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THE UNIFORM TRANSPARENT GRAVITATIONAL LENS

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SUMMARY

An elementary discussion is given of the gravitational deflection of light due to radially and cylindrically symmetric masses. The effect of the deflection on apparent luminosity of distant sources is also considered. All results are limited to weak, static, asymptotically flat gravitational fields. Emphasis here is on observations of sources aligned behind the disc of the deflecting mass so that the possible transparency of this mass is decisive in image formation. Detailed calculations are made for the simple case of the uniformly dense transparent sphere and comparisons are made with the opaque mass sphere. If they have the same mass and radius, the maximum light deflection produced by these lenses is nearly equal. However, their effects on the area of narrow light beams may be quite different. The uniform transparent sphere does not produce multiple images of one source and, to first order, introduces no image distortion.

1. INTRODUCTION

The lens-like amplification of apparent luminosity by the gravitational field of a star has been developed in detail and has been widely applied. The conventional stellar lens analysis deals only with photons which propagate in a weak external Schwarzschild field. Thus every photon reaching the observer from the amplified source is assumed to have been deflected from its Euclidean path by a small angle given by the familiar relation

$$\theta(b) \approx K_1 b^{-1}; \quad K_1 \equiv \frac{4Gm}{c^2}. \tag{1}$$

In this small deflection situation the ' collision parameter ', b, can be approximated by the Euclidean distance of closest approach of the photon to the mass centre. The meaning of $\theta(b)$ is apparent in Fig. 1 which is the usual Euclidean embedding of the 2-space projection of a ' weak Schwarzschild ' null geodesic. Of course all angles have been exaggerated for clarity. The gravitational deflection actually occurs continuously (as in the dotted path), but for weak fields the photon path can be approximated by the extensions of the initial and final tangents back to the star. The angle between these lines defines $\theta(b)$ and the conventional amplifying properties of the opaque stellar lens follow from equation (1) (Einstein 1936; Liebes 1964; Refsdal 1964a).

There are at least two cases when equation (1) does not apply. If the deflecting mass is extremely dense and nearby, then the observer can see photons which passed through the part of the stellar field where the curvature is too large for equation (1) to apply. This situation is discussed by Darwin (1959, 1962), Metzner (1963) and

Atkinson (1965). Another possibility occurs if the deflecting mass is effectively transparent and large enough for observations to be made through it. All deflections may be small but, because the observed photons did not remain external to the deflector, equation (1) cannot be used and the usual stellar lens properties do not apply. This paper deals with this case. It will be found that the properties of this 'transparent' lens are rather different from those of the corresponding opaque lens.

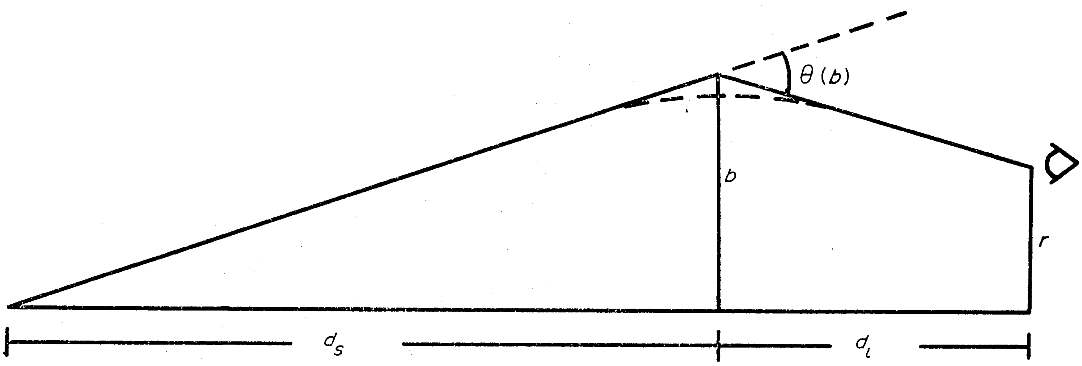

FIG. 1. *The Euclidean embedding of the 2-space projection of a 'weak Schwarzschild' null geodesic.*

Possible physical prototypes of the transparent lens developed here are a dust free giant elliptical galaxy and a populous symmetric cluster of galaxies. The observational effects of these objects has been calculated by treating them as opaque stellar lenses (see, e.g. Barnothy 1966; Kantowski 1969; Refsdal 1964b, 1970; Sadeh 1967). Certainly this treatment is justified irrespective of the possible transparency of the deflector as long as the observed light remains at all times outside the deflecting mass. However, if for some observer the source is aligned behind the apparent disc of the deflector, then for this observer the transparency of the deflector plays a decisive role in image formation. For a deflecting mass as extensive as a rich cluster of galaxies the required source alignment would not be rare or transient. Of course the approximation of a cluster of galaxies by a uniform transparent mass sphere requires some justification. However, such specific applications of the transparent lens in observational cosmology are pursued in a forthcoming paper. The present paper is chiefly concerned with deriving the imaging properties of the uniform transparent mass sphere and with comparing this lens with the opaque stellar lens.

2. THE DEFLECTION ANGLE RELATION FOR THE UNIFORM TRANSPARENT LENS

For ease of comparison with the stellar lens this analysis of the transparent lens is also based on the deflection angle relation $\theta(b)$. For a transparent mass of radius a, $\theta(b<a)$ is clearly dependent on the mass distribution in the deflector. In many applications a variety of centrally condensed mass distributions would be of interest. However, here we make the simple assumption that the lens is approximated as a uniformly dense static sphere. As in the stellar lens case the surrounding universe is assumed asymptotically flat. If, in addition, we assume that the transparent mass has small mean density and is of less than cosmological scale, then the

metric may be approximated as

$$g_{00} = 1 + \Delta g_0 \tag{2}$$

$$g_{0a} = 0 \tag{3}$$

$$g_{ab} = -\delta_{ab}(1 - \Delta g_0) \tag{4}$$

where

$$\Delta g_0 \equiv \frac{2\Phi(r)}{c^2} \tag{5}$$

(see, e.g. Adler, Basin & Schiffer 1965).

For a radially symmetric mass distribution, the Newtonian gravitational potential Φ takes the familiar form

$$\Phi(r \geqslant a) = -\frac{Gm}{r} \tag{6}$$

$$\Phi(r \leqslant a) = -\frac{Gm(r)}{r} \tag{7}$$

where m is the total mass and $m(r)$ is the mass contained within a sphere of radius r.

In a space whose metric may be approximated in this form the deflection angle may be calculated using a very general approximation technique due to Plebanski (1960). Plebanski's technique is based on the eikonal equation (or geometric optics in a pseudo-Riemannian space) and is applicable to any asymptotically flat space-time whose metric can be expressed as the flat metric plus a small correction in each component.

For the metric given above Plebanski's light deflection formula takes the simple form

$${}^f t^m = {}^0 t^m - (\delta^{mn} - {}^0 t^m\, {}^0 t^n) \int_{-\infty}^{\infty} dx^0 \Delta g_{0,n}(x^0, p^s + {}^0 t^s x^0). \tag{8}$$

In equation (8) ${}^f t^m$ is the 'final' ($x^0 \to \infty$) unit tangent of a light ray which at a large negative time before deflection had an 'initial' unit tangent ${}^0 t^m$. In the argument of the integrand p^s is that part of the 'initial' photon position vector which is orthogonal to ${}^0 t^m$. The coordinates represent time and space as measured in Minkowski space comoving with the deflecting mass. Thus this argument is the flat space position vector of the photon at time x^0/c. The Plebanski formula is correct to first order in Δg_0.

It is convenient here to choose an orthogonal coordinate system with the origin at the centre of the deflecting mass. Because of the spherical symmetry of the mass no particular orientation of the coordinate triad can be distinguished and calculations based on a convenient orientation will be completely general. As shown in Fig. 2 an orientation is chosen such that ${}^0 t^m = [1, 0, 0]$ and $p^s = [0, 0, b]$. With this choice $p^3 \equiv b$ plays the role of a photon collision parameter and by the small deflection assumption, b is also approximately the distance of closest approach of the ray to the mass centre. The time is chosen to be zero when the photon crosses the $x^1 = 0$ plane.

Spherical symmetry or explicit calculation shows that the ray will remain in the plane containing the initial tangent vector and the mass centre. Thus, in these

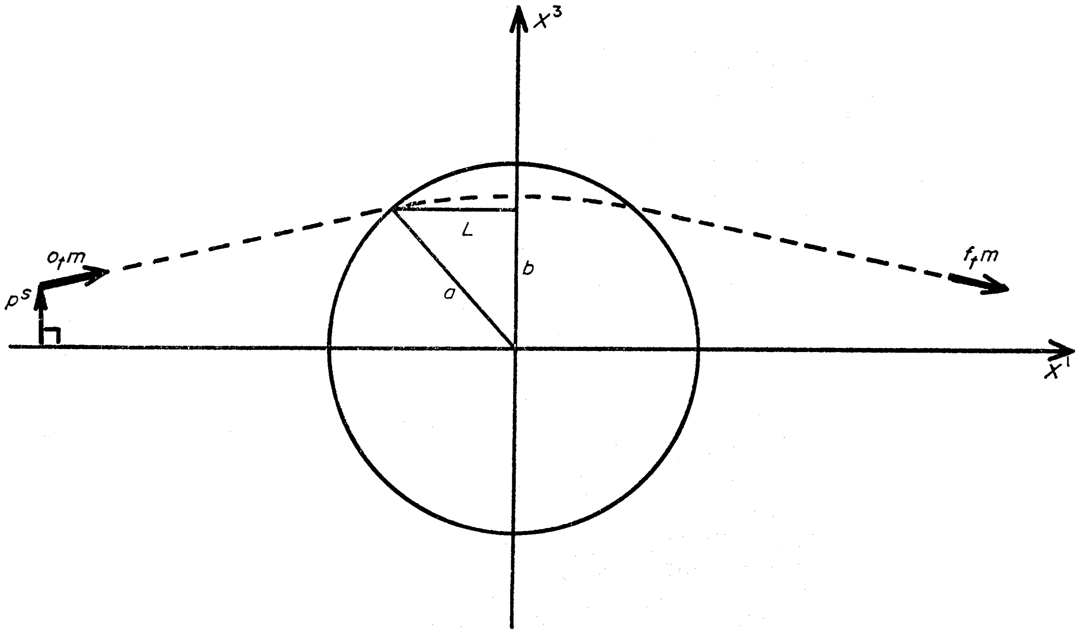

FIG. 2. *The orthogonal coordinate system chosen for calculation of the transparent lens ray deflection.*

coordinates, the only deflection which can occur is in the $-x^3$ direction. The small deflection approximation implies $^ft^3 - {}^0t^3 \approx -\theta(b)$. Equation (8) becomes

$$\theta(b \leqslant a) = 2\int_{-\infty}^{-L} dx^0 \Delta g_{0,3}(r \geqslant a) + \int_{-L}^{L} dx^0 \Delta g_{0,3}(r \leqslant a). \tag{9}$$

The distance r is the Euclidean distance of the photon from the mass centre and in the present coordinates is

$$r(x^0, b) = [(x^0)^2(\hat{e}_1)^2 + b^2(\hat{e}_3)^2]^{1/2}$$

where $\hat{e}_i$ is the unit vector in the x^i direction. The integration limit L is the value of x^0 corresponding to $r = a$, the mass radius. It is apparent in Fig. 2 that $L = (a^2 - b^2)^{1/2}$. Using a metric perturbation based on equations (6) and (7) the integration results in

$$\theta(b \leqslant a) = \frac{4Gm}{c^2 b}\left[1 - \frac{L}{a}\right] + \frac{4Gb}{c^2}\int_0^L \frac{m(r)}{r^3}\, dx. \tag{9a}$$

For constant density this becomes

$$\theta(b \leqslant a) = \frac{4Gm}{c^2}\left[\frac{1}{b} - \frac{(a^2 - b^2)^{3/2}}{a^3 b}\right]. \tag{10}$$

This deflection is a smooth function which increases monotonically from a minimum of $\theta(b = 0) = 0$ to a single maximum at $b = (\frac{3}{4})^{1/4}\, a$. This maximum value of $\theta(b)$ is only slightly greater than the maximum opaque stellar deflection, $\theta(b = a) = K_1 a^{-1}$. Of course, for $b \geqslant a$ the deflection is given by equation (1). For $b \ll a$ equation (10) becomes

$$\theta(b \ll a) \approx \left(\frac{6Gm}{a^2c^2}\right) b = \frac{3}{2} K_1 a^{-2} b \equiv K_2 b. \tag{11}$$

Because $d^2\theta/db^2 < 0$ for $0 < b < a$, the 'exact' deflection is always less than this linear approximation. However, for $b < a/2$ the approximation error increases very slowly with b. At $b = a/3$ the error is 2 per cent and at $b = a/2$ it is 7 per cent. For most purposes of application equation (11) may be considered a valid approximation of $\theta(b)$ not just for $b \ll a$ but for $b < a/2$. This behaviour of $\theta(b)$ and its physical basis are clarified in the next section.

3. A NEWTONIAN MODEL OF GRAVITATIONAL LIGHT DEFLECTION APPLIED TO SPHERICAL AND CYLINDRICAL MASSES

In the deflection discussion above, the photon is deflected with respect to a Euclidean straight line simply because it must follow a null geodesic in a curved space-time. In the weak field approximation this space-time curvature is a function only of the Newtonian gravitational potential Φ (whose argument is distance in a flat 3-space). This allows the deflection to be interpreted as a Newtonian gravitational scattering process in a fictitious flat 3-space. As an intuitive aid, the language of this process is adopted in this section. However, the Newtonian language is not essential to the argument. For example, the discussion is easily recast in terms of a relativistic Fermat's Principle of stationary travel time (Weyl 1922). The simplicity of the argument depends on the fact that Φ obeys Poisson's equation—not that $\nabla\Phi$ be called a force.

It follows almost immediately from requiring a null interval and $\Phi/c^2 \ll 1$ in the weak field metric that a photon in such a space-time is deflected as if it were a Newtonian particle with speed c in a Euclidean space with an effective gravitational potential 2Φ. In order to account for this relativistic curvature factor a Newtonian could perhaps assign the photon an anomalous passive gravitational mass twice that of slow particles having the same inertial mass.

Considered as a Newtonian scattering problem the small angle of deflection of the photon can be approximated by the fractional transverse change in the photon velocity vector and this is given by the time integral of the transverse force along the photon trajectory. For small deflections this path integral may be calculated assuming the photon moves along its undeflected path with speed c.

If the localized mass distribution has radial symmetry then only $m(r)$, the mass inside a sphere of radius r, exerts a radial force at r and

$$\theta(b) = \frac{2G}{c^2}\int_{-\infty}^{\infty} \frac{m(r)}{r^2}\frac{b}{r}\,dx. \tag{9b}$$

The coordinates are chosen here to agree with the derivation of equation (9a) and this result is obviously in agreement with (9a).

Further physical insight into the dependence of the deflection on the mass distribution is provided by applying Gauss' Law to an axially symmetric mass. Consider an undeflected light ray parallel to the symmetry axis and at a distance b. By rotation of the undeflected ray generate a circular cylinder of radius b about the mass symmetry axis and cap the ends arbitrarily to form a closed surface. According to Gauss' Law, the total normal Newtonian flux into this surface is $4\pi Gm(b)$ where $m(b)$ is the total mass enclosed by the surface. For a cylindrical surface of length l extended sufficiently far beyond a localized mass, the flux through the end surfaces is negligible and the mean normal force over the cylindrical part of the surface is

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$2Gm(b)/bl$. By cylindrical symmetry this must also be the mean normal Newtonian force across each undeflected ray generating the surface. The effective photon deflecting force is twice this. Thus, over a sufficiently long distance l, the photon is deflected by an angle

$$\theta(b) = \frac{4Gm(b)}{c^2 b}. \tag{9c}$$

Gauss' Law requires that the mean flux over the closed surface due to mass exterior to this surface vanish. Again, if the mean outward normal flux through the end surfaces is zero, axial symmetry requires that the mean flux across each generator (light ray) vanish. Thus, axially symmetric mass situated beyond b does not contribute to the small total deflection of a ray initially parallel to the axis at a distance b.

In the special case of radial symmetry all lines through the centre are axes of cylindrical symmetry and this relation then applies to all rays. Simple solid geometry demonstrates that equation (9c) is another version of (9a). For a constant density right circular cylinder of length $2a$, equation (9c) is identical with the small b approximation given in equation (11). This demonstrates that this approximation treats the column density of mass seen along rays at small b as a constant.

For some purposes equations (9a) or (9b) are more useful than (9c) because, by changing the limits of integration, the spherical treatment yields information on where the deflection occurs. This is useful in studying lens effects when the source is near or inside the deflector.

4. GENERAL EFFECTS OF MASS ON BEAM AREA

In the deflection discussion of the previous section emphasis was on the total mass within the cylinder generated by rotating the undeflected light ray about a symmetry axis. However, if we are interested specifically in the effect of the light deflection on the area of a narrow beam of light, it is only the mass enclosed by the beam which is important. For small deflections and narrow beams, the mass outside of the beam has only a second order effect on beam area. This is apparent in the results of the final section where it is shown that, for small b, the effect of the uniform transparent sphere on beam area is first order in the parameter $\theta/b \propto \rho a$. However, if the observed beam passes entirely outside of the mass sphere the first order area effect cancels and the second order effect depends on $(\theta/b)^2 \propto m^2/b^4$.

In terms of ray geometry, the area effect is insensitive to mass outside of the beam because it is not the deflection itself which changes the beam area but rather the difference in the deflection of the rays forming the boundary of the beam.

This feature of the effect of mass on beam area is easily seen to be a general property of weak gravitational fields. Simply apply Gauss' Divergence Theorem to the coordinate light velocity $\vec{v}(x, y, z)$ and the closed surface formed by cutting a narrow bundle of light rays with two plane surfaces orthogonal to the central ray of the bundle. Consider two planes a differential distance ds apart as measured along the central ray. If the area of cross-section, δS, as defined by each of these planes is small enough to ignore terms in $(\delta S)^2$, then Gauss' Law takes the familiar first order form

$$\frac{d\delta S}{\delta S} \approx \frac{1}{c} \nabla . \vec{v}\, ds.$$

It has also been assumed here that $|\vec{v}| \approx c$ on both end surfaces. This is consistent with the assumption that $\Phi/c^2 \ll 1$ (small deflections) and requires further that longitudinal effects on $\vec{v}$ vanish. Under these dynamical conditions, a beam with no initial divergence satisfies

$$\nabla.\vec{v} = -\frac{2}{c}\int_{s_0}^{s} \nabla^2\Phi \, ds' = -\frac{8\pi G}{c}\int_{s_0}^{s} \rho(s')\, ds'.$$

Thus, first order gravitational effects on beam area depend only on the column density as seen along the central ray of the beam.

In the following sections the lens effects of the uniformly dense, transparent mass sphere will be examined in detail. This brief gaussian treatment serves to show that the central part of the lens, approximated as a region of constant column density, has particularly simple effects on beam area. Furthermore, the first order effect on beam area is not changed by the addition of a massive core to an otherwise uniform sphere unless the beam passes through the core. In this sense, the assumption of uniform density is not so special ss it first appears.

This gaussian analysis could be the basis of a complete treatment of weak field lens effects. Indeed, it is only a special case of the optical scalar equations of General Relativity (see, e.g. Kristian & Sachs 1966; Penrose 1966; Zipoy 1966). However, in the sections to follow, we choose to analyse the effects of spherical gravitational lenses by an elementary ray construction. Because this ray analysis is based on spherical symmetry it is easily generalized for application to homogeneous, isotropic cosmological spaces.

5. THE EFFECT OF A SPHERICAL LENS ON APPARENT LUMINOSITY

An easy way to calculate the effect of the light deflection on the apparent luminosity of distant sources is to examine the effect of this deflection on the cross-sectional area of a narrow bundle of light from the source. Because the transparent lens under study here is spherically symmetric we can use a geometric construction identical to that used by Refsdal (1964a) in his analysis of the stellar lens and its effect on bundle area.

Such a narrow bundle of light coming from a source at S and passing through a weakly deflecting but otherwise arbitrary spherical lens is shown in Fig. 3. After deflection this bundle reaches the observer with an area A_d. If the beam had propagated undeflected through a Euclidean space and had an area A_l in the median plane of the lens it would have intersected the plane of A_d with an area $A_e = n^2 A_l$ where $n \equiv (d_s + d_l)/d_s$.

In cylindrical coordinates the bundle area A_l can be written as $A_l = b(db)\, d\phi_l$ and the observed bundle area is $A_d = r(dr)\, d\phi_d$. The spherical symmetry of the lens means that all rays lie entirely in a plane containing the mass centre. Therefore $d\phi_l = d\phi_d$. The deflection then results in a smaller bundle area which is related to the undeflected area by

$$\frac{A_e}{A_d} = n^2 \frac{b}{r}\left(\frac{dr}{db}\right)^{-1}. \tag{12}$$

This analysis assumes that no caustics may be observed in the ray bundle. The geometry of Fig. 3 shows that this is equivalent to $dr/db > 0$.

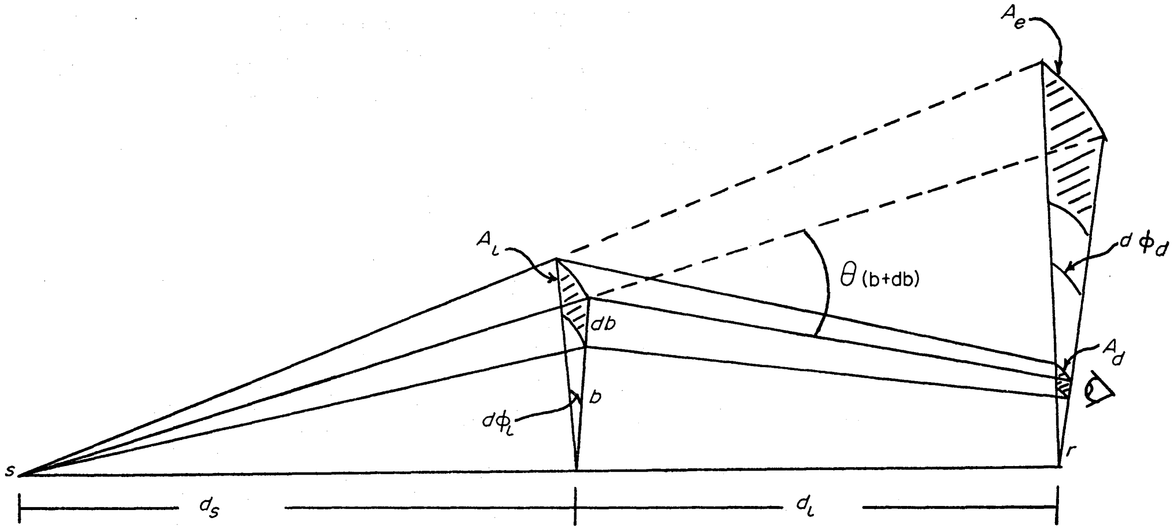

FIG. 3. *Cylindrical coordinates used to analyse the area of a narrow bundle of light which has been deflected by a spherical lens.*

For small deflections it is apparent from the Euclidean triangles of Fig. 3 that

$$r = nb - d_l\theta(b) \tag{13}$$

and

$$\frac{A_e}{A_d} = n^2 \frac{b}{r}\left(n - d_l \frac{d\theta}{db}\right)^{-1}. \tag{14}$$

In the context of relativistic geometric optics it has been shown that the family of observers who all see the same photon frequency in this bundle find

$$\frac{L_d}{L_e} = \frac{A_e}{A_d} \equiv E(b, d_l, d_s) \tag{15}$$

where L_d is the apparent luminosity of the source as seen by an observer whose local bundle area is A_d (Kristian & Sachs 1966). The luminosity ratio, E, will be referred to here as the amplification factors of the lens.

6. FOCUSING EFFECTS OF THE UNIFORM TRANSPARENT LENS

The results of applying the 'exact' deflection equation (10) in equation (14) are quite complicated. However it is not difficult to show that the lens described by equation (10) has the following simple properties.

(1) The amplification factor, $E(b < a)$, is a monotonically decreasing function of b.

(2) For small b this amplification factor is approximately independent of b.

In proof of the first property consider a fixed source and lens and the class of observers with fixed d_l but arbitrary b. For these observers

$$E(b) \propto \left(\frac{r}{b}\frac{dr}{db}\right)^{-1}. \tag{16}$$

Under the two conditions that $r(b = 0) = 0$ and that dr/db is strictly increasing it follows necessarily that r/b is also strictly increasing with b. It is also clear that both

dr/db and r/b must approach an identical minimum as b approaches 0. Thus $E(b)$, the inverse of their product, is a strictly decreasing function of b.

The first condition for this result holds for all spherical lenses (no deflection of radial light rays). For all lenses described by equation (14) (small deflections) the second condition is equivalent to the requirement that $d\theta/db$ be strictly decreasing. (Recall that the 'no caustic' condition is $dr/db > 0$.) Thus a sufficient (but not necessary) condition that $E(b)$ be monotonically decreasing for any weak spherical lens is $d^2\theta/db^2 < 0$. In particular this condition is satisfied for the uniform transparent lens whose deflection relation is given in equation (10). This further implies that the maximum $E(b)$ for the uniform lens results from the use of the small b approximation for $\theta(b)$ which is given in equation (11).

Equation (14) implies that any weak spherical lens with a linearly increasing $\theta(b)$ is characterized by a constant $E(b)$. As a special case, the amplification factor for the uniform transparent lens is approximately constant in the domain of validity of equation (11). The resulting lens is discussed in detail in the following section.

7. FOCAL EFFECTS OF THE CENTRAL TRANSPARENT LENS COMPARED WITH THE STELLAR LENS

For simplicity and because it yields the maximum amplification factor of the lens the remainder of this study of the uniform density transparent lens is based on equation (11). This approximation applies only for $b < a/2$ and thus the following results are valid only for observations made through the central half of this lens. However, in application the central region may be large enough to have many distant sources visible in its background.

Substitution from equation (11) into equation (14) yields the simple b independent relation

$$E(b < a/2) \approx n^2(n - K_2 d_l)^{-2}. \tag{17}$$

Apparently, if the source is very distant ($n \to 1$) and there is no observed caustic ($d_l < K_2^{-1}$), the effect of the lens increases with d_l. The conditions required for this result are seen more clearly by using equation (15) to rewrite (17) as

$$L_a\left(b<\frac{a}{2}\right) = E\left(b<\frac{a}{2}\right)\frac{L}{(d_l+d_s)^2} \approx \frac{L}{d_s^2}[1+d_l(d_s^{-1}-K_2)]^{-2}. \tag{18}$$

Here L is the intrinsic luminosity of the source and we assume that there is no redshift. Thus if $d_s > K_2^{-1} > d_l$ the apparent luminosity of the source actually increases as the observer moves away from the lens (and source). Although Einstein (1936) considered a similar result for the stellar lens 'curious', this is simply the requirement that a narrow pencil of light from the source be converging after it passes through the central transparent lens. Of course equation (18) is valid only in a flat static universe. In application the essentially Euclidean distances d_s and d_l may be of cosmological scale. In this case d_s and d_l must be generalized to cosmological luminosity distances corrected for redshift effects. Such a generalization will be given in a forthcoming paper.

Equation (17) implies that the central part of the transparent lens has a focal length

$$f_t = K_2^{-1} = \left(\frac{c^2}{6G}\right)\left(\frac{a^2}{m}\right). \tag{19}$$

Although this result follows several stages of approximation, the focal length terminology seems more justified here than for the opaque stellar lens. Even in first order the stellar lens condenses the light from a distant point source onto a semi-infinite line rather than onto a point. This can easily be seen in the ' primary image ' amplification factor for the opaque stellar lens which results from the use of equation (1) in (14).

$$E_s(b>0, d_l, d_s) = n^2[n^2-(d_l K_1 b^{-2})^2]^{-1}. \tag{20}$$

(In spite of the different form of this relation which results from a different choice of variable, it is equivalent to equation (17) of Refsdal (1964a) or equation (15) of Liebes (1964).)

Although the focal length of the stellar lens is b dependent, it is perhaps useful to define the ' minimum ' focal length as the caustic distance for any two parallel rays which skirt the stellar limb. For these rays $b = a$ and

$$f_s^{\min} = a^2 K_1^{-1} = 3/2 f_t. \tag{21}$$

The reference above to the ' primary image ' is necessary as a caution here because this simple derivation of E_s assumes that there is no observed caustic ($d_l < f_s^{\min}$) and thus does not reveal the possibility of a secondary image of one source. Because this image is seen at a smaller b than the primary and crosses the source-lens axis before reaching the observer it cannot be observed unless $d_l > f_s^{\min}$. The existence of a secondary image under certain conditions is seen in the quadratic nature of equation (13) when $\theta(b)$ is given by equation (1). The resulting images are discussed clearly in the papers of Refsdal (1964a) and Liebes (1964).

If the stellar deflector is extremely dense, then not only can a luminous secondary image be seen by a nearby observer but both images also have large geometric distortion. As a spherical source approaches perfect alignment behind such a dense stellar lens the two distorted images of this source coalesce and at $b = 0$ the image becomes a luminous annular ring around the deflector. Of course, such spectacular relativistic effects depend on rare and transient alignments behind unusual deflectors.

In contrast to the stellar lens, the uniform transparent lens produces only one image of a given source seen through the lens. This property is easily understood by considering Fig. 1 and equation (13) in light of the general behaviour of $\theta(b)$ discussed after equation (10). If $\theta(b)$ increased as b^2 or faster, then multiple images would be possible. However, the rate of increase of $\theta(b)$ for this lens actually is everywhere less than linear.

It is also characteristic of the uniform transparent lens that it produces no observable image distortion. Penrose (1966) has given an interesting interpretation of image distortion due to gravitational lenses. From his viewpoint the focusing of light by the opaque stellar lens is dominated by the Weyl conformal tensor of the stellar field and this results in astigmatic focusing. The action of the central, uniform, transparent lens may be described as Ricci tensor focusing resulting in distortion-free magnification. This is to be expected from the conformal flatness of the uniform density mass sphere (Schwarzschild interior) (Buchdahl 1971).

This may also be understood from simple geometric considerations. As discussed above, the light deflection due to a spherical lens moves all image points radially away from the deflection centre ($d\phi_l = d\phi_d$). This stretches the image in a direction perpendicular to this radial image motion. The stretching is proportional to the change in the apparent radial distance of the image centre from the deflection centre. The central, uniform, transparent lens also stretches the image by the same factor along the radial direction. This is true wherever the deflection $\theta(b)$ is proportional to b, the apparent radial distance to the deflection centre. For larger b the radial image stretching decreases and for $b > (\frac{3}{4})^{1/4}(a)$ it becomes compression. However, because practical candidates for transparent gravitational lenses must be extended masses with small mean density, the distortions produced at these large values of b will not be large enough to observe.

Finally it should be noted that no annular images are produced by the transparent lens. A spherical source seen with $b = 0$ has a full disc image.

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NOTE ADDED IN PROOF

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