotes weight $\frac{1}{2}(-\frac{1}{2})$, $\ddagger\,(=)$ denotes weight $1(-1)$, etc. The most general supersymmetry algebra has p left-handed (real) supersymmetry generators Q^I_+, $I = 1, \ldots, p$ and q right-handed (real) supersymmetry generators $Q^{I'}_-$, $I' = 1, \ldots, q$, where

$$\begin{aligned} \{Q^I_+, Q^J_+\} &= \delta^{IJ} P_{\ddagger} \\ \{Q^{I'}_-, Q^{J'}_-\} &= \delta^{I'J'} P_{=} \\ \{Q^I_+, Q^{J'}_-\} &= 0 \end{aligned} \tag{3.49}$$

with

$$P_{\ddagger} = \frac{1}{2}(P_0 + P - 1); \qquad P_{=} = \frac{1}{2}(P_0 - P_1).$$

The simplest supersymmetric σ-model has (1,0) supersymmetry. It is described (in flat (1,0) superspace $\Sigma^{(1,0)}$) by the action

$$I = -i \int d^2x d\theta^+ \left(g_{ij}(\phi) + b_{ij}(\phi)\right) D_+\phi^i \partial_= \phi^j \tag{3.50}$$

$$= \int d^2x \left((g_{ij} + b_{ij}) \partial_{\ddagger} \phi^i \partial_= \phi^j + \frac{i}{2} g_{ij} \lambda^i_+ \nabla^{(+)}_= \lambda^j_+ \right), \tag{3.51}$$

where ϕ is a map from $\Sigma^{(1,0)} \to \mathcal{M}$, the σ-model target space, which is equipped with a Riemannian metric g and a locally-defined two-form b. The superfield ϕ has the expansion

$$\phi^i(x,\theta) = \phi^i(x) + \theta^+ \lambda^i_+(x) \tag{3.52}$$

and we use the same notation for the superfield as for its first component (it will be clear from the context which is meant). The covariant derivative $\nabla^{(+)}_=$ appearing in (3.51) is defined by

$$\nabla^{(+)}_= \lambda^i_+ = \partial_= \phi^j \Gamma^{i(+)}_{jk} \lambda^k_+ \tag{3.53}$$

where

$$\Gamma^{i(\pm)}_{jk} = \mathring{\Gamma}^i_{jk} \pm \frac{1}{2} H^i_{jk}; \qquad H_{ijk} = 3\partial_{[i} b_{jk]} \tag{3.54}$$

and $\mathring{\Gamma}$ is the Levi-Civita connection for the metric g. The Wess-Zumino-Witten term involving b is crucial to this model for anomaly cancellation [69,70].

The second left-handed supersymmetry is defined by

$$\delta_\zeta \phi^i = \zeta \mathcal{I}^i_j D_+ \phi^j \tag{3.55}$$

for some tensor $\mathcal{I}$ on $\mathcal{M}$. The algebra of supersymmetry transformation closes if $\mathcal{I}$ is a complex structure [71,69],

$$\mathcal{I}^2 = -1; \qquad N^i_{jk}(\mathcal{I}) \equiv 2(\mathcal{I}^i_l \partial_{[j} \mathcal{I}^l_{k]} - \mathcal{I}^l_{[j} \partial_{|l|} \mathcal{I}^i_{k]}) = 0. \tag{3.56}$$

The action is invariant under (3.55) if g is an hermitean metric and $\Gamma^{(+)}$ is a complex connection,

$$\mathcal{I}_{ij} = -\mathcal{I}ji; \qquad \nabla^{(+)}_i \mathcal{I}^j_k = 0. \tag{3.57}$$

Note that $\mathcal{M}$ is *not* a Kähler manifold (at least with respect to the above tensors) because $\Gamma^{(+)}$ is not torsion-free.

Further left-handed supersymmetries require the existence of additional complex structures $\mathcal{I}_r$, $r = 1, 2, \ldots, p-1$ and in order for the action to be invariant, (3.57) must hold for each of these separately. Algebraic closure requires that each pair of complex structures must anticommute and so any two define a third,

$$\mathcal{I}_r \mathcal{I}_s + \mathcal{I}_s \mathcal{I}_r = 0, \quad r \neq s \quad \Longrightarrow \quad \mathcal{I}_1 \mathcal{I}_2 = \mathcal{I}_3, \quad \text{etc.} \tag{3.58}$$

Thus, there are only two cases: the complex case $p = 2$ and the quaternionic case $p = 4$, because $p = 3 \Rightarrow p = 4$ and $p > 4$ implies that $\mathcal{M}$ is reducible.

The basic model can be extended by the inclusion of a right-handed ("Yang-Mills") sector [73]. One then introduces a set of m real right-handed spinor superfields ψ^A_- which transform under a representation of $SO(m)$, or of some subgroup of $SO(m)$. These fields are associated with a real rank m vector bundle E over the target space $\mathcal{M}$ (ψ is a section of $\phi^* E \otimes S_-$, where S_- is the bundle of right-handed spinors over $\Sigma^{(1,0)}$). The action for this sector is

$$I = \int d^2x d\theta^+ h_{AB} \psi^A_- \nabla_+ \psi^B_-, \tag{3.59}$$

where the covariant derivative is defined by

$$\nabla_+ \psi^A_- = D_+ \psi^A_- + D_+ \phi^i \mathcal{A}_i{}^A_B \psi^B_-, \tag{3.60}$$

in which $h_{AB}(\phi)$ is a fibre metric on E and $\mathcal{A}_i{}^A_B(\phi)$ is a connection. A second supersymmetry transformation may be defined by [74]

$$\delta_\zeta \psi^A_- = -\delta_\zeta \phi^i \mathcal{A}_i{}^A_B \psi^B_- + \zeta \mathcal{I}^A_B \nabla_+ \psi^B_- \tag{3.61}$$

and the algebra closes if, among other requirements, $\mathcal{I}^A_B$ is a complex structure on the fibre. The action is invariant if h is hermitean and if $F(\mathcal{A})$, the curvature two-form associated with $\mathcal{A}$, is a (1,1)-form, i.e. if

$$\mathcal{I}^k_{[i}F_{j]k}{}^A{}_B = 0. \tag{3.62}$$

In the (4,0) case, similar conditions must be satisfied and the bundle E must admit a quaternionic structure.

Models with $p, q \geq 1$ supersymmetry can be constructed from the general (1,0) model formed by combining (3.50) and (3.59), but it is easier to write them in terms of (1,1) superfields. The basic action is

$$I = \int d^2x d\theta^+ d\theta^- (g_{ij} + b_{ij}) D_+\phi^i D_-\phi^j \tag{3.63}$$

where

$$D_+^2 = i\partial_{\ddagger}, \qquad D_-^2 = i\partial_{=}, \qquad \{D_+, D_-\} = 0. \tag{3.64}$$

Additional supersymmetry transformations are defined by

$$\delta_\zeta \phi^i = \zeta^+ \mathcal{I}^{(+)i}_j D_+\phi^j + \zeta^- \mathcal{I}^{(-)i}_j D_-\phi^j. \tag{3.65}$$

The left-handed and right-handed complex structures are not in general identified. For a model with (p,q) supersymmetry, $(p-1)$ left-handed and $(q-1)$ right-handed complex structures are required. Such models exist for $p, q \leq 4$ and their geometry has been studied in refs. [75,74]. If the algebra (3.65) is to close without the addition of further auxiliary fields, then the left and right-handed complex structures must commute. This is not possible for the original $N = 4$ model ($(p,q) = (4,4)$) with hyperkähler target space [72].

In order to apply the non-renormalization theorem, we have to ascertain how much supersymmetry can be linearly realized off shell. In fact, for almost all of the models listed above, it is possible to realize all of the supersymmetries without the addition of further auxiliary fields. The exceptional cases are those (p,q) models with $p, q \geq 2$, where the algebra of extended supersymmetry transformation does not close off shell. However, in these cases, harmonic superfields are available [35,76] and they lead to the same power-counting results.

We illustrate the procedure in the (2,0) case [77]. Let θ_0^+ and θ_1^+ denote the odd coordinates of $\Sigma^{(2,0)}$, where θ_0^+ is the original coordinate for $\Sigma^{(1,0)}$. The *constrained* (2,0) σ-model field ϕ satisfies

$$D_{1+}\phi^i = \mathcal{I}^i_j D_{0+}\phi^j \tag{3.66}$$

or, in complex notation,

$$\bar{\Delta}_+\phi^i = i\mathcal{I}^i_j \bar{\Delta}_+\phi^j, \qquad \bar{\Delta}_+ \equiv D_{0+} - iD_{1+}. \tag{3.67}$$

Since $\mathcal{I}$ is a complex structure, there are holomorphic coordinates available on $\mathcal{M}$ $(z^\alpha, \bar{z}^{\bar{\alpha}};\ \ \alpha = 1, \ldots, n = \dim \mathcal{M})$. In these coordinates, (3.67) becomes simply the chirality condition

$$\bar{\Delta}_+\phi^\alpha = 0. \tag{3.68}$$

The extended supersymmetry variations can similarly be converted into constraints on extended superfields in all other models that fulfil the conditions stated above. However, the constraints do not in general linearise in special coordinate systems. This is because there are a number of complex structures involved and, although they are separately integrable, they are not in general simultaneously integrable (i.e., there are no coordinate systems in which all of the complex structures are simultaneously constant). This makes it difficult in practice to quantize the general model using extended superfields explicitly.

Nonetheless, we need only know that extended superfields can be used in principle in order to apply the non-renormalization theorem. We observe that the actions (3.50), (3.59) and (3.63) can be interpreted as (p,q)-extended actions of the "superaction" type (i.e. integrals over submanifolds of superspace [43]) if the superfields are taken to be (p,q)-extended superfields that satisfy appropriate constraints and the θ's appearing explicitly in the measure and the derivatives are taken to be θ_0's. This procedure works because the necessary constraints are actually just transcriptions into extended superfields of the equations for the extended supersymmetry transformations of the model from the forms originally written in terms of (1,0) or (1,1) superfields.

We can use the background field method as before. We write the total field ϕ as the sum of a constrained background field φ and a quantum field Q. The constraints on Q are solved in terms of a quantum prepotential X which is unconstrained. The allowable counterterms corresponding to divergences in graphs with no external quantum lines must be full superspace integrals of local functionals of ϕ (note that $\dim \phi = 0$) of the form

$$\Delta I = \int d^2x (d\theta^+)^p (d\theta^-)^q L_{p,q}. \tag{3.69}$$

Since the measure has dimension $\frac{1}{2}(p+q-4)$ and Lorentz weight $(p-q)$, we have

$$\begin{aligned} \dim(L_{p,q}) &= \frac{1}{2}(4-p-q) \\ \text{Lorentz weight}\,(L_{p,q}) &= q-p. \end{aligned} \tag{3.70}$$

For $p \geq q$, we may write schematically

$$L_{p,q} = (\text{diff. operators})\, F_{p,q}(\phi). \tag{3.71}$$

The differential operators in this expression must have dimension at least $\frac{1}{2}(p-q)$ in order to have the correct Lorentz weight, so

$$\dim F_{p,q} = 2-p \geq 0. \tag{3.72}$$

$F_{p,q}$ could also involve further differential operators in Lorentz scalar combinations; these would make the dimension even larger. Equation (3.72) cannot be satisfied for $p > 2$ and so we conclude that (p,q) σ-models with $p > 2$, $q \leq p$ are ultraviolet finite. Of course, $p > 2 \Rightarrow p = 4$, so the statement is really that $(4,q)$ models are ultraviolet finite for $0 \leq q \leq 4$ [77].

We conclude this section with a brief discussion of the $N = 2$ model ($(p,q) = (2,2)$) on a Kähler target manifold. This model is relevant to the compactification of ten-dimensional string theories on Calabi-Yau spaces (i.e. on Ricci-flat Kähler manifolds). These models were originally conjectured to be finite [72,55], and indeed have been explicitly computed to be finite through the three-loop order [78]. According to the power-counting argument given above, these models ought not be expected to be finite, but this argument does not make any specific reference to the geometrical aspects of the σ-model. In (2,2) superfields, the action for this model is (note that $\mathcal{I}^{(+)} = \mathcal{I}^{(-)}$; $b = 0$)

$$I = \int d^2x d^4\theta K(\phi^a, \bar{\phi}^{\bar{a}}), \tag{3.73}$$

where K is the Kähler potential of the target manifold,

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \tag{3.74}$$

and the ϕ^a are chiral superfields,

$$\bar{D}_\alpha \phi^a = 0, \qquad \alpha = 1,2 \tag{3.75}$$

where α is a 2-component spinor index.

It is not possible to use the standard background-quantum split based upon geodesics here, since chirality would then imply that the target space was locally flat. However, it is possible to quantize the model covariantly by introducing a prepotential for ϕ^a [59]. As a consequence of the Feynman rules developed using this technique, the counterterms at $\ell \geq 2$ loops are required to be integrals of globally-defined scalar integrands T over the full $N = 2$ superspace. The renormalization corrections that these counterterms give to the metric $g_{a\bar{b}}$ (or rather to the Kähler 2-form $\omega = -2ig_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^{\bar{b}}$) are cohomologically trivial, but they need not vanish, and indeed it was eventually found that these models do diverge at $\ell = 4$ loops [78].

$N = 2$ superspace perturbation theory also provides a second way of understanding the finiteness of the $N = 4$ (i.e (4,4)) models on hyperkähler spaces. Consider a compact Kähler manifold with complex structure $\mathcal{I}$, Kähler metric g, Kähler form ω and Ricci-form ρ, where

$$\begin{aligned} \omega = -2ig_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^{\bar{b}}; \qquad \rho &= -2iR_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^{\bar{b}} \\ d\omega = d\rho &= 0. \end{aligned} \tag{3.76}$$

Yau's theorem [79] states that if ρ' is a closed (1,1)-form cohomologous to ρ, then there is a unique g' and $\mathcal{I}'$ with ω' cohomologous to ω and with ρ' equal to the Ricci-form of g'. As a corollary, if the first Chern class of $\mathcal{M}$ vanishes so that ρ is cohomologous to the zero (1,1)-form, and if we take $\rho' = 0$, then for a given cohomology class of $\mathcal{I}$, there is a unique Ricci-flat metric. If $\mathcal{M}$ is hyperkähler, then it is necessarily Ricci-flat, so that the model is one-loop finite, since the one-loop correction to the metric is just the Ricci tensor [58,55]. Since the higher-loop counterterms are cohomologically trivial, it follows that $\omega^{(0)}$ is cohomologous to ω, where

$$\omega^{(0)} = -2ig^{(0)}_{a\bar{b}}d\phi^a \wedge d\bar{\phi}^{\bar{b}} \tag{3.77}$$

and $g^{(0)}$ is the bare metric. Assuming that $g^{(0)}$ is also hyperkähler, i.e. that $N = 4$ supersymmetry is unbroken, then $g^{(0)}$ is Ricci-flat and hence by Yau's theorem $g = g^{(0)}$ and the theory is finite. This proof can be extended to the non-compact case as well.

The above argument is a variant of the counterterm argument given in refs. [80]. The first indication that the $N = 4$ σ-model theory was finite was given in ref. [72]. It was conjectured that the non-renormalization theorem should be applicable in ref. [30].

The standard non-renormalization theorem was applied to the twisted (4,4) model in ref. [81] and the (4,4) and (4,0) models were analysed in harmonic superspace in refs. [36] and [76] respectively.

IV — $N = 1$ SUPERSYMMETRIC GAUGE MODELS

The special ultraviolet properties of extended super Yang-Mills models with matter coupling in four dimensions can be generalized to models with various forms of soft supersymmetry breaking. For example, one may start from one of the $N = 2$ finite supersymmetric models and quantize it in $N = 1$ superfields, taking care to add finite local counterterms as necessary in order to ensure the continued cancellation of all the gauge-invariant divergences. Addition of $N = 1$ supersymmetric mass terms to the starting Lagrangian will not then disturb the cancellation of divergences proportional to the highest-dimension operators (i.e. dimension-two in superspace or dimension-four in components). Moreover, the non-renormalization theorem for $N = 1$ supersymmetry will ensure that all the added mass terms are not themselves renormalized [45,46]. That the addition of mass terms does not affect the renormalization of the highest dimension operators can most easily be seen using a mass-independent regularization and subtraction scheme such as dimensional reduction.

The existence of finite models with $N = 1$ supersymmetry constructed by the above "soft breaking" of extended supersymmetry naturally led to a search for more general finite $N = 1$ supersymmetric models. The most general Lagrangian for a set of chiral superfields ϕ^a coupled to $N = 1$ super Yang-Mills theory with a compact simple gauge group $\mathcal{G}$ is

$$I = \int d^4x\Big(\int d^4\theta\bar{\phi}_a(e^{gV})^a_b\phi^b - \frac{1}{g^2C_2(\mathcal{G})}\mathrm{tr}\int d^2\theta W^\alpha W_\alpha + \frac{1}{3!}\int d^2\theta d_{abc}\phi^a\phi^b\phi^c + \text{h.c.} + \text{gauge fixing} + \text{ghost terms}\Big), \quad (4.1)$$

where the superfield ϕ^a transforms according to a redicible representation of $\mathcal{G}$ with representation matrices $(\mathcal{R}_i)^a_b$, g is the coupling constant and d_{abc} is a set of $\mathcal{G}$-invariant Yukawa couplings, totally symmetric in the indices a, b, c. The indices a, b, c run over the irreducible representations A and their individual representation indices s, i.e. $a = (A, s)$.

Thus, $(\mathcal{R}_i)^a_b = (\mathcal{R}^A_i)^s_t$, where i labels the particular element of the Lie algebra. Then the following conditions hold:

$$[\mathcal{R}_i, \mathcal{R}_j] = if_{ijk}\mathcal{R}_k, \qquad (\mathcal{R}_i)^a_b(\mathcal{R}_i)^b_c = C_2(A)\delta^a_c,$$
$$(\mathcal{R}_i)^a_b(\mathcal{R}_j)^b_a = \delta_{ij}\sum_A T(A), \qquad f_{ijk}f_{lkj} = C_2(\mathcal{G})\delta_{il}, \quad (4.2)$$

where $C_2(A)$ is the quadratic Casimir for the irreducible representation A and $C_2(\mathcal{G})$ is the quadratic Casimir for the adjoint representation.

The model defined by (4.1) is finite at the one-loop order if the representations A and the gauge and Yukawa couplings are chosen so that [48]

$$\sum_A T(A) = 3C_2(\mathcal{G}) \quad (4.3)$$

$$S^B_A = \delta^B_A C_2(A), \quad (4.4)$$

where S^B_A is defined by

$$d_{acd}\bar{d}^{bcd} \equiv 2S^b_a g^2 = 2\delta^t_s S^B_A g^2. \quad (4.5)$$

The class of $N = 1$ supersymmetric models satisfying (4.3-4.5) includes the finite $N = 2$ models discussed in §3.3 (see eq. (3.20)). As in the case of the finite $N = 2$ models, the relations (4.3-4.5) imply that there is only one independent coupling constant in the model, i.e. the Yukawa couplings are determined by the gauge coupling. A large set of $N = 1$ supersymmetric models satisfying the constraints (4.3-4.5) has been constructed in refs. [54], including some whose representation structure is consistent with phenomenological constraints.

Although one might expect that an $N = 1$ model constrained by (4.3-4.5) would diverge at two loops, this turns out not to be the case [48]. For the gauge coupling constant, this persistence of finiteness follows from an argument [82] based upon power counting in superspace using a particular implementation of the background field method and also using regularization by dimensional reduction [83,84]. In this implementation of the background field method, the superspace covariant-derivative algebra that is done in evaluating superspace Feynman diagrams must be carried out while keeping the full background superspace gauge invariance manifest. In this fashion, a superspace Feynman diagram may be reduced to a single integral over the fermionic coordinates, contracted with an ordinary Feynman momentum-space integral. The result of doing the calculation in this way is that the divergences must be written as full-superspace integrals of background-gauge-invariant expressions that are constructed from the *vectorial* part of the superspace gauge connection $\mathcal{A}_a$, and they may not involve the spinorial connections $\mathcal{A}_\alpha$, $\mathcal{A}_{\dot\alpha}$ explicitly. Thus, naïvely, there are no allowable dimension-two integrands for the $N = 1$ super Yang-Mills superspace counterterms (as with other background-field-method discussions in gauge theories, the one-loop graphs are a special case).

In regularization by dimensional reduction, it *is* possible to construct an invariant counterterm consistent with the above requirements, since the projection $\mathcal{A}_{\hat a}$ of $\mathcal{A}_a$ into the ϵ-dimensional subspace between 4 and $4-\epsilon$ dimensions transforms covariantly in the adjoint representation of the gauge group. Thus, $\int d^4x d^4\theta \mathrm{tr} \mathcal{A}_{\hat a}\mathcal{A}_{\hat a}$ is an admissible structure in this regularization scheme [82], although it is clearly of order ϵ (in fact, it can be shown to be proportional to $\epsilon \int d^4x d^2\theta \mathrm{tr} W_\alpha W^\alpha + \mathrm{c.c}$). The net effect is that the $1/\epsilon$ gauge-coupling-constant renormalization at $\ell \geq 2$ loops is determined by a diagram that naïvely would appear to be of order $1/\epsilon^2$. In a model that is finite at $\ell - 1$ loops, this divergence vanishes. It is not clear what the implications of this type of argument are in other regularization schemes, such as Pauli-Villars.

For the Yukawa couplings, the persistence of finiteness at two loops is less apparent, but it has been shown to occur by explicit calculation [48]. The natural expectation at higher loops would be for the special cancellations in the Yukawa couplings to cease at three loops, with the gauge coupling constant consequently diverging at four loops. Explicit three-loop calculations have shown that this is typically the case [49]. However, it is possible to impose further constraints over and above (4.3-4.5) in a suitably complicated model, in order to make the three-loop Yukawa divergences cancel. It would seem to be possible to continue this procedure to even higher orders at the cost of increasing complexity in the model.

Instead of demanding finiteness separately in each loop order, one could alternatively try to arrange for cancellations between different loop orders, in analogy with the critical $d = 2$ non-linear σ-models with vanishing β-functions that are of interest in connection with string theory. In order to do this in a $d = 4$ super Yang-Mills theory, one could attempt to modify the constraints (4.3-4.5) by $\hbar$-dependent terms [50,51]. This causes an imbalance in the divergence cancellations in the one-loop two-point diagram with external chiral matter fields, but since the changes to (4.3-4.5) are $\hbar$-dependent, these changes will not disturb the cancellation of the $\mathcal{O}(\hbar)$ divergences guaranteed by the $\mathcal{O}(\hbar^0)$ constraints (4.3-4.5). The point to making such finite changes in the relations between the gauge and Yukawa coupling constants is that one can then try to cancel the new

one-loop (but higher order in $\hbar$) divergences against the divergences in true higher-loop diagrams defined from the original $\mathcal{O}(\hbar^0)$ Lagrangian satisfying (4.3-4.5).

At each loop order, finiteness will be preserved if the anomalous dimensions of the chiral multiplets are made to vanish by suitable re-adjustments of the relations between the Yukawa couplings and the gauge coupling. At each order, there are new terms needed in an expansion of the Yukawa couplings in powers of the gauge coupling that are necessary for the anomalous dimensions of the chiral multiplets to vanish to that order. This iterative construction can be carried out because the equation determining the new terms at any order always has the new terms contracted into the same matrix [50,51]. This matrix is invertible, provided that the number of independent irreducible chiral multiplets is less than or equal to the number of Yukawa couplings. There are no further constraints required for the vanishing of the gauge coupling constant's β-function, because, using the arguments of ref. [82], this is automatically zero as a consequence of the lower order cancellations (i.e. the vanishing of the gauge β-function and of all the anomalous dimensions in lower orders).

The above discussion relies heavily on the special properties of regularization by dimensional reduction. Although this regularization scheme is known to be reliable at low loop orders, it suffers from well-known inconsistencies [84] and thus does not allow rigorous conclusions to be drawn. Nonetheless, the above results have been confirmed using the more rigorous BPHZ formalism [52,53]. The BPHZ approach differs somewhat from that outlined above in that it focuses primarily on the consistency conditions necessary if one tries to impose a set of relations between renormalized coupling constants. These consistency conditions are necessary for such relations to hold independently of the choice of the renormalization scale μ. In the models in question, one wishes to reduce all the Yukawa couplings to functions of the gauge coupling constant g, i.e. to set $d_{abc} = d_{abc}(g)$. For consistency, this requires

$$\beta_{abc} \equiv \mu \frac{\partial d_{abc}}{\partial \mu} = \beta(g) \frac{d\, d_{abc}}{dg}. \tag{4.6}$$

The result proved in [52,53] is that any solution of (4.6) that also satisfies (4.4) in lowest order, with gauge-group representations chosen to satisfy (4.3), leads to vanishing gauge and Yukawa coupling β-functions to all orders in perturbation theory. The function $d_{abc}(g)$ is constructed in perturbation theory by solving (4.6) iteratively, subject to the lowest-order conditions (4.3-4.5).

The framework of iteratively-constructed finite $N = 1$ super Yang-Mills - matter models may be applied to the extended supersymmetric theories discussed earlier. For extended supersymmetric models, the iterative solution to (4.6) is in fact given by the *tree-level* relations (4.4) between the gauge and Yukawa coupling constants, which are required to hold by the extended supersymmetry. In order for an $N = 2$ model to be finite, it also must satisfy the constraint (4.3) on the representation content, which is equivalent to (3.24). In $N = 4$ super Yang-Mills theory, the constraint (4.3) is itself a consequence of the $N = 4$ extended supersymmetry. An advantage of this approach is that it does not require that the chiral multiplets transform according to real representations of the Yang-Mills gauge group.

REFERENCES

[1] I. Waller, *Z. Physik* **62** (1930) 673;
J.R. Oppenheimer, *Phys. Rev.* **35** (1930) 461.

[2] Willis E. Lamb, Jr. and Robert E. Retherford, *Phys. Rev.* **72** (1947) 241.

[3] Many of these are collected in J. Schwinger, ed. *Quantum Electrodynamics* (Dover, New York, 1958).
Some others are: A. Salam, *Phys. Rev.* **82** (1951) 217;
J.C. Ward, *Proc. Phys. Soc.* **A64** (1951) 54.

[4] E.C.G. Stueckelberg and A. Peterman, *Helv. Phys. Acta* **26** (1953) 499;
M. Gell-Mann and F.E. Low, *Phys. Rev.* **95** (1954) 1300;
N.N. Bogoliubov and D.V. Shirkov, *Introduction to the theory of quantized fields* (Interscience, New York, 1959);
K. Symanzik, *Comm. Math. Phys.* **18** (1970) 227;
C.G. Callan, *Phys. Rev.* **D2** (1970) 1541;
K. Wilson, *Phys. Rev.* **179** (1969) 1499.

[5] For reviews of the applications of spontaneously broken supersymmetry to particle physics, see R. Barbieri and S. Ferrara, *Surveys in High Energy Physics* **4** (1983) 33;
H.P.A. Nilles, *Phys. Rep.* **110** (1984) 1;
P. Fayet, in *Supersymmetry and Supergravity '84*, eds. B. de Wit, P. Fayet and P. van Nieuwenhuizen (World Scientific, Singapore, 1984).

[6] J. Wess and B. Zumino, *Nucl. Phys.* **B70** (1974) 39.

[7] J. Wess and B. Zumino, *Phys. Lett.* **49B** (1974) 52;
J. Iliopoulos and B. Zumino, *Nucl. Phys.* **B76** (1974) 310;
S. Ferrara, J. Iliopoulos and B. Zumino, *Nucl. Phys.* **B77** (1974) 41.

[8] D. Capper and G. Leibbrandt, *Nucl. Phys.* **B85** (1975) 492;
K. Fujikawa and W. Lang, *Nucl. Phys.* **B88** (1975) 61;
R. Delbourgo, *Nuovo Cimento* **25A** (1975) 646;
S. Ferrara and O. Piguet, *Nucl. Phys.* **B93** (1975) 261.

[9] M.T. Grisaru, W. Siegel and M. Roček, *Nucl. Phys.* **B159** (1979) 42.

[10] J. Wess and B. Zumino, *Nucl. Phys.* **B78** (1974) 1;
A. Salam and J. Strathdee, *Phys. Lett.* **51B** (1974) 353;
S. Ferrara and B. Zumino, *Nucl. Phys.* **B79** (1974) 413.

[11] P. Fayet, *Nucl. Phys.* **B113** (1976) 135.

[12] F. Gliozzi, J. Scherk and D. Olive, *Nucl. Phys.* **B122** (1977) 253;
L. Brink, J. Schwarz and J. Scherk, *Nucl. Phys.* **B121** (1977) 77.

[13] S. Ferrara and B. Zumino, in ref. [10].

[14] D.R.T. Jones, *Phys. Lett.* **72B** (1977) 199;
E. Poggio and H. Pendleton, *Phys. Lett.* **72B** (1977) 200.

[15] O. Tarasov and A.A. Vladimirov, *Phys. Lett.* **96B** (1980) 94;
M.T. Grisaru, M. Roček and W. Siegel, *Phys. Rev. Lett.* **45** (1980) 1063;
W. Caswell and D. Zanon, *Phys. Lett.* **100B** (1980) 152.

[16] C. Montonen and D. Olive, *Phys. Lett.* **72B** (1977) 117;
H. Osborn, *Phys. Lett.* **83B** (1979) 321.

[17] M.F. Sohnius and P.C. West, *Phys. Lett.* **100B** (1981) 45;
S. Ferrara and B. Zumino, unpublished.

[18] M.T. Grisaru, M. Roček and W. Siegel, in ref. [15];
K.S. Stelle, in *Quantum Structure of Space and Time*, eds. M.J. Duff and C.J. Isham (Cambridge University Press, 1972).

[19] M. Roček and W. Siegel, *Phys. Lett.* **105B** (1981) 275.

[20] V. Rivelles and J. G. Taylor, *J. Phys. A: Math. Gen.* **15** (1982) 163.

[21] B.S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[22] J. Honerkamp, *Nucl. Phys.* **B36** (1972) 130.

[23] G. 't Hooft, in *Proc. 12th Winter School of Theoretical Physics In Karpacz, Acta. Univ. Wratisl.* **38** (1975);
B.S. DeWitt, in *Quantum Gravity 2*, eds. C.J. Isham, R. Penrose and D.W. Sciama (Clarendon Press, Oxford, 1981);
D.G. Boulware, *Phys. Rev.* **D23** (1981) 389.

[24] L. Abbot, *Nucl. Phys.* **B185** (1981) 189.

[25] M.T. Grisaru and W. Siegel, *Nucl. Phys.* **B201** (1982) 292.

[26] P. Fayet, *Nucl. Phys.* **B113** (1976) 135;
M.F. Sohnius, *Nucl. Phys.* **B138** (1978) 109.

[27] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* **B214** (1983) 519.

[28] R. Grimm, M.F. Sohnius and J. Wess, *Nucl. Phys.* **B113** (1978) 275.

[29] L. Mezincescu, J.I.N.R. Report P2-12572 (1979);
S.J. Gates and W. Siegel, *Nucl. Phys.* **B195** (1982) 39.

[30] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* **B236** (1984) 125.

[31] J. Koller, *Phys. Lett.* **124B** (1983) 324; *Nucl. Phys.* **B222** (1983) 319.

[32] W. Siegel and S.J. Gates, Jr., *Nucl. Phys.* **B189** (1981) 295.

[33] L. Brink, O. Lindgren and B. Nilsson, *Nucl. Phys.* **B212** (1983) 401.

[34] S. Mandelstam, *Nucl. Phys.* **B213** (1983) 149;
L. Brink, O. Lindgren and B. Nilsson, *Phys. Lett.* **123B** (1983) 328.

[35] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, *Class. Quantum Grav.* **1** (1984) 469; **2** (1985) 155.

[36] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Class. Quantum Grav.* **2** (1985) 601; 617.

[37] P.S. Howe, K.S. Stelle and P.C. West, *Phys. Lett.* **124B** (1983) 55.

[38] P.S. Howe, K.S. Stelle and P.C. West, *Class. Quantum Grav.* **2** (1985) 815.

[39] P.S. Howe and K.S. Stelle, *Phys. Lett.* **137B** (1984) 175.

[40] N. Marcus and A. Sagnotti, *Phys. Lett.* **135B** (1984) 85;
Nucl. Phys. **B256** (1985) 77.

[41] S. Deser, J. H. Kay and K.S. Stelle, *Phys. Rev. Lett.* **38** (1977) 527.

[42] R.E. Kallosh, *Phys. Lett.* **99B** (1981) 122.

[43] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* **B191** (1981) 445.

[44] P.S. Howe, H. Nicolai and A. Van Proeyen, *Phys. Lett.* **112B** (1982) 446.

[45] A. Parkes and P.C. West, *Phys. Lett.* **122B** (1983) 365;
Nucl. Phys. **B222** (1983) 269;
A. Namazie, A. Salam and J. Strathdee, *Phys. Rev.* **D28** (1983) 1481;
J.J. Van der Bij and Y.P. Yao, *Phys. Lett.* **125B** (1983) 171;
S. Rajpoot, J.G. Taylor and M. Zaimi, *Phys. Lett.* **127B** (1983) 347.

[46] A.J. Parkes and P.C. West, *Phys. Lett.* **127B** (1983) 353.

[47] L. Girardello and M.T. Grisaru, *Nucl. Phys.* **B194** (1982) 55.

[48] A.J. Parkes and P.C. West, *Phys. Lett.* **138B** (1984) 99;
Nucl Phys. **B256** (1985) 340.

[49] A.J. Parkes, *Phys. Lett.* **156B** (1985) 73;
D.R.T. Jones and A.J. Parkes, *Phys. Lett.* **160B** (1985) 267.

[50] D.I. Kazakov, *Phys. Lett.* **179B** (1986) 952.

[51] A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Nucl. Phys.* **B281** (1987) 72.

[52] O. Piguet and K. Sibold, *Nucl. Phys.* **B248** (1984) 336;
Nucl. Phys. **B249** (1984) 396.

[53] C. Luccesi, O. Piguet and K. Sibold, *Helv. Phys. Acta* **61** (1988) 321.

[54] S. Hamidi, J. Patera and J.H. Schwarz, *Phys. Lett.* **141B** (1984) 349;
X.D. Jiang and X.J. Zhou, *Commun. Theor. Phys.* **5** (1986) 179;
F.X. Dong, X.D. Jiang and X.J. Zhou *J. Phys.* **A19** (1986) 3863.

[55] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *Ann. Phys. (N.Y.)* **134** (1981) 85.

[56] G.A. Vilkovisky, in *Quantum Theory of Gravity*, ed. S.M. Christensen (Adam Hilger, Bristol, 1984);
B.S. DeWitt, in *Quantum Field Theory and Quantum Statistics*, eds. I.A. Batalin, C.J. Isham and G.A. Vilkovisky (Adam Hilger, Bristol, 1987).

[57] H. Kluberg-Stern and J.B. Zuber, *Phys. Rev.* **D12** (1975) 482; 3159.

[58] D. Friedan, *Phys. Rev. Lett.* **45** (1980) 1057; *Ann. Phys. (N.Y.)* **163** (1985) 318.

[59] P.S. Howe, G. Papadopoulos and K.S. Stelle, *Nucl. Phys.* **B296** (1988) 26.

[60] S. Mukhi, *Nucl. Phys.* **B264** (1986) 640.

[61] P. Breitenlohner, D. Maison and K.S. Stelle, *Phys. Lett.* **134B** (1984) 63.

[62] S. Adler and W.A. Bardeen, *Phys. Rev.* **182** (1969) 1517;
S. Adler, in *Lectures in Elementary Particles and Quantum Field Theory*, vol. I, eds. S. Deser, M.T. Grisaru and H. Pendleton (M.I.T. Press, 1970);
W.A. Bardeen, *Nucl. Phys.* **B75** (1974) 246.

[63] J. Wess and B. Zumino, *Phys. Lett.* **66B** (1977) 361.

[64] S.J. Gates, Jr., K.S. Stelle and P.C. West, *Nucl. Phys.* **B169** (1980) 347;
P.S. Howe, *Nucl. Phys.* **B199** (1982) 309.

[65] P.S. Howe and U. Lindstrom, *Nucl. Phys.* **B181** (1981) 487.

[66] W. Siegel, *Nucl. Phys.* **B177** (1981) 325.

[67] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* **B192** (1981) 332.

[68] E. Cremmer and B. Julia, *Nucl. Phys.* **B159** (1979) 141.

[69] C.M. Hull and E. Witten, *Phys. Lett.* **160B** (1985) 398.

[70] G. Moore and P. Nelson, *Comm. Math. Phys.* **100** (1985) 83.

[71] B. Zumino, *Phys. Lett.* **87B** (1979) 205.

[72] L. Alvarez-Gaumé and D.Z. Freedman, *Comm. Math. Phys.* **80** (1981) 443.

[73] D. Gross, J. Harvey, E. Martinec and R. Rohm, *Nucl. Phys.* **B257** (1986) 75.

[74] P.S. Howe and G. Papadopoulos, *Class. Quantum Grav.* (in press).

[75] S.J. Gates, C.M. Hull and M. Roček, *Nucl. Phys.* **B248** (1984) 157.

[76] E. Sokatchev and K.S. Stelle, *Class. Quantum Grav.* **4** (1987) 501.

[77] P.S. Howe and G. Papadopoulos, *Nucl. Phys.* **B289** (1987) 264.

[78] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, *Phys. Lett.* **173B** (1986) 423;
Nucl. Phys. **B277** (1986) 388; 409.

[79] S.T. Yau, *Proc. Natl. Acad. Sci.* **74** (1977) 1798.

[80] C.M. Hull, *Nucl. Phys.* **B260** (1985) 182;
L. Alvarez-Gaumé and P. Ginsparg, *Comm. Math. Phys.* **102** (1985) 311.

[81] J. Grundberg, A. Karlhede, U. Lindstrom and G. Theodorikis, *Nucl. Phys.* **B282** (1987) 142.

[82] M.T. Grisaru, B. Milewski and D. Zanon, *Phys. Lett.* **155B** (1985) 357.

[83] W. Siegel, *Phys. Lett.* **84B** (1979) 193.

[84] W. Siegel, *Phys. Lett.* **94B** (1980) 37.