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THE UNIT TANGENT SPHERE BUNDLE OF A COMPLEX SPACE FORM

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Dedicated to Professor K. Sekigawa on the occasion of his sixtieth birthday

ABSTRACT. In this paper, we study the unit tangent sphere bundles $T_1M(4c)$ of complex space forms $M(4c)$ with constant holomorphic sectional curvature $4c$. In particular, we determine $T_1M(4c)$ whose Ricci tensors satisfy the Einstein-like conditions.

1. Introduction

By making use of the decomposition of the covariant derivative $\nabla\rho$ of the Ricci (0,2)-tensor ρ, A. Gray([7]) introduced two interesting classes $\mathfrak{A}$ and $\mathfrak{B}$ of Riemannian manifolds which lie between the class of Ricci-parallel Riemannian manifolds and the one of Riemannian manifolds of constant scalar curvature, namely,

1. the class $\mathfrak{A}$ of Riemannian manifolds whose Ricci tensors are Codazzi tensors, i.e., $(\nabla_X\rho)(Y,Z) = (\nabla_Y\rho)(X,Z)$ for all vector fields X, Y, Z on the manifold.
2. the class $\mathfrak{B}$ of Riemannian manifolds whose Ricci tensors are Cyclic parallel (or Killing tensors), i.e., $(\nabla_X\rho)(Y,Z) + (\nabla_Y\rho)(Z,X) + (\nabla_Z\rho)(X,Y) = 0$ for all vector fields X, Y, Z on the manifold.

On the other hand, it is known that the tangent bundle TM of a Riemannian manifold M admits a natural Riemannian metric $\tilde{g}$, called the Sasaki metric (cf. [1], [2]). The unit tangent sphere bundle T_1M, considered as a hypersurface of TM, inherits a Riemannian metric g which is homothetically changed for normalization from the induced

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metric. D. E. Blair ([3]) proved that T_1M is locally symmetric if and only if M is flat manifold or is 2-dimensional and of constant curvature 1. Recently, in [4] it was proved that the unit tangent sphere bundles $T_1M(c)$ of spaces of constant curvature c has a Codazzi-type Ricci tensor if and only if $c = 0$ or $n = 2$ and $c = 1$, i.e., if and only if $T_1M(c)$ is locally symmetric. Also, they proved that $T_1M(c)$ has a cyclic parallel Ricci tensor if and only if $n = 2$ or $c \in \{0, 1\}$. In this paper, we prove the corresponding results in the case that the base manifold is a complex space form. Namely, we prove

THEOREM A. *Let $M(4c)$ be a complex space form with constant holomorphic sectional curvature $4c$. Then $T_1M(4c)$ is class of $\mathfrak{A}$ if and only if $c = 0$, in this case $T_1M(4c)$ is locally symmetric.*

THEOREM B. *Let $M(4c)$ be a complex space form with constant holomorphic sectional curvature $4c$. $T_1M(4c)$ is of class $\mathfrak{B}$ if and only if $c = 0$ or $c = 1$.*

REMARK. The standard Riemannian structure of the unit tangent sphere bundle of a complex projective space $\mathbb{C}P^n$ with the Fubini-Study metric is not Ricci-parallel, but has the cyclic parallel Ricci tensor.

2. The contact Riemannian structures of the unit tangent sphere bundle

All manifolds in the present paper are assumed to be connected and of class C^∞. The basic facts and fundamental formulae about tangent bundles are well-known (cf. [6], [8], [10]). We only briefly review notation and definitions. Let $M = (M, G)$ be an n-dimensional Riemannian manifold and let TM denote its tangent bundle with the projection $\pi : TM \to M$, $\pi(x, u) = x$. For a vector $X \in T_xM$, we denote by X^H and X^V, *the horizontal lift* and *the vertical lift*, respectively. Then we can define a Riemannian metric $\tilde{g}$, the *Sasaki metric*, on TM in a natural way. That is,

$$\tilde{g}(X^H, Y^H) = \tilde{g}(X^V, Y^V) = G(X, Y) \circ \pi, \quad \tilde{g}(X^H, Y^V) = 0$$

for all vector fields X and Y on M. Also, a natural almost complex structure tensor J of TM is defined by $JX^H = X^V$ and $JX^V = -X^H$. Then we easily see that $(TM; \tilde{g}, J)$ is an almost Hermitian manifold. We note that J is integrable if and only if (M, G) is locally flat ([6]).

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian manifold* and is denoted by $M = (M; \eta, g)$. Now we consider the unit tangent sphere bundle (T_1M, g'), which is an isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^V$. For $X \in T_xM$, we define the *tangential lift* of X to $(x, u) \in T_1M$ by

$$X^T_{(x,u)} = X^V_{(x,u)} - G(X, u)N_{(x,u)}.$$

Clearly, the tangent space $T_{(x,u)}T_1M$ is spanned by vectors of the form X^H and X^T where $X \in T_xM$. We put

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Then we find $g'(X, \phi' Y) = 2d\eta'(X, Y)$. By taking $\xi = 2\xi'$, $\eta = \frac{1}{2}\eta'$, $\phi = \phi'$, and $g = \frac{1}{4}g'$, we get the standard contact Riemannian structure (ϕ, ξ, η, g). Indeed, we easily check that these tensors satisfy (2.1). The tensors ξ and ϕ are explicitly given by

$$\begin{aligned} \xi &= 2u^H, \\ \phi X^T &= -X^H + \frac{1}{2}G(X, u)\xi, \\ \phi X^H &= X^T \end{aligned}$$

where X and Y are vector fields on M. From now we consider $T_1M = (T_1M; \eta, g)$ with the standard contact Riemannian structure. We arrange fundamental formulas without proofs, which are needed for the proofs of our Theorem. (cf. [3], [4], [5], [9]). We denote by ∇ and R, the Levi-Civita connection and the Riemannian curvature tensor associated with g, respectively. We, also, denote by D and K, the Levi-Civita connection and the Riemannian curvature tensor associated with G, respectively. Then we have

$$
\begin{aligned}
\nabla_{X^T} Y^T &= -G(Y,u)X^T,\\
\nabla_{X^T} Y^H &= \frac{1}{2}(K(u,X)Y)^H,\\
\nabla_{X^H} Y^T &= (D_X Y)^T + \frac{1}{2}(K(u,Y)X)^H,\\
\nabla_{X^H} Y^H &= (D_X Y)^H - \frac{1}{2}(K(X,Y)u)^T,
\end{aligned} \tag{2.2}
$$

and

(2.3)

$$
\begin{aligned}
R(X^T,Y^T)Z^T &= -g'(X^T,Z^T)Y^T + g'(Z^T,Y^T)X^T,\\
R(X^T,Y^T)Z^H &= \{K(X-G(X,u)u, Y-G(Y,u)u)Z\}^H\\
&\quad + \frac{1}{4}\{[K(u,X),K(u,Y)]Z\}^H\\
R(X^H,Y^T)Z^T &= -\frac{1}{2}\{K(Y-G(Y,u)u, Z-G(Z,u)u)X\}^H\\
&\quad - \frac{1}{4}\{K(u,Y)K(u,Z)X\}^H\\
R(X^H,Y^T)Z^H &= \frac{1}{2}\{K(X,Z)(Y-G(Y,u)u)\}^T\\
&\quad - \frac{1}{4}\{K(X,K(u,Y)Z)u\}^T\\
&\quad + \frac{1}{2}\{(D_X K)(u,Y)Z\}^H,\\
R(X^H,Y^H)Z^T &= \{K(X,Y)(Z-G(Z,u)u)\}^T\\
&\quad + \frac{1}{4}\{K(Y,K(u,Z)X)u - K(X,K(u,Z)Y)u\}^T\\
&\quad + \frac{1}{2}\{(D_X K)(u,Z)Y - (D_Y K)(u,Z)X\}^H,\\
R(X^H,Y^H)Z^H &= (K(X,Y)Z)^H + \frac{1}{2}\{K(u,K(X,Y)u)Z\}^H\\
&\quad - \frac{1}{4}\{K(u,K(Y,Z)u)X - K(u,K(X,Z)u)Y\}^H
\end{aligned}
$$

$$+\frac{1}{2}\{(D_Z K)(X,Y)u\}^T$$

for all vector fields X, Y and Z on M.

3. The unit tangent sphere bundle of complex space form

Let M be a complex n-dimensional Kählerian manifold with almost complex structure J and metric G. For each 2-plane p in the tangent space $T_x(M)$, the sectional curvature $H(p)$ is defined by

$$H(p) = G(K(X,Y)Y,X),$$

where $\{X,Y\}$ is an orthonormal basis for p. If p is invariant by J, then $H(p)$ is called the *holomorphic sectional curvature* determined by p. The holomorphic sectional curvature $H(p)$ is given by

$$H(p) = G(K(X,JX)JX,X),$$

where X is a unit vector in p. If $H(p)$ is a constant for all holomorphic planes p in $T_x(M)$ and for any point $x \in M$, then M is called a *space of constant holomorphic sectional curvature* or simply, a *complex space form*. (Sometimes, a complex space form is defined as a simply connected and complete one.) Then it is well-known that the curvature tensor of a complex space form is expressed in a nice form. Namely, we have (cf. [11]),

PROPOSITION 3.1. *A Kählerian manifold M is of constant holomorphic sectional curvature $4c$ (i.e., M is a complex space form of constant holomorphic sectional curvature $4c$) if and only if*

(3.1)
$$\begin{aligned} K(X,Y)Z = c\{&G(Y,Z)X - G(X,Z)Y \\ &+ G(JY,Z)JX - G(JX,Z)JY + 2G(X,JY)JZ\} \end{aligned}$$

for any vector fields X, Y and Z on M.

Let $(M(4c),G)$ be a $2n$-dimensional complex space form of constant holomorphic sectional curvature $4c$. We consider the unit tangent sphere bundle $(T_1M(4c),g)$ of a complex space form $M(4c)$. We compute the

Levi Civita connection ∇ and the Riemannian curvature tensor R of $(T_1M(4c), g)$. Namely, from (2.2), (2.3) and (3.1), we obtain

(3.2)
$$
\begin{aligned}
\nabla_{X^T}Y^T &= -G(Y,u)X^T,\\
\nabla_{X^T}Y^H &= \frac{c}{2}\Big\{\frac{1}{2}G(X,Y)\xi - G(Y,u)X^H + G(JX,Y)(Ju)^H\\
&\qquad + G(JY,u)(JX)^H + 2G(JX,u)(JY)^H\Big\},\\
\nabla_{X^H}Y^T &= (D_XY)^T + \frac{c}{2}\Big\{\frac{1}{2}G(X,Y)\xi - G(X,u)Y^H + G(X,JY)(Ju)^H\\
&\qquad + 2G(JY,u)(JX)^H + G(JX,u)(JY)^H\Big\},\\
\nabla_{X^H}Y^H &= (D_XY)^H - \frac{c}{2}\Big\{G(Y,u)X^T - G(X,u)Y^T + 2G(X,JY)(Ju)^T\\
&\qquad + G(JY,u)(JX)^T - G(JX,u)(JY)^T\Big\}
\end{aligned}
$$

and further we have

(3.3)
$$
\begin{aligned}
R(X^T,Y^T)Z^T &= (G(Y,Z) - G(Y,u)G(Z,u))X^T\\
&\quad - (G(X,Z) - G(X,u)G(Z,u))Y^T,
\end{aligned}
$$

$$
\begin{aligned}
&R(X^T,Y^T)Z^H\\
&= \Big(c - \frac{c^2}{4}\Big)\Big\{(G(Y,Z) - G(Y,u)G(Z,u))\Big(X^H - \frac{1}{2}G(X,u)\xi\Big)\\
&\quad - (G(X,Z) - G(X,u)G(Z,u))\Big(Y^H - \frac{1}{2}G(Y,u)\xi\Big)\\
&\quad + (G(JY,Z) + G(Y,u)G(JZ,u))((JX)^H - G(X,u)(Ju)^H)\\
&\quad - (G(JX,Z) + G(X,u)G(JZ,u))((JY)^H - G(Y,u)(Ju)^H)\Big\}\\
&\quad + 2c\{G(X,JY) - G(X,u)G(JY,u) + G(X,u)G(JX,u)\}(JZ)^H\\
&\quad + \frac{c^2}{4}\Big\{\frac{1}{2}(G(JX,Z)G(JY,u) - G(JY,Z)G(JX,u)\\
&\quad - 2G(JX,Y)G(JZ,u))\xi\\
&\quad - G(JY,u)G(JZ,u)X^H + G(JX,u)G(JZ,u)Y^H\\
&\quad + \big(G(Y,Z)G(JX,u) - G(X,Z)G(JY,u)\\
&\quad - 2G(JX,Y)G(Z,u)\big)(Ju)^H\\
&\quad - G(JY,u)G(Z,u)(JX)^H + G(JX,u)G(Z,u)(JY)^H\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&R(X^H, Y^T)Z^T \\
&= \Big(\frac{c^2}{4} - \frac{c}{2}\Big)(G(X,Z) - G(X,u)G(Z,u))\Big(Y^H - \frac{1}{2}G(Y,u)\xi\Big) \\
&\quad + \frac{c}{2}(G(X,Y) - G(X,u)G(Y,u))\Big(Z^H - \frac{1}{2}G(Z,u)\xi\Big) \\
&\quad + \Big(\frac{c^2}{4} - \frac{c}{2}\Big)(G(X,JZ) + G(JX,u)G(Z,u))((JY)^H - G(Y,u)(Ju)^H) \\
&\quad + \frac{c}{2}(G(X,JY) + G(JX,u)G(Y,u))((JZ)^H - G(Z,u)(Ju)^H) \\
&\quad + \frac{c^2}{8}G(X,u)(G(Y,Z) - G(Y,u)G(Z,u))\xi \\
&\quad - \frac{c^2}{4}G(JX,u)(G(Y,Z) - G(Y,u)G(Z,u))(Ju)^H \\
&\quad - c(G(Y,JZ) + G(JY,u)G(Z,u) - G(Y,u)G(JZ,u))(JX)^H \\
&\quad - \frac{c^2}{4}\Big\{\frac{1}{2}(-3G(X,JZ)G(JY,u) + G(JX,u)G(Y,JZ) \\
&\quad + 2G(JX,Y)G(JZ,u))\xi \\
&\quad - 4G(JY,u)G(JZ,u)X^H + 3G(JX,u)G(JZ,u)Y^H \\
&\quad + \big(3G(X,Z)G(JY,u) - G(JY,Z)G(X,u) \\
&\quad + 2G(X,Y)G(JZ,u)\big)(Ju)^H \\
&\quad - 3G(X,u)G(JZ,u)(JY)^H - 2G(JX,u)G(JY,u)Z^H \\
&\quad - 2G(X,u)G(JY,u)(JZ)^H\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&R(X^H, Y^T)Z^H \\
&= \Big(\frac{c}{2} - \frac{c^2}{4}\Big)(G(Y,Z) - G(Y,u)G(Z,u))X^T \\
&\quad - \frac{c^2}{4}G(X,u)G(Z,u)Y^T \\
&\quad - \frac{c}{2}(G(X,Y) - G(X,u)G(Y,u))Z^T \\
&\quad + \Big(\frac{c}{2} - \frac{c^2}{4}\Big)(G(Y,JZ) - G(Y,u)G(JZ,u))(JX)^T \\
&\quad - \frac{c^2}{4}G(JX,u)G(Z,u)(JY)^T \\
&\quad - \frac{c}{2}(G(JX,Y) - G(JX,u)G(Y,u))(JZ)^T
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3c^2}{4} G(JY,u)G(JZ,u)X^T - \frac{c^2}{4} G(JX,u)G(JZ,u)Y^T \\
&- \frac{c^2}{2} G(JX,u)G(JY,u)Z^T + \frac{3c^2}{4} G(JY,u)G(Z,u)(JX)^T \\
&+ \Big(c\, G(X,JZ) + \frac{c^2}{4} G(X,u)G(JZ,u)\Big)(JY)^T \\
&+ \frac{c^2}{2} G(X,u)G(JY,u)(JZ)^T - c\, G(X,JZ)G(Y,u)(Ju)^T \\
&+ \frac{c^2}{4}\Big\{2G(X,Y)G(JZ,u) + 4G(X,Z)G(JY,u) \\
&\qquad + 3G(Y,Z)G(JX,u) + 2G(X,JY)G(Z,u) \\
&\qquad + 3G(JY,Z)G(X,u)\Big\}(Ju)^T,
\end{aligned}
$$

$$
\begin{aligned}
&R(X^H,Y^H)Z^T \\
= &\Big(c - \frac{c^2}{4}\Big)\big\{(G(Y,Z) - G(Y,u)G(Z,u))X^T \\
&\qquad - (G(X,Z) - G(X,u)G(Z,u))Y^T\big\} \\
&+ \Big(c - \frac{c^2}{4}\Big)\big\{(G(JY,Z) - G(JY,u)G(Z,u))(JX)^T \\
&\qquad - (G(JX,Z) - G(JX,u)G(Z,u))(JY)^T\big\} \\
&+ 2c\, G(X,JY)(JZ)^T - 2c\, G(X,JY)G(Z,u)(Ju)^T \\
&+ \frac{c^2}{4}\Big\{G(JX,u)G(JZ,u)Y^T - G(JY,u)G(JZ,u)X^T \\
&\qquad - G(X,u)G(JZ,u)(JY)^T + G(Y,u)G(JZ,u)(JX)^T \\
&\qquad - (G(X,Z)G(JY,u) - G(Y,Z)G(JX,u) \\
&\qquad + G(X,JZ)G(Y,u) - G(Y,JZ)G(X,u))(Ju)^T \\
&\qquad + 2(G(X,u)G(JY,u) - G(Y,u)G(JX,u))(JZ)^T\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&R(X^H,Y^H)Z^H \\
= &\Big(c\, G(Y,Z) - \frac{3c^2}{4} G(Y,u)G(Z,u)\Big)X^H \\
&- \Big(c\, G(X,Z) - \frac{3c^2}{4} G(X,u)G(Z,u)\Big)Y^H \\
&+ \frac{3c^2}{8}\Big(G(X,Z)G(Y,u) - G(Y,Z)G(X,u)\Big)\xi
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(c\,G(JY,Z) - \frac{3c^2}{4}G(JY,u)G(Z,u)\Big)(JX)^H \\
&- \Big(c\,G(JX,Z) - \frac{3c^2}{4}G(JX,u)G(Z,u)\Big)(JY)^H \\
&+ \frac{3c^2}{4}\Big(G(Y,Z)G(JX,u) - G(X,Z)G(JY,u)\Big)(Ju)^H \\
&- \frac{3c^2}{4}G(JY,u)G(JZ,u)X^H + \frac{3c^2}{4}G(JX,u)G(JZ,u)Y^H \\
&- \Big\{\frac{c^2}{2}G(JY,u)G(Z,u) - \frac{5c^2}{4}G(Y,u)G(JZ,u) \\
&\qquad - c^2G(Y,JZ)\Big\}(JX)^H \\
&+ \Big\{\frac{c^2}{2}G(JX,u)G(Z,u) - \frac{5c^2}{4}G(X,u)G(JZ,u) \\
&\qquad - c^2G(X,JZ)\Big\}(JY)^H \\
&- \Big\{\frac{5c^2}{2}G(X,u)G(JY,u) - \frac{5c^2}{2}G(JX,u)G(Y,u) \\
&\qquad - (2c-2c^2)G(X,JY)\Big\}(JZ)^H \\
&- \Big\{\frac{5c^2}{8}G(JY,Z)G(JX,u) - \frac{5c^2}{8}G(JX,Z)G(JY,u) \\
&\qquad - \frac{5c^2}{4}G(JX,Y)G(JZ,u)\Big\}\xi \\
&- \Big\{\frac{5c^2}{4}G(JY,Z)G(X,u) - \frac{5c^2}{4}G(JX,Z)G(Y,u) \\
&\qquad - \frac{5c^2}{2}G(JX,Y)G(Z,u)\Big\}(Ju)^H.
\end{aligned}
$$

Next, we determine the Ricci tensor ρ of $(T_1M(4c), g)$ and its first covariant derivative. To calculate these tensors at the point $(x,u) \in T_1M(4c)$, let $E_1, \ldots, E_n = u, JE_1, \ldots, JE_n = Ju$ be an orthonormal basis of T_xM. Then $2E_1^T, \ldots, 2E_{n-1}^T, 2(JE_1)^T, \ldots, 2(JE_n)^T, 2E_1^H, \ldots, 2E_n^H = \xi, 2(JE_1)^H, \ldots, 2(JE_n)^H$ is an orthonormal basis for $T_{(x,u)}T_1M$. Then ρ is given by

$$
\rho(X,Y) = \sum_{i=1}^{n-1} R(2E_i^T, X, Y, 2E_i^T) + \sum_{i=1}^{n} R(2(JE_i)^T, X, Y, 2(JE_i)^T)
$$

$$+\sum_{i=1}^{n} R(2E_i^H, X, Y, 2E_i^H) + \sum_{i=1}^{n} R(2(JE_i)^H, X, Y, 2(JE_i)^H).$$

Thus by using (3.3) we see that

(3.4)

$$\begin{aligned}\rho(X^T, Y^T) &= (2n - 2 + c^2)(G(X,Y) - G(X,u)G(Y,u)) \\ &\quad + c^2(2n+5)G(JX,u)G(JY,u), \\ \rho(X^H, Y^H) &= c(2n + 2 - 3c)G(X,Y) - c^2(n+4)G(X,u)G(Y,u) \\ &\quad - c^2(n+4)G(JX,u)G(JY,u), \\ \rho(X^T, Y^H) &= 0.\end{aligned}$$

Furthermore, by using (3.2) we obtain

(3.5)

$$\begin{aligned}&(\nabla_{Z^T}\rho)(X^T, Y^T) \\ &= c^2(2n+5)\{(G(JX,Z) - G(JX,u)G(Z,u) \\ &\qquad + G(X,u)G(JZ,u))G(JY,u) \\ &\qquad + (G(JY,Z) - G(JY,u)G(Z,u) \\ &\qquad + G(Y,u)G(JZ,u))G(JX,u)\}, \\ &(\nabla_{Z^T}\rho)(X^H, Y^H) \\ &= \frac{1}{2}c^2(c-2)(n+4)\{(G(X,Z) - G(X,u)G(Z,u))G(Y,u) \\ &\qquad + (G(Y,Z) - G(Y,u)G(Z,u))G(X,u) \\ &\qquad + (G(JX,Z) - G(JX,u)G(Z,u))G(JY,u) \\ &\qquad + (G(JY,Z) - G(JY,u)G(Z,u))G(JX,u)\}, \\ &(\nabla_{Z^H}\rho)(X^T, Y^H) \\ &= \frac{c^3}{2}(n+6)((G(X,Z) - G(X,u)G(Z,u))G(Y,u) \\ &\quad - c^3(G(X,Y) - G(X,u)G(Y,u))G(Z,u) \\ &\quad + \frac{c^3}{2}(n+6)((G(X,JZ) - G(X,u)G(JZ,u))G(JY,u) \\ &\quad - c^3(G(X,JY) - G(X,u)G(JY,u))G(JZ,u) \\ &\quad + \frac{c^3}{2}(7n+22)\{G(JX,u)G(Y,u)G(JZ,u)\end{aligned}$$

$$
\begin{aligned}
&- G(JX,u)G(JY,u))G(Z,u)\} \\
&+ c\{c^2-(n+1)c+(n-1)\}\{G(X,Z)G(Y,u)-G(X,Y)G(Z,u) \\
&\quad - G(JX,Z)G(JY,u)+G(JX,Y)G(JZ,u)\} \\
&- c\{(2n+9)c^2-2(n+1)c+2(n-1)\}G(JY,Z)G(JX,u),
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{Z^T}\rho)(X^T,Y^H) &= 0,\\
(\nabla_{Z^H}\rho)(X^T,Y^T) &= 0,\\
(\nabla_{Z^H}\rho)(X^H,Y^H) &= 0.
\end{aligned}
$$

Then, taking account of (3.5), we have at once the following:

THEOREM 3.2. *$T_1M(4c)$ is Ricci-parallel* ($\nabla\rho = 0$) *if and only if M is flat* ($c = 0$).

Now, we first prove Theorem A. We put $\bar{X} = X^T + X^H, \bar{Y} = Y^T + Y^H$ and $\bar{Z} = Z^T + Z^H$. Then, together with (3.5), a long but straightforward calculation gives

(3.6)

$$
\begin{aligned}
&(\nabla_{\bar{Z}}\rho)(\bar{X},\bar{Y}) - (\nabla_{\bar{X}}\rho)(\bar{Z},\bar{Y}) \\
&= (\nabla_{Z^T+Z^H}\rho)(X^T+X^H,Y^T+Y^H) \\
&\quad - (\nabla_{X^T+X^H}\rho)(Z^T+Z^H,Y^T+Y^H) \\
&= (\nabla_{Z^T}\rho)(X^T,Y^T) + (\nabla_{Z^T}\rho)(X^T,Y^H) + (\nabla_{Z^T}\rho)(X^H,Y^T) \\
&\quad + (\nabla_{Z^T}\rho)(X^H,Y^H) + (\nabla_{Z^H}\rho)(X^T,Y^T) + (\nabla_{Z^H}\rho)(X^T,Y^H) \\
&\quad + (\nabla_{Z^H}\rho)(X^H,Y^T) + (\nabla_{Z^H}\rho)(X^H,Y^H) - (\nabla_{X^T}\rho)(Z^T,Y^T) \\
&\quad - (\nabla_{X^T}\rho)(Z^T,Y^H) - (\nabla_{X^T}\rho)(Z^H,Y^T) - (\nabla_{X^T}\rho)(Z^H,Y^H) \\
&\quad - (\nabla_{X^H}\rho)(Z^T,Y^T) - (\nabla_{X^H}\rho)(Z^T,Y^H) - (\nabla_{X^H}\rho)(Z^H,Y^T) \\
&\quad - (\nabla_{X^H}\rho)(Z^H,Y^H) \\
&= c^2(2n+5)\{2(G(JX,Z)-G(JX,u)G(Z,u) \\
&\quad + G(X,u)G(JZ,u))G(JY,u) + (G(JY,Z)-G(JY,u)G(Z,u) \\
&\quad + G(Y,u)G(JZ,u))G(JX,u) - (G(X,JY)-G(X,u)G(JY,u) \\
&\quad + G(JX,u)G(Y,u))G(JZ,u)\} \\
&\quad + \frac{1}{2}c^2(c-2)(n+4)\{(G(X,Z)-G(X,u)G(Z,u))G(Y,u) \\
&\quad + (G(Y,Z)-G(Y,u)G(Z,u))G(X,u)
\end{aligned}
$$

$$+ (G(JX, Z) - G(JX, u)G(Z, u))G(JY, u)$$
$$+ (G(JY, Z) - G(JY, u)G(Z, u))G(JX, u)\}.$$

Suppose that $T_1M(4c)$ have Codazzi-type Ricci tensors. Then from (3.6) we have

$$c^2(2n+5) = 0,$$
$$\frac{1}{2}c^2(c-2)(n+4) = 0.$$

Hence, we see that $c = 0$. Conversely, we easily see that $T_1M(0)$ satisfies (3.6). Thus we have proved Theorem A.

Before we prove Theorem B, we remark in general that the polarization and the symmetry of $\nabla\rho$ yields that the cyclic parallel Ricci tensor condition is equivalent to $(\nabla_Z\rho)(Z, Z) = 0$ for all vector field Z. We now prove Theorem B. By using (3.5) and $\bar{X} = X^T + Y^H$, we obtain

(3.7)
$$\begin{aligned}
&(\nabla_{X^T+Y^H}\rho)(X^T + Y^H, X^T + Y^H) \\
&= (\nabla_{X^T}\rho)(X^T, X^T) + (\nabla_{X^T}\rho)(X^T, Y^H) + (\nabla_{X^T}\rho)(Y^H, X^T) \\
&\quad + (\nabla_{X^T}\rho)(Y^H, Y^H) + (\nabla_{Y^H}\rho)(X^T, X^T) + (\nabla_{Y^H}\rho)(X^T, Y^H) \\
&\quad + (\nabla_{Y^H}\rho)(Y^H, X^T) + (\nabla_{Y^H}\rho)(Y^H, Y^H) \\
&= 2c^2(c-1)(n+4)\{(G(X, Y) - G(X, u)G(Y, u))G(Y, u) \\
&\quad + (G(X, JY) - G(X, u)G(JY, u))G(JY, u)\}.
\end{aligned}$$

From (3.7), we see that it is necessary and sufficient condition for $T_1M(4c)$ to have cyclic parallel Ricci tensors that $c = 0$ or $c = 1$.

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