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THE UNIFORMIZATION OF COMPACT KÄHLER SURFACES OF NEGATIVE CURVATURE

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1. Introduction

One of the major accomplishments in the theory of Riemann surfaces is the uniformization theorem which roughly says that the universal covering of a compact Riemann surfaces of genus greater than one is analytically equivalent to the unit disc $\{z \mid |z| < 1\}$. The higher dimensional analog is one of the central problems in hyperbolic complex analysis. In this section we summarize one direction of this research recently moved forward by differential geometers. The starting point for us is a theorem of H. Wu [30] given below.

Theorem 1.1. *Let M be a compact complex Kähler manifold of nonpositive sectional curvature. Then its universal covering is a Stein manifold.*

For a long time, examples of compact complex Kähler manifolds of negative sectional curvature known to us were only compact quotients of the unit ball in C^n, until recently Mostow and Siu discovered a compact Kähler surface of negative sectional curvature which is not uniformized by the ball [20]. A perhaps more natural and nontrivial generalization of hyperbolic Riemann surfaces for algebraic geometers and complex analysis is the notion of negative tangent bundle in the sense of H. Grauert.

Definition. Let M be a compact complex manifold. The tangent bundle $T(M)$ of M is said to be negative if it is a strongly pseudo-convex manifold whose only exceptional variety is the zero section.

The concept of negative tangent bundle is intimately related to that of bisectional curvature described below (see [8], [6]). Let M be a Kähler manifold, and R its Riemannian curvature tensor. Given two complex planes σ and σ' in $T_p(M)$, $p \in M$, we define the bisectional curvature $H(\sigma, \sigma')$ by $H(\sigma, \sigma') = R(X, JX, Y, JY)$, where J is the complex structure tensor of M, $X \in \sigma$, $Y \in \sigma'$. Furthermore, by Bianchi identity we have the following relation

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY).$$

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It is not difficult to show that the tangent bundle of a compact Kähler manifold of negative bisectional curvature must be negative in the sense of Grauert. Now we are in a position to state our main problem.

Conjecture. *If the tangent bundle of a compact complex manifold M is negative, then there are strong restrictions on the homotopy-type of M.*

The noncompact counterpart of our conjecture is a well-known result of Andreotti and Frankel (for a proof, see [18]).

Theorem 1.2. *Let M be a Stein manifold of complex dimension n. Then $H_i(M) = 0$ for all $i > n$.*

Along the line of the classical uniformization theorem it is natural for us to raise the question of the existence of a simply-connected compact Kähler manifold with negative bisectional curvature. It should be remarked here that this question can be reduced to complex two dimensional case by the process indicated below. Let M be a compact complex n-dimensional Kähler manifold with negative bi-curvature. It is projective algebraic by Kodaira embedding theorem. We denote by N a nonsingular hyperplane section of M. If we assume $\dim_C M \geqslant 3$, then we have $\pi_1(M) = \pi_1(N)$ by the Lefschetz theorem on hyperplane sections. Since a subbundle of a negative bundle is negative, $T(N)$ is negative. Combining these two observations (1) $\pi_1(M) = \pi_1(N)$ if $\dim_C M \geqslant 3$, (2) $T(N)$ is negative if $T(M)$ is, we therefore prove our assertion claimed above. This problem constitutes the main ingredient of our conjecture. Nontrial examples of compact complex manifolds of negative tangent bundles are not easy to construct explicitly (see §6 for an example). It is therefore natural for us to consider those compact quotients covered by bounded domains in C^m. Even for a bounded domain $\tilde{M}$ in C^m it is in general rather difficult to determine whether $\mathrm{Aut}(\tilde{M})$ has enough discrete subgroup to form a compact quotient. Following are some known examples in C^2:

(1) $B_2 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$,

(2) $\Delta_2 = \{(z_1, z_2) \mid |z_1| < 1, |z_2| < 1\}$,

(3) the universal coverings of "Kodaira surfaces", [14], [7].

Both B_2 and Δ_2 are symmetric domains in C^2. The Chern numbers c_1^2, c_2 of the compact complex quotient manifolds of B_2 and Δ_2 satisfy the identities $c_1^2 = 3c_2$ and $c_1^2 = 2c_2$ respectively. These are known as the Klein-Clifford forms which had been studied by Borel, Hirzebruch and others [3], [11].

A converse statement is a uniformization theorem of Shing-Tung Yau [33] as a consequence of his resolution of Calabi conjecture.

Theorem 1.3 *Let M be a compact complex n-dimensional manifold with first Chern class negative definite. Then the inequality $(-1)^n c_1^{n-2} c_2(M) \geqslant (-1)^n \frac{1}{2} n(n+1) c_1^n(M)$ is always true, and equality holds if and only if M is holomorphically covered by the unit ball.*

When n is equal to two, we have $c_1^2 \leqslant 3c_2$ which was independently proved by Miyaoka [19] and Yau [33]. Significant study of the conjecture concerning the above inequality for Chern numbers of complex surfaces of general type was started by Van de Ven, and much later a remarkable progress was achieved by F. Bogomolov [2]. Yau's method is particularly striking since he is able to obtain more information when $c_1^2 = 3c_2$ as well as a natural generalization for higher dimensional case.

In [14] Kodaira showed that the signature ($= \frac{1}{3}(c_1^2 - 2c_2)$) of "Kodaira surfaces" is always positive, and from his explicit formula one can easily further see that "$c_1^2 \neq 3c_2$". Following the discussion above we can conclude that there are nonsymmetric bounded domains in C^2 which have enough discrete subgroups of their automorphism groups to form compact quotients. The following global result concerning negative bisectional curvature is due to Paul Yang [32].

Theorem 1.4. *The polydisc $\Delta_n (n > 1)$ does not admit a complete Kähler metric with its bisectional curvature $H(\sigma, \sigma')$ bounded between two negative constants $-c^2$ and $-d^2$, i.e.,*

$$-c^2 \leqslant H(\sigma, \sigma') \leqslant -d^2.$$

The immediate consequence is that if M ($\dim_C M > 1$) is a compact Kähler manifold with negative bisectional curvature, then its universal covering cannot be a polydisc. Yang's theorem indicates the sharp distinctions between a ball and a polydisc in C^n in terms of the concept of bisectional curvature.

The following is our main result.

Theorem 1.5. *Let M be a compact Kähler surface which is hyperbolic in the sense of* [12]. *Suppose that*

1. *$M = \tilde{M}/\Gamma$, where $\tilde{M}$ is the universal covering of M, and Γ is a discrete subgroup of the identity component of* $\mathrm{Aut}(\tilde{M})$ *acting freely on $\tilde{M}$,*

2. *Γ is not isomorphic to the fundamental groups of a compact real surface. Then $\tilde{M}$ is either biholomorphic to $B_2 = \{(z_1, z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 < 1\}$ or $\Delta_2 = \{(z_1, z_2) \mid |z_1| < 1, |z_2| < 1\}$.*

Remarks. 1. Our conditions imply that $\tilde{M}$ must be noncompact and Γ is infinite.

2. It should be stressed that the assumption on Γ that it is contained in the identity component of $\mathrm{Aut}(\tilde{M})$ is a very strong restriction.

The immediate consequences related to the context of our discussion above are the following corollaries.

Corollary 1. *Let M be a Kähler surface of negative sectional curvature and $M = \tilde{M}/\Gamma$ where $\tilde{M}$ is the universal covering of M, and Γ is a discrete subgroup of* $\mathrm{Aut}^0(\tilde{M})$ *(identity component of* $\mathrm{Aut}(\tilde{M})$*) acting freely on $\tilde{M}$. Then $\tilde{M}$ is biholomorphic to the unit ball in C^2.*

Corollary 2. *Let $M = \tilde{M}/\Gamma$ be a compact complex surface such that $\tilde{M}$ is a bounded domain in C^2 with the assumption that the boundary $\partial\tilde{M}$ of $\tilde{M}$ is a topological three-dimensional manifold, and $\Gamma \subset \text{Aut}^0(\tilde{M})$ is a discrete subgroup acting on $\tilde{M}$ freely. Then $\tilde{M}$ is a bounded symmetric domain in C^2.*

Corollary 3. *Let $M = \tilde{M}/\Gamma$ be a compact Kähler surface of negative bisectional curvature. Suppose that $\Gamma \subset \text{Aut}^0(\tilde{M})$ acts freely on $\tilde{M}$, and Γ is not isomorphic to the fundamental group of a compact real surface. Then $\tilde{M}$ is biholomorphic to the unit ball in C^2.*

Corollary 4. *Let $M = \tilde{M}/\Gamma$ be a compact complex surface such that $\tilde{M}$ is a bounded domain in C^2. Suppose that $\Gamma \subset \text{Aut}^0(\tilde{M})$ acts freely on $\tilde{M}$, and Γ is not isomorphic to the fundamental groups of a compact real surface. Then $\tilde{M}$ is a bounded symmetric domain.*

2. A theorem on homogeneous complex manifolds

Theorem 2.1. *Let $D = G/H$ be a homogeneous complex manifold, where G is a connected Lie group acting on D effectively, and H is the isotopy subgroup of G. If there exists a discrete subgroup $\Gamma \subset G$ such that $M = D/\Gamma$ is a compact complex manifold of negative definite first Chern class, then D is a bounded symmetric domain in C^n ($n = \dim_C D$).*

Proof. Since the first Chern class of M is negative definite, it follows that M is measure hyperbolic. D is also measure hyperbolic as it is a covering of M.

Let $E_D = |E_D|\, dz_1 \wedge dz_2 \wedge \cdots \wedge d\bar{z}_n$ be the differential Eisenman-Kobayashi measure on D. Since D is homogeneous, locally $|E_D|$ is a smooth function. Our aim is to show that the associated two-form (or Ricci form) or E_D, namely dS_D^2 (or $\text{Ric}(E_D)) = -\Sigma_{i,j}(\partial^2 \ln |E_D|/\partial z_i \partial\bar{z}_j)\, dz_i \wedge d\bar{z}_j$, is negative definite.

Since the first Chern class of M is negative definite, it is well known that we can always construct a nonzero smooth volume form $V = |V|\, dw_1 \wedge dw_2 \wedge \cdots \wedge d\bar{w}_n$ on M such that $-\Sigma_{i,j}(\partial^2 \ln |V|/\partial w_i \partial\bar{w}_j)\, dw_i \wedge d\bar{w}_j$ is negative definite. We denote by $E_M = |E_M|\, dw_1 \wedge dw_2 \wedge \cdots \wedge d\bar{w}_n$ the differential Eisenman-Kobayashi measure on M. It is easy to check that $E_D = \pi^*(E_M)$, where $\pi\colon D \to D/\Gamma$ is the covering projection. Thus $|E_M|$ is also locally a smooth function.

However, it is not hard to see that $f = |E_M|/|V|$ is a positive smooth function which is globally defined on M. Taking the logarithm of f, we have $\ln f = \ln |E_M| - \ln |V|$. Since M is compact, $\ln f$ takes a minimum at some $p \in M$. Thus $d(\ln f)(p) = 0$, and the matrix $(\partial^2(\ln f)/\partial w_i \partial\bar{w}_j(p))$ is positive semidefinite. However,

$$\frac{\partial^2 \ln f}{\partial w_i \partial \bar{w}_j}(p) = \frac{\partial^2 \ln(E_M)}{\partial w_i \partial \bar{w}_j}(p) - \frac{\partial^2 \ln |V|}{\partial w_i \partial \bar{w}_j}(p).$$

It follows that, for any nonzero vector $t \in T_p(M)$,

$$\left(\frac{\partial^2 \ln(E_M)}{\partial w_i \partial \overline{w}_j}(p)\right)(t, \bar{t}) \geqslant \left(\frac{\partial^2 \ln |V|}{\partial w_i \partial \overline{w}_j}(p)\right)(t, \bar{t}) > 0,$$

which proves $-(\partial^2 \ln(E_M)/\partial w_i \partial \overline{w}_j(p))$ is a negative-definite $n \times n$ matrix. Now let $q \in D$ such that $\pi(q) = p$. Obviously $-\partial^2 \ln |E_D|/\partial z_i \partial \bar{z}_j(q))$ is also negative definite since $\pi^*(E_M) = E_D$. Our claim follows from the fact that D is homogeneous.

The conclusion that D is a bounded symmetric domain in C^n is a consequence of the following known facts:

1. If a Lie group G contains a discrete subgroup Γ such that G/Γ is compact, then G is unimodular.

2. **Theorem of Hano** [9]. If the Ricci curvature of a Kählerian homogeneous space of a connected unimodular Lie group is nondegenerate, and the group acts effectively on the space, then the group is semisimple.

In our case, $ds^2 = \sum_{i,j}(\partial^2 \ln |E_D|/\partial z_i \partial \bar{z}_j)\, dz_i \cdot d\bar{z}_j$ is a G-invariant Kähler metric on G/H. It is easy to see that the Ricci tensor of ds^2 is nondegenerate since it is only a constant multiple of ds^2.

3. **Theorem of Kozul** [17]. Let G/H be a homogeneous complex manifold with G a connected semi-simple Lie group. Suppose there exists a G-invariant volume form $V = |V|\, dz_1 \wedge dz_2 \wedge \cdots \wedge d\bar{z}_n$ such that its associated Ricci form $-\sum_{i,j}(\partial^2 \ln |V|/\partial z_i \partial \bar{z}_j)\, dz_1 \wedge d\bar{z}_j$ is negative definite. Then G/H is a hermitian bounded symmetric domain in C^n.

Our proof of Theorem 2.1 is therefore complete.

Note. In a letter to the author, Professor Piatetskii Shapiro kindly informed us the following theorem of his (*Geometry of classical domains and automorphic functions*, Gorden and Breach, New York, Vol. 8, 1969):

Let $D = G/H$ be a homogeneous complex manifold with an invariant volume $V = |V|\, dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ whose Ricci form $-\sum(\partial^2 \ln |V|/\partial z_i \partial \bar{z}_j)\, dz_1 \wedge d\bar{z}_j$ is negative definite. Then D is a bounded homogeneous domain in C^n.

Depending on the validity of this statement one can shorten the argument given above.

3. Two remarks on compact hyperbolic surfaces

In this section we shall prove the following two results concerning hyperbolic manifolds.

Theorem 3.1. *Let M be a compact Kähler surface of hyperbolic type in the sense of* [12]. *Then the first Chern class of M is negative definite.*

Theorem 3.2. *Let M be a compact Kähler surface of hyperbolic type. Then the Euler characteristic of M is positive.*

Proof of Theorem 3.1. According to the classification theory of Enriques-Kodaira, compact complex surfaces free of exceptional curves can be divided into seven classes:

(1) the class of projective plane and ruled surfaces,

(2) the class of $K3$-surfaces,

(3) the class of complex tori,

(4) the class of minimal elliptic surfaces with $b_1 = 0(2)$, $P_{12} \geqslant 0$, $K \neq 0$,

(5) the class of minimal algebraic surfaces with $P_2 > 0$, $C_1^2 > 0$,

(6) the class of minimal elliptic surfaces with $b_1 \equiv 1$ (or 2), $P_{12} > 0$,

(7) the class of minimal surfaces with $b_1 = 1$, $P_{12} = 0$.

Class (6) and (7) are nonKähler surfaces (since their first Betti numbers are odd), and hence not of our interest. Thus the set of all compact Kähler hyperbolic surfaces are contained in classes (1), ...,

(5). However, classes (1), (3), (4) are not hyperbolic since they contain either rational or elliptic curves. In order to complete our proof of Theorem 3.1 we have to prove the following:

Claim 1. *$K3$-surfaces are not hyperbolic.*

Claim 2. *The first Chern class of an algebraic surface of general type which does not contain any rational curve is negative definite.*

Proof of Claim 1. We need two known results to conclude this fact:

1. **Theorem of Brody** [4]. Let M_0 be a compact hyperbolic manifold. Then any sufficiently small local deformation M_t of M_0 is also hyperbolic (i.e., Hyperbolicity is stable under local deformation).

2. It was proved in [13, Theorem 18] that for any $K3$-surface M_0, there exists an arbitrarily small deformation M_t or M_0 which is an elliptic $K3$ surface.

However, an elliptic surface contains a lot of elliptic curves, hence cannot be hyperbolic. This proves Claim 1.

Proof of Claim 2. The following result is known in the theory of surfaces [16], [21].

Let M be an algebraic surface of general type. Then for sufficiently large m, $H^0(M, mK)$ has enough sections to define the pluri-canonical map

$$\phi_m: M \to CP_N, \quad \text{where } N = \dim_C |mK|,$$

such that:

1. the pluri-canonical system $|mK|$ has no base point, and ϕ_m is a holomorphic map,

2. if $m \geqslant 6$, then ϕ_m is biholomorphic modulo ε, where ε is the union of all nonsingular rational curves E such that $KE = 0$.

We remark that ϕ_m is said to be biholomorphic modulo ε if ϕ_m is biholomorphic on $M - \varepsilon$, and $\phi_m^{-1}\phi_m(z)$ is a connected component of ε which contains the given point z.

Now, if M is assumed to be void of rational curves, it follows immediately that ε is an empty set. Thus the pluri-canonical map ϕ_m is biholomorphic and defines an embedding of M into CP_N; in other words, mK is very ample if m is sufficiently large. By a theorem of Kodaira, it implies immediately that K is a positive line bundle over M; i.e., the first Chern class of M is negative definite.

Proof of Theorem 3.2. One way to verify this fact goes as follows. By Theorem 3.1 the first Chern class of M is negative definite. A Theorem of Yau [33] says that M admits an Einstein-Kähler metric. The Chern-Gauss-Bonnet theorem furnishes us with the following formula for any Riemannian metric on M:

$$X(M) = \frac{1}{2^4\pi^2 2!}\int_M \sum \varepsilon_{ijkl}\Omega^i_j \wedge \Omega^k_l.$$

The curvature tensor R is a symmetric linear transformation

$$R\colon \Lambda^2(T^*(M)) \to \Lambda^2(T^*(M))$$

such that

$$R = (l_i \wedge l_j) = \Omega^i_j = \tfrac{1}{2}\sum R_{ijkl} l_k \wedge l_l$$

relative to a local orthonormal basis $\{l_i\}$ of one-forms. A very useful canonical form of curvature tensor for an Einstein metric on four-dimensional manifold was given by Berger [1]. In summary, there exists an orthonormal basis $\{l_1, l_2, l_3, l_4\}$ of one-forms such that $\{l_1 \wedge l_2, l_1 \wedge l_3, l_1 \wedge l_4, l_2 \wedge l_3, l_2 \wedge l_4, l_3 \wedge l_4\}$ constitute a basis for $\Lambda^2(T^*(M))$ such that the curvature tensor (with respect to this set of basis) assume the following simple expression,

$$R = \begin{pmatrix} K & H \\ H & K \end{pmatrix},$$

where

$$K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}.$$

By a computation one can get

$$X(M) = \frac{1}{2^4\pi^2 2!}\int_M \sum_{i=1}^{3} \left(|k_i|^2 + |h_i|^2\right) * 1.$$

Therefore $X(M) \geqslant 0$, and $X(M) = 0$ if M is flat. Hence Theorem 3.2 is proved.

Remark. There are many methods to verify Theorem 3.2; for the recent development of the relationship between Chern numbers on complex surfaces of general type one can consult [2], [19], [33]. Actually all one needs is the fact $ac_2 \geqslant c_1^2$ for some $a > 0$ (Van de Ven's old estimate ($a = 8$) is sufficient for our purpose). The proof given here is probably a "bad proof".

4. Proof of the main theorem

Before embarking on the proof we first make two remarks here (with the same notation as in Theorem 1.5):

(i) Since $\tilde{M}$ is complete-hyperbolic, $\mathrm{Aut}(\tilde{M})$ acts properly on $\tilde{M}$ in the sense that if K_1 and K_2 are relatively compact subsets of $\tilde{M}$, then $(K_1, K_2) = \{g \in \mathrm{Aut}(\tilde{M}) \mid g(K_1) \cap K_2 \neq \varnothing\}$ is relatively compact in $\mathrm{Aut}\,\tilde{M}\}$ (see [12], [31]).

(ii) $\tilde{M}$ admits a G-invariant (remind: $G = \mathrm{Aut}(\tilde{M})$) Riemannian metric (see [24, Theorem 4.3.1, p. 316]).

It is well known that the only homogeneous bounded domains in C^2 are B_2 and Δ_2. From Theorems 2.1 and 3.1 it suffices for us to prove that $\tilde{M}$ is homogeneous. We break our proof into several steps.

First of all, we claim that Γ must be infinite. If Γ is finite, then $\tilde{M}$ is compact. Thus $\mathrm{Aut}(\tilde{M})$ consists of only finitely many elements since $\tilde{M}$ is compact hyperbolic, [12], [31]. Moreover, since $\Gamma \subset \mathrm{Aut}^0(\tilde{M})$, it follows immediately $\Gamma \cong \{1\}$. This is obviously a contradiction to the fact that $\{1\}$ is the fundamental group of S^2. Then by recalling that $\Gamma \subset \mathrm{Aut}^0(\tilde{M})$ is a discrete subgroup, and G^0 acts properly on $\tilde{M}$, we see that the orbits $G^0(p)$ are *noncompact closed* submanifolds in $\tilde{M}$ with $\dim G^0(p) \geqslant 1$ for all $p \in \tilde{M}$. We have to keep this fact in mind in our proof given below.

Secondly, if we denote by $\pi\colon \tilde{M} \to M = \tilde{M}/\Gamma$ the canonical projection, it is obvious that $\pi^{-1}(V) = G^0(p)$, where $V = G^0(p)/\Gamma$. Thus V must be a closed subset in M with respect to the quotient topology since $G^0(p)$ is closed. Since M is compact, $V = G^0(p)/\Gamma$ is a compact submanifold of M. We shall use this fact in the sequel of our proof.

(1) Suppose there exist a point $p \in \tilde{M}$ such that $\dim_R G^0(p) = 1$. By the above argument we know that $G^0(p)$ is a closed noncompact connected one-dimensional submanifold of $\tilde{M}$. It follows that $G^0(p)$ must be homeomorphic to R^1. Since Γ acts on $\tilde{M}$ freely, Γ also acts freely on $G^0(p)$. Furthermore, $V = G^0(p)/\Gamma$ is a compact one-dimensional manifold. Consequently the only possibility if $V \cong S^1$ and $\Gamma \cong Z$. We further observe that the first Betti number

of M must be even since it is a Kähler manifold. This is a contradiction since the rank of $H_1(M, Z) = \Gamma/[\Gamma, \Gamma]$ is equal to one if $\Gamma \cong Z$.

(2) Suppose there exists a point $p \in \tilde{M}$ such that $\dim_R G^0(p) = 2$. By Remark (ii) there exists a G-invariant Riemannian metric dS^2 on $\tilde{M}$. With this metric, $(G^0(p), dS^2)$ is a noncompact homogeneous two-dimensional Riemannian manifold. There are three cases:

Case 1. $(G^0(p), dS^2)$ is isometric to the upper-half plane with Poincaré metric whose sectional curvature is negative constant.

Case 2. $(G^0(p), dS^2)$ is isometric to $\mathbf{R}^2$ with flat metric.

In both Cases 1 and 2, $V = G^0(p)/\Gamma$ is a compact two-dimensional manifold which is topologically covered by R^2. Therefore Γ is isomorphic to the fundamental group of a compact surface. This is a contradiction to assumption 2 of our theorem.

Case 3. $(G^0(p), dS^2)$ is isometric to the flat cylinder $S^1 \times \mathbf{R}$.

In this case, $V = G^0(p)/\Gamma$ is a compact surface embedded in M. Since $S^1 \times R$ is flat, V is either a flat torus or Klein bottle with the inherited metric form $(G^0(p), dS^2)$. From the fact that Γ acts freely on $S^1 \times R$ as isometries, it is not hard to see that $\Gamma \cong F \oplus Z$, where F is a finite abelian group. Since the rank of $H_1(M, Z) = \Gamma/[\Gamma, \Gamma]$ is equal to one if $\Gamma \cong F \oplus Z$, it contradicts the assumption that M is Kähler again.

(3) Suppose that *for all* $p \in \tilde{M}$, $\dim_R G^0(p) = 3$.

As before, $G^0(p)$ is a closed three-dimensional submanifold of $\tilde{M}$ and invariant under Γ, and $G^0(p)/\Gamma$ is a compact three-dimensional submanifold of $M = \tilde{M}/\Gamma$ for all $p \in \tilde{M}$. In this way we have a smooth codimenson-one foliation of M, whose leaves are compact three-dimensional manifolds. This would give rise to a nonvanishing vector field on M or on a double covering of M, and the Euler characteristic of M is thus equal to zero, a contradiction to Theorem 3.2.

(4) There exists a point $p \in \tilde{M}$ such that $\dim_R G^0(p) = 4$.

We note that G^0 acts properly on $\tilde{M}$, so that $G^0(p)$ is a four-dimensional closed orbit in $\tilde{M}$. Since $\dim_R \tilde{M} = 4$, it is led to conclude that $G^0(p) = \tilde{M}$, i.e., $\tilde{M}$ is homogeneous. *This is the only possibility.*

The proof of our main theorem is therefore complete.

5. Proofs of Corollaries 1 $\cdots$ 4

Corollary 1 is an immediate consequence of the following facts:

1. A complete Riemannian manifold of nonpositive sectional curvature is a $K(\Gamma, 1)$ space.

2. There is no compact orientable four-dimensional $K(\Gamma, 1)$ manifold with $\Gamma \cong$ fundamental groups of real compact surfaces.

3. A complete hermitian manifold of strongly negative holomorphic sectional curvature is complete hyperbolic [12], [13].

4. Theorem of Paul Yang (Theorem 1.4).

5. Our main result.

We only have to verify fact 2. If Γ is the fundamental group of a $K(\Gamma, 1)$ space, it is a standard fact that $H^i(\Gamma, Z) = H^i(M, Z)$. Hence we have $H^i(\Gamma, Z) = 0$ for all $i \geqslant 5$ and $H^4(\Gamma, Z) = Z$ (M is compact orientable). If Γ is the fundamental group of a real compact surface S, there are two possibilities:

(a) If S is topologically covered by R^2, then $H^i(\Gamma, Z) = 0$ for all $i > 2$. This contradicts our previous conclusion $H^4(\Gamma, Z) = H^4(M, Z) = Z$.

(b) If S is topologically covered by S^2, then $H^i(\Gamma, Z)$ satisfies certain periodical property if $\Gamma \ncong \{1\}$. In particular, we have $H^i(\Gamma, Z) \neq 0$ for some $i \geqslant 5$ if $\Gamma \ncong \{1\}$ is a contradiction to the fact $H^i(\Gamma, Z) = H^i(M, Z) = 0$ for $i \geqslant 5$. Finally if $\Gamma \cong \{1\}$, then $H^4(\Gamma, Z) = 0$ contradicting $H^4(\Gamma, Z) = H^4(M, Z) = Z$.

The main steps to prove Corollary 2 are an application of fact 2 mentioned above, the well known fact that compact quotients of bounded domains are complete hyperbolic, and the following lemma.

Lemma 5.1. *Let $\tilde{M}$ be a bounded domain in C^n whose boundary $\partial\tilde{M}$ is a $(2n-1)$-topological manifold. If there exists a discrete subgroup $\Gamma \subset \mathrm{Aut}(\tilde{M})$ such that $M = \tilde{M}/\Gamma$ is compact, then $\pi_j(\tilde{M}) = 0, j \geqslant 1$.*

Proof. We denote by $\partial\tilde{M}$ the boundary of M as usual, which is a compact topological $(2n-1)$-manifold embedded in $R^{2n} = C^n$. Let d be the distance function from the origin in R^{2n}. Since d is a continuous function which is defined on $\partial\tilde{M}$, suppose that d assumes a maximum at $p \in \partial\tilde{M}$, so that we can write $d(o, p) = r > 0$. If we draw a sphere with center at o and radius r, namely $S_r = \{x \in R^{2n} \mid d(o, x) = r\}$, it is elementary to show there exists no complex analytic subvariety of positive dimension sitting on S_r. Since $\partial\tilde{M}$ is more convex than S_r at p, one can also easily verify that there exists no complex analytic subvariety passing through p of positive dimension lying on $U \cap \partial\tilde{M}$, where U is an open neighborhood of p in C^n. To be precise, there exists no nontrivial holomorphic map $g\colon \Delta = \{z \in C \mid |z| < \varepsilon\} \to U \cap \partial\tilde{M} \subset C^n$ such that $g(o) = p$ where "o" is the origin of Δ.

From our assumption that there exists a discrete subgroup $\Gamma \subset \mathrm{Aut}(\tilde{M})$ such that $\tilde{M}/\Gamma$ is compact, we can always find a compact set $K \subset \tilde{M}$ so that for each $x \in \tilde{M}$ there are $y \in K$ and $g \in \Gamma$ satisfying $g(y) = x$ (K is called a fundamental domain of $M = \tilde{M}/\Gamma$, [23]). Let $\{p_i\}$ be a sequence of points in $\tilde{M}$ converging to the point $p \in \partial\tilde{M}$ fixed above. There correspond a sequence of points $\{x_i\}$ in K and $\{g_i\} \subset \Gamma$ such that $g_i(x_i) = p_i$ for all i. Since K is a compact set, there exists a point $x \in K$ such that $\{x_i\}$ converges to x (passing to a subsequence if necessary). We claim that $\{g_i(x)\}$ would converge to p. To prove this claim we observe that $d_c(g_i(x), g_i(x_i)) = d_c(x, x_i)$, where d_c denotes the Carathéodory distance function on $\tilde{M}$ which is invariant under biholomorphisms [23]. Since $\{x_i\}$ converges to x and d_c is a continuous bounded function on $K \times K$, $d_c(x, x_i)$ tends to zero as i approaches infinity. This implies that $d(g_i(x), g_i(x_i))$ will also approach zero as i goes to infinity. We recall the fact $\{g_i(x_i)\} \to p$ and observe the inequality $d_c \geqslant s \cdot d$ holds on $\tilde{M}$, where s is a positive constant, and d is the Euclidean distance function. Now it is trivial to see that $\{g_i(x)\}$ must converge to p. This provides a proof of our claim.

By normal family argument we can prove $\{g_i\}$ converges on compacta (passing to a subsequence if necessary) to a bounded holomorphic function g: $\tilde{M} \to C^n$ such that $g(x) = p$, $g(\tilde{M}) \subset \partial\tilde{M}$. g must be a constant map from $\tilde{M}$ to C^n, otherwise g would map an open set containing x onto a complex analytic variety of positive dimension lying on $\partial\tilde{M}$ passing through the point p.

To complete our proof we draw a sphere S in $\tilde{M}$ to represent a nontrivial class of $\pi_j(\tilde{M})$, $j \geqslant 1$. Since $\{g_i\} \to g$, where g is the above constant map such that $g(\tilde{M}) = p$, it is clear $g_i(S) \subset U \cap \tilde{M}$ for sufficiently large i. We can choose such a "U" so that $U \cap \tilde{M}$ is contractible since $\partial\tilde{M}$ has been assumed to be a topological $(2n - 1)$-manifold. This would imply that $g_i(S)$ can be deformed into one point within the region $U \cap \tilde{M}$ for large i. Nevertheless, the fact that g_i is a biholomorphism gives rise to a contradiction to our assumption that S represents a nontrivial class in $\pi_j(\tilde{M})$.

This completes the proof that $\pi_j(\tilde{M}) = 0$ for all $j \geqslant 1$.

The proof of Corollary 4 is again a trivial consequence of the fact that compact quotients of bounded domains in C^n are complete hyperbolic. As for Corollary 3, one simply has to invoke the theorem of Yang (Theorem 1.4) that Δ_2 does not admit any complete Kähler metric whose bisectional curvature is pinched between two negative numbers.

6. An example and additional comments

As a final remark we shall give the following example which is related to our negative tangent bundle conjecture mentioned in the introduction.

1. *The construction of a compact Kähler surface of negative holomorphic bisectional curvature admitting no Riemannian metric of nonpositive sectional curvature*. Let N be a compact quotient of the complex 3-ball. By the Kodaira embedding theorem, N is a complex submanifold of some complex projective number space P_n. Let H be a hyperplane in P_n such that $M = N \cap H$ is nonsingular. From the monotonicity property of the holomorphic bisectional curvature it follows that the Kähler metric induced by N on M has negative holomorphic bisectional curvature. By the Lefschetz hyperplane section theorem, $\pi_1(M) = \pi_1(N)$. Since $Z = H_6(N, Z) = H_6(\pi_1(N), Z) = H_6(\pi_1(M), Z) \neq H_6(M, Z) = 0$, it follows that M cannot be an Eilenberg-MacLane space and therefore cannot carry a Riemannian metric of nonpositive sectional curvature.

2. *The nonvanishing of $H^1(M, T_M)$ for the manifold M constructed above.* Consider a family of hyperplanes H_t in P_n close to H. Let $M_t = N \cap H_t$. Any two distinct M_t, $M_{t'}$ cannot be biholomorphic, because any biholomorphic map $\phi: M_t \to M_{t'}$ is a harmonic map from M_t to N which, together with the harmonic map $M_t \hookrightarrow N$, contradicts the uniqueness theorem of Hartman [10] on harmonic maps for target manifolds with negative sectional curvature. Hence M is not rigid, and $H^1(M, T_M)$ cannot be zero. This interesting phenomenon is significant, because of the result of Siu [24] on its rigidity in the case of compact Kähler manifold of strongly negative curvature. It shows that such a rigidity theorem fails for compact Kähler surfaces of negative holomorphic bisectional curvature.

This example has already been described in [29], but the author wants to add one remark here concerning the Hirzebruch index of $M = N \cap H$. If we embed N in P_n by its canonical system, it should not be difficult to give an estimate of the Hirzebruch index of M. For most of such M the index is a negative number. In this way one could obtain a complex surface of ample cotangent bundle with negative index which is not an Eilenberg-MacLane space. This type of surfaces are rather interesting, and the author will discuss this matter in a separate paper.

Finally, the author would like to take this opportunity to point out that one of the questions he asked in [29] (problem 5) is a known result. It is very easy to prove that if X is a compact Kähler manifold with its sectional curvature pinched between -1 and $-\frac{1}{4}$, then it is in fact of constant holomorphic sectional curvature. In particular, X is covered holomorphically by the unit ball. Moreover, problem 3(b) is in general false following from a nontrivial topological fact.

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