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The Universal Connection of an Arbitrary System (*)(**).

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Abstract. – *Using the theory of smooth spaces, we generalize the notion of finite dimensional system of connections on a fiber bundle to the concept of arbitrary system of connections. Then we study the universal connection of a regular system and the universal curvature.*

Introduction.

The concept of a finite dimensional system of connections on a fiber bundle $E \to B$ was introduced by M. MODUGNO [8]. He also defined the universal connection of such a system and studied the universal curvature, which is related with an earlier idea by P. L. GARCIA [5]. We point out that the concept of smooth space by A. FRÖLICHER [4], enables us to study arbitrary (i.e. infinite dimensional as a rule) systems of connections on E. That is why we start with a review of the Frölicher's theory. However, since we do not need some technical complicated parts of the whole theory, we just present a simplified version of the notion of smooth space, which is sufficient for our purposes. Next we describe the basic properties of smooth bundles, which represent the most frequent type of functional spaces appearing in differential geometry. Then we study the system $C \to B$ of all connections on E and its tangent bundle in the sense of [3]. A regular system of connections on E is a subbundle $D \subset C$ such that the tangent space TD behaves well. We study the universal connection Λ_D of D in the form of lifting map, because it seems to be reasonable to avoid jets in the case of infinite dimensional base space. In order to define the curvature of Λ_D, we develop a new approach to the curvature of a classical connection on E. Then we prove that the universal curvature of D has all basic properties of the universal curvature of a finite dimensional system. In

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conclusion we discuss one of the most interesting infinite dimensional systems of connections, the system of all polynomial connections on an affine bundle.

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1. - Smooth spaces.

We present a simplified version of a theory by A. FRÖLICHER [4]. Our approach is based on the concept of a smooth curve only, for this is sufficient for our purposes. The relations of our approach to the Frölicher's theory are explained in Remark 1.2 below. By $\mathcal{M}\mathrm{ap}\,(A_1, A_2)$ we denote the set of all maps between two sets A_1 and A_2. If M and N are two classical manifolds, then $C^\infty(M, N) \subset \mathcal{M}\mathrm{ap}\,(M, N)$ means the set of all maps of class C^∞

DEFINITION 1.1. - A smooth space is a set S together with a subset $C_S \subset \mathcal{M}\mathrm{ap}\,(\mathbb{R}, S)$ satisfying the following two conditions:

(i) each constant map $\mathbb{R} \to S$ belongs to C_S,

(ii) if $\gamma \in C_S$ and $\delta \in C^\infty(\mathbb{R}, \mathbb{R})$, then $\gamma \circ \delta \in C_S$.

Each element of C_S is called a *smooth curve* on S.

A trivial example of a smooth space is a classical manifold M of class C^∞ with $C_M = C^\infty(\mathbb{R}, M)$.

Let (S_1, C_{S_1}) and (S_2, C_{S_2}) be two smooth spaces.

DEFINITION 1.2. - A map f: $S_1 \to S_2$ is said to be smooth, if $f \circ \gamma$ is a smooth curve on S_2 for every smooth curve γ on S_1.

Thus, we have defined the category $\mathbf{S}$ of smooth spaces and smooth maps. Clearly, the meaning of the above assumption (ii) is that each smooth curve γ on S is a smooth map $\gamma: \mathbb{R} \to S$. The smooth maps between two classical manifolds M and N coincide with the C^∞-maps by virtue of the following deep analytical result due to BOMAN [1].

PROPOSITION 1.1. - *Let f: $M \to N$ be a map such that $f \circ \gamma \in C^\infty(\mathbb{R}, N)$ for each $\gamma \in C^\infty(\mathbb{R}, M)$. Then f is a map of class C^∞.*

In other words, $C^\infty \subset \mathbf{S}$ is a full subcategory.

A specific feature of the category $\mathbf{S}$ is that each subset $Q \subset S$ of an $\mathbf{S}$-object (S, C_S) is an $\mathbf{S}$-object, provided we define C_Q as the subset of all $\gamma \in C_S$ satisfying $\gamma(\mathbb{R}) \subset Q$. The smooth structure on the product $S_1 \times S_2$ of two smooth spaces is defined by requiring that a smooth curve on $S_1 \times S_2$ is a pair (γ_1, γ_2), $\gamma_1 \in C_{S_1}$, $\gamma_2 \in C_{S_2}$ (and analogously for a product of arbitrary many smooth spaces).

The following definition points out the most important difference between the categories C^∞ and $\mathbf{S}$, which consists in the fact that the set $\mathbf{S}(S_1, S_2)$ of all smooth maps between two smooth spaces S_1, S_2 is a smooth space as well.

DEFINITION 1.3. – A curve $\gamma: \mathbb{R} \to \mathcal{S}(S_1, S_2)$ is said to be smooth, if the associated map $\tilde{\gamma}: \mathbb{R} \times S_1 \to S_2$, $\tilde{\gamma}(t, x) = \gamma(t)(x)$, is smooth.

One verifies easily that both requirements from Definition 1.1 are fulfilled. More generally, one deduces easily the following assertion. Let S_3 be a third smooth space.

PROPOSITION 1.2. – *A map* $f: S_3 \to \mathcal{S}(S_1, S_2)$ *is smooth, iff the associated map* $\tilde{f}: S_3 \times S_1 \to S_2$, $\tilde{f}(z, x) = f(z)(x)$, *is smooth.*

REMARK 1.1. – In the sequel, we shall need the smooth structure on different spaces of C^∞-maps. But we find it interesting to present another example of an infinite dimensional smooth space, which is closely related with the classical differential geometry. The set $J^\infty(M, N)$ of jets of order ∞ between two manifolds M, N is the projective limit of the infinite sequence

$$J^1(M, N) \leftarrow J^2(M, N) \leftarrow \ldots \leftarrow J^r(M, N) \leftarrow \ldots$$

with respect to the jet projections $\pi^r_{r-1}: J^r(M, N) \to J^{r-1}(M, N)$. Hence a curve $\gamma: \mathbb{R} \to J^\infty(M, N)$ is a sequence of curves $\gamma_r: \mathbb{R} \to J^r(M, N)$ satisfying $\pi^r_{r-1} \circ \gamma_r = \gamma_{r-1}$. One can define a smooth curve γ by requiring that each γ_r belongs to $C^\infty(\mathbb{R}, J^r(M, N))$. Then $J^\infty(M, N)$ becomes a smooth space.

In some cases the smooth curves on a set S can be defined by using certain real valued functions on S.

DEFINITION 1.4. – Let $F \subset \mathcal{M}\mathrm{ap}(S, \mathbb{R})$ be a non-empty subset. Then we define $C(F) \subset \mathcal{M}\mathrm{ap}(\mathbb{R}, S)$ to be the set of all $\gamma: \mathbb{R} \to S$ satisfying $\phi \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\phi \in F$.

One verifies easily that $C(F)$ endows S with the structure of a smooth space, which is said to be generated by F. For example, the C^∞-curves on $\mathbb{R}^m$ are generated by linear functions on $\mathbb{R}^m$.

If we have a smooth space (S, C_S), then the smooth functions on S are defined as the elements of $\mathcal{S}(S, \mathbb{R})$. By definition, C_S is a subset of $C(\mathcal{S}(S, \mathbb{R}))$.

DEFINITION 1.5. – A smooth space (S, C_S) is said to be closed with respect to smooth functions, if $C_S = C(\mathcal{S}(S, \mathbb{R}))$.

In other words, a curve $\gamma: \mathbb{R} \to S$ is smooth, iff its composition with every smooth function belongs to $C^\infty(\mathbb{R}, \mathbb{R})$.

Consider two smooth spaces (S_1, C_{S_1}) and (S_2, C_{S_2}).

PROPOSITION 1.3. – *If* (S_2, C_{S_2}) *is closed with respect to smooth functions, then a map* $f: S_1 \to S_2$ *is smooth, iff* $\phi \circ f \in \mathcal{S}(S_1, \mathbb{R})$ *for all* $\phi \in \mathcal{S}(S_2, \mathbb{R})$.

PROOF. – Since $\mathcal{S}$ is a category, $f \in \mathcal{S}(S_1, S_2)$ and $\phi \in \mathcal{S}(S_2, \mathbb{R})$ imply $\phi \circ f \in \mathcal{S}(S_1, \mathbb{R})$. Conversely, let $f: S_1 \to S_2$ be a map such that $\phi \circ f \in \mathcal{S}(S_1, \mathbb{R})$ for all $\phi \in \mathcal{S}(S_2, \mathbb{R})$. For every $\gamma \in C_{S_1}$ we have $\phi \circ (f \circ \gamma) = (\phi \circ f) \circ \gamma$, so that the composition of $f \circ \gamma$ with all smooth functions on S_2 is smooth. Since S_2 is closed with respect to smooth functions, $f \circ \gamma$ belongs to C_{S_2}. Hence f is smooth.

Thus, if we consider the smooth spaces closed with respect to smooth functions, then the smooth maps can be characterized in terms of smooth functions as well.

REMARK 1.2. – Our concept of a smooth spaces closed with respect to smooth functions is equivalent to the definition of smooth space by A. FRÖLICHER [4]. His approach has several advantages of general character. However, the characterization of smooth functions in terms of smooth curves is usually a complicated problem, and we do not need it. That is why we decided to avoid smooth functions in this paper.

We conclude this section by introducing the concept of finite dimensional submanifold of a smooth space S.

DEFINITION 1.6. – A finite dimensional submanifold of S is a subset $W \subset S$ together with a classical manifold M and a bijection $i\colon M \to W$ such that the classical smooth curves on M correspond to the smooth curves on W.

Clearly, the pair (M, i) is determinated up to a C^∞-isomorphism. Indeed, if $\bar{i}\colon \overline{M} \to W$ is another bijection with the property from Definition 1.6, then $\bar{i}^{-1} \circ i\colon M \to \overline{M}$ is a bijection preserving the smooth curves. Hence $\bar{i}^{-1} \circ i$ is a C^∞-isomorphism by the Boman theorem.

2. – Smooth bundles.

Let $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ be two classical fiber bundles with standard fibers Q_1 and Q_2. The set $\mathcal{F}(E_1, E_2) = \bigcup_{x \in B} C^\infty(E_{1x}, E_{2x})$ of all C^∞-maps from a fiber of E_1 into the fiber of E_2 over the same base point is endowed with a canonical projection $p\colon \mathcal{F}(E_1, E_2) \to B$. The structure of a smooth space on $\mathcal{F}(E_1, E_2)$ can be introduced by a simple modification of Definition 1.3. In general, if M is a manifold and $f\colon M \to \mathcal{F}(E_1, E_2)$ is a map such that $p \circ f\colon M \to B$ is of class C^∞, we can construct the induced bundle $(p \circ f)^* E_1$ and the associated map $\tilde{f}\colon (p \circ f)^* E_1 \to E_2$, $\tilde{f}(x, y) = f(x)(y)$, $(x, y) \in (p \circ f)^* E_1$, [3].

DEFINITION 2.1. – A curve $\gamma\colon \mathbb{R} \to \mathcal{F}(E_1, E_2)$ is said to be smooth, if $p \circ \gamma\colon \mathbb{R} \to B$ and $\tilde{\gamma}\colon (p \circ f)^* E_1 \to E_2$ are C^∞-maps.

The following assertion is direct consequence of Proposition 1.1.

PROPOSITION 2.1. – *A map $f\colon M \to \mathcal{F}(E_1, E_2)$ is smooth, iff $p \circ \gamma$ and $\tilde{f}$ are C^∞-maps.*

The smooth space $p\colon \mathcal{F}(E_1, E_2) \to B$ is the simplest example of a functional space derived from classical fiber bundles. In general, let S and Q be smooth spaces, let B a classical manifold and $p\colon S \to B$ be a surjective smooth map. The following definition modifies the standard requirement of local triviality to such a situation.

DEFINITION 2.2. – We say that S is a smooth bundle with standard fiber Q, if for every $x \in B$ there exists a neighbourhood U and an S-isomorphism ϕ: $p^{-1}(U) \to U \times Q$ satisfying $pr_1 \circ \phi = p | p^{-1}(U)$.

Clearly, p: $\mathcal{F}(E_1, E_2) \to B$ is a smooth bundle with standard fiber $C^\infty(Q_1, Q_2)$. If $\bar{p}$: $\bar{S} \to \bar{B}$ is another smooth bundle, a morphism $S \to \bar{S}$ is a pair of smooth maps f: $S \to \bar{S}, f_0$: $B \to \bar{B}$ satisfying $\bar{p} \circ f = f_0 \circ p$. We denote by $\mathcal{FS}$ the category of smooth bundles and their morphisms.

A subbundle of a smooth bundle p: $S \to B$ is a subset $D \subset S$ such that $p | D$: $D \to B$ is a smooth bundle.

In particular, a finite dimensional subbundle of S is a finite dimensional submanifold $W \subset S$, which is a classical fiber bundle over B.

3. – Systems of connections.

In the theory of finite dimensional systems of connections by MODUGNO [8], a connection on a fiber bundle p: $E \to B$ is interpreted as section Γ: $E \to J^1 E$ of the first jet prolongation of E. However, for our theory of arbitrary systems of connections we find it more suitable to study a connection in the form of its lifting map Γ: $E \times_B TB \to TE$, which is linear in the second factor and satisfies $(\pi_E, Tp) \circ \Gamma = id_{E \times_B TB}$, provided π_E: $TE \to E$ is the bundle projection of the tangent bundle. Under such an approach, an element of connection on E over $x \in B$ is a section c: $E_x \times T_x B \to (TE)_x$ linear in the second factor.

DEFINITION 3.1. – The set q: $C \to B$ of all elements of connection on E will be called *the system of all connections* on E.

The inclusion $C \subset \mathcal{F}(E \times_B TB \to B, TE \to B)$ defines the structure of a smooth space on C. One sees easily that $C \to B$ is a smooth bundle, the standard fiber of which is a subset $H \subset C^\infty(Q \times \mathbb{R}^m, TQ)$, $m = \dim B$, Q = the standard fiber of E. An element $c \in H$ is characterized by $\pi_Q \circ c = pr_1$ and by linearity in the second factor. Thus, if x^i, y^p are some local fiber coordinates on E and X^i, Y^p are the induced coordinates on TE, then the coordinate form of an element $c \in C$ is

$$y^p = y^p, \; X^i = X^i, \; Y^p = F_i^p(y) X^i \tag{1}$$

Every smooth section γ: $B \to C$ defines a connection Γ: $E \times_B TB \to TE$ by $\Gamma(y, X) = \gamma(p(y))(y, X)$. Write e: $C \times_B E \times_B TB \to TE$ for evaluation map

$$e(c, y, X) = c(y, X). \tag{2}$$

A finite dimensional subbundle Z of C is, in fact, a finite dimensional system of connections in the sense of MODUGNO. If x^i, z^a are some local fiber coordinates on Z with the same coordinates x^i on B, then the coordinate expression of the evaluation map e_Z of the system Z is

$$Y^p = F_i^p(x, y, z) X^i$$

Every C^∞-section $\xi\colon B \to Z$ defines a connection on E of the form

$$Y^p = F^p_i(x, y, \xi(x))X^i$$

In [3] we have defined the general concept of the tangent bundle of $\mathcal{F}(E_1, E_2)$ with two projections $\pi\colon T\mathcal{F}(E_1, E_2) \to \mathcal{F}(E_1, E_2)$ and $Tp\colon T\mathcal{F}(E_1, E_2) \to TB$. For every $X \in TB$ over $x \in B$, $T_X E_1 = (Tp_1)^{-1}(X)$ or $T_X E_2 = (Tp_2)^{-1}(X)$ is an affine bundle with derived vector bundle $T(E_{1x})$ or $T(E_{2x})$, respectively. Each vector $A = \partial/\partial t|_0 \gamma(t)$ tangent to a smooth curve $\gamma\colon \mathbb{R} \to \mathcal{F}(E_1, E_2)$ over $Tp(A) = \partial/\partial t|_0(p \circ \gamma) = X$ can be interpreted as an affine bundle morphism $\tilde{A}\colon T_X E_1 \to T_X E_2$ over $\gamma(0) = \pi(A)\colon E_{1x} \to E_{2x}$, the derived linear morphism of which is $T(\gamma(0))\colon T(E_{1x}) \to T(E_{2x})$.

We are going to describe the subset $TC \subset T\mathcal{F}(E \times_B TB \to B, TE \to B)$ of all vectors tangent to the curves lying in C. We shall characterize the additional structure of C in three steps. First of all, consider two fiber bundles $q_1\colon G_1 \to E$ and $q_2\colon G_2 \to E$, so that the total projections are $p \circ q_1\colon G_1 \to B$ and $p \circ q_2\colon G_2 \to B$. Denote by $\mathcal{F}^E(G_1, G_2) \subset \mathcal{F}(G_1 \to B, G_2 \to B)$ the set of all C^∞-maps $\phi\colon G_{1x} \to G_{2x}$ over the identity of E_x. Then one sees directly that an element $A \in T\mathcal{F}^E(G_1, G_2)$ over $X \in TB$ is characterized by the fact that $\tilde{A}\colon T_X G_1 \to T_X G_2$ is projectable over the identity of $T_X E$.

Assume further that G_1 and G_2 are vector bundles over E and denote by $\mathcal{L}\mathcal{F}^E(G_1, G_2) \subset \mathcal{F}^E(G_1, G_2)$ the set of all linear morphisms $G_{1x} \to G_{2x}$ over the identity of E_x. Let x^i, y^p be local fiber coordinates on E and z^a or w^α be some additional linear coordinates on G_1 or G_2, respectively. A curve γ on $\mathcal{L}\mathcal{F}^E(G_1, G_2)$ is of the form $x^i(t)$, $y^p = y^p$, $w^\alpha = f^\alpha_a(y, t)z^a$. Hence the associated map $\tilde{A}$ of $A = \partial/\partial t|_0 \gamma$ is

$$\dot{y}^p = \dot{y}^p, \qquad \dot{w}^\alpha = \frac{\partial \phi^\alpha_a}{\partial y^p}\dot{y}^p z^a + \phi^\alpha_a(y)\,\dot{z}^a + \Phi^\alpha_a(y)\,z^a \tag{3}$$

with $\phi^\alpha_a(y) = f^\alpha_a(y, 0)$ and $\Phi^\alpha_a(y) = \partial f^\alpha_a(y, 0)/\partial t$, provided the dot denotes the induced tangent coordinates. The map $\phi = \gamma(0)$ is of the form

$$y^p = y^p,\ w^\alpha = \phi^\alpha_a(y)\,z^a \tag{4}$$

so that the coordinate expression of $T\phi$ is

$$\dot{y}^p = \dot{y}^p,\ \dot{w}^\alpha = \frac{\partial \phi^\alpha_a}{\partial y^p}\dot{y}^p z^a + \phi^\alpha_a(z)\,\dot{z}^a \tag{5}$$

We know that $\tilde{A}$ is over $id_{T_X E}$, affine and the derived linear map is $T\phi$. In coordinates, this means

$$\dot{y}^p = \dot{y}^p,\ \dot{w}^\alpha = \frac{\partial \phi^\alpha_a}{\partial y^p}\dot{y}^p z^a + \phi^\alpha_a(y)\,\dot{z}^a + \Phi^\alpha(y, z) \tag{6}$$

For every $Y \in T_X E$, $T_Y G_1 = (Tq_1)^{-1}$ and $T_Y G_2 = (Tq_2)^{-1}$ are vector bundles and $\tilde{A}$ induces a restricted map $\tilde{A}_Y\colon T_Y G_1 \to T_Y G_2$. If we require that each $\tilde{A}_Y$ is linear, we obtain $\Phi^\alpha(y, z) = \Phi^\alpha_a(y)\,z^a$. This implies (3). Hence the linearity of each $\tilde{A}_Y$ characterizes $T\mathcal{L}\mathcal{F}^E(G_1, G_2)$.

Assume finally we have a surjective linear morphism $\mu\colon G_2 \to G_1$ over id_E. Then we define

$$\mathcal{LF}^E_\mu(G_1, G_2) \subset \mathcal{LF}^E(G_1, G_2)$$

to be the subset of all ϕ satisfying $\mu \circ \phi = id$. We have $\dim G_2 \geqslant \dim G_1$ and we can choose the coordinates in such a way that $w^a = (z^a, v^\lambda)$. Then we verify easily that the elements of $T\mathcal{LF}^E_\mu(G_1, G_2)$ are characterized by $\phi^a_b(y) = \delta^a_b = \Phi^a_b(y)$.

In the case of the system C of all connections on E, we have

$$C = \mathcal{LF}^E_{(\pi_E,\, Tp)}(E \times_B TB,\, TE)\,.$$

Hence the above results imply that the coordinate form of $A \in TC$ over $c \in C$ is (1) and

$$\dot y^p = \dot y^p,\ \dot X^i = \dot X^i,\ \dot Y^p = \frac{\partial F^p_i}{\partial y^q}\,\dot y^q X^i + \Phi^p_i(y) X^i + F^p_i(y)\,\dot X^i \tag{7}$$

Now we define the main subject of the paper.

DEFINITION 3.2. – *A system of connections* on E is a smooth subbundle $D \subset C$. We say that D is regular, if TD is a vector subbundle of TC.

If $i\colon D \to C$ is the inclusion of a regular system, we denote by $Ti\colon TD \to TC$ the inclusion of the tangent bundle.

4. – The universal connection.

The concept of universal connection of a finite dimensional system of connections $Z \to B$ was introduced by MODUGNO [8]. This is a connections Λ_Z on $Z \times_B E \to Z$, so that its lifting map is of the form $(Z \times_B E) \times_Z TZ \to T(Z \times_B E)$. But $(Z \times_B E) \times_Z TZ \to E \times_B TZ$, so that we can also write $\Lambda_Z = E \times_B TZ \to TE \times_{TB} TZ$.

First of all, we generalize this concept to the system $q\colon C \to B$ of all connection on B. We have $Tq\colon TC \to TB$ and $\pi_C\colon TC \to C$.

DEFINITION 4.1. – The map

$$\Lambda\colon E \times_B TC \to TE \times_{TB} TC, \qquad \Lambda(y, A) = (e(\pi_C(A), y, Tq(A)), A)$$

is called *the universal connection of the system of all connections* on E.

We also say that Λ is the universal connection of E.

The map $(C \times_B E) \times_C TC \to T(E \times_B C)$, $((\pi_C(A), y), A)(e(\pi_C(A), y, Tq(A)), A)$ can be interpreted as a lifting map, i.e. we have a connection on $(E \times_B C) \to C$.

DEFINITION 4.2. – *The universal connection of a regular system* $q\colon D \to B$ is a map $\Lambda_D\colon E \times_B TD \to TE \times_{TB} TD$ defined by

$$\Lambda_D(y, A) = (e(\pi_D(A), y, Tq(A)), A) \tag{8}$$

In other words, the universal connection of D is the restriction of the universal connection Λ of E to D. In the case of a finite dimensional system $Z \to B$ we obtain the universal connection Λ_Z by MODUGNO.

5. - Another approach to the classical curvature.

The curvature of a classical connection Γ: $E \times_B TB \to TE$ is an antisymmetric map $C\Gamma$: $E \times_B TB \times_B TB \to VE$, where VE is the vertical tangent bundle of E. We present an original construction of $C\Gamma$ in a way, which can be generalized to regular systems of connections. For every manifolds M, write $\pi_M^1 = \pi_{TM}$: $TTM \to TM$, $\pi_M^2 = T\pi_M$: $TTM \to TM$ and denote by $\Delta M \subset TTM \times_M TTM$ the set of all pairs (ξ, η) satisfying

$$\pi_M^1 \xi = \pi_M^2 \eta \quad \text{and} \quad \pi_M^2 \xi = \pi_M^1 \eta \tag{9}$$

Let κ_M be the canonical involution of TTM. The strong difference $\xi \div \eta$ of $(\xi, \eta) \in \Delta M$ is, in fact, the difference $\xi - \kappa_M \eta$ identified with an element of TM, [6]. Hence $\div$ is a map $\Delta M \to TM$.

Taking into account the tangent map $T\Gamma$: $TE \times_{TB} TTB \to TTE$, we construct

$$\bar{\Gamma}\colon E \times_B TTB \to TTE \quad \text{by} \quad \bar{\Gamma}(y, \xi) = T\Gamma(\Gamma(y, \pi_B^2 \xi), \xi)\,.$$

For every $(y, X_1, X_2) \in E \times_B TB \times_B TB$, consider any $\xi \in TTB$ satisfying $X_1 = \pi_B^1 \xi$ and $X_2 = \pi_B^2 \xi$. Then one verifies easily that $\bar{\Gamma}(y, \xi)$ and $\bar{\Gamma}(y, \kappa_B \xi)$ satisfy the condition (9).

PROPOSITION 5.1. - *The strong difference*

$$\bar{\Gamma}(y, \xi) \div \bar{\Gamma}(y, \kappa_B \xi) \tag{10}$$

does not depend on the choice of ξ over X_1 and X_2. The value of the induced map at (y, X_1, X_2) coincides with the curvature $C\Gamma(y, X_1, X_2)$.

PROOF. - This can be proved by direct evaluation.

6. - The universal curvature.

Analogously to [3], the tangent map Te of the evaluation map e: $C \times_B E \times_B TB \to TE$ should be a map Te: $TC \times_{TB} TE \times_{TB} TTB \to TTE$ defined by

$$Te(A, Y, \xi) = \frac{\partial}{\partial t}\bigg|_0 e(c(t), y(t), X(t)) \tag{11}$$

for $A = \partial/\partial t|_0 c(t)$, $Y = \partial/\partial t|_0 y(t)$ and $\xi = \partial/\partial t|_0 X(t)$, where $c(t)$, $y(t)$ and $X(t)$ are over the same curve on B. But we must prove that Te is well defined, i.e. (11) not depend on the generating curves. This is a consequence of the following proposition, which also gives the geometric interpretation of Te. We recall that $A \in TC$ over $X \in TB$ is characterized by the associated map $\tilde{A}$: $T_X(E\times_B TB)\to T_X TE$. Clearly, we have $T_X(E\times_B TB)=T_X E\times T_X TB$.

PROPOSITION 6.1. – *It holds*

$$Te(A, Y, \xi)=\tilde{A}(Y, \xi) \tag{12}$$

PROOF. – Let $x^i(t)$ be the coordinate expression of the underlying curve on B and $\phi^p_i(y, t)$, $y^p(t)$ and $X^i(t)$ be the additional coordinate expressions of $c(t)$, $y(t)$ and $X(t)$. Then the coordinate form of $e(c(t), y(t), X(t))$ is

$$Y^p=\phi^p_i(y(t), t)X^i(t)$$

By differentiating, we obtain

$$\dot{Y}^p=\frac{\partial F^p_i(y)}{\partial y^q}\,\dot{y}^q X^i+\Phi^p_i(y)\,X^i+F^p_i(y)\,\dot{X}^i \tag{13}$$

with $F^p_i(y)=\phi^p_i(y, 0)$ and $\Phi^p_i(y)=\partial\phi^p_i(y, 0)/\partial t$ By (7), this is the coordinate expression of $\tilde{A}(y, \xi)$.

We introduce the curvature $C\Lambda$ of the universal connection of E as a map $C\Lambda$: $E\times\times_B(TC\times_C TC)\to VE$ similarly to Section 5. For every $(Y, A_1, A_2)\in E\times_B(TC\times_C TC)$, consider any $\xi\in TTB$ satisfying $Tq(A_1)=\pi^1_B\xi$ and $Tq(A_2)=\pi^2_B\xi$. Write $c=\pi_C A_1==\pi_C A_2$. Using the coordinate expressions, we deduce that $\tilde{A}_1(e(c, y, Tq(A_1)), \xi)$, $\tilde{A}_2(e(c, y, Tq(A_2)), \kappa_B\xi)\in TTE$ satisfy the condition for the existence of strong difference (9).

PROPOSITION 6.2. – *The strong difference*

$$\tilde{A}_1(e(c, y, Tq(A_1)), \xi)\div\tilde{A}_2(e(c, y, Tq(A_2)), \kappa_B\xi) \tag{14}$$

does not depend on the choice of ξ *and belongs to* $VE\subset TE$.

PROOF. – Let x^i, A^i, $F^p_i(y)$, $\Phi^p_i(y)$ and x^i, B^i, $F^p_i(y)$, $\psi^p_i(y)$ be the coordinate expressions of A_1 and A_2 and let x^i, A^i, B^i, ξ^i the coordinates of ξ. By (13) we find the following coordinate form of (14)

$$\frac{\partial F^p_i(y)}{\partial y^q}F^q_j(y)(A^iB^j-A^jB^i)+\Phi^p_i(y)B^i-\psi^p_i(y)\,A^i \tag{15}$$

together with the zero vector on the base. This proves our assertion.

DEFINITION 6.1. – The map $C\Lambda$: $E\times_B(TC\times_C TC)\to VE$ defined by (14) is called the curvature of the universal connection of E.

We also say that $C\Lambda$ is *the universal curvature* of E.

We remark that (15) can be interpreted as the coordinate expression of the universal curvature of E.

Consider a regular system of connections $i\colon D \to C$. Then we define its universal curvature $C\Lambda_D\colon E \times_B (TD \times_D TD) \to VE$ by the same formula (14). This implies directly

PROPOSITION 6.3. – *It holds*

$$C\Lambda_D = C\Lambda \circ (id_E \times_B (Ti \times_D Ti)) .$$

The universal connection Λ_Z of a finite dimensional system $i\colon Z \to C$, which is a classical fibered manifold $Z \to B$, is a classical connection on $Z \times_B E \to Z$. Hence we can apply the classical definition of curvature to Λ_Z. The following result can be deduced by direct evaluation.

PROPOSITION 6.4. – *The classical curvature of* Λ_Z *coincides with* $\Lambda \circ (id_E \times_B (Ti \times \times_Z Ti))$.

In particular, a connection Γ on E represents a trivial system given by a smooth section $B \to C$. Its universal connection on $E \to B$ coincides with Γ itself. In this case, Proposition 6.4 yields.

COROLLARY 6.1. – *The curvature of every connection on* E *is induced from the universal curvature of* E.

7. – The system of polynomial connections.

An interesting example of an infinite dimensional system of connections are the polynomial connections on an affine bundle.

Assume $p\colon E \to B$ is a classical affine bundle. Then $J^1E \to B$ is an affine bundle as well. The following definition is due to K. MARATHE and M. MODUGNO [7].

DEFINITION 7.1. – A connection $\Gamma\colon E \to J^1E$ is called polynomial, if each restriction $\Gamma_x\colon E_x \to J^1_xE$ is a polynomial map.

Even $Tp\colon TE \to TB$ is an affine bundle. The lifting form of Γ_x is a map

$$E_x \times T_xB \to (TE)_x \tag{16}$$

Clearly, Γ_x is polynomial, iff (16) is a polynomial map for each $X \in T_xB$. An element of connection (16) with such a property will be called a polynomial element of connection on the affine bundle $E \to B$. Hence the coordinate form of (16) is

$$Y^p = F^p_i(y)\, X^i \tag{17}$$

where F^p_i are same polynomials on E_x.

Let $P \to B$ denote the smooth bundle of all polynomial elements of connection on E. One deduces directly that the elements of TP are of the form (17) and

$$\dot{Y}^p = \frac{\partial F^p_i(y)}{\partial y^q}\, \dot{y}^q X^i + \Phi^p_i(y)\, X^i + F^p_i(y)\, \dot{X}^i \tag{18}$$

where Φ_i^p are another polynomials on E_x. This implies that P is a regular system of connections in the sense of Definition 3.2.

By Definition 4.2, the universal connection Λ_P is expressed by (8) with polynomial A. By Proposition 6.3, the universal curvature $C\Lambda_P$ is given by (15) with polynomial F_i^p, Φ_i^p and Ψ_i^p.

REFERENCES

[1] J. BOMAN, *Differentiability of a function and of its composition with a function of one variable*, Math. Scand., **20** (1967), pp. 249-268.

[2] A. CABRAS, *On the universal connection of a system of connections*, Note di Matematica, VIII (1987), pp. 173-209.

[3] A. CABRAS - I. KOLÁŘ, *Connections on some functional bundles*, to appear in Czechoslovak Math. J.

[4] A. FRÖLICHER, *Smooth Structures*, Category theory 1981, LNM, **962**, Springer-Verlag (1982), pp. 69-81.

[5] P. L. GARCIA, *Connections and 1-jet fibre bundles*, Rend. Sem. Mat. Univ. Padova, **47** (1972), pp. 227-242.

[6] I. KOLÁŘ - P. MICHOR - J. SLOVÁK, *Natural Operations in Differential Geometry*, Springer-Verlag (1993).

[7] K. MARATHE - M. MODUGNO, *Polynomial connections on affine bundles*, Tensor, N.S., **50** (1991), pp. 35-46.

[8] M. MODUGNO, *New results on the theory of connections: systems, overconnections and prolongations*, Differential Geometry and Its Applications, Proceedings, D. Reidel Publishing Company (1987), pp. 243-269.