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## The unified treatment of the non-relativistic bound state solutions, thermodynamic properties and expectation values of exponential-type potentials.

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### Abstract

In this paper, we obtained the analytical  $N$ -dimensional bound state solutions of exponential-type potential functions using the semi-classical WKB approximation method with an improved approximation to the orbital centrifugal barrier. Furthermore, we derived the partition function with the associated thermodynamic properties. We calculated the expectation values ( $\langle 1/r^2 \rangle$ ) via the Hellmann-Feynman theorem. Our results for the energy levels are in excellent agreement compared to the best available results obtained by other methods in the existing literature. The variations of the thermodynamic functions with the temperature parameter ( $\beta$ ) agree very well with the other results obtained under different potential energy functions. The expectation values conformed to the ones obtained by another approach in previous work. This work demonstrates the exactness of the leading order WKB approximation.

**Keywords:** Manning-Rosen potential, Eckart potential, Hulthén Potential, Thermodynamic properties, WKB approximation method.

### 1. Introduction

In quantum mechanics and applied physics, the solutions to the wave equations play important roles in understanding the properties of physical systems. Of these properties, the eigensolutions may serve as a starting point for the derivation of the other properties such as the expectation values [1-6] thermodynamic properties [6-20], including the molecular structures [21, 22]. Several interaction potential functions have been proposed and solved with the wave equations. However, the solution of some potential functions with the wave equations for the s-wave case ( $l = 0$ ) are exactly solvable but other potential functions do not permit the rovibrational bound states solutions of systems owing to the presence of the centrifugal barrier term in the effective potential. The impossibility can be circumvented by the use of approximations and numerical techniques. Approximations that have proven to be very effective are the Pekeris-type [23] and the short-range potential approximation proposed by Greene and Aldrich [24]. In this present work, we studied the  $N$ -dimensional radial Schrödinger equation (SE) under the interaction of a four parameter exponential potential function given by

$$U(r) = \frac{V_1 e^{-r/b} + V_2 e^{-2r/b}}{(1 - e^{-r/b})^2} - \left( \frac{V_3 e^{-r/b}}{1 - e^{-r/b}} \right) \quad (1)$$

The potential in (1) has a minimum value at the equilibrium position obtained from  $(dU/dr)_{r=r_e} = 0$

$$U_{min}(r_e) = -\frac{(V_1 - V_3)^2}{4(V_1 + V_2)} \quad (2)$$

where

$$r_e = b \ln \left( \frac{V_1 + 2V_2 + V_3}{V_3 - V_1} \right) \quad (3)$$

The force constant is obtained from the second derivative of Eq. (1) at  $r = r_e$

$$\left( \frac{d^2U}{dr^2} \right)_{r=r_e} = K_e = \frac{(V_1 - V_3)^2 (V_1 + 2V_2 + V_3)^2}{8b^2 (V_1 + V_2)^3} \quad (4)$$

where  $V_i (i = 1 - 3)$  are adjustable potential parameters and  $b$  is a screening parameter having a potential range of  $1/b$ . The potential in (1), reduces to the respective Eckart-type [25], Manning-Rosen [26], and the Hulthén [27] potentials with the proper adjustment of  $V_i (i = 1 - 3)$

$$U_{EKP}(r) = \frac{B e^{-r/b}}{(1 - e^{-r/b})^2} - \frac{\alpha' e^{-r/b}}{1 - e^{-r/b}}, \quad V_1 = B, \quad V_3 = \alpha', \quad V_2 = 0. \quad (5)$$

$$U_{MRP}(r) = \frac{\alpha(1 - \alpha)\hbar^2 e^{-2r/b}}{2\mu b^2 (1 - e^{-r/b})^2} - \frac{\hbar^2 A e^{-r/b}}{2\mu b^2 (1 - e^{-r/b})}, \quad V_2 = \frac{\alpha(1 - \alpha)\hbar^2}{2\mu b^2}, \quad V_3 = \frac{A\hbar^2}{2\mu b^2}, \quad V_1 = 0 \quad (6)$$

$$U_{HP}(r) = -\frac{A\hbar^2 e^{-r/b}}{2\mu b^2 (1 - e^{-r/b})}, \quad V_2 = V_1 = 0, \quad V_3 = \frac{A\hbar^2}{2\mu b^2} \quad (7)$$

The exponential-type potentials such as those given by the Eqs. (1, 5-7) are of significant importance in quantum mechanics and applied physics and have been extensively studied with both the relativistic and non-relativistic wave equations [28-59]. Due to the non-trivial properties of the rovibrational bound states solutions, several approximate methods have been applied to deal with the centrifugal barrier term appearing in the effective potential. To test the exactness of the different approximate techniques, some authors [30, 43-45, 47-55] compared the rovibrational energy spectra to the ones obtained by numerical integration [60] and also with MATHEMATICA package [61].

The main aim of this present work is to use the Semi-classical WKB approximation method to obtain the bound state solutions of the  $N$ -dimensional radial SE under the interaction of a four parameter exponential-type potential function. The energy eigenvalues depend on the dimensionality number  $N$ . Also, due to the inter-dimensional degeneracy of eigenvalues, we can also reproduce eigenvalues of an upper or a lower dimensional system from the obtained eigenvalues of a lower or an upper dimensional system by mapping  $(n, l, N) \rightarrow (n, l \pm 1, N \mp 2)$  [46]. An improved Pekeris-type approximation scheme is applied to the centrifugal barrier term [44, 47, 49, 57, 58] due to the fact that the energy spectra obtained by the frequently used Greene-Aldrich approximation are less accurate for long-range potential. Furthermore, the thermodynamic properties and expectation values ( $\langle 1/r^2 \rangle$ ) of the general potential are reported for the first time to the best of our knowledge. Thus the use of the WKB method also allows one to compare the energy levels with the other eigensolutions techniques where similar approximation schemes were applied. This method allows one to obtain the bound state solution directly from the WKB quantization condition without solving or transforming the SE into a more regular form which is common in some analytical approaches [62].

The organization of the remaining part of the paper is as follows. In section 2, the analytical bound states solutions of the modeled potential in  $N$ -dimensional space is presented using the WKB method. In section 3, the analytical solution of the thermodynamic properties and the expectation values of the system are presented. Section 4, contains the numerical analysis and the comparison of the present results with the ones obtained in existing literature. Concluding remarks are given in section 5.

## 2. WKB solution for the hyper-radial Schrödinger equation

The WKB approximation was originally proposed to solve the one-dimensional time independent SE in the limiting case of large quantum numbers [63]. The WKB wave function and probability density diverge at the classical turning points where the particle's velocity is zero. This phenomenon has no place in quantum mechanics which requires that the wave function be well-behaved, finite and continuous [64]. This drawback has been circumvented by using the so-called connection formula described in quantum mechanics textbooks [65, 66]. Furthermore, it has been shown in Refs. [63, 67-73] that the leading order approximation reproduces the correct energy eigenvalues of quantum systems provided that the orbital centrifugal barrier term is transformed from  $l(l+1)\hbar^2/2\mu r^2$  to  $(l+\frac{1}{2})^2\hbar^2/2\mu r^2$ . The standard quantization condition for the two turning point problem is given by [62-73]

$$\int_{r_1}^{r_2} P(r) dr = \pi\hbar\left(n + \frac{1}{2}\right), \quad r_1 < r < r_2 \quad n = 0, 1, 2, \dots \quad (8)$$

The turning points are obtained from Eq. 8 by solving the equation  $P(r) = 0$ .

The semi-classical wave function in the leading  $\hbar$  approximation has the form of

$$\psi^{WKB}(r) = \frac{C}{\sqrt{P(r)}} \exp\left[\pm \frac{i}{\hbar} \int P(r) dr\right]. \quad (9)$$

where  $C$  is a normalization constant.

In order to obtain the momentum in  $N$ -dimensional space, we write the hyper-radial SE as [74, 75, 76]

$$\frac{d^2\psi(r)}{dr^2} + \frac{N-1}{r} \frac{d\psi(r)}{dr} + \left[ \frac{2\mu}{\hbar^2} (E - U(r)) - \frac{l(l+N-2)}{r^2} \right] \psi(r) = 0 \quad (10)$$

where  $l, N$ , and  $\mu$  are the respective angular momentum quantum number, the dimensionality number, and the reduced mass

If we substitute the transformation  $\psi(r) = R(r) r^{(1-N)/2}$  into (10), then we obtain [16, 74]:

$$\frac{d^2R}{dr^2} + \left[ \frac{2\mu}{\hbar^2} (E - U(r)) - \frac{\left(l + \frac{N-2}{2}\right)^2 - \frac{1}{4}}{r^2} \right] R(r) = 0 \quad (11)$$

Equation (11) can be expressed in terms of the momentum eigenvalue relation with

$\left(l + \frac{N-2}{2}\right)^2 - \frac{1}{4}$  replaced by  $\left(l + \frac{N-2}{2}\right)^2$ . This replacement is equivalent to the Langer's modification [73] in  $N$ -dimensional space by simply adding  $1/4$  to the centrifugal barrier term such that the general transformation  $(l(l+1) \rightarrow (l + \frac{1}{2})^2)$  can be retrieved for  $N = 3$ .

$$\left[-i\frac{\hbar d}{dr}\right]^2 R(r) = 2\mu\left[E - U(r) - \frac{(l + (N-2)/2)^2 \hbar^2}{2\mu r^2}\right] R(r), \quad (12)$$

where the classical momentum is obtained as

$$P(r) = \sqrt{2\mu\left[E - U(r) - \frac{(l + (N-2)/2)^2 \hbar^2}{2\mu r^2}\right]} \quad (13)$$

By inserting Eqs. (13) and (1) into the quantization integral given by Eq. (8), we obtained:

$$\int_{r_1}^{r_2} \sqrt{2\mu\left[E_{nl} - \left(\frac{V_1 e^{-r/b} + V_2 e^{-2r/b}}{(1 - e^{-r/b})^2} - \left(\frac{V_3 e^{-r/b}}{(1 - e^{-r/b})}\right)\right) - \frac{(l + (N-2)/2)^2 \hbar^2}{2\mu r^2}\right]} dr = \pi\hbar\left(n + \frac{1}{2}\right) \quad (14)$$

To obtain the analytical solution of (14), we used the improved approximation to the orbital centrifugal barrier term  $(1/r^2)$  given by [47, 49, and 57]

$$\frac{1}{r^2} \sim \frac{1}{b^2} \left[ c_0 + \frac{e^{-r/b}}{1 - e^{-r/b}} + \frac{e^{-2r/b}}{(1 - e^{-r/b})^2} \right] \quad (15)$$

where the parameter  $c_0 = 1/12$  is a shift determined from expansion procedures [47, 57]. Using the transformation  $y = 1 - 2e^{-r/b} \in [-1, 1]$  in Eq. (14) with the change of variable from  $r \rightarrow y$  we obtained:

$$\sqrt{\frac{2H\mu b^2 y^2}{\hbar^2} \int_{y_1}^{y_2} \frac{dy}{1 - y^2} \sqrt{-y^2 + Fy + G}} = \pi\left(n + \frac{1}{2}\right), \quad -1 < y_1 < y_2 < 1, \quad (16)$$

where the quantities  $F, G, H$  are given as

$$F = \frac{2(V_1 + V_2 + L + E_{nl} - Lc_0)}{H}$$

$$G = \frac{2V_1 + V_2 - V_3 + 2L + Lc_0 - E_{nl}}{H}$$

$$H = V_2 + V_3 - E_{nl} + Lc_0$$

The notation  $L$  is defined as

$$L = \frac{(l + (N-2)/2)^2 \hbar^2}{2\mu b^2}$$

The classical turning points  $y_1, y_2$  are obtained from solution of the quadratic equation  $-y^2 + Fy + G = 0$ .

$$y_1 = \frac{F - \sqrt{F^2 - 4G}}{2}$$

$$y_2 = \frac{F + \sqrt{F^2 - 4G}}{2}$$

We can rewrite Eq. (16) in a regular form of

$$\sqrt{\frac{2H\mu b^2 y^2}{\hbar^2}} \int_{y_1}^{dy} \sqrt{(y_2 - y)(y - y_1)} = \pi \left( n + \frac{1}{2} \right) \quad (17)$$

Performing the integration of (17) with the use of the standard integral given in Ref. [72] we obtained:

$$E_{nl} = V_2 + V_3 + Lc_0 - \frac{\hbar^2 \left( n + \frac{1}{2} + \sqrt{\frac{2\mu b^2}{\hbar^2} (V_2 + V_1 + L)} + \frac{\frac{2\mu b^2}{\hbar^2} (V_2 + V_3)}{2 \left( n + \frac{1}{2} + \sqrt{\frac{2\mu b^2}{\hbar^2} (V_2 + V_1 + L)} \right)} \right)^2}{2\mu b^2} \quad (18)$$

Furthermore, Eq. (18) can be simplified as

$$E_{nl} = \Lambda_0 - \frac{\hbar^2 \left( \frac{n + \Delta}{2} + \frac{Q}{2(n + \Delta)} \right)^2}{2\mu b^2} \quad (19)$$

where

$$\Lambda_0 = V_2 + V_3 + Lc_0$$

$$\Delta = \frac{1}{2} + \sqrt{\frac{2\mu b^2}{\hbar^2} (V_2 + V_1 + L)}$$

$$Q = \frac{2\mu b^2}{\hbar^2} (V_2 + V_3)$$

### 3. Properties of the modeled potential.

#### 3.1 Partition function and thermodynamic properties

Having obtained the energy eigenvalues, we can now obtain the thermal properties of the potential function under consideration starting from the partition function given by

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_n} \quad (20)$$

where,  $\lambda$  is the upper bound of the vibrational quantum number obtained from the numerical solution of  $\frac{dE_n}{dn} = 0$ .  $\beta = \frac{1}{kT}$ , where  $k$  and  $T$  are the respective Boltzmann constant and absolute temperature. In the classical limit, the summation in Eq.(20) can be replaced with an integral:

$$Z(\beta) = \int_0^{\lambda} e^{-\beta E_n} dn \quad (21)$$

The energy equation given by Eq.(19) can be simplified further as

$$E_n = - \left( \Lambda_1 \rho^2 + \frac{\Lambda_2}{\rho^2} \right) - \delta \quad (22)$$

where,

$\Lambda_1 = \frac{\hbar^2}{8\mu b^2}$ ,  $\Lambda_2 = \Lambda_1 Q^2$ ,  $\delta = 2 \Lambda_1 Q - \Lambda_0$ ,  $\rho = n + \Delta$ . The quantities  $\Lambda_0$ ,  $\Delta$ , and  $Q$  have their usual meanings as those given in Eq. (19).

The maximum vibrational quantum number is obtained as  $\lambda = \sqrt{Q} - \Delta$ .

Inserting Eq. (22) into (21) and performing the integration with MAPLE package, we obtained:

$$Z(\beta) = \frac{e^{\Lambda_3 \beta} \sqrt{\pi} \left[ e^{4\beta \sqrt{\Lambda_1 \Lambda_2}} [\operatorname{erf}(\Lambda_4 \sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5 \sqrt{\beta}) + 1] \right]}{\Lambda_6 \sqrt{\beta}} \quad (23)$$

where,

$$\begin{aligned}\Lambda_3 &= \delta - 2\sqrt{\Lambda_1\Lambda_2}, \\ \Lambda_4 &= \sqrt{-\Lambda_1}\left(\rho + \frac{Q}{\rho}\right), \quad \Delta \leq \rho \leq \lambda + \Delta \\ \Lambda_5 &= \sqrt{-\Lambda_1}\left(\rho - \frac{Q}{\rho}\right), \quad \Delta \leq \rho \leq \lambda + \Delta \\ \Lambda_6 &= \sqrt{-\frac{2\hbar^2}{\mu b^2}}\end{aligned}$$

Using Eq. (23), we obtained the thermodynamic properties of the modeled potential as follows:

(a) Vibration mean energy :

$$U(\beta) = -\frac{\partial \ln Z(\beta)}{\partial \beta} = -\frac{\sqrt{\pi}e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1][4\beta\sqrt{\Lambda_1\Lambda_2} + \Lambda_3\beta - 1/2] + \sqrt{\pi}(\Lambda_3\beta - 1/2)[\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]}{\beta\sqrt{\pi}[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]} - \frac{(\Lambda_4\sqrt{\beta} e^{\beta(4\sqrt{\Lambda_1\Lambda_2} - \Lambda_4^2)} + \Lambda_5\sqrt{\beta} e^{-\beta\Lambda_5^2})}{\beta\sqrt{\pi}[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]} \quad (24)$$

(b) Vibrational specific heat capacity:

$$C(\beta) = k\beta^2\left(\frac{\partial^2 \ln Z(\beta)}{\partial \beta^2}\right) = -\frac{2k\beta^2\Lambda_4^3 e^{\beta(8\sqrt{\Lambda_1\Lambda_2} - 2\Lambda_4^2)}}{2\beta^2\pi[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]^2} - k\left(\frac{4\Lambda_4 e^{\beta(4\sqrt{\Lambda_1\Lambda_2} - \Lambda_4^2)}\left(\Lambda_5\beta^2 e^{-\beta\Lambda_5^2} + \frac{1}{2}\left[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}(\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1)(\Lambda_4^2\beta + \frac{1}{2}) + \beta\sqrt{\pi}[\Lambda_4^2\beta - 8\beta\sqrt{\Lambda_1\Lambda_2} + \frac{1}{2}][\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]\right)\right)}{2\beta^{1/2}\pi[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]^2}\right) - k\left(\frac{2\Lambda_5^2\beta^2 e^{-2\beta\Lambda_5^2} + \Lambda_5\beta\sqrt{\pi}e^{-\beta\Lambda_5^2}\left\{(2\Lambda_5^2\beta + 16\beta\sqrt{\Lambda_1\Lambda_2} + 1)e^{4\beta\sqrt{\Lambda_1\Lambda_2}}(\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1) + (2\Lambda_5^2\beta + 1)[\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]\right\}\right)}{2\beta^{1/2}\pi[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]^2}\right) + k\left(\frac{\pi\sqrt{\beta} e^{8\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1]^2 + \pi e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1][\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1](32\beta^{5/2}\Lambda_1\Lambda_2 + 2\sqrt{\beta}) + \pi\sqrt{\beta} [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]^2}{2\beta^{1/2}\pi[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]^2}\right) \quad (25)$$

(c) Vibrational free energy:

$$F(\beta) = -kT \ln Z(\beta) = -\frac{1}{\beta} \ln \frac{e^{\Lambda_3\beta}\sqrt{\pi}[e^{4\beta\sqrt{\Lambda_1\Lambda_2}}[\operatorname{erf}(\Lambda_4\sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5\sqrt{\beta}) + 1]]}{\Lambda_6\sqrt{\beta}} \quad (26)$$

(d) Vibrational entropy:

$$\begin{aligned}
S(\beta) &= k \ln Z(\beta) - k\beta \frac{\partial \ln Z(\beta)}{\partial \beta} = k \ln \frac{e^{\Lambda_3 \beta} \sqrt{\pi} [e^{4\beta \sqrt{\Lambda_1 \Lambda_2}} [\operatorname{erf}(\Lambda_4 \sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5 \sqrt{\beta}) + 1]]}{\Lambda_6 \sqrt{\beta}} \\
&- k \left( \frac{e^{4\beta \sqrt{\Lambda_1 \Lambda_2}} [\operatorname{erf}(\Lambda_4 \sqrt{\beta}) - 1] [4\beta \sqrt{\Lambda_1 \Lambda_2} + \Lambda_3 \beta - 1/2] + (\Lambda_3 \beta - 1/2) [\operatorname{erf}(\Lambda_5 \sqrt{\beta}) + 1]}{[e^{4\beta \sqrt{\Lambda_1 \Lambda_2}} [\operatorname{erf}(\Lambda_4 \sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5 \sqrt{\beta}) + 1]]} \right) \\
&\frac{k(\Lambda_4 \sqrt{\beta} e^{\beta(4\sqrt{\Lambda_1 \Lambda_2} - \Lambda_4^2)} + \Lambda_5 \sqrt{\beta} e^{-\beta \Lambda_5^2})}{\sqrt{\pi} [e^{4\beta \sqrt{\Lambda_1 \Lambda_2}} [\operatorname{erf}(\Lambda_4 \sqrt{\beta}) - 1] + [\operatorname{erf}(\Lambda_5 \sqrt{\beta}) + 1]]}
\end{aligned} \tag{27}$$

### 3.2 Expectation values for $1/r^2$ using the Hellmann-Feynman theorem.

The expectation values for the inversely quadratic position space  $\langle 1/r^2 \rangle$  are computed using the Hellmann-Feynman theorem [6, 77]. Suppose that the Hamiltonian of a quantum system depend on a certain parameter say  $q$ , and if we take the respective energy eigenvalue and wave function of the Hamiltonian as  $E_{nl}(q)$  and  $\psi_{nl}(q)$ , then the Hellmann-Feynman relation can be written as [6, 77].

$$\frac{\partial E_{nl}(q)}{\partial q} = \psi_{nl}(q) \left| \frac{\partial H(q)}{\partial q} \right| \psi_{nl}(q) \tag{28}$$

The respective Hamiltonian and energy eigenvalues for the system are given by

$$\begin{aligned}
H &= -\frac{\hbar^2 d^2}{2\mu dr^2} + \frac{\hbar^2 \left(l + \frac{N-2}{2}\right)^2}{2\mu r^2} + \frac{V_1 e^{-r/b} + V_2 e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{V_3 e^{-r/b}}{1 - e^{-r/b}} \\
E_{nl} &= \Lambda_0 - \frac{\hbar^2}{2\mu b^2} \left(\frac{n+\Delta}{2} + \frac{Q}{2(n+\Delta)}\right)^2
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\Lambda_0 &= V_2 + V_3 + Lc_0 \\
\Delta &= \frac{1}{2} + \sqrt{\frac{2\mu b^2}{\hbar^2} (V_2 + V_1 + L)} \\
Q &= \frac{2\mu b^2}{\hbar^2} (V_2 + V_3)
\end{aligned}$$

If we let  $q = l$  in Eq. (28) then we can obtain the expectation value for  $1/r^2$ :

$$1/r^2 = \frac{2\mu}{(2l + N - 2) \hbar^2} \left( \frac{\partial E_{nl}(l)}{\partial l} \right) = \frac{1}{b^2} \left( c_0 - \frac{1}{2\Delta - 1} \left( \frac{n+\Delta}{2} - \frac{Q^2}{2(n+\Delta)^3} \right) \right) \tag{30}$$

## 4. Numerical results and discussions

We have obtained the rovibrational energy levels, thermodynamic functions and expectation value  $\langle 1/r^2 \rangle$  of the modeled exponential-type potential function. If we set  $V_2 = 0, V_1 = B, V_3 = \alpha'$  in Eq.(1), the potential reduces to the Eckart-type potential given by Eq. (5) and the corresponding energy eigenvalue equation in  $N$ -dimensional space for any arbitrary angular momentum quantum number  $l$  becomes

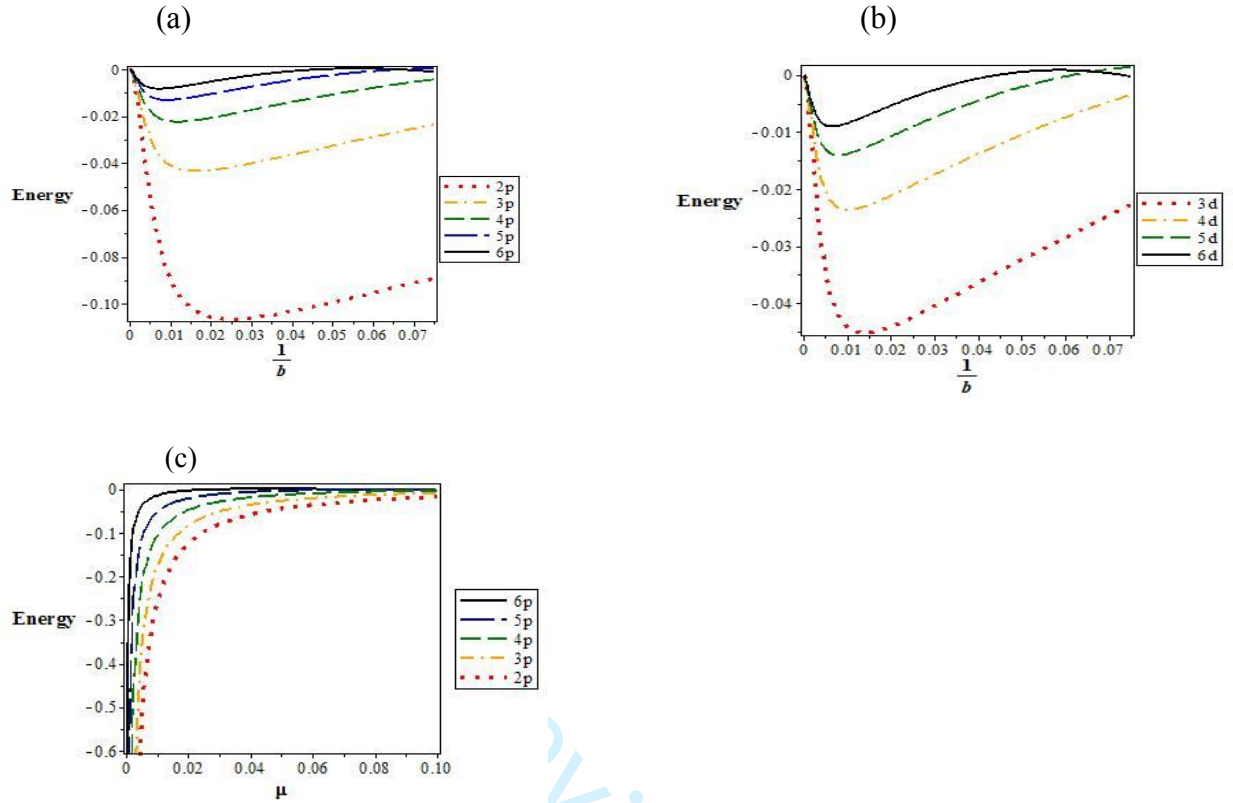


$$E_{nl} = \alpha' + \frac{(l + (N-2)/2)^2 \hbar^2 c_0}{2\mu b^2} - \frac{\hbar^2}{2\mu b^2} \left( \frac{\left( n + \frac{1}{2} + \sqrt{\frac{2\mu b^2 B}{\hbar^2} + \left( l + \frac{N-2}{2} \right)^2} \right)}{2} + \frac{\frac{2\mu b^2 \alpha'}{\hbar^2}}{2 \left( n + \frac{1}{2} + \sqrt{\frac{2\mu b^2 B}{\hbar^2} + \left( l + \frac{N-2}{2} \right)^2} \right)} \right)^2 \quad (31)$$

Also, we computed rovibrational energy levels for different quantum states, screening parameters and potential depth B shown in Table 1. Our results are in excellent agreement with the most accurate ones obtained by other analytical methods [49, 50] and MATHEMATICA programme [61]. Furthermore, we examined the variation of the energy levels for the  $p$  and  $d$  states of the Eckart-type potential with both the screening parameter and reduced mass shown in Figures 1(a-c). The energy variation with increasing screening parameter for each quantum state decreases to a minimum before increasing linearly. While the energy spectrum increases as the reduced mass increases and converge.

**Table 1.** Comparison of the ro-vibrational energy spectra ( $-E_{nl}$ ) in atomic units ( $\mu = \hbar = 1$ ) for the 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states for the Eckart-type potential function with  $\alpha' = \frac{1}{b}$ ,  $B = 0.0001$ ,  $B = 5 \times 10^{-5}$ .

States	$1/b$	$B = 0.0001$				$B = 5 \times 10^{-5}$			
		Present	Stanek [50]	Diaf [49]	Lucha et al.[61]	Present	Stanek [50]	Diaf [49]	Lucha et al. [61]
2p	0.025	0.1008293	0.1008306	0.1008358	0.1008358	0.1064672	0.1064616	0.1064736	0.1064737
	0.050	0.0978090	0.0978345	0.0978350	0.0978358	0.0993893	0.0994116	0.0994153	0.0994162
	0.075	0.0883557	0.0884174	0.0884143	0.0884183	0.0890658	0.0891252	0.0891243	0.0891284
	0.100	0.0782685	0.0783776	0.0783726	0.0783854	0.0786640	0.0787732	0.0787681	0.0787809
	0.150	0.0588048	0.0590651	0.0590391	0.0591059	0.0589724	0.0592326	0.0592067	0.0592734
3p	0.025	0.0401182	0.0401232	0.0401247	0.0401250	0.0418332	0.0418361	0.0418397	0.0418400
	0.050	0.0322172	0.0322435	0.0322432	0.0322482	0.0326702	0.0326956	0.0326962	0.0327011
	0.075	0.0234716	0.0235330	0.0235301	0.0235553	0.0236629	0.0237237	0.0237215	0.0237464
	0.100	0.0156724	0.0157843	0.0157766	0.0158588	0.0157699	0.0158817	0.0158740	0.0159559
	0.150	0.0036992	0.0039725	0.0039335	0.0044091	0.0037289	0.0040020	0.0039632	0.0044376
3d	0.025	0.0413570	0.0413638	0.0413635	0.0413642	0.0424517	0.0424587	0.0424581	0.0424588
	0.050	0.0321610	0.0321936	0.0321870	0.0321973	0.0324374	0.0324700	0.0324634	0.0324736
	0.075	0.0226868	0.0227792	0.0227453	0.0227991	0.0228025	0.0228947	0.0228610	0.0229146
	0.100	0.0140837	0.0142976	0.0141878	0.0143675	0.0141424	0.0143561	0.0142465	0.0144257
	0.150	0.0036992	0.0039725	0.0039335	0.0044091	0.0037289	0.0040020	0.0039632	0.0044376
4p	0.025	0.0184556	0.0184616	0.0184621	0.0184632	0.0191712	0.0191762	0.0191776	0.0191787
	0.050	0.0106730	0.0106995	0.0106990	0.0107159	0.0108424	0.0108687	0.0108685	0.0108852
	0.075	0.0043555	0.0044170	0.0044141	0.0045059	0.0044137	0.004475	0.0044723	0.0045636
	0.100	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
	0.150	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
4d	0.025	0.0189125	0.0189196	0.0189189	0.0189216	0.0193662	0.0193735	0.0193727	0.0193753
	0.050	0.0103912	0.010428	0.0104172	0.0104603	0.0104945	0.0105312	0.0105205	0.0105633
	0.075	0.0034645	0.0035778	0.0035230	0.0037658	0.0034997	0.0036127	0.0035583	0.0038001
	0.100	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
	0.150	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
4f	0.025	0.0190118	0.0190212	0.0190182	0.0190220	0.0193423	0.0193516	0.0193488	0.0193525
	0.050	0.0098250	0.0098979	0.0098510	0.0099137	0.0098992	0.0099719	0.0099252	0.0099876
	0.075	0.0020783	0.0023963	0.0021369	0.0025081	0.0021036	0.0024207	0.0021621	0.0025321
	0.100	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
	0.150	0.0002754	0.0003883	0.0003795	0.0007212	0.0002938	0.0004067	0.0003980	0.000738
5p	0.025	0.0086758	0.008682	0.0086822	0.0086850	0.0090229	0.0090287	0.0090293	0.009032
5d	0.025	0.0088439	0.0088512	0.0088503	0.0088576	0.0090633	0.0090707	0.0090698	0.0090768
5f	0.025	0.0088113	0.0088225	0.0088178	0.0088297	0.0089709	0.008982	0.0089774	0.0089892
5g	0.025	0.0086732	0.0086919	0.0086797	0.0086943	0.0087985	0.0088169	0.0088049	0.0088194
6p	0.025	0.0037749	0.0037813	0.0037813	0.0037873	0.0039526	0.0039588	0.0039548	0.0039648
6d	0.025	0.0038102	0.0038176	0.0038176	0.0038327	0.0039225	0.0039299	0.0039299	0.0039447
6f	0.025	0.0037172	0.0037296	0.0037296	0.0037525	0.0037989	0.0038111	0.0038111	0.0038337
6g	0.025	0.0035448	0.0035709	0.0035709	0.0035919	0.0036090	0.0036347	0.0036347	0.0036554



**Figure 1.** (a-b) shows the variations of the bound states energy of the Eckart-type potential for  $p$  and  $d$ -states with the screening parameter  $1/b$  ( $B = 5 \times 10^{-5}$ ). (c) Shows the variations of the  $p$ -states energy levels with the reduced mass  $\mu$  ( $1/b = 0.025$ ,  $B = 5 \times 10^{-5}$ ).

Also, if we insert  $V_1 = 0$  into Eq.(1), we obtained the Manning-Rosen potential function given by Eq. (6) and the corresponding  $N$ -dimensional energy level equation is given by

$$E_{nl} = \frac{A\hbar^2}{2\mu b^2} + \frac{\alpha(1-\alpha)\hbar^2}{2\mu b^2} + \frac{\left(l + \frac{N-2}{2}\right)^2 \hbar^2 c_0}{2\mu b^2} - \frac{\hbar^2 \left( n + \frac{1}{2} + \sqrt{\alpha(1-\alpha) + \left(l + \frac{N-2}{2}\right)^2} \right)^2}{2\mu b^2} + \frac{A + \alpha(1-\alpha)}{2\left( n + \frac{1}{2} + \sqrt{\alpha(1-\alpha) + \left(l + \frac{N-2}{2}\right)^2} \right)^2} \quad (32)$$

With  $\alpha = 0$  in Eq.(32), we retrieved the Hulthén potential energy eigenvalue equation in  $N$ -dimensional space:

$$E_{nl} = \frac{A\hbar^2}{2\mu b^2} + \frac{(l + (N-2)/2)^2 \hbar^2 c_0}{2\mu b^2} - \frac{\hbar^2 \left( n + \frac{1}{2} + l + \frac{N-2}{2} \right)^2}{2\mu b^2} + \frac{A}{2\left( n + \frac{1}{2} + l + \frac{N-2}{2} \right)^2} \quad (33)$$

If we set  $A = 2Z\mu e^2 b / \hbar^2$  with  $N = 3$ , for  $\frac{1}{b} \ll 1$ , we obtained the Coulomb's energy equation given by [47, 78,79]

$$E_{nl} = -\frac{\mu}{2\hbar^2} \frac{Z^2 e^4}{(n+l+1)^2} \quad (34)$$

We can also obtain Eq. (34) from (31) with  $\alpha' = \frac{Ze^2}{b}$ ,  $B = 0$ , for  $N = 3$ ,  $\frac{1}{b} \ll 1$ .

We computed the energy levels of the Manning-Rosen potential for both a hypothetical system ( $N = 2 - 4$ ) and also for diatomic molecules ( $N = 3$ ) numerically with different screening parameters and potential depth using Eq. (32). Our results (see Tables 2-5) are in good agreements with the ones obtained previously by other methods [44, 46, 47,] and MATHEMATICA programme [61]. We examined the variations of the energy levels with the potential range  $1/b$ , reduced mass  $\mu$  and the potential depth as shown in Figures 2 (a-d). The energy level spacing increases with increasing potential range parameter while the variation of the energy spectra increases with an increase in the reduced mass before converging. The variations of the energy levels with the potential depth decreases to minimum value for small  $\alpha$  and thereafter converge to a constant value as  $\alpha$  increases. The trend agrees very well with the results in Ref. [48].

**Table 2.** Comparison of the ro-vibrational energy spectra ( $-E_{nl}$ ) in atomic units ( $\mu = \hbar = 1$ ) for the 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states for the Manning-Rosen potential function with  $\alpha = 0.75$ ,  $\alpha = 1.5$ ,  $A = 2b$ ,  $N = 3$

State	$1/b$	$\alpha = 0.75$				$\alpha = 1.5$			
		Present	Ikhdaïr [47]	Falaye et al [44]	Lucha et al. [61]	Present	Ikhdaïr [47]	Falaye et al. [44]	Lucha et al. [61]
2p	0.025	0.120520755	0.1205279	0.120527265	0.1205271	0.089964295	0.0899715	0.090022888	0.0899708
	0.050	0.108188423	0.1082170	0.108214465	0.1082151	0.080012836	0.0800414	0.080247211	0.0800400
	0.075	0.096384688	0.0964490	0.096443282	0.0964469	0.070505903	0.0705703	0.068220746	0.0705701
	0.100	0.085109550	0.0852240			0.061443493	0.0615579		
3p	0.025	0.045871101	0.0458783	0.045877611	0.0458779	0.036906504	0.0369137	0.036965098	0.0369134
	0.050	0.035032827	0.0350610	0.035058868	0.0350633	0.027237557	0.0272662	0.027471932	0.0272696
	0.075	0.025483615	0.0255480	0.025542209	0.0255654	0.018857673	0.0189220	0.019385017	0.0189474
	0.100	0.017223466	0.0173380			0.011766851	0.0118813		
3d	0.025	0.044767183	0.0447756	0.044773693	0.0447743	0.039471709	0.0394801	0.039634470	0.0394789
	0.050	0.033657202	0.0343082	0.033683244	0.0336930	0.029411881	0.0294456	0.030062923	0.0294496
	0.075	0.023651969	0.0251168	0.023710563	0.0237621	0.020347250	0.0204232	0.021812093	0.0204663
4p	0.025	0.020802227	0.0208094	0.020808737	0.0208097	0.017166322	0.0171735	0.017224917	0.0171740
	0.050	0.011694811	0.0119292	0.011720853	0.0117365	0.008867499	0.0088961	0.009101874	0.0089134
	0.075	0.004949970	0.0054773		0.0050945	0.003020497	0.0030849		0.0031884
4d	0.025	0.020292787	0.0203012	0.020299297	0.0203017	0.018202151	0.0182106	0.018364912	0.0182115
	0.050	0.010923116	0.0109570	0.010949158	0.0109904	0.009443703	0.0094775	0.010094745	0.0095167
	0.075	0.003739860	0.0038160		0.0040331	0.002815981	0.0028919		0.0031399
4f	0.025	0.019969653	0.0199801	0.019976163	0.0199797	0.018603263	0.0186136	0.018922274	0.0186137
	0.050	0.010152313	0.0101940	0.010178355	0.0102393	0.009309208	0.0093507	0.010585249	0.0094015
	0.075	0.002222396	0.0023160		0.0026443	0.001781650	0.0018749		0.0022307
5p	0.025	0.009799009	0.0098062	0.009805520	0.0098079	0.008072190	0.0080793	0.008130783	0.0080816
5d	0.025	0.009500927	0.0095094	0.009507438	0.0095141	0.008527484	0.0085359	0.008690244	0.0085415
5f	0.025	0.009264665	0.0092751	0.009271176	0.0092825	0.008643195	0.0086536	0.008962206	0.0086619
5g	0.025	0.009012478	0.0090254	0.009018988	0.0090330	0.008593660	0.0086066	0.009121004	0.0086150
6p	0.025	0.004346541	0.0043537	0.004149868	0.0043583	0.003474827	0.0034820	0.003533421	0.0034876
6d	0.025	0.004143358	0.0041518	0.003952747	0.0041650	0.003658157	0.0036666	0.003820917	0.0036813
6f	0.025	0.003946237	0.0039566	0.003722009	0.0039803	0.003641621	0.0036520	0.003960631	0.0036774
6g	0.025	0.003715499	0.0037284	0.003722009	0.0037611	0.003514842	0.0035278	0.004042185	0.0035623

**Table 3.** Comparison of the two and four dimensional ro-vibrational energy spectra ( $-E_{nl}$ ) in atomic units ( $\mu = \hbar = 1$ ) for the 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states for the Manning-Rosen potential

States	$1/b$	$N = 2, \alpha = 0.75$		$N = 2, \alpha = 1.5$		$N = 4, \alpha = 0.75$		$N = 4, \alpha = 1.5$	
		Present	Ikhdairet al. [46]	Present	Ikhdairet al. [46]	Present	Ikhdairet al. [46]	Present	Ikhdairet al. [46]
2p	0.025	0.241061686	0.241087728	0.140923023	0.140949065	0.070630524	0.07073469	0.058794695	0.058898861
	0.050	0.227842510	0.227946676	0.131633162	0.131737328	0.058927418	0.059344084	0.048637489	0.049054156
	0.075	0.214939499	0.215173874	0.122602490	0.122836866	0.048015339	0.048952839	0.039171606	0.040109106
	0.100	0.202352652	0.202769319	0.113831012	0.114247678	0.037894290	0.039560954	0.030397050	0.032063712
3p	0.025	0.074253072	0.074279113	0.051907390	0.051933432	0.030105655	0.030209821	0.025964179	0.026068346
	0.050	0.062709397	0.062813564	0.042038383	0.042142549	0.019978910	0.020395577	0.016675382	0.017092049
	0.075	0.052074227	0.052308602	0.033139044	0.03337342	0.011565416	0.012502916	0.009065736	0.010003237
	0.100	0.042347560	0.042764227	0.025209376	0.025626042	0.004865173	0.006531840	0.003135240	0.004801908
3d	0.025	0.070630523	0.070734690	0.058794695	0.058898861	0.029599281	0.029833656	0.026993902	0.027228277
	0.050	0.058927418	0.059344084	0.048637489	0.049054156	0.019109709	0.020047209	0.017239269	0.018176769
	0.075	0.048015339	0.048952839	0.039171606	0.040109106	0.010090295	0.012199670	0.008838598	0.010947973
	0.100	0.031422081	0.031448122	0.023355900	0.023381941	0.014076185	0.014180352	0.012253431	0.012357598
4p	0.025	0.021441565	0.021545731	0.014501967	0.014606136	0.005880329	0.006296995	0.004655693	0.005072360
	0.050	0.013275759	0.013510134	0.007651092	0.007885467	0.000632715	0.001570215	0.000040705	0.000978205
	0.075	0.030105655	0.030209821	0.025964179	0.026068346	0.013777448	0.014011823	0.012658607	0.012892982
	0.100	0.019978910	0.020395577	0.016675382	0.017092049	0.005225313	0.006162813	0.004556847	0.005494347
4d	0.025	0.011565416	0.012502916	0.009065736	0.010003237	0.000616664	0.001492711	0.000905253	0.001204122
	0.050	0.029599280	0.029833656	0.026993902	0.027228277	0.013512707	0.013929374	0.012765472	0.013182139
	0.075	0.019109709	0.020047209	0.017239269	0.018176769	0.004430689	0.006097355	0.004058223	0.005724889
	0.100	0.010090295	0.012199670	0.008838598	0.010947973	-0.002294703	0.001455297	-0.002416837	0.001333163
5p	0.025	0.014706028	0.014732070	0.011074919	0.011100961	0.006486861	0.007127957	0.005591584	0.006175251
5d	0.025	0.014076185	0.014180352	0.012253431	0.012357598	0.006272376	0.006506751	0.005732645	0.005967020
5f	0.025	0.013777448	0.014011823	0.012658607	0.012892982	0.006048823	0.006465489	0.005696540	0.006113207
5g	0.025	0.013512707	0.013929374	0.012765472	0.013182139	0.005789917	0.006440958	0.005552963	0.006204004
6p	0.025	0.006840277	0.006866319	0.005030169	0.005056211	0.002630647	0.002734814	0.002182294	0.002286461
6d	0.025	0.006486861	0.005435481	0.005591584	0.005695750	0.002457472	0.002691847	0.002190127	0.002424502
6f	0.025	0.006272376	0.006506751	0.005732645	0.005967020	0.002254151	0.002670817	0.002082369	0.002499036
6g	0.025	0.006048823	0.006465489	0.005696540	0.006113207	0.002007276	0.002658317	0.001894332	0.002545374

function with  $\alpha = 0.75$ ,  $\alpha = 1.5$  and  $A = 2b$ .

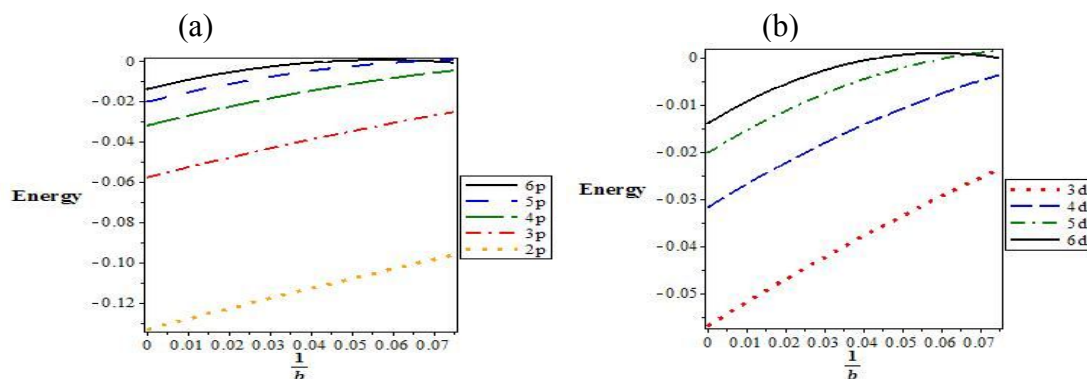
**Table 4.** Comparison of the ro-vibrational energy spectra ( $-E_{nl}$ ) of HCl molecule (in eV) for the 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states for the Manning-Rosen potential function with  $N = 3$ ,  $\alpha = 0.75$ ,  $\alpha = 1.5$ ,  $A = 2b$ ,  $\mu = 0.9801045$  amu. We used the conversions  $\hbar c = 1973.29\text{eV}\text{\AA}$  and  $1\text{amu} = 931.494028\text{MeV}/c^2$ .

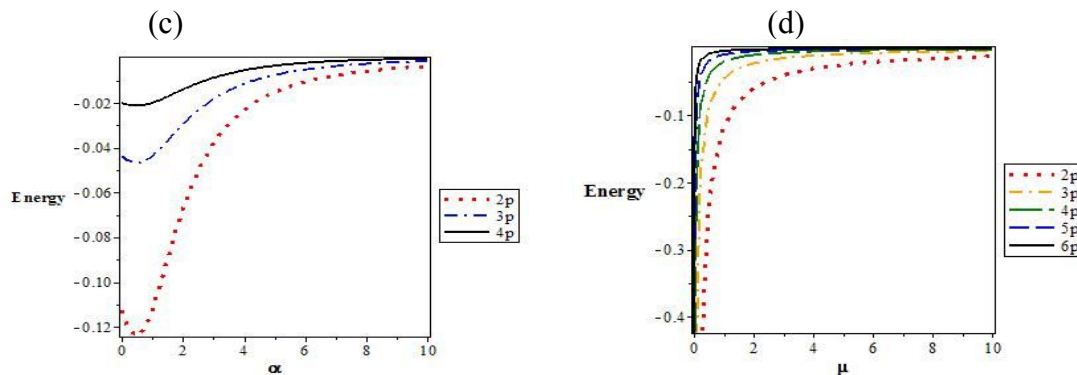
State	$1/b$	HCl/ $\alpha$ , $\alpha = 0.75$			HCl/ $\alpha$ , $\alpha = 1.5$		
		Present	Ikhdairet [47]	Falaye et al. [44]	Present	Ikhdairet [47]	Falaye et al. [44]
2p	0.025	5.140332524	5.14059	5.142509481	3.837068486	3.83734	3.838763931
	0.050	4.614346065	4.61553	4.617162023	3.412628692	3.41382	3.415000656
	0.075	4.110904786	4.11362	4.114923639	3.007148316	3.00987	3.010759359
	0.100	3.630008702	3.63486		2.620627347	2.42549	
3p	0.025	1.956448998	1.95674	1.957449620	1.574099859	1.57439	1.574959215
	0.050	1.494185615	1.49539	1.495848780	1.161709475	1.16292	1.163249798
	0.075	1.086902052	1.08964	1.089803634	0.804298904	0.80704	0.807096074
	0.100	0.734598318	0.73947		0.501868166	0.50674	
3d	0.025	1.909365780	1.90971	1.910349004	1.683508472	1.68385	1.684408248
	0.050	1.435513830	1.43694	1.437155317	1.254446585	1.25588	1.257865749
	0.075	1.008780506	1.01201	1.011653222	0.867830856	0.87106	0.870651496
	0.100	0.887236083	0.88753	0.887841665	0.732161092	0.73246	0.732709379
4p	0.025	0.498795575	0.50001	0.500090979	0.378207828	0.37942	0.379458678
	0.050	0.211121240	0.21387		0.128827264	0.13157	
	0.075	0.865507962	0.86586	0.872318429	0.776340222	0.77669	0.794019001
	0.100	0.465881998	0.46732	0.494277550	0.402783518	0.40422	0.438304965
4d	0.025	0.159508811	0.16275		0.120104448	0.12334	
	0.050	0.851725967	0.85216	0.862429785	0.793448071	0.79388	0.81401901
	0.075	0.433006455	0.43477	0.484277089	0.397047155	0.39881	0.398304965
	0.100	0.094787429	0.098765		0.075989188	0.079967	

5p	0.025	0.417937691	0.41824	0.418369884	0.344287078	0.34459	0.344692059
5d	0.025	0.405224192	0.40558	0.405651686	0.363705845	0.36406	0.364118001
5f	0.025	0.395147377	0.395586	0.395571148	0.368641054	0.36908	0.369055033
5g	0.025	0.384391324	0.38494	0.384811122	0.366528326	0.367077	0.366941526
6p	0.025	0.185384380	0.18569	0.185730652	0.148204899	0.14851	0.148537434
6d	0.025	0.176718416	0.17708	0.177061486	0.156024110	0.15638	0.156359533
6f	0.025	0.168311002	0.168752	0.168650965	0.155318811	0.155759	0.155653975
6g	0.025	0.158469783	0.15902	0.158806110	0.149911561	0.150462	0.150244728

**Table 5.** Comparison of the ro-vibrational energy spectra ( $-E_{nl}$ ) of CO molecule (in eV) for the 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states for the Manning-Rosen potential function with  $N = 3$ ,  $\alpha = 0.75$ ,  $\alpha = 1.5$ ,  $A = 2b$ ,  $\mu = 6.8606719$  a.m.u. We used the conversions  $\hbar c = 1973.29\text{eV}\text{\AA}$ , and  $1\text{a.m.u.} = 931.494028\text{MeV}/c^2$ .

State	$1/b$	CO/ $\alpha$ , $\alpha = 0.75$			CO/ $\alpha$ , $\alpha = 1.5$		
		Present	Ikhdaire [47]	Falaye et al. [44]	Present	Ikhdaire [47]	Falaye et al. [44]
2p	0.025	0.734339597	0.734377	0.734650594	0.548157403	0.548196	0.548399611
	0.050	0.659198021	0.659367	0.659600305	0.487522619	0.487693	0.487861475
	0.075	0.587277213	0.587664	0.587851341	0.429596349	0.429985	0.430112216
	0.100	0.518577177	0.519270		0.374378588	0.375073	
3p	0.025	0.279495142	0.279536	0.279638089	0.224873362	0.224915	0.224996128
	0.050	0.213456942	0.213629	0.213694539	0.165959938	0.166133	0.166179986
	0.075	0.155273071	0.155664	0.155687586	0.114900842	0.115292	0.115300441
	0.100	0.104943528	0.105640		0.071696075	0.072393	
3d	0.025	0.272768910	0.272818	0.272909371	0.240503299	0.240553	0.240631840
	0.050	0.205075186	0.205279	0.205309686	0.179208212	0.179412	0.179433155
	0.075	0.144112753	0.144574	0.144523144	0.123976914	0.124439	0.124379866
4p	0.025	0.126749113	0.126792	0.126835625	0.104595350	0.104638	0.104673678
	0.050	0.071257130	0.071431	0.071442189	0.054030159	0.054204	0.054208853
	0.075	0.030160439	0.030552		0.018404055	0.018796	
4d	0.025	0.123645069	0.123695	0.121760833	0.110906709	0.110957	0.113432271
	0.050	0.066555149	0.066760	0.062040189	0.057541002	0.057746	0.056901204
	0.075	0.022787171	0.023250		0.017157927	0.01762	
4f	0.025	0.121676195	0.121738	0.121760833	0.113350709	0.113413	0.113432271
	0.050	0.061858602	0.062111	0.062040189	0.056721515	0.056974	0.061047654
	0.075	0.013541179	0.014109		0.010855693	0.011424	
5p	0.025	0.059705903	0.059749	0.059767645	0.049184296	0.049227	0.049242151
5d	0.025	0.057889673	0.057941	0.057950744	0.051958429	0.052009	0.052017309
5f	0.025	0.056450116	0.056513	0.056510655	0.052663465	0.052726	0.052722606
5g	0.025	0.054913523	0.054992	0.054973495	0.052361644	0.05244	0.052420673
6p	0.025	0.026483713	0.026527	0.026533181	0.0221172312	0.0221216	0.021219818
6d	0.025	0.025245707	0.025297	0.025294718	0.022289352	0.022341	0.02233727
6f	0.025	0.024044638	0.024108	0.024093204	0.022188595	0.022252	0.022236475
6g	0.025	0.022638737	0.022717	0.022686784	0.021416123	0.021495	0.021463719





**Figure 2.** (a-b) show the variations of the bound ( $p,d$ ) states energy of the Manning-Rosen potential with the screening parameter  $1/b$  ( $\alpha=0.75$ ). (c-d) shows the variations of bound state energy with the potential depth  $\alpha$  ( $1/b = 0.025$ ) and with the reduced mass  $\mu$  ( $1/b = 0.025, \alpha = 0.75$ ).

Furthermore, by using Eq. (33) for the case  $N = 3$ , we obtained the energy spectra of the Hulthén potential function numerically for various potential range parameters as shown in Table 6. Our results are in excellent agreement with the ones obtained by other analytical methods in existing literature [40, 51, 54, and 55] and numerical integration [60].

**Table 6.** Comparison of energy spectra ( $-E_{nl}$ ) of the Hulthén potential as a function of the screening parameter  $1/b$  for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f, 6g, 6h, 7i, 8k, 9l, 10m states with  $N = 3$ ,  $A = 2b$  in atomic units ( $\mu = \hbar = 1$ ).

State	$1/b$	Present	Bayrak et al.[40]	Ikhdaïr [54]	Stanek [51]	Varshni [60]	Jia et al.[55]
2p	0.025	0.1127539	0.1127605	0.1127611	0.1127604	0.1127605	0.1126344
	0.050	0.1010156	0.1010425	0.1010442	0.101042	0.1010425	0.1009128
	0.075	0.0897852	0.0898478	0.0898495	0.0898453	0.0898478	0.0898350
	0.100	0.0790625	0.0791794	0.0791769	0.0791717	0.0791794	0.0794011
	0.150	0.0591406	0.0594415	0.0593981	0.0594007	0.0594415	0.0604650
	0.200	0.0412500	0.0418861	0.0417078	0.0417491	0.0418860	0.0441045
3p	0.025	0.0253906	0.0266111	0.0261059	0.0262466	0.0266111	0.0303195
	0.050	0.0115625	0.0137900	0.0125925	0.0129347	0.0137900	0.0191101
	0.075	0.0437001	0.0437069	0.0437072	0.0437066	0.0437069	0.0436848
	0.100	0.0331337	0.0331645	0.0331623	0.0331602	0.0331645	0.0332390
	0.150	0.0238563	0.0239397	0.0239207	0.0239173	0.0239397	0.0242183
	0.200	0.0158681	0.0160537	0.0159825	0.0159798	0.0160537	0.0166227
3d	0.025	0.0037587	0.0044663	0.0040162	0.0040316	0.0044663	0.0057067
	0.050	0.0435959	0.0436030	0.0436044	0.0436028	0.0436030	0.0435371
	0.075	0.0327170	0.0327532	0.0327508	0.0327495	0.0327532	0.0329817
	0.100	0.0229188	0.0230307	0.0229948	0.0230109	0.0230307	0.0238893
	0.150	0.0142014	0.0144842	0.0143364	0.0144147	0.0144842	0.0162600
	0.200	0.0119414	0.0119489	0.0119486	0.011948	0.0119489	0.0119625
4p	0.025	0.0110156	0.0110582	0.0110442	0.0110422	0.0110582	0.0111938
	0.050	0.0044727	0.0046219	0.0045370	0.0045340	0.0046219	0.0049439
	0.075	0.0003125	0.0007549	0.0004269	0.0004252	0.000755	0.0012128
	0.100	0.0198372	0.0198463	0.0198457	0.0198444	0.0198462	0.0198877
	0.150	0.0105990	0.0106674	0.0106327	0.0106355	0.0106674	0.0110819
	0.200	0.0035352	0.0038345	0.0036111	0.0036479	0.0038345	0.0048327
4f	0.025	0.0196810	0.0196911	0.0196914	0.0196903	0.0196911	0.0197756
	0.050	0.0099740	0.0100620	0.0100154	0.0100463	0.0100620	0.0109150



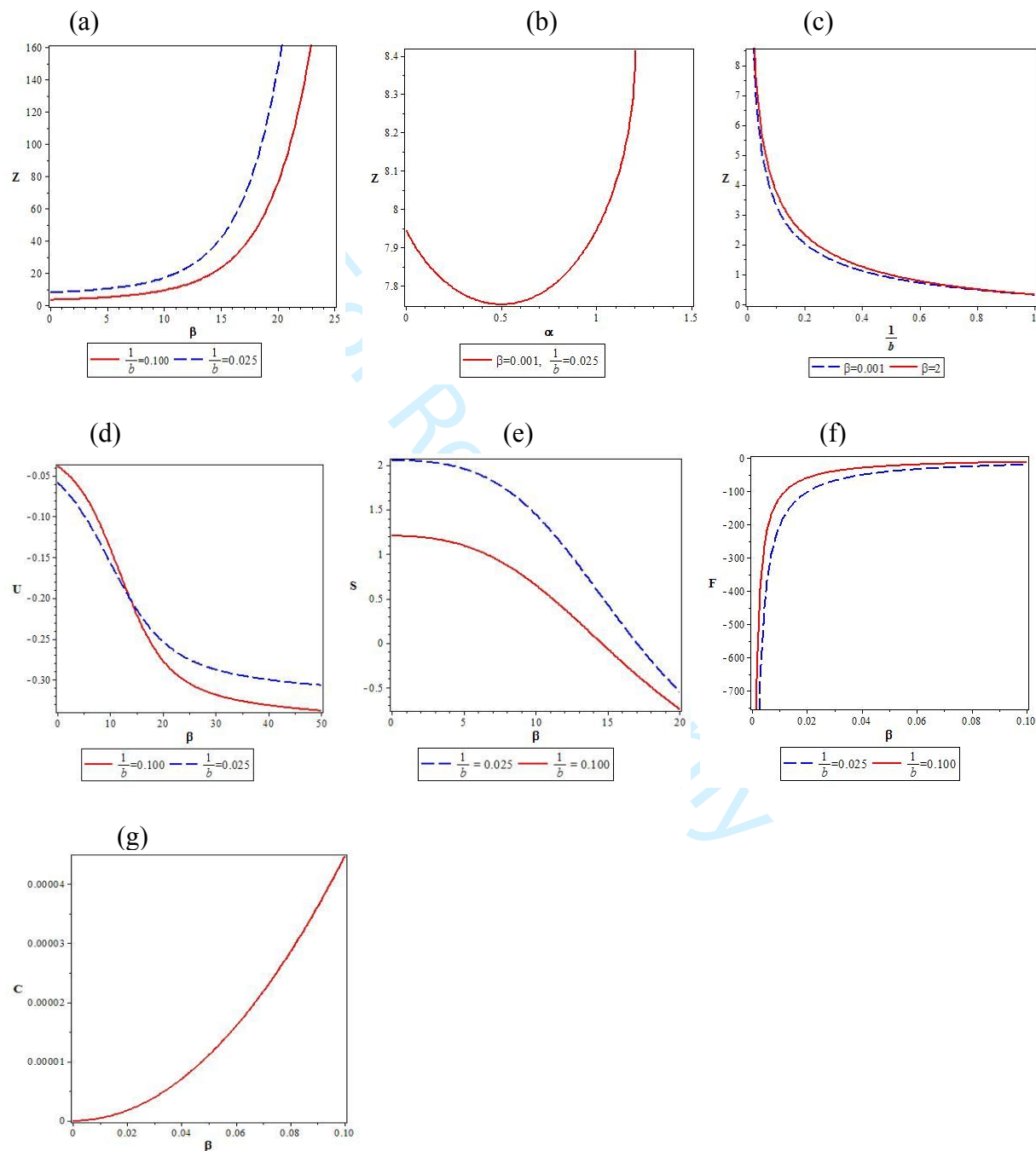
	0.075	0.0021289	0.0025563	0.0022222	0.0024452	0.0025563	0.0046682
5p	0.025	0.0093945	0.0094036	0.0094017	0.0094011	0.0094036	0.0094325
	0.05	0.0025781	0.0026490	0.0026067	0.0026047	0.0026490	0.0027900
5d	0.025	0.0092904	0.0093037	0.0092988	0.0092977	0.0093037	0.0093914
5f	0.025	0.0091341	0.0091521	0.0091445	0.0091451	0.0091521	0.0093298
5g	0.025	0.0089258	0.0089465	0.0089387	0.0089441	0.0089465	0.0092480
6p	0.025	0.0041428	0.0041548	0.0041500	0.0041493	0.0041548	0.0041899
6d	0.025	0.0040386	0.0040606	0.0040471	0.004046	0.0040606	0.0041671
6f	0.025	0.0038824	0.0039168	0.0038927	0.0038945	0.0039168	0.0042014
6g	0.025	0.0036740	0.0037201	0.0036870	0.0036996	0.0037201	0.0040876
6h	0.025	0.0034136	0.0034654				
7i	0.010	0.0056405	0.0056442				
8k	0.010	0.0033781	0.0033852				
9l	0.010	0.0018843	0.0018981				
10m	0.010	0.0008740	0.0009003				

On the properties of the exponential-type potentials, we found the expectation values of  $\langle 1/r^2 \rangle$  for a hypothetical system under the Manning-Rosen, Eckart-type and the Hulthén potentials (see Table 7). Our results for the 2p, 3p, 4p and 3d quantum states for the Manning Rosen potential function is in good agreement with the results obtained by Roy [48] using the Generalized pseudo-spectral (GPS) method. We note that the slight differences in  $\langle 1/r^2 \rangle$  could be as a result of the different approach applied and also due to the fact that the wave functions are not exact. Furthermore, we obtained the partition function  $Z(\beta)$  and the thermodynamic properties of the system under study. Using the Manning-Rosen potential as a model, the partition function and thermal properties are plotted against the chosen independent variables such as the potential range  $1/b$ , potential depth  $\alpha$  and temperature parameter  $\beta$  as shown in Figures 3(a-g) with a fixed potential depth of  $\alpha = 0.75$  for the  $1/b$  and  $\beta$  variables. In Figure 3a, the partition function  $Z(\beta)$  increases gradually with increasing  $\beta$  for both the short and long range potential parameter before diverging at a critical value of  $\beta$ . This result is similar to the trend obtained in Refs.[8, 12 and 20]. In Figure 3b, the partition function exhibits a parabolic behaviour with the potential depth by decreasing to zero at a turning point of  $\alpha = 0.5$  and increases thereafter. In Figure 3c, the partition function is characterized by an inverse relationship with the  $1/b$  parameter. Furthermore, in Figure 3d, the vibrational mean energy  $U(\beta)$  decreases with  $\beta$ . This trend is in good agreement with the results reported in the existing literature [8, 12, 14, 16 and 20]. In Figure 3e, the vibrational entropy  $S(\beta)(k = 1)$  decreases monotonically with increasing  $\beta$  similar to the results reported in Refs.[8, 16 and 20] and in some cases in Ref. [14]. In Figure 3f, the vibrational free energy  $F(\beta)$  increases with increasing  $\beta$ . This trend is in excellent agreement with the results obtained in Refs. [8, 12 and 20] and also in some cases in Ref.[14]. Also, the specific heat capacity  $C(\beta)(k = 1)$  shown in Figure 3g increases with the temperature parameter  $\beta$ . This trend is in consonance with the results obtained for the HCL and CrH molecules in Ref. [14] and also in Ref. [20] for small  $\beta$  parameter.

**Table 7.** Expectation values of  $1/r^2$  in atomic units ( $\mu = \hbar = 1$ ) for the Manning-Rosen, Hulthén and Eckart-type potentials for the case  $N = 3$ .

States	Manning-Rosen Potential		Hulthén potential		Eckart-type potential
	$A = 2b, \alpha = 0.75, 1/b = 0.1$	$A = 2b, \alpha = 1.5, 1/b = 0.1$	$A = 2b, 1/b = 0.1$	$A = 2b, 1/b = 0.1$	$B = 5 \times 10^{-5}, \alpha' = 1/b, 1/b = 0.1$
2p	Present 0.091611	Roy [48] 0.091584	Present 0.053498	Roy [48] 0.053464	0.080833 0.080237

3p	0.022718	0.022569	0.014575	0.014398	0.020525	0.020394
4p	0.005186	0.004549	0.002924	0.002030	0.004583	0.004544
3d	0.013159	0.013003	0.010930	0.010760	0.012648	0.012607



**Figure 3.** (a-c). Show the respective variations of the partition function of the Manning-Rosen potential with temperature parameter  $\beta$ , potential depth  $\alpha$  and screening parameter  $1/b$ . (d) Shows the variation of the vibrational mean energy with temperature parameter  $\beta$ . (e) The variation of the vibrational entropy with temperature parameter  $\beta$ . (f) The variation of the free energy with temperature parameter  $\beta$ . (g) The variation of the specific heat capacity with temperature parameter  $\beta$ .



## 5. Conclusion

In this paper, we obtained the eigenenergies of exponential-type potentials such as the Manning-Rosen, Hulthén, Eckart-type including the Coulomb's potential in  $N$ -dimensions using the semi-classical WKB approximation approach. Furthermore, we obtained the partition function and the associated thermodynamic properties including the expectation values of the modeled potential. Using the Manning-Rosen potential as a model, we examined the variations of the partition function against the temperature parameter  $\beta$ , potential depth and range. Also, the variations of the vibrational mean energy, vibrational free energy, vibrational entropy and the specific heat capacity with temperature parameter  $\beta$  were studied for a non-physical system. Our results for the energy spectra such as those given in Tables 1-6, are in excellent agreement with the best available results obtained by other analytical methods and numerical approaches in the literature. The thermodynamic properties are similar to the ones obtained with different potential energy functions in the existing literature. The expectation values of  $1/r^2$  obtained for the 2p, 3p, 4p and 3d quantum states using the Hellmann-Feynman theorem conform to the ones obtained by Roy [48] using the GPS method. This work further demonstrates the exactness of the leading order WKB approximation and thus may be a popular resource material for applied science students and professionals.

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