The undulatory motion of a chain of particles in a resistive medium

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The motion of a finite chain of identical bodies along a straight line in a resistive medium is studied. The major aim of this study is to investigate the fundamental properties of such systems, in particular, their ability to move from a state of rest and sustain the motion at constant average velocity in media with different resistance properties and the influence of the control strategy on the motion. The motion is excited and controlled by changing the distances between the bodies of the chain. For a given friction law, the necessary and sufficient conditions for the system to be able to move from rest are established.

1 Introduction

Since recently, attention of researchers has been drawn by mobile systems that move in a resistive environment due to change in their configuration or motion of internal masses. This mechanism could account for the locomotion of some limbless animals, e.g., snakes or worms.

A number of publications [5, 7–9, 11, 14, 16–18, 20–23] deal with snake-like planar mechanisms designed as a chain of rigid links connected to one another by revolute joints with drives located at them. Two types of strategies for the motion along a rough plane were considered, the dynamic strategy and the quasi-static strategy. The dynamic strategy [5, 7–9, 11, 14, 21, 22] involves alternating fast and slow phases of motion. During the slow phases, some of the links are moving, while the other links are kept unmoved on the plane due to dry friction forces. As a result, the center of mass of the system shifts, the magnitude and direction of the shift depending on the change in the system's configuration. In the fast phases, the control torques at the joints are much higher than the dry friction torques and the duration of the motion is small, due to which the center of mass of the system virtually does not move. By combining the slow and fast phases, the snake-like linkage can be moved to any prescribed position. The quasi-static strategy [6, 16–18, 23] involves only slow motions. This strategy is inferior to the dynamic strategy in terms of the speed of motion but surpasses it in terms of the energy consumption.

Worm-like mobile mechanisms designed as a chain of links connected by powered prismatic (translational) joints are investigated in [10, 25, 26, 28]. Studies [25, 28] deal with the analysis of the rectilinear motion along a rough plane of chains of bodies connected to one another by means of elastic elements in the case where the normal pressure does not change and the system is driven by harmonically varying forces acting between the bodies. The asymmetry in the force of friction necessary for the motion of the system in a prescribed direction is provided by the dependence of the coefficient of friction on the direction of the velocity of the system's components. This can be provided, for example, by covering the contact surfaces of the robot with scaly or pinned plates with an appropriate orientation of scales (pins).

Mechanisms of another type consist of a body with movable internal masses [3, 4, 12, 13, 19, 24–27]. The inertial masses interact with the body by means of forces generated and controlled by drives but do not interact with the environment. When the control force is applied to an internal mass, the reaction force is applied to the body and changes its velocity, which affects the resistance force exerted on the body by the environment. The issue of the optimal control of the motion of a body with movable internal masses was considered in [3, 12, 13, 19]. In [4], the motion of a single body with an unbalanced vibration exciter along a straight line on a rough horizontal plane was studied. A difference in the coefficients of forward and backward friction was not assumed. The dynamic asymmetry was provided by the phase shift between the horizontal and vertical components of the driving force produced by the exciter. The motion of a similar system along an inclined plane

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was investigated in [24]. A mobile system that consists of two identical modules connected by a spring was considered in [27]. Each module contains an unbalanced vibration exciter. The exciters rotate with the same frequency in one direction but have a phase shift. It was shown that when the excitation frequency passes through the resonance with the natural frequency of the system, the direction of the motion changes.

This principle of motion can be used for some kind of mobile robots. An advantage of these robots against conventional mobile machines is that the snake-like and worm-like robots do not need propelling devices such as wheels, legs, or caterpillars, can be readily made hermetic, and do not involve protruding components, which enables them to be used for motion in "vulnerable" media. Such robots can be used for motion in narrow slots and tubes to produce various technical operations in these media.

In this paper, the motion of a finite chain of identical bodies along a straight line in a resistive medium is studied. The motion is excited and controlled by changing the distances between the bodies of the chain.

2 Description of the mechanical system

The subject matter of the investigation is the motion of a chain of identical particles along a horizontal straight line in a resistive medium (environment) (Fig. 1). The motion is excited and controlled by changing the distances between the particles. The medium acts on the ith particle with the force $F(\dot{x}_i)$, depending on the velocity $\dot{x}_i$ of this particle relative to the environment that is assumed to be unmoved in an inertial reference frame. The force F is the force of friction between the particle and the medium. We will consider two kinds of friction, viscous friction and dry friction.

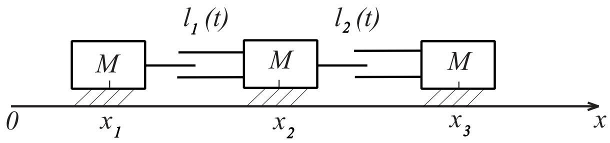

Fig. 1 Mechanical model.

By *viscous friction* we understand the resistance characterized by a continuous, monotonically decreasing function $F(\dot{x}_i)$, vanishing at $\dot{x}_i = 0$. A typical example of the viscous friction is the power-law friction characterized by

$$F(\dot{x}_i) = \begin{cases} \mu_-|\dot{x}_i|^\alpha & \text{if } \dot{x}_i \le 0, \\ -\mu_+|\dot{x}_i|^\alpha & \text{if } \dot{x}_i > 0, \end{cases} \tag{1}$$

where μ_- and μ_+ are positive coefficients of friction resisting the leftward ($\dot{x} < 0$) and rightward ($\dot{x} > 0$) motion, respectively, and $\alpha > 0$ is the power exponent of the resistance law.

By *dry friction* we understand the resistance characterized by the law:

$$F(\dot{x}_i) = \begin{cases} k_-, & \text{if } \dot{x}_i < 0 \text{ or } \dot{x}_i = 0, \text{ and } \Phi < -k_- \\ -k_+, & \text{if } \dot{x}_i > 0 \text{ or } \dot{x}_i = 0, \text{ and } \Phi > k_+ \\ -\Phi, & \text{if } \dot{x}_i = 0 \text{ and } -k_- < \Phi < k_+, \end{cases} \tag{2}$$

where k_- and k_+ are positive quantities characterizing the maximum magnitude of the friction force resisting the leftward ($\dot{x} < 0$) and rightward ($\dot{x} > 0$) motion, respectively, and Φ is the resultant of the forces, other than frictional ones, applied to the particle. The friction characterized by this law is sometimes called *Coulomb's anisotropic friction*. Technically, the anisotropy can be provided by appropriately structuring the contact surface of the body, for example, by covering it with a "hairy" material with inclined hairs. The classical Coulomb law is a particular case of Eq. (2) for $k_- = k_+$.

By *discontinuous friction* we understand the resistance characterized by a decreasing function $F(\dot{x}_i)$, which is discontinuous at $\dot{x}_i = 0$, has the left-hand limit and the right-hand limit at this point, and these limits satisfy the inequalities

$$\lim_{\dot{x}_i \to -0} F(\dot{x}_i) > 0, \quad \lim_{\dot{x}_i \to +0} F(\dot{x}_i) < 0. \tag{3}$$

We assume that the change of the distances between the particles in the chain is produced by forces of interaction between the particles, which are internal forces for the entire system. The only external forces acting on the system are the forces of friction.

Let the chain consist of n particles of mass M each. The particles are numbered consecutively from the left end to the right end of the chain. Denote the absolute coordinate of the ith particle by x_i and the difference between the coordinates of adjacent particles $j+1$ and j by l_j, i.e.,

$$l_j = x_{j+1} - x_j, \quad j = 1, \ldots, n-1. \tag{4}$$

The quantity $|l_j|$ is the distance between particles j and $j+1$. Based on Eq. (4), the coordinates x_2 to x_n can be expressed in terms of x_1 and the quantities l_j:

$$x_i = x_1 + \sum_{j=1}^{i-1} l_j, \quad i = 2, \dots, n. \tag{5}$$

The absolute coordinate X of the center of mass of the chain and its velocity V are defined by

$$X = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad V = \dot{X}. \tag{6}$$

The motion of the center of mass is governed by the equation

$$nM\dot{V} = \sum_{i=1}^{n} F(\dot{x}_i). \tag{7}$$

Using Eqs. (4) and (5), the coordinates x_i of the particles can be expressed in terms of X and l_j:

$$x_i = X + \frac{1}{n} \sum_{j=1}^{n-1} j l_j - \sum_{j=i}^{n-1} l_j, \quad i = 1, \dots, n-1, \quad x_n = X + \frac{1}{n} \sum_{j=1}^{n-1} j l_j. \tag{8}$$

Substitute these relations into Eq. (7) to obtain

$$\dot{V} = \frac{1}{nM} \sum_{i=1}^{n-1} F\left(V + \frac{1}{n} \sum_{j=1}^{n-1} j\dot{l}_j - \sum_{j=i}^{n-1} \dot{l}_j\right) + \frac{1}{nM} F\left(V + \frac{1}{n} \sum_{j=1}^{n-1} j\dot{l}_j\right). \tag{9}$$

For the linear resistance law characterized by the function $F(z) = -kz$, where k is the coefficient of friction, the last equation becomes

$$\dot{V} = -\frac{1}{M} kV \tag{10}$$

and, hence, the center of mass of the system moves as a mass point in a medium with linear viscous friction, irrespective of the relative motion of the particles of the chain. Therefore, the motion of the chain in a medium with a linear resistance law cannot be controlled by changing the distances between its particles.

In what follows, we assume that the system moves in a medium with nonlinear resistance law and is controlled kinematically, i.e., the quantities $l_j(t)$ are prescribed functions of time. Then Eq. (9) becomes a first-order time-varying equation for V.

Proposition. If the distances between the particles change synchronously, i.e., all $l_j(t)$ are identical, and the function F is odd, then the solution of Eq. (9) subject to the initial condition $V(0) = 0$ is $V(t) \equiv 0$. This means that the center of mass of the chain in an isotropic medium cannot start moving from a state of rest.

For $l_i(t) = l(t)$, $i = 1, \dots, n-1$, Eq. (9) becomes

$$\dot{V} = \frac{1}{nM} \sum_{i=1}^{n} F\left(V - \left(\frac{n+1}{2} - i\right) \dot{l}(t)\right). \tag{11}$$

This equation can be rewritten as

$$\dot{V} = \frac{1}{nM} \sum_{i=1}^{n/2} \left[F\left(V - \left(\frac{n+1}{2} - i\right) \dot{l}(t)\right) + F\left(V + \left(\frac{n+1}{2} - i\right) \dot{l}(t)\right)\right] \tag{12}$$

for even n and as

$$\dot{V} = \frac{1}{nM} \sum_{i=1}^{(n-1)/2} \left[F\left(V - \left(\frac{n+1}{2} - i\right) \dot{l}(t)\right) + F\left(V + \left(\frac{n+1}{2} - i\right) \dot{l}(t)\right)\right] + \frac{1}{nM} F(V) \tag{13}$$

for odd n. If F is an odd function, then the right-hand sides of these equations vanish for $V = 0$ and, hence, $V(t) \equiv 0$ is a solution of Eq. (11). Due to the uniqueness theorem, this is the only solution corresponding to zero initial condition for V. This completes the proof.

Remark. In the proof of the proposition, we referred to the uniqueness theorem for Eq. (11). The conventional uniqueness theorem [1] for the initial-value problem

$$\dot{x} = f(x, t), \quad x(t_0) = x_0 \tag{14}$$

assumes that the function $f(x, t)$ is defined in a neighborhood of the point (x_0, t_0) in the (xt)-space, is continuous in this neighborhood, and satisfies the Lipschitz conditions with respect to x. For Eq. (11), these conditions may be violated, because we admit power-law friction laws of Eq. (1) and the dry-friction law of Eq. (2). The function $F(z)$ of Eq. (1) does not satisfy the Lipschitz condition at $z = 0$ for $\alpha < 1$, and the function $F(z)$ of Eq. (2) is discontinuous at $z = 0$. Nevertheless, the solution of Eq. (11) subject to the initial condition $V(0) = 0$ is uniquely defined. Equation (11) has the form $\dot{V} = f(V, t)$, where $f(0, t) \equiv 0$ and $f(V, t) \operatorname{sgn} V < 0$. Apparently, $V(t) \equiv 0$ is a solution of this equation. Assume that there exists another solution $\widetilde{V}(t)$ satisfying the initial condition $\widetilde{V}(0) = 0$. The function $\widetilde{V}(t)$ must satisfy the inequality $\widetilde{V}\dot{\widetilde{V}} > 0$ at some t, which contradicts the inequality $f(V, t) \operatorname{sgn} V < 0$.

The proposition implies an important corollary that to enable the chain to move "as a whole", the distances between the particles should be changed asynchronously. It follows, in particular, that the center of mass of a system of two identical particles in an isotropic medium cannot start moving from a state of rest.

3 Undulatory excitation of motion

We will consider the motion excited by a periodic change in the relative position of adjacent particles according to the law

$$l_i(t) = a\,(t + (i - 1)\,\tau)\,, \quad i = 1, \ldots, n - 1\,, \tag{15}$$

where $a(z)$ is a periodic function and τ is a parameter of dimension of time. Equation (4) implies that the function $l_i(t)$ defines the motion of particle $i + 1$ relative to particle i. We will say that particles i and $i + 1$ form the ith pair of adjacent particles of the chain. From (15) it follows that the relative motion of particles in the ith pair repeats the relative motion of particles in the $(i + 1)$st pair with a time shift of τ. Expression (15) can be reduced to the relation

$$l_i(t) = f\left(i + \frac{t}{\tau}\right), \quad \text{where } f(z) = a\,(\tau\,(z - 1))\,, \tag{16}$$

which describes a deformation wave running forward (for $\tau < 0$) or backward (for $\tau > 0$) through the chain with a velocity of $1/|\tau|$. For that reason, we will call the excitation mode defined by Eq. (15) and the corresponding motion of the chain the undulatory excitation and the undulatory motion, respectively.

4 Nondimensionalization

Introduce the dimensionless variables

$$t^* = \frac{t}{T}, \quad l_i^* = \frac{l_i}{L},\ i = 1, \ldots, n - 1\,, \quad V^* = \frac{VT}{L}, \quad F^* = \frac{F}{F_0}\,, \tag{17}$$

where T is the period of the functions $l_i(t)$, L is a length characteristic of the change in the distances between the particles, and F_0 is a quantity that has a dimension of force and characterizes the magnitude of the friction force. For example, L can be defined by

$$L = \max_i \left[\max_t l_i(t) - \min_t l_i(t)\right]. \tag{18}$$

If the function $|F(v)|$ is bounded, its upper bound can be taken as F_0. For example, for the dry-friction law of Eq. (2), we can define $F_0 = \max(k_-,\, k_+)$.

In what follows, we deal only with the dimensionless form of Eq. (9) and omit the asterisks identifying the dimensionless variables. The nondimensionalized Eq. (9) can be represented as

$$\dot{V} = \varepsilon G(V,t),$$
$$G(V,t) = \frac{1}{n}\left[\sum_{i=1}^{n-1} F\left(V + \frac{1}{n}\sum_{j=1}^{n-1} ju_j - \sum_{j=i}^{n-1} u_j\right) + F\left(V + \frac{1}{n}\sum_{j=1}^{n-1} ju_j\right)\right], \tag{19}$$

where

$$\varepsilon = \frac{F_0 T^2}{ML}, \quad u_i(t) = \dot{l}_i, \quad i = 1, \ldots, n-1\,. \tag{20}$$

Since the functions $l_i(t)$ are periodic (with period 1), the functions $u_i(t)$ are also periodic (with the same period) and have zero averages, i.e.,

$$\int_0^1 u_i(t)dt = 0, \quad i = 1, \ldots, n-1\,. \tag{21}$$

5 Method of averaging

We assume that the parameter ε is small ($\varepsilon \ll 1$) and that the function $G(V,t)$ on the right-hand side of Eq. (19) has an order of unity. The smallness of ε means that the maximum magnitude of the force of friction is small in comparison with the characteristic magnitude of the force that should be applied to an individual particle to produce a prescribed change in the distance from the adjacent particle during a prescribed time. Then the *method of averaging* [2] can be applied to Eq. (19). According to this method, the time-varying Eq. (19) is replaced by the time-invariant averaged equation

$$\dot{v} = \varepsilon \bar{G}(v), \tag{22}$$

where

$$\bar{G}(v) = \frac{1}{n}\int_0^1 \left[\sum_{i=1}^{n-1} F\left(v + \frac{1}{n}\sum_{j=1}^{n-1} ju_j - \sum_{j=i}^{n-1} u_j\right) + F\left(v + \frac{1}{n}\sum_{j=1}^{n-1} ju_j\right)\right] dt. \tag{23}$$

Let $V(t)$ and $v(t)$ be the solutions of Eqs. (19) and (22), respectively, subject to the identical initial conditions $V(0) = V_0$ and $v(0) = V_0$. Then

$$|V(t) - v(t)| = O(\varepsilon) \tag{24}$$

for $t \in [0, \xi]$, where $\xi \sim 1/\varepsilon$. Thus, the solution $v(t)$ of the averaged equation (22) provides an acceptable approximation to the solution of the primary equation (19). The relationship (24) was proved by Bogolyubov and Mitropolskii [2] for systems in which the function $G(V,t)$ is Lipschitz continuous with respect to V.

In what follows in this paper, we will study the motion of the chain of particles in the environments characterized by the resistance laws that are not necessarily described by Lipschitz continuous functions. This is the case, for example, for the viscous friction law of Eq. (1) with $\alpha < 1$, which does not satisfy the Lipschitz condition at $\dot{x}_i = 0$, or for the dry friction law of Eq. (2), which is discontinuous at $\dot{x}_i = 0$. Accordingly, the function $G(V,t)$ in Eq. (19) for these resistance laws is not Lipschitz-continuous in V. The practice shows, however, that the class of systems to which the averaging technique can be applied is rather wider than the systems the right-hand sides of which are Lipschitz continuous with respect to the state variables. Bogolyubov and Mitropolskii [2] used the method of averaging for a single-degree-of-freedom system with dry friction in an illustrative example to demonstrate that the behavior of the averaged system adequately reflects the behavior of the primary one. In [15], the accuracy of Eq. (24) was proved for a class of systems, in which the function $G(V,t)$ is discontinuous in V. This class covers, in particular, the systems that are subject to Coulomb's friction and move without sticking or the sticking intervals are ε-small.

Based on this observation, we will apply the method of averaging to Eq. (19) for all resistance laws that will be considered, even if mathematical theorems that prove the accuracy of the approximation of the solutions of the primary equations of motion by the solutions of the averaged equations are unavailable. The solution of the averaged equations can be obtained in closed form, while the primary equations, as a rule, can be solved only numerically, which complicates the qualitative analysis of the behavior of the system. To prove that the solutions of the averaged equations adequately reflect the system's behavior, we will solve numerically the primary equations to compare the respective solutions.

In what follows, the motion of the center of mass of the chain is studied on the basis of the solution of the averaged equation.

6 The motion of the center of mass of the chain according to the averaged equation

6.1 Continuous $F(z)$

If the function $F(z)$, which characterizes the friction law, is continuous, then the function $\bar{G}(v)$ of Eq. (22) has one and only one root. It was assumed previously that the function $F(z)$ is defined for any $v \in (-\infty, +\infty)$, monotonically decreases, and vanishes at $z = 0$. Hence, $F(z) > 0$ for $z < 0$ and $F(z) < 0$ for $z > 0$. Therefore, the function $\bar{G}(v)$ monotonically decreases, is positive when v is negative and has large enough absolute value, and is negative when v is positive and large enough. If $F(z)$ is continuous, then $\bar{G}(v)$ is also continuous, and, hence, has a root v_*, which is uniquely defined.

The value v_* is a steady-state solution of Eq. (22). Since $\bar{G}(v)$ is a monotonically decreasing function, $\bar{G}(v) < 0$ for $v > v_*$ and $\bar{G}(v) > 0$ for $v < v_*$. Therefore, with reference to Eq. (22), $\dot{v} > 0$ for $v < v_*$ and $\dot{v} < 0$ for $v > v_*$ and, hence, the solution v_* is asymptotically stable.

Thus, if the function $F(z)$ is continuous, then

- the system can start moving from a state of rest, if and only if $\bar{G}(0) \neq 0$;
- if $\bar{G}(0) > 0$ ($\bar{G}(0) < 0$), then the system moves rightward (leftward) with the velocity v monotonically converging to the steady-state value v_*.

6.2 Discontinuous $F(z)$

Let the friction between the chain and the environment be discontinuous, which implies that the function $F(z)$ monotonically decreases and has a discontinuity at $z = 0$ characterized, in accordance with Eq. (3), by the inequalities

$$F(-0)F(+0) < 0, \quad F(-0) > 0, \tag{25}$$

where $F(-0)$ and $F(+0)$ are the left-hand limit and the right-hand limit of the function $F(z)$ at $z = 0$, respectively. The function $G(V, t)$ of Eq. (19) monotonically decreases in V, since the function $F(z)$ decreases. Therefore, the function $\bar{G}(v)$ of Eq. (23) monotonically decreases in v and, in addition, $\bar{G}(v)v < 0$ for $|v|$ large enough. The function $\bar{G}(v)$ may have points of discontinuity, the inequality $\bar{G}(v+0) < \bar{G}(v-0)$ holding for all such points. Then the function $\bar{G}(v)$ has one and only one point v_*, which either satisfies the relation $\bar{G}(v_*) = 0$ or is defined as a discontinuity point at which $\bar{G}(v_* - 0) \geq 0$ and $\bar{G}(v_* + 0) \leq 0$.

As follows from the equation $\dot{v} = \varepsilon \bar{G}(v)$, the value v_* is a steady-state solution of this equation, if $\bar{G}(v)$ is continuous at v_*. This solution is stable, since $\bar{G}(v)$ decreases, due to which any solution starting in a small neighborhood of v_* approaches this value.

If v_* is a discontinuity point of $\bar{G}(v)$, then all solutions starting in a small neighborhood of v_* approach this point. Therefore, it is reasonable to regard the value v_* as a stable steady-state solution even if $\bar{G}(v)$ is discontinuous at $v = v_*$.

Thus, for discontinuous friction,

- the system can move from a state of rest if and only if $v = 0$ is not a steady-state velocity, i.e., $\bar{G}(-0)\bar{G}(+0) > 0$;
- if $\bar{G}(-0) > 0$ and $\bar{G}(+0) > 0$ ($\bar{G}(-0) < 0$ and $\bar{G}(+0) < 0$), then the system accelerates rightward (leftward) monotonically approaching the steady state velocity v_*.

In the subsequent sections, the dynamics of a three-particle system moving in a resistive medium is studied. An interest in this system is motivated primarily by the fact that three is the minimal number of particles that a chain must have to be able to move in an isotropic medium due to the change in the distances between the particles. A three-particle system is easier to study than systems with larger number of particles and, hence, the results are simpler to represent and to interpret.

7 Three-particle system

For a three-particle system, the function $\bar{G}(v)$ in Eq. (23) becomes

$$\bar{G}(v) = \frac{1}{3}\int_0^1 \left[F\left(v - \frac{2}{3}u_1(t) - \frac{1}{3}u_2(t)\right) + F\left(v + \frac{1}{3}u_1(t) - \frac{1}{3}u_2(t)\right) + F\left(v + \frac{1}{3}u_1(t) + \frac{2}{3}u_2(t)\right)\right] dt. \tag{26}$$

We will consider the motion of the center of mass of the chain governed by the averaged equation (Eq. (22)) for a specific type of the undulatory excitation of Eq. (15). Since the motion of the center of mass characterizes the motion of the chain "as a whole", we will frequently use the terms the *motion of the center of mass of the chain (system)* and the *motion of the chain (system)* as synonyms.

If the motion is excited in accordance with the undulatory law of Eq. (15), the expression of Eq. (26) can be represented as

$$\bar{G}(v) = \frac{1}{3}\int_0^1 \left[F\left(v - \frac{2}{3}u(t) - \frac{1}{3}u(t+\tau)\right) + F\left(v + \frac{1}{3}u(t) - \frac{1}{3}u(t+\tau)\right) + F\left(v + \frac{1}{3}u(t) + \frac{2}{3}u(t+\tau)\right)\right] dt, \quad u(t) = \dot{a}(t). \tag{27}$$

In what follows, we confine ourselves to the case of $\tau > 0$. As follows from Eq. (27), the case of $\tau < 0$ for the excitation function $u(t)$ is equivalent to the case of $\tau > 0$ for the excitation function $-u(t)$.

7.1 Piecewise linear undulatory excitation

Consider the excitation law of Eq. (15) for the 1-periodic piecewise linear function $a(t)$ the restriction of which on the time interval $[0,\, 1]$ is given by

$$a(t) = \begin{cases} 0, & t \in [0,\, \tau), \\ t - \tau, & t \in [\tau,\, (1+\tau)/2), \\ -t + 1, & t \in [(1+\tau)/2,\, 1], \end{cases} \tag{28}$$

where $0 \le \tau < 1$. Then $l_1(t) = a(t)$ and $l_2(t) = a(t+\tau)$. The restriction of the function $l_2(t)$ on the time interval $[0,\, 1]$ is expressed as

$$l_2(t) = \begin{cases} t, & t \in [0,\, (1-\tau)/2), \\ -t + 1 - \tau, & t \in [(1-\tau)/2,\, 1-\tau), \\ 0, & t \in [1-\tau,\, 1], \end{cases} \tag{29}$$

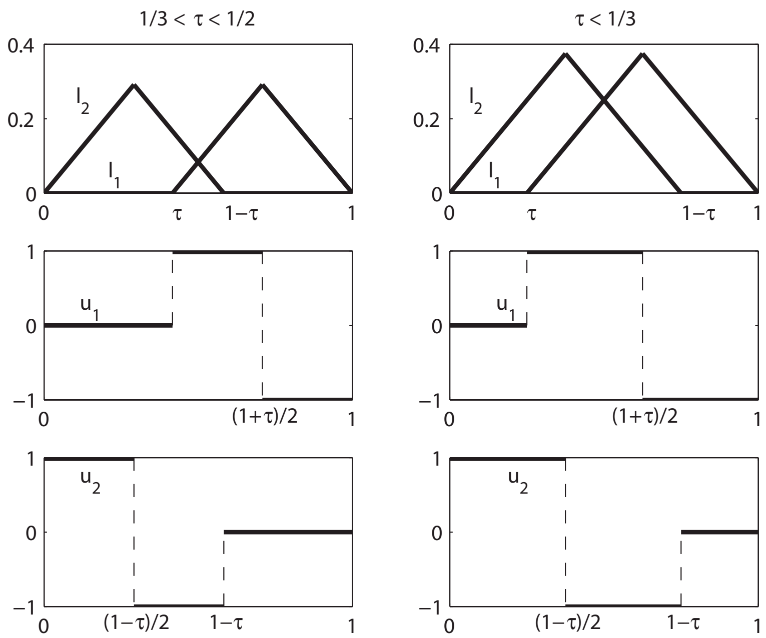

Fig. 2 Functions $l_1(t), l_2(t), u_1(t)$, and $u_2(t)$.

The functions $l_1(t)$ and $l_2(t)$ are piecewise linear and, hence, the derivatives $u_1(t)$ and $u_2(t)$ occurring in Eq. (26) are piecewise constant 1-periodic functions that are related by $u_1(t) = u(t)$ and $u_2(t) = u(t+\tau)$, where $u(t) = \dot{a}(t)$, and are restricted on the time interval $[0,\ 1]$ as follows:

$$u_1(t) = \begin{cases} 0, & t \in [0,\ \tau), \\ 1, & t \in [\tau,\ (1+\tau)/2), \\ -1, & t \in [(1+\tau)/2,\ 1]. \end{cases} \tag{30}$$

$$u_2(t) = \begin{cases} 1, & t \in [0,\ (1-\tau)/2), \\ -1, & t \in [(1-\tau)/2,\ 1-\tau), \\ 0, & t \in [1-\tau,\ 1]. \end{cases} \tag{31}$$

The functions $l_1(t)$, $l_2(t)$, $u_1(t)$, and $u_2(t)$ are shown in Fig. 2. Substituting these functions into Eq. (26) yields the right-hand side of the averaged equation of motion

$$\bar{G}(v) = \frac{1}{3} \begin{cases} [F(v+1) + F(v) + F(v-1)]\,(1-3\tau) \\ +\,[3F(v+2/3) + 6F(v-1/3)]\,\tau, & 0 \le \tau \le 1/3, \\ [F(v+2/3) + 2F(v-1/3)]\,(2-3\tau) \\ +\,[2F(v+1/3) + F(v-2/3)]\,(3\tau-1), & 1/3 < \tau \le 1/2, \\ [2F(v-1/3) + 2F(v+1/3) + F(v-2/3) \\ +F(v+2/3)]\,(1-\tau) + 3F(v)(2\tau-1), & 1/2 < \tau < 1. \end{cases} \tag{32}$$

An important characteristic of the system and the excitation law is the steady-state velocity v_*, defined as a solution of the equation $\bar{G}(v) = 0$. As was proved in Sect. 6, the steady-state velocity is uniquely defined for a given friction law and a given excitation law. The excitation law of Eqs. (28) – (31) depends on the only parameter τ. By adjusting this parameter one can control the motion of the system, in particular, its steady-state velocity.

7.1.1 Isotropic viscous friction

The isotropic viscous friction is characterized by an odd monotonically decreasing function $F(z)$. As was shown in Sect. 6, for continuous $F(z)$, the system can start moving from a state of rest if and only if $\bar{G}(0) \neq 0$. For odd function $F(z)$, $\bar{G}(0)$ is defined by

$$\bar{G}(0) = \begin{cases} [F(2/3) - 2F(1/3)]\,\tau, & 0 \le \tau \le 1/3, \\ [F(2/3) - 2F(1/3)]\,(1-2\tau), & 1/3 < \tau \le 1/2, \\ 0, & 1/2 < \tau < 1. \end{cases} \tag{33}$$

From this expression it follows that $\bar{G}(0) \neq 0$ if and only if

$$F(2/3) - 2F(1/3) \neq 0 \quad \text{and} \quad 0 < \tau < 1/2. \tag{34}$$

If $F(2/3) > 2F(1/3)$, the system moves rightward from a state of rest; if $F(2/3) < 2F(1/3)$, the system moves leftward; if $F(2/3) = 2F(1/3)$, the system cannot move from a state of rest. The starting acceleration $\dot{v}(0)$ depends on the time shift τ. The maximum absolute value of the starting acceleration occurs for $\tau = 1/3$.

Power-law friction. The power-law friction is characterized by the function

$$F(z) = -k|z|^{\alpha} \operatorname{sgn} z, \quad k > 0, \quad \alpha > 0. \tag{35}$$

Then

$$F(2/3) - 2F(1/3) = k\left[2\left(\frac{1}{3}\right)^{\alpha} - \left(\frac{2}{3}\right)^{\alpha}\right]. \tag{36}$$

This expression is positive, equal to zero, and negative for $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, respectively. Accordingly, the system moves rightward for $\alpha < 1$, moves leftward for $\alpha > 1$, and cannot move from a state of rest for $\alpha = 1$. The case of $\alpha = 1$ corresponds to linear friction. In this case, Eq. (32) implies $\bar{G}(v) \equiv -k\varepsilon v$. The last equation is the nondimensionalized Eq. (10). Hence, for linear friction law, the averaged equation of motion coincides with the exact equation of motion.

Calculate the steady-state velocity v_* for $\tau = 1/3$, in which case the starting acceleration of the system is a maximum. The quantity v_* is a root of the equation $\bar{G}(v) = 0$, which, taking into account (32) for $\tau = 1/3$ and (35), can be reduced to the form

$$\left|v + \frac{2}{3}\right|^{\alpha} \operatorname{sgn}\left(v + \frac{2}{3}\right) + 2\left|v - \frac{1}{3}\right|^{\alpha} \operatorname{sgn}\left(v - \frac{1}{3}\right) = 0. \tag{37}$$

This equation has only one solution, which lies in the interval $(-2/3,\ 1/3)$ and is expressed by

$$v_* = \frac{2^{1/\alpha} - 2}{3\left(2^{1/\alpha} + 1\right)}. \tag{38}$$

The steady-state velocity is positive for $\alpha < 1$, equal to zero for $\alpha = 1$, and negative for $\alpha > 1$. The value v_* approaches $1/3$ as $\alpha \to 0$ and approaches $-1/6$ as $\alpha \to \infty$.

Quadratic friction. Quadratic friction is a particular case of the power-law friction for $\alpha = 2$. The system subject to the quadratic friction moves leftward ($v < 0$) from a state of rest until it reaches a negative steady-state velocity v_*. Solve equation $\bar{G}(v) = 0$ to find the steady-state velocity. For $F(z) = -k|z|^2 \operatorname{sgn} z$, the function $\bar{G}(v)$ of Eq. (32) is a piecewise polynomial function. In each of the intervals $(-\infty,\ -1)$, $(-1,\ -2/3]$, $(-2/3,\ -1/3]$, $(-1/3,\ 0]$, $(0,\ 1/3]$, $(1/3,\ 2/3]$, $(2/3,\ 1]$, and $(1,\ +\infty)$, this function is a quadratic polynomial. For these intervals, the corresponding quadratic equations should be solved. For the quadratic friction, it suffices to consider only the intervals that lie on the negative side of the number axis. Since the steady-state velocity exists and is uniquely defined for any specific friction law, it turns out that for fixed τ, the quadratic equations have a root only in one of the intervals listed above. Finally, the steady state velocity is expressed by

$$v_*(\tau) = \begin{cases} 2(1-\tau) - \sqrt{4(1-\tau)^2 + 2\tau/3}, & 0 \le \tau \le 1/3, \\ \dfrac{4 - \sqrt{18(1-2\tau)^2 + 16}}{9(1-2\tau)}, & 1/3 < \tau < 1/2, \\ 0, & 1/2 \le \tau < 1. \end{cases} \tag{39}$$

Differentiation of this expression with respect to τ shows that $dv_*(\tau)/d\tau < 0$ for $0 \le \tau \le 1/3$ and $dv_*(\tau)/d\tau > 0$ for $1/3 < \tau < 1/2$. Therefore, the function $v_*(\tau)$ has the global minimum at $\tau = 1/3$, which means that this value of the time shift provides a maximum for the absolute value of the steady-state velocity of the system's motion. It was proved earlier that $\tau = 1/3$ provides a maximum for the absolute value of the starting acceleration of the system for any continuous isotropic friction law.

7.1.2 Isotropic discontinuous friction

Isotropic discontinuous friction is characterized by a decreasing odd function $F(z)$, which is discontinuous at $z = 0$ and satisfies the relationships

$$F(+0) = -F(-0), \quad F(-0) > 0, \tag{40}$$

where $F(-0)$ and $F(+0)$ are the left-hand and right-hand limits of the function $F(z)$ at the point $z = 0$, respectively. In this case, the function $\bar{G}(v)$ of Eq. (32) is discontinuous at $v = 0$, if $0 \le \tau < 1/3$ or $1/2 < \tau < 1$.

The equation $\dot{v} = \varepsilon\bar{G}(v)$ subject to zero initial condition $v(0) = 0$ has a nonzero solution if and only if

$$\bar{G}(-0)\bar{G}(+0) > 0. \tag{41}$$

This implies that for the system to be able to move from a state of rest, both one-sided limits $\bar{G}(-0)$ and $\bar{G}(+0)$ should be nonzero and coincide in sign. The system moves rightward, if these limits are positive, and moves leftward, if they are negative.

For the isotropic discontinuous friction, the expressions for $\bar{G}(-0)$ and $\bar{G}(+0)$ corresponding to $\bar{G}(v)$ of Eq. (32) are given by

$$\bar{G}(\pm 0) = \begin{cases} [F(2/3) - 2F(1/3)]\,\tau + F(\pm 0)(1/3 - \tau), & 0 \le \tau \le 1/3, \\ [F(2/3) - 2F(1/3)]\,(1 - 2\tau), & 1/3 < \tau \le 1/2, \\ F(\pm 0)(2\tau - 1), & 1/2 < \tau < 1. \end{cases} \tag{42}$$

If $F(2/3) - 2F(1/3) = 0$, the inequality (41) does not hold for any τ. If $F(2/3) - 2F(1/3) \neq 0$, this inequality holds for

$$\tau_* < \tau < 1/2, \quad \tau_* = \frac{F(-0)}{3\,[F(-0) + |F(2/3) - 2F(1/3)|]} \tag{43}$$

and does not hold for τ from the remaining portion of the interval $[0,\,1)$. Thus, for discontinuous isotropic friction, the necessary and sufficient condition for the system to be able to move from a state of rest is given by the inequalities

$$F(2/3) - 2F(1/3) \neq 0, \quad \tau_* < \tau < 1/2. \tag{44}$$

These inequalities are valid also for isotropic continuous friction and coincide with those of (34), since in that case $F(-0) = 0$.

Coulomb's friction. For Coulomb's friction, $F(z) = -k\,\mathrm{sgn}\,z$ for $z \neq 0$. Then from (44) it follows that the system can move from a state of rest if and only if $1/6 < \tau < 1/2$. Since $F(2/3) - 2F(1/3) = k > 0$, the system will move rightward. To find the steady-state velocity v_* consider the function $\bar{G}(v)$ of Eq. (32). For Coulomb's friction, this function is defined for $0 < \tau < 1/2$ as follows:

$$\bar{G}(v) = \frac{k}{3} \begin{cases} 3, & \text{if } v < -1, \\ 1 + 6\tau, & \text{if } -1 < v < -2/3, \\ 1, & \text{if } -2/3 < v < 0, \\ 6\tau - 1, & \text{if } 0 < v < 1/3, \\ -1 - 6\tau, & \text{if } 1/3 < v < 1, \\ -3, & \text{if } v > 1, \end{cases} \qquad 0 \le \tau \le 1/3, \tag{45}$$

$$\bar{G}(v) = \frac{k}{3} \begin{cases} 3, & \text{if } v < -2/3, \\ -1 + 6\tau, & \text{if } -2/3 < v < -1/3, \\ 3 - 6\tau, & \text{if } -1/3 < v < 1/3, \\ 6\tau - 5, & \text{if } 1/3 < v < 2/3, \\ -3, & \text{if } v > 2/3. \end{cases} \qquad 1/3 < \tau \le 1/2, \tag{46}$$

As was shown in Sect. 6.2, the steady-state velocity is either a root of the function $\bar{G}(v)$ or a point of discontinuity at which $\bar{G}(v-0) \ge 0$ and $\bar{G}(v+0) \le 0$. For $1/6 < \tau < 1/2$, the only such a point is $v_* = 1/3$. Starting from zero velocity, the system moves with constant acceleration

$$\dot{v} = \begin{cases} \varepsilon k(2\tau - 1/3), & \text{if} \quad 1/6 < \tau \le 1/3, \\ \varepsilon k(1 - 2\tau), & \text{if} \quad 1/3 < \tau < 1/2, \end{cases} \tag{47}$$

and reaches the steady-state velocity at the instant

$$t_* = \begin{cases} \dfrac{1}{\varepsilon k(6\tau - 1)}, & \text{if} \quad 1/6 < \tau \le 1/3, \\ \dfrac{1}{3\varepsilon k(1 - 2\tau)}, & \text{if} \quad 1/3 < \tau < 1/2. \end{cases} \tag{48}$$

The minimum value of t_* corresponds to $\tau = 1/3$, and this value can be regarded as an optimal parameter of the excitation law of Eqs. (28)–(31).

Continuous approximation of Coulomb's friction. Consider the friction law defined by the function

$$F(z) = -k\frac{2}{\pi}\arctan(\beta z), \quad k > 0, \quad \beta > 0. \tag{49}$$

This function, for large β, is frequently used as a continuous approximation of Coulomb's friction law $F(z) = -k\,\mathrm{sgn}\,z$. We will find the steady-state velocity for the friction law of Eq. (49) for $\tau = 1/3$. Substitute $F(z)$ of Eq. (49) into Eq. (32) to obtain

$$\bar{G}(v) = -k\frac{2}{3\pi}\left[\arctan\left(\beta\left(v + 2/3\right)\right) + 2\arctan\left(\beta\left(v - 1/3\right)\right)\right]. \tag{50}$$

The steady-state velocity v_* is a solution of the equation $\bar{G}(v) = 0$. The function $\bar{G}(v)$ of Eq. (50) is continuous, monotonically decreases, is positive for $v = 0$ and negative for $v = 1/3$. Hence, it has a unique root $v_* \in (0, 1/3)$. The equation $\bar{G}(0) = 0$ implies

$$\beta^2\left(3v - 1\right)^2\left(3v + 2\right) - 81v = 0. \tag{51}$$

For $\beta \gg 1$, this equation has three solutions with the following asymptotic behavior:

$$v_1 = \frac{1}{3} - \frac{1}{\beta} + O\left(\frac{1}{\beta^2}\right), \quad v_2 = \frac{1}{3} + \frac{1}{\beta} + O\left(\frac{1}{\beta^2}\right), \quad v_3 = -\frac{2}{3} + O\left(\frac{1}{\beta^2}\right). \tag{52}$$

Only the solution v_1 belongs to the interval $(0,\,1/3)$ and, hence, represents the steady-state velocity v_*. As $\beta \to \infty$, this solution approaches $1/3$, which is the steady-state velocity for the system subject to Coulomb's friction. The other two solutions are inadmissible.

7.2 Anisotropic friction

We will consider two types of anisotropic friction laws: piecewise linear friction and anisotropic dry friction.

7.2.1 Piecewise linear friction

Piecewise linear friction law is represented by Eq. (1) for $\alpha = 1$. For this law, expression (32) for $\bar{G}(v)$ becomes

$$\bar{G}(v) = \frac{1}{3}\begin{cases} -3\mu_- v, & v \le -1, \\ (\mu_- - \mu_+)(1 - 3\tau) - [\mu_+(1 - 3\tau) + \mu_-(2 + 3\tau)]\,v, & -1 < v \le -2/3, \\ (\mu_- - \mu_+)(1 - \tau) - (\mu_+ + 2\mu_-)\,v, & -2/3 < v \le 0, \\ (\mu_- - \mu_+)(1 - \tau) - [\mu_+(2 - 3\tau) + \mu_-(1 + 3\tau)]\,v, & 0 < v \le 1/3, \\ (\mu_- - \mu_+)(1 - 3\tau) - [\mu_+(2 + 3\tau) + \mu_-(1 - 3\tau)]\,v, & 1/3 < v \le 1, \\ -3\mu_+ v, & v > 1, \end{cases} \tag{53}$$

for $0 \le \tau < 1/3$,

$$\bar{G}(v) = \frac{1}{3}\begin{cases} -3\mu_- v, & v \le -2/3, \\ \frac{2}{3}(\mu_- - \mu_+)(2 - 3\tau) - [\mu_+(2 - 3\tau) + \mu_-(1 + 3\tau)]\,v, & -2/3 < v \le -1/3, \\ \frac{2}{3}(\mu_- - \mu_+) - 3\left[\mu_+\tau + \mu_-(1 - \tau)\right]v, & -1/3 < v \le 1/3, \\ \frac{2}{3}(\mu_- - \mu_+)(3\tau - 1) - [\mu_+(4 - 3\tau) + \mu_-(3\tau - 1)]\,v, & 1/3 < v \le 2/3, \\ -3\mu_+ v, & v > 2/3, \end{cases} \tag{54}$$

for $1/3 \le \tau < 1/2$,

$$\bar{G}(v) = \frac{1}{3}\begin{cases} -3\mu_- v, & v \le -2/3, \\ \frac{2}{3}(\mu_- - \mu_+)(1-\tau) - [\mu_+(1-\tau) + \mu_-(2+\tau)]\, v, & -2/3 < v \le -1/3, \\ \frac{4}{3}(\mu_- - \mu_+)(1-\tau) - 3[\mu_+(1-\tau) + \mu_-\tau]\, v, & -1/3 < v \le 0, \\ \frac{4}{3}(\mu_- - \mu_+)(1-\tau) - 3[\mu_+\tau + \mu_-(1-\tau)]\, v, & 0 < v \le 1/3, \\ \frac{2}{3}(\mu_- - \mu_+)(1-\tau) - [\mu_+(2+\tau) + \mu_-(1-\tau)]\, v, & 1/3 < v \le 2/3, \\ -3\mu_+ v, & v > 2/3, \end{cases} \tag{55}$$

for $1/2 \le \tau < 1$.

The starting acceleration of the system is

$$\bar{G}(0) = \frac{1}{3}\begin{cases} (\mu_- - \mu_+)(1-\tau), & 0 \le \tau < 1/3, \\ \frac{2}{3}(\mu_- - \mu_+), & 1/3 \le \tau < 1/2, \\ \frac{4}{3}(\mu_- - \mu_+)(1-\tau), & 1/2 \le \tau < 1, \end{cases} \tag{56}$$

which implies that the system moves leftward, if $\mu_- < \mu_+$, and rightward, if $\mu_- > \mu_+$, i.e., the motion occurs in the direction of lower friction. The maximum starting acceleration corresponds to $\tau = 0$.

Denote $\mu = \mu_+/\mu_-$. Then the steady-state velocity of the system (the solution of the equation $\bar{G}(v) = 0$) is expressed as follows:

$$v_* = \begin{cases} \frac{(1-\mu)(1-3\tau)}{1-3\tau+\mu(2+3\tau)}, & \mu \le \frac{2-6\tau}{5-6\tau}, \\ \frac{(1-\mu)(1-\tau)}{1+3\tau+\mu(2-3\tau)}, & \frac{2-6\tau}{5-6\tau} < \mu \le 1, \\ \frac{(1-\mu)(1-\tau)}{2+\mu}, & 1 < \mu \le \frac{7-3\tau}{1-3\tau}, \\ \frac{(1-\mu)(1-3\tau)}{\mu(1-3\tau)+2+3\tau}, & \mu > \frac{7-3\tau}{1-3\tau}, \end{cases} \tag{57}$$

for $0 \le \tau < 1/3$,

$$v_* = \begin{cases} \frac{2}{3}\frac{(1-\mu)(3\tau-1)}{\mu(4-3\tau)+3\tau-1}, & \mu \le \frac{3\tau-1}{3\tau+2}, \\ \frac{2}{9}\frac{(1-\mu)}{\mu\tau+1-\tau}, & \frac{3\tau-1}{3\tau+2} < \mu \le \frac{5-3\tau}{2-3\tau}, \\ \frac{2}{3}\frac{(1-\mu)(2-3\tau)}{\mu(2-3\tau)+1+3\tau}, & \mu > \frac{5-3\tau}{2-3\tau}, \end{cases} \tag{58}$$

for $1/3 \le \tau < 1/2$,

$$v_* = \begin{cases} \frac{2}{3}\frac{(1-\mu)(1-\tau)}{\mu(2+\tau)+1-\tau}, & \mu \le \frac{1-\tau}{4-\tau}, \\ \frac{4}{9}\frac{(1-\mu)(1-\tau)}{\mu\tau+1-\tau}, & \frac{1-\tau}{4-\tau} < \mu \le 1, \\ \frac{4}{9}\frac{(1-\mu)(1-\tau)}{\mu(1-\tau)+\tau}, & 1 < \mu \le \frac{4-\tau}{1-\tau}, \\ \frac{2}{3}\frac{(1-\mu)(1-\tau)}{\mu(1-\tau)+2+\tau}, & \mu > \frac{4-\tau}{1-\tau}, \end{cases} \tag{59}$$

for $1/2 \le \tau < 1$.

The quantity v_* is a continuous function of μ and τ. For any fixed τ, the function v_* monotonically decreases as μ increases from 0 to $+\infty$, changing from 1 to -1, if $0 < \tau \leq 1/3$, or from $2/3$ to $-2/3$, if $1/3 \leq \tau < 1$. The steady-state velocity v_* is equal to zero for $\mu = 1$. This implies that the system in the steady-state mode moves forward ($v_* > 0$) for $0 < \mu < 1$ and backward ($v_* < 0$) for $\mu > 1$, i.e., the motion occurs in the direction of lower friction.

7.2.2 Anisotropic dry friction

Anisotropic dry friction law is defined by Eq. (2). Then expression (32) for $\bar{G}(v)$ becomes

$$\bar{G}(v) = \frac{1}{3}\begin{cases} 3k_-, & v < -1, \\ 2k_- - k_+ + 3(k_- + k_+)\tau, & -1 < v < -2/3, \\ 2k_- - k_+, & -2/3 < v < 0, \\ k_- - 2k_+ + 3(k_- + k_+)\tau, & 0 < v < 1/3, \\ k_- - 2k_+ - 3(k_- + k_+)\tau, & 1/3 < v < 1, \\ -3k_+, & v > 1, \end{cases} \tag{60}$$

for $0 \leq \tau < 1/3$,

$$\bar{G}(v) = \frac{1}{3}\begin{cases} 3k_-, & v < -2/3, \\ k_- - 2k_+ + 3(k_- + k_+)\tau, & -2/3 < v < -1/3, \\ 3k_- - 3(k_- + k_+)\tau, & -1/3 < v < 1/3, \\ -k_- - 4k_+ + 3(k_- + k_+)\tau, & 1/3 < v < 2/3, \\ -3k_+, & v > 2/3, \end{cases} \tag{61}$$

for $1/3 \leq \tau < 1/2$,

$$\bar{G}(v) = \frac{1}{3}\begin{cases} 3k_-, & v < -2/3, \\ 2k_- - k_+ + (k_- + k_+)\tau, & -2/3 < v < -1/3, \\ -3k_+ + 3(k_- + k_+)\tau, & -1/3 < v < 0, \\ 3k_- - 3(k_- + k_+)\tau, & 0 < v < 1/3, \\ k_- - 2k_+ - (k_- + k_+)\tau, & 1/3 < v < 2/3, \\ -3k_+, & v > 2/3, \end{cases} \tag{62}$$

for $1/2 \leq \tau < 1$.

Denote $\kappa = k_+/k_-$. Then the steady-state velocity for the three-particle chain subject to anisotropic dry friction (defined as a discontinuity point v_* of the function $\bar{G}(v)$ at which $\bar{G}(v_* - 0) \geq 0$ and $\bar{G}(v_* + 0) \leq 0$) is expressed as

$$v_* = \begin{cases} 1, & 0 < \kappa < \dfrac{1-3\tau}{2+3\tau}, \\ (1/3,\ 1)\,, & \kappa = \dfrac{1-3\tau}{2+3\tau}, \\ 1/3, & \dfrac{1-3\tau}{2+3\tau} < \kappa < \dfrac{1+3\tau}{2-3\tau}, \\ (0,\ 1/3)\,, & \kappa = \dfrac{1+3\tau}{2-3\tau}, \\ 0, & \dfrac{1+3\tau}{2-3\tau} < \kappa < 2, \\ (-2/3,\ 0)\,, & \kappa = 2, \\ -2/3, & 2 < \kappa < \dfrac{2+3\tau}{1-3\tau}, \\ (-1,\ -2/3)\,, & \kappa = \dfrac{2+3\tau}{1-3\tau}, \\ -1, & \kappa > \dfrac{2+3\tau}{1-3\tau}, \end{cases} \tag{63}$$

for $0 \le \tau < 1/3$,

$$v_* = \begin{cases} 2/3, & 0 < \kappa < \dfrac{3\tau-1}{4-3\tau}, \\ (1/3,\ 2/3)\,, & \kappa = \dfrac{3\tau-1}{4-3\tau}, \\ 1/3, & \dfrac{3\tau-1}{4-3\tau} < \kappa < \dfrac{1-\tau}{\tau}, \\ (-1/3,\ 1/3)\,, & \kappa = \dfrac{1-\tau}{\tau}, \\ -1/3, & \dfrac{1-\tau}{\tau} < \kappa < \dfrac{1+3\tau}{2-3\tau}, \\ (-2/3,\ -1/3)\,, & \kappa = \dfrac{1+3\tau}{2-3\tau}, \\ -2/3, & \kappa > \dfrac{1+3\tau}{2-3\tau}, \end{cases} \tag{64}$$

for $1/3 \le \tau < 1/2$,

$$v_* = \begin{cases} 2/3, & 0 < \kappa < \dfrac{1-\tau}{2+\tau}, \\ (1/3,\ 2/3)\,, & \kappa = \dfrac{1-\tau}{2+\tau}, \\ 1/3, & \dfrac{1-\tau}{2+\tau} < \kappa < \dfrac{1-\tau}{\tau}, \\ (0,\ 1/3)\,, & \kappa = \dfrac{1-\tau}{\tau}, \\ 0, & \dfrac{1-\tau}{\tau} < \kappa < \dfrac{\tau}{1-\tau}, \\ -1/3, & \dfrac{\tau}{1-\tau} < \kappa < \dfrac{2+\tau}{1-\tau}, \\ (-2/3,\ -1/3)\,, & \kappa = \dfrac{2+\tau}{1-\tau}, \\ -2/3, & \kappa > \dfrac{2+\tau}{1-\tau}, \end{cases} \tag{65}$$

for $1/2 \le \tau < 1$.

Notice that for the anisotropic dry friction, v_* is not a single-valued function of κ and τ in the entire admissible domain of these parameters. On some isolated curves in the $\kappa\tau$-plane, the steady-state velocity v_* can take on any value from appropriate intervals defined in Eqs. (63)–(65). For example, for $\kappa = (1 - 3\tau)/(2 + 3\tau)$ in Eq. (63), any value from the interval $(1/3,\ 1)$ is a steady-state velocity. We will illustrate this observation for $\tau = 1/3$, in which case the averaged equation coincides with the exact primary equation and has the form (see also Subsect. 7.3)

$$\dot{v} = \frac{\varepsilon}{3}\left[F\left(v + \frac{2}{3}\right) + 2F\left(v - \frac{1}{3}\right)\right]. \tag{66}$$

Any value of v at which the right-hand side of this equation vanishes is a steady-state velocity. For $v > 1/3$ ($v < -2/3$) both terms on the right-hand side are negative (positive) according to the friction law of Eq. (2). The value $v = -2/3$ is a steady-state velocity for $\kappa \geq 2$. For $v = -2/3$, the first term in the square brackets on the right-hand side of Eq. (66) is not uniquely defined but lies in the interval $[-k_+,\ k_-]$, while the second term is equal to $2k_-$. For the sum of the two terms to be able to vanish, the value $-2k_-$ must belong to the interval $[-k_+,\ k_-]$, which is equivalent to the inequality $\kappa \geq 2$. In a similar way, one can prove that $v = 1/3$ is a steady-state velocity for $\kappa \leq 2$. For $-2/3 < v < 1/3$, we have $F\left(v + \frac{2}{3}\right) + 2F\left(v - \frac{1}{3}\right) = -k_+ + 2k_-$. This expression is equal to zero for $k_+ = 2k_-$, or $\kappa = 2$. Hence, for $\kappa = 2$, any v from the interval $(-2/3,\ 1/3)$ is a steady-state velocity. This agrees with Eq. (64) for $\tau = 1/3$.

As was the case for the motion in the environment with the piecewise linear friction law, for any fixed τ, the function v_* decreases (stepwise) as κ increases from 0 to $+\infty$, changing from 1 to -1, if $0 < \tau \leq 1/3$, or from $2/3$ to $-2/3$, if $1/3 \leq \tau < 1$. However, for the environment characterized by an anisotropic dry friction law, the steady-state motion of the system does not necessarily occur in the direction of lower friction. For example, for $1/6 < \tau \leq 1/3$, the motion with the steady-state velocity $v_* = 1/3$ may occur for $1 \leq \kappa < (1 + 3\tau)/(2 - 3\tau)$.

7.3 Comparison of the solutions of the exact and the averaged equations

For the three-particle system the exact equation of motion of the center of mass (Eq. (19)) has the form

$$\dot{V} = \frac{\varepsilon}{3}\left[F\left(V - \frac{2}{3}u_1(t) - \frac{1}{3}u_2(t)\right) + F\left(V + \frac{1}{3}u_1(t) - \frac{1}{3}u_2(t)\right) + F\left(V + \frac{1}{3}u_1(t) + \frac{2}{3}u_2(t)\right)\right]. \tag{67}$$

We will compare the solution of this equation with the solution of the averaged equation $\dot{v} = \varepsilon\bar{G}(v)$ for the control functions $u_1(t)$ and $u_2(t)$ specified by Eqs. (30) and (31). For this case, the function $\bar{G}(v)$ is defined by Eq. (32). Both these equations will be subjected to the same initial condition $V(0) = v(0) = 0$. The exact equation may coincide with the averaged equation for a specific choice of the control parameters. This is the case, for example, for $\tau = 1/3$ in Eqs. (30)

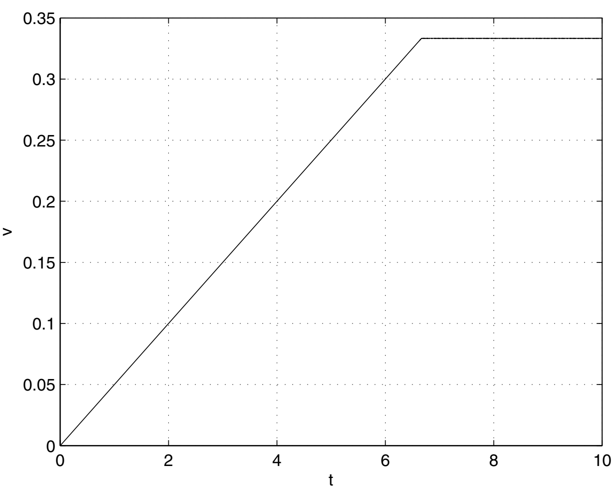

Fig. 3 v and V vs. t ($k_- = 1, k_+ = 1.5, \tau = 1/3$).

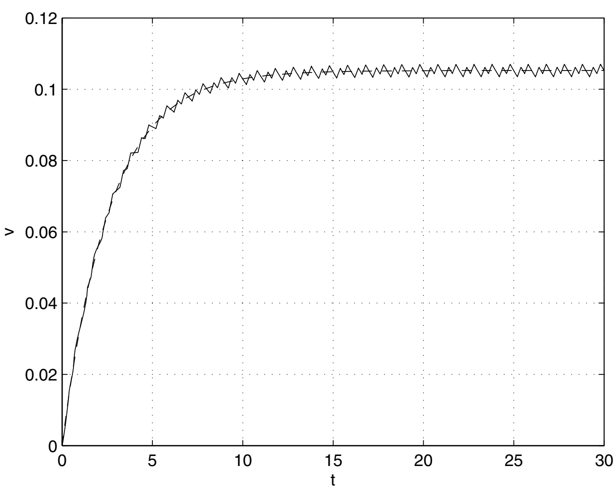

Fig. 4 v and V vs. t ($\mu_- = 1.5, \mu_+ = 1, \tau = 1/5$).

and (31). Both the exact and the averaged equations then become

$$\dot{v} = \frac{\varepsilon}{3}\left[F\left(v+\frac{2}{3}\right)+2F\left(v-\frac{1}{3}\right)\right]. \tag{68}$$

The solution of the last equation for $F(z)$ defined by Eq. (2), which corresponds to an anisotropic dry friction, for zero initial conditions is shown in Fig. 3.

The parameters ε, k_-, and k_+ were taken as follows: $\varepsilon = 0.3$, $k_- = 1$, and $k_+ = 1.5$. The motion in this case occurs in the direction of higher friction.

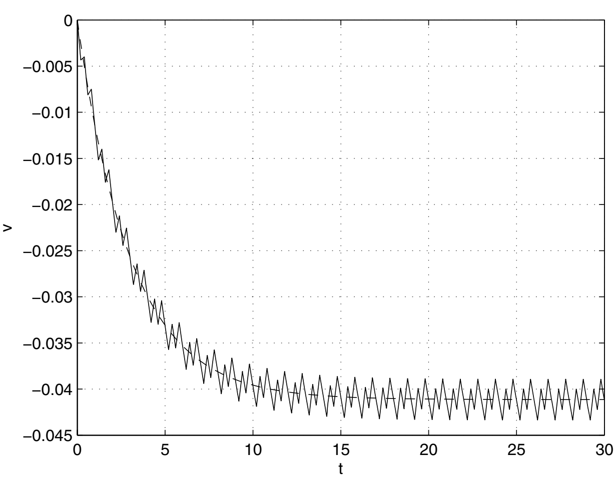

Fig. 5 v and V vs. t ($\alpha = 2, k = 1, \tau = 1/5$).

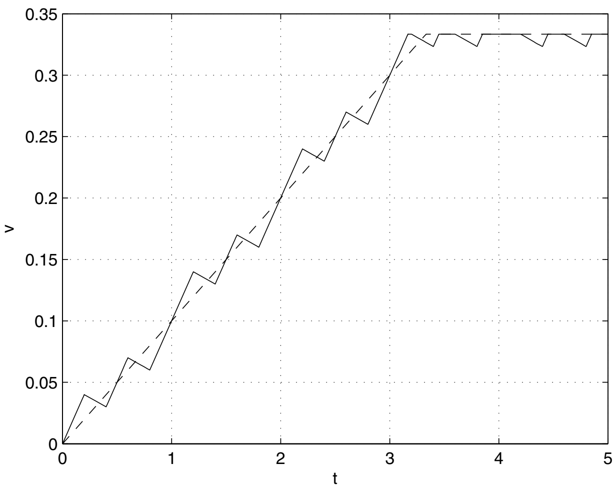

Fig. 6 v and V vs. t ($k_- = 1.5, k_+ = 1, \tau = 1/5$).

Figs. 4, 5, and 6 show the solutions of the exact equation (67) and the corresponding averaged equation for $\varepsilon = 0.3$ and $\tau = 1/5$. The solid and dashed curves correspond to the exact and the averaged equations, respectively. For $\tau = 1/5$, the exact equation becomes

$$\dot{V} = \frac{\varepsilon}{3}\begin{cases} 2F(V-1/3)+F(V+2/3), & t\in[0,\frac{1}{5})\cup[\frac{2}{5},\frac{3}{5})\cup[\frac{4}{5},1], \\ F(V-1)+F(V)+F(V+1), & t\in[\frac{1}{5},\frac{2}{5})\cup[\frac{3}{5},\frac{4}{5}) \end{cases} \tag{69}$$

and the averaged equation in accordance with (32) is given by

$$\dot{v} = \frac{\varepsilon}{15}\left[2F(v-1)+2F(v)+2F(v+1)+6F(v-1/3)+3F(v+2/3)\right]. \tag{70}$$

Fig. 4 corresponds to the piecewise-linear friction law of Eq. (1) for $\alpha = 1$, $\mu_- = 1.5$, and $\mu_+ = 1$. Fig. 5 corresponds to the quadratic-law friction defined by Eq. (35) for $\alpha = 2$ and $k = 1$. Fig. 6 corresponds to the anisotropic dry friction law of Eq. (2) for $k_- = 1.5$ and $k_+ = 1$.

8 Conclusion

The motion of a chain of identical particles can be controlled by changing the distances between the particles only in an environment with a nonlinear resistance law. The chain cannot be controlled in any environment if the distances between all adjacent particles change synchronously. According to the averaged equations of motion, a three-body chain subjected to an undulatory excitation with piecewise-linear law of change of the distances between the particles can start moving from a state of rest if the velocity of propagation of the excitation wave is sufficiently large. The motion of the chain in an environment with an anisotropic dry friction law can occur in the direction of higher friction for certain excitation and friction parameters.

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