THE UNIQUENESS OF THE (COMPLETE) NORM TOPOLOGY

BY B. E. JOHNSON

Communicated by Richard Arens, March 7, 1967

In this paper we show that every semisimple Banach algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ has the uniqueness of norm property, that is we show that if $\mathfrak{A}$ is a Banach algebra with each of the norms $\|\ \|$, $\|\ \|'$ then these norms define the same topology. This result is deduced from a maximum property of the norm in a primitive Banach algebra (Theorem 1).

In the following $\boldsymbol{F}$ is a field which may be taken throughout as $\boldsymbol{R}$, the real field, or $\boldsymbol{C}$, the complex field. If $\mathfrak{X}$ is a normed space then $\mathfrak{B}(\mathfrak{X})$ will denote the space of bounded linear operators on $\mathfrak{X}$.

LEMMA 1. *Let F, G be closed subspaces of the Banach space E such that $F+G=E$. Then there exists $L>0$ such that if $x\in E$ then there is an $f\in F$ with*

(i) $\|f\|\leqq L\|x\|$.

(ii) $x-f\in G$.

PROOF. The map $(f, g)\rightarrow f+g$ is a continuous map of $F\oplus G$ onto E and so is open by the open mapping theorem [**1**, p. 34]. Thus there is $\delta>0$ such that if $y\in E$ with $\|y\|<\delta$ then there are f', $g'\in G$ with $\|f'\|$, $\|g'\|\leqq 1$ and $f'+g'=y$. The result of the lemma then follows if we take $L=\delta^{-1}$, $y=x\|x\|^{-1}\delta$ and $f=f'L\|x\|$.

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra over $\boldsymbol{F}$ and let $\mathfrak{X}$ be a normed space over $\boldsymbol{F}$. Suppose that $\mathfrak{X}$ is a faithful strictly irreducible left $\mathfrak{A}$-module and that the maps $\xi\rightarrow a\xi$ from $\mathfrak{X}$ into $\mathfrak{X}$ are continuous for each $a\in\mathfrak{A}$. Then there exists a constant M such that*

$$\|a\xi\|' \leqq M\|a\|\cdot\|\xi\|'$$

for all $a\in\mathfrak{A}$, $\xi\in\mathfrak{X}$, where $\|\cdot\|$ is the norm in $\mathfrak{A}$ and $\|\cdot\|'$ the norm in $\mathfrak{X}$.

The theorem asserts that the natural map $\mathfrak{A}\rightarrow\mathfrak{B}(\mathfrak{X})$ is continuous. It is a much stronger version of [**4**, Theorem 2.2.7] but applicable only to primitive algebras. It would be interesting to know how far it can be generalized.

PROOF. If $\xi\in\mathfrak{X}$ and $a\rightarrow a\xi$ $(\mathfrak{A}\rightarrow\mathfrak{X})$ is continuous then the map $a\rightarrow ab\rightarrow ab\xi$, being a composition of continuous maps, is continuous. Since $\mathfrak{X}$ is strictly irreducible, if $\xi\neq 0$ we can, by a suitable choice of b, make $b\xi$ any particular vector in $\mathfrak{X}$ and so if $a\rightarrow a\xi$ is continuous for one nonzero ξ it is continuous for all ξ in $\mathfrak{X}$. We shall deduce a contradiction by assuming $a \to a\xi$ continuous only for $\xi = 0$ and hence show that all these maps are continuous. We assume $\mathfrak{X} \neq \{0\}$ since this case is trivial.

The $\mathfrak{A}$-module $\mathfrak{X}$ is of infinite dimension over $\boldsymbol{F}$ since otherwise, as $\mathfrak{X}$ is faithful, $\mathfrak{A}$ would be a finite dimensional algebra and any linear map $\mathfrak{A} \to \mathfrak{X}$ would be continuous. Since $\mathfrak{X}$ is a strictly irreducible $\mathfrak{A}$-module the norm on $\mathfrak{A}$ determines a complete norm $\|\cdot\|$ on $\mathfrak{X}$ [**4**, Theorem 2.2.6] and so the centralizer $\mathfrak{D}$ of $\mathfrak{A}$ on $\mathfrak{X}$ is isomorphic with $\boldsymbol{R}$, $\boldsymbol{C}$ or the quarternions [**4**, Lemma 2.4.4] and in any case is of finite dimension over $\boldsymbol{F}$. Since $\mathfrak{X}$ is of infinite dimension over $\boldsymbol{F}$ it is of infinite dimension over $\mathfrak{D}$. We can thus choose a linearly independent (over $\mathfrak{D}$) sequence $\xi_1, \xi_2, \cdots$ from $\mathfrak{X}$ with $\|\xi_i\|' = 1$.

We now show that for each K, $\epsilon > 0$ and for each positive integer m there is $x \in \mathfrak{A}$ such that

(i)′ $\|x\| < \epsilon$.

(ii)′ $x\xi_1 = x\xi_2 = \cdots = x\xi_{m-1} = 0$.

(iii)′ $\|x\xi_m\|' > K$.

Put $J_i = \{a; a \in \mathfrak{A}, a\xi_i = 0\}$, then [**3**, p. 6, Theorem 2] J_i is a maximal modular left ideal and $I = (J_1 \cap J_2 \cdots \cap J_{m-1}) + J_m$ is a left ideal containing J_m. Since $\xi_1, \cdots, \xi_m$ are linearly independent over $\mathfrak{D}$ we can find, by the density theorem [**3**, p. 28], $y \in \mathfrak{A}$ such that $y\xi_1 = y\xi_2 = \cdots = y\xi_{m-1} = 0$ and $y\xi_m = \xi_m \neq 0$. We have $y \in I$, $y \notin J_m$ so that I contains J_m properly and, by maximality of J_m, $I = \mathfrak{A}$. Take the number L given by applying Lemma 1 with $E = \mathfrak{A}$, $F = J_1 \cap J_2 \cdots \cap J_{m-1}$, $G = J_m$. By the discontinuity of the map $x \to x\xi_m$ we can find $x_0 \in \mathfrak{A}$ satisfying (i)′ with ϵ replaced by ϵ/L and (iii)′. Then, by Lemma 1, there exists $x \in J_1 \cap J_2 \cdots \cap J_{m-1}$ (so that (ii)′ holds for x), such that $x_0 - x \in J_m$ (i.e. $x_0\xi_m = x\xi_m$) and $\|x\| \leqq L\|x_0\| < \epsilon$.

Now choose, by induction, a sequence $x_1, x_2, \cdots$ in $\mathfrak{A}$ such that

(i)° $\|x_n\| < 2^{-n}$.

(ii)° $x_n\xi_1 = \cdots = x_n\xi_{n-1} = 0$.

(iii)° $\|x_n\xi_n\|' \geqq n + \|x_1\xi_n + \cdots + x_{n-1}\xi_n\|'$.

Put $z_i = \sum_{n>i} x_n$. Since $x_n \in J_i$ for $n > i$ and J_i is closed in $\mathfrak{A}$ we see that $z_i \in J_i$, that is $z_i\xi_i = 0$, and $z_0 = x_1 + \cdots + x_i + z_i$. Thus

$$\begin{aligned} \|z_0\xi_i\|' &= \|x_1\xi_i + \cdots + x_i\xi_i + z_i\xi_i\|' \\ &\geqq \|x_i\xi_i\|' - \|x_1\xi_i + \cdots + x_{i-1}\xi_i\|' \\ &\geqq i, \end{aligned}$$

using (iii)°. Since $\|\xi_i\|' = 1$ this contradicts the hypothesis that $\xi \to z_0\xi$ is a bounded linear operator in $\mathfrak{X}$.

We have shown that $(a, \xi) \to a\xi$ is continuous $(\mathfrak{A}, \|\ \|) \to (\mathfrak{X}, \|\ \|')$

for each $\xi \in \mathfrak{X}$. The result of the theorem now follows since we also have that $(a, \xi) \rightarrow a\xi$ is continuous for fixed a (by hypothesis) and so by [2, p. 38, Proposition 2] $(a, \xi) \rightarrow a\xi$ is jointly continuous.

THEOREM 2. *Let $\mathfrak{A}$ be a semisimple algebra over **R** or **C**. Let $\| \;\|$, $\| \;\|'$ be norms on $\mathfrak{A}$ such that $(\mathfrak{A}, \| \;\|)$ and $(\mathfrak{A}, \| \;\|')$ are Banach algebras. Then the norms $\| \;\|$, $\| \;\|'$ define the same topology on $\mathfrak{A}$.*

PROOF. By [4, Chapter 2, §5, in particular p. 74] it is enough to prove the result for primitive $\mathfrak{A}$. Thus we are in the position of Theorem 1 with $\mathfrak{X} = \mathfrak{A}/J$ for some maximal modular left ideal J in $\mathfrak{A}$. We denote the quotient norms on $\mathfrak{X}$ obtained from $\| \;\|$ and $\| \;\|'$ on $\mathfrak{U}$ by the same symbols. Suppose $\|x_n\| \rightarrow 0$ and $\|x_n - y\|' \rightarrow 0$ $(x_n, y \in \mathfrak{A})$. Then for each $\xi \in \mathfrak{X}$ we have $\|x_n\xi - y\xi\|' \rightarrow 0$. However using Theorem 1 we see that $\|x_n\| \rightarrow 0$ implies $\|x_n\xi\|' \rightarrow 0$ so that $y\xi = 0$ for each $\xi \in \mathfrak{X}$ and, since the representation is faithful, $y = 0$. The closed graph theorem [1, p. 37] then shows that the identity map $(\mathfrak{A}, \| \;\|) \rightarrow (\mathfrak{A}, \| \;\|')$ is continuous and the result follows by arguing with $\| \;\|$ and $\| \;\|'$ interchanged.

REFERENCES

1. N. Bourbaki, *Éléments de mathématique, Espaces vectoriels topologiques*, Chapters I–II, Act. Sci. Ind. 1189, Hermann, Paris, 1953.

2. ———, *Éléments de mathématique, Espaces vectoriels topologiques*, Chapters III–V, Act. Sci. Ind. 1229, Hermann, Paris, 1955.

3. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

4. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, New York, 1960.

THE UNIVERSITY, NEWCASTLE UPON TYNE, ENGLAND.