Journal of Inequalities and Applications
a SpringerOpen Journal

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The upper bound estimation on the spectral norm of r-circulant matrices with the Fibonacci and Lucas numbers

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Abstract
Let us define $A = \mathrm{Circ}_r(a_0, a_1, \ldots, a_{n-1})$ to be a $n \times n$ r-circulant matrix. The entries in the first row of $A = \mathrm{Circ}_r(a_0, a_1, \ldots, a_{n-1})$ are $a_i = F_i$, or $a_i = L_i$, or $a_i = F_iL_i$, or $a_i = F_i^2$, or $a_i = L_i^2$ $(i = 0, 1, \ldots, n-1)$, where F_i and L_i are the ith Fibonacci and Lucas numbers, respectively. This paper gives an upper bound estimation of the spectral norm for r-circulant matrices with Fibonacci and Lucas numbers. The result is more accurate than the corresponding results of S Solak and S Shen, and of J Cen, and the numerical examples have provided further proof.

Keywords: r-circulant matrices; Fibonacci number; Lucas number; spectral norm; estimation

1 Introduction

For $n > 0$, the Fibonacci sequence $\{F_n\}$ is defined by $F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$ and $F_1 = 1$. If we start by zero, then the sequence is given by

$$\begin{array}{cccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \end{array} \tag{1}$$

If we deduce from F_{n+1} that $L_{n+1} = L_n + L_{n-1}$, and let $L_0 = 2$, $L_1 = 1$, then we obtain the Lucas sequence $\{L_n\}$,

$$\begin{array}{cccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & \cdots \end{array} \tag{2}$$

Furthermore, the sequences $\{F_n\}$ and $\{L_n\}$ satisfy the following recursion:

$$F_n + L_n = 2F_{n+1}. \tag{3}$$

Definition 1.1 A matrix A is an r-circulant matrix if it is of the form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ ra_{n-1} & a_0 & \cdots & a_{n-3} & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ra_2 & ra_3 & \cdots & a_0 & a_1 \\ ra_1 & ra_2 & \cdots & ra_{n-1} & a_0 \end{pmatrix}.$$

Obviously, the elements of this r-circulant matrix are determined by its first row elements $a_0, a_1, \ldots, a_{n-1}$ and the parameter r, thus we denote $A = \mathrm{Circ}_r(a_0, a_1, \ldots, a_{n-1})$. Especially when $r = 1$, we obtain $A = \mathrm{Circ}(a_0, a_1, \ldots, a_{n-1})$.

Definition 1.2 A matrix A is called a symmetric r-circulant matrix if it is of the form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & ra_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-2} & a_{n-1} & \cdots & ra_{n-4} & ra_{n-3} \\ a_{n-1} & ra_0 & \cdots & ra_{n-3} & ra_{n-2} \end{pmatrix}.$$

Obviously, the elements of this r-circulant matrix are determined by its first row elements $a_0, a_1, \ldots, a_{n-1}$ and the parameter r; thus we denote $A = \mathrm{SCirc}_r(a_0, a_1, \ldots, a_{n-1})$. Especially when $r = 1$, we obtain $A = \mathrm{SCirc}(a_0, a_1, \ldots, a_{n-1})$.

For any $A = (a_{ij})_{m\times n}$, the well-known spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i\big(A^H A\big)},$$

in which $\lambda_i(A^H A)$ is the eigenvalue of $A^H A$ and A^H is the conjugate transpose of matrix A.

Define the maximum column length norm $c_1(\cdot)$ and the maximum row length norm $r_1(\cdot)$ of any matrix A by

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}$$

and

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2},$$

respectively.

Let A, B, and C be $m \times n$ matrices. If $A = B \circ C$, then in accordance with [1] we have

$$\|A\|_2 \leq r_1(B)c_1(C) \tag{4}$$

and

$$\|A\|_2 \leq \|B\|_2\|C\|_2. \tag{5}$$

Here, we define $B = (b_{ij})_{m\times n}$, $C = (c_{ij})_{m\times n}$, and we let $B \circ C$ be the Hadamard product of B and C.

In recent years, many authors (see [2–6]) were concerned with r-circulant matrices associated with a number sequence. References [2–4] calculate and estimate the Frobenius norm and the spectral norm of a circulant matrix where the elements of the r-circulant matrix are Fibonacci numbers and Lucas numbers; the authors found more accurate results of the upper bound estimated, and the numerical examples also have provided further proof.

Theorem 1.3 (see [2]) *Let* $A = \mathrm{Circ}(F_0, F_1, \dots, F_{n-1})$ *be a circulant matrix, then we have*

$$\|A\|_2 \le F_n F_{n-1},$$

where $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Theorem 1.4 (see [3]) *Let* $A = \mathrm{Circ}(L_0, L_1, \dots, L_{n-1})$ *be a circulant matrix, then we have*

$$\|A\|_2 \le \begin{cases} \sqrt{[F_nF_{n-1} + 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + 4] \times [F_nF_{n-1} + 4F_{n-1}^2 + 4F_{n-1}F_{n-2} + 4]}, & n \text{ odd}, \\ \sqrt{[F_nF_{n-1} + 4F_{n-1}^2 + 4F_{n-1}F_{n-2}] \times [F_nF_{n-1} + 4F_{n-1}^2 + 4F_{n-1}F_{n-2} - 3]}, & n \text{ even}, \end{cases}$$

where $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Theorem 1.5 (see [4]) *Let* $A = \mathrm{Circ}_r(F_0, F_1, \dots, F_{n-1})$ *be a r-circulant matrix, in which* $|r| \ge 1$, *and then*

$$\|A\|_2 \le |r| F_n F_{n-1},$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Theorem 1.6 (see [4]) *Let* $A = \mathrm{Circ}_r(L_0, L_1, \dots, L_{n-1})$ *be a r-circulant matrix and* $|r| \ge 1$, *then we obtain*

$$\|A\|_2 \le \begin{cases} \sqrt{(5|r|^2F_nF_{n-1} + 4)(5F_nF_{n-1} + 1)}, & n \text{ odd}, \\ \sqrt{[5|r|^2F_nF_{n-1} + 4(1 - |r|^2)](5F_nF_{n-1} - 3)}, & n \text{ even}, \end{cases}$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

2 Main results

Theorem 2.1 *Let* $A = \mathrm{Circ}(F_0, F_1, \dots, F_{n-1})$ *be a circulant matrix, then we have*

$$\|A\|_2 \le \sqrt{(n-1)F_nF_{n-1}},$$

where $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Proof Since $A = \mathrm{Circ}(F_0, F_1, \dots, F_{n-1})$ is a circulant matrix, let the matrices B and C be

$$B = \begin{pmatrix} F_0 & 1 & \cdots & 1 \\ 1 & F_0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & F_0 \end{pmatrix}, \qquad C = \begin{pmatrix} F_0 & F_1 & \cdots & F_{n-2} & F_{n-1} \\ F_{n-1} & F_0 & \cdots & F_{n-3} & F_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_2 & F_3 & \cdots & F_0 & F_1 \\ F_1 & F_2 & \cdots & F_{n-1} & F_0 \end{pmatrix},$$

we get $A = B \circ C$.

For

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{n-1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \max_j \sqrt{\sum_{i=1}^{n} |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} F_s^2} = \sqrt{F_n F_{n-1}}.$$

From (4), we have

$$\|A\|_2 \leq \sqrt{(n-1)F_n F_{n-1}}.$$ □

Corollary 2.2 *Let* $A = \mathrm{SCirc}(F_0, F_1, \ldots, F_{n-1})$ *be a symmetric circulant matrix, then we have*

$$\|A\|_2 \leq \sqrt{(n-1)F_n F_{n-1}},$$

where $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Corollary 2.3 *Let* $A = \mathrm{Circ}(F_0^2, F_1^2, \ldots, F_{n-1}^2)$ *be a circulant matrix, then we have*

$$\|A\|_2 \leq (n-1)F_n F_{n-1},$$

where $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Proof Since $A = \mathrm{Circ}(F_0^2, F_1^2, \ldots, F_{n-1}^2)$ is a circulant matrix, if the matrices $B = \mathrm{Circ}(F_0, F_1, \ldots, F_{n-1})$, we get $A = B \circ B$; thus from (5) and Theorem 2.1 we obtain

$$\|A\|_2 \leq (n-1)F_n F_{n-1}.$$ □

Theorem 2.4 *Let* $A = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$ *be a circulant matrix, then we have*

$$\|A\|_2 \leq \begin{cases} \sqrt{5nF_n F_{n-1} + 4n}, & n \text{ odd}, \\ \sqrt{5nF_n F_{n-1}}, & n \text{ even}, \end{cases}$$

where $\|\cdot\|_2$ *is the spectral norm and* L_n *denotes the Lucas number.*

Proof Since $A = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$ is a circulant matrix, let the following matrices be defined:

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} L_0 & L_1 & \cdots & L_{n-2} & L_{n-1} \\ L_{n-1} & L_0 & \cdots & L_{n-3} & L_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_2 & L_3 & \cdots & L_0 & L_1 \\ L_1 & L_2 & \cdots & L_{n-1} & L_0 \end{pmatrix},$$

then $A = B \circ C$.

We have

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{n}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} L_s^2} = \sqrt{\sum_{s=0}^{n-1} (F_s + 2F_{s-1})^2}.$$

Here

$$\sum_{s=0}^{n-1} F_s^2 = F_n F_{n-1}, \qquad \sum_{s=0}^{n-1} F_s F_{s-1} = \begin{cases} F_{n-1}^2, & n \text{ odd}, \\ F_{n-1}^2 - 1, & n \text{ even}, \end{cases} \qquad \sum_{s=0}^{n-1} F_{s-1}^2 = F_{n-1}F_{n-2} + 1,$$

thus

$$c_1(C) = \begin{cases} \sqrt{5F_nF_{n-1} + 4}, & n \text{ odd}, \\ \sqrt{5F_nF_{n-1}}, & n \text{ even}, \end{cases}$$

and from (4) we obtain

$$\|A\|_2 \le \begin{cases} \sqrt{5nF_nF_{n-1} + 4n}, & n \text{ odd}, \\ \sqrt{5nF_nF_{n-1}}, & n \text{ even}. \end{cases}$$ □

Corollary 2.5 *Let* $A = \mathrm{SCirc}(L_0, L_1, \ldots, L_{n-1})$ *be a symmetric circulant matrix, then we have*

$$\|A\|_2 \le \begin{cases} \sqrt{5nF_nF_{n-1} + 4n}, & n \textit{ odd}, \\ \sqrt{5nF_nF_{n-1}}, & n \textit{ even}, \end{cases}$$

where $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Corollary 2.6 *Let* $A = \mathrm{Circ}(L_0^2, L_1^2, \ldots, L_{n-1}^2)$ *be circulant matrices, then*

$$\|A\|_2 \le \begin{cases} 5nF_nF_{n-1} + 4n, & n \textit{ odd}, \\ 5nF_nF_{n-1}, & n \textit{ even}, \end{cases}$$

where $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Proof Since $A = \mathrm{Circ}(L_0^2, L_1^2, \ldots, L_{n-1}^2)$ is a circulant matrix, if the matrices $B = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$, we get $A = B \circ B$; thus from (5) and Theorem 2.4, we obtain

$$\|A\|_2 \le \begin{cases} 5nF_nF_{n-1} + 4n, & n \text{ odd}, \\ 5nF_nF_{n-1}, & n \text{ even}. \end{cases}$$ □

Corollary 2.7 *Let* $A = \operatorname{Circ}(F_0L_0, F_1L_1, \dots, F_{n-1}L_{n-1})$ *be circulant matrices, then*

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)nF_nF_{n-1}(5F_nF_{n-1}+4)}, & n \text{ odd}, \\ \sqrt{5(n-1)n}F_nF_{n-1}, & n \text{ even}, \end{cases}$$

where $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Proof Since $A = \operatorname{Circ}(F_0L_0, F_1L_1, \dots, F_{n-1}L_{n-1})$ is a circulant matrix, if the matrices $B = \operatorname{Circ}(F_0, F_1, \dots, F_{n-1})$ and $C = \operatorname{Circ}(L_0, L_1, \dots, L_{n-1})$, we get $A = B \circ C$; thus from (5), Theorems 2.1, and 2.4, we obtain

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)nF_nF_{n-1}(5F_nF_{n-1}+4)}, & n \text{ odd}, \\ \sqrt{5(n-1)n}F_nF_{n-1}, & n \text{ even}. \end{cases}$$

□

Theorem 2.8 *Let* $A = \operatorname{Circ}_r(F_0, F_1, \dots, F_{n-1})$ *be a r-circulant matrix, in which* $|r| \geq 1$, *and then*

$$\|A\|_2 \leq \sqrt{(n-1)|r|^2F_nF_{n-1}},$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Proof Since $A = \operatorname{Circ}_r(F_0, F_1, \dots, F_{n-1})$ is a r-circulant matrix, let B and C, respectively, be

$$B = \begin{pmatrix} F_0 & 1 & 1 & \cdots & 1 \\ r & F_0 & 1 & \cdots & 1 \\ r & r & F_0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & F_0 \end{pmatrix}, \qquad C = \begin{pmatrix} F_0 & F_1 & F_2 & \cdots & F_{n-1} \\ F_{n-1} & F_0 & F_1 & \cdots & F_{n-2} \\ F_{n-2} & F_{n-1} & F_0 & \cdots & F_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_1 & F_2 & F_3 & \cdots & F_0 \end{pmatrix},$$

then $A = B \circ C$.

For

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{(n-1)|r|^2}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} F_s^2} = \sqrt{F_nF_{n-1}},$$

from (4), we have

$$\|A\|_2 \leq \sqrt{(n-1)|r|^2F_nF_{n-1}}.$$

□

Corollary 2.9 *Let* $A = \mathrm{SCirc}_r(F_0, F_1, \ldots, F_{n-1})$ *be a symmetric r-circulant matrix, in which* $|r| \geq 1$, *and then*

$$\|A\|_2 \leq \sqrt{(n-1)|r|^2 F_n F_{n-1}},$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the nth Fibonacci number.*

Corollary 2.10 *Let* $A = \mathrm{Circ}_r(F_0^2, F_1^2, \ldots, F_{n-1}^2)$ *be a r-circulant matrix, while* $|r| \geq 1$, *then we obtain*

$$\|A\|_2 \leq (n-1)|r|F_n F_{n-1},$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm and* F_n *denotes the Fibonacci number.*

Proof Since $A = \mathrm{Circ}_r(F_0^2, F_1^2, \ldots, F_{n-1}^2)$ is a r-circulant matrix, if the matrices $B = \mathrm{Circ}_r(F_0, F_1, \ldots, F_{n-1})$ and $C = \mathrm{Circ}(F_0, F_1, \ldots, F_{n-1})$, we get $A = B \circ C$; thus from (5), Theorems 2.1, and 2.8, we obtain

$$\|A\|_2 \leq (n-1)|r|F_n F_{n-1}. \qquad \square$$

Corollary 2.11 *Let* $A = \mathrm{Circ}_r(F_0 L_0, F_1 L_1, \ldots, F_{n-1} L_{n-1})$ *be a r-circulant matrix, while* $|r| \geq 1$, *then we obtain*

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)n|r|^2 F_n F_{n-1}(5F_n F_{n-1} + 4)}, & n \text{ odd}, \\ F_n F_{n-1}\sqrt{5|r|^2(n-1)n}, & n \text{ even}, \end{cases}$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Proof Since $A = \mathrm{Circ}_r(F_0 L_0, F_1 L_1, \ldots, F_{n-1} L_{n-1})$ is a r-circulant matrix, if the matrices $B = \mathrm{Circ}_r(F_0, F_1, \ldots, F_{n-1})$ and $C = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$, we get $A = B \circ C$; thus from (5), Theorems 2.4, and 2.8, we obtain

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)n|r|^2 F_n F_{n-1}(5F_n F_{n-1} + 4)}, & n \text{ odd}, \\ F_n F_{n-1}\sqrt{5|r|^2(n-1)n}, & n \text{ even}. \end{cases} \qquad \square$$

Theorem 2.12 *Let* $A = \mathrm{Circ}_r(L_0, L_1, \ldots, L_{n-1})$ *be a r-circulant matrix and* $|r| \geq 1$, *then we obtain*

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1} + 4}, & n \text{ odd}, \\ \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1}}, & n \text{ even}, \end{cases}$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ *is the spectral norm, and* L_n *and* F_n *denote the nth Lucas and Fibonacci numbers, respectively.*

Proof Since $A = \mathrm{Circ}_r(L_0, L_1, \ldots, L_{n-1})$ is a r-circulant matrix, let B and C, respectively, be

$$B = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r & r & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} L_0 & L_1 & \cdots & L_{n-2} & L_{n-1} \\ L_{n-1} & L_0 & \cdots & L_{n-3} & L_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_2 & L_3 & \cdots & L_0 & L_1 \\ L_1 & L_2 & \cdots & L_{n-1} & L_0 \end{pmatrix},$$

and then $A = B \circ C$.

We have

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{(n-1)|r|^2 + 1}$$

and

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{in}|^2} = \sqrt{\sum_{s=0}^{n-1} L_s^2} = \sqrt{\sum_{s=0}^{n-1} (F_s + 2F_{s-1})^2},$$

in which

$$\sum_{s=0}^{n-1} F_s^2 = F_n F_{n-1}, \qquad \sum_{s=0}^{n-1} F_{s-1} F_s = \begin{cases} F_{n-1}^2, & n \text{ odd}, \\ F_{n-1}^2 - 1, & n \text{ even}, \end{cases} \qquad \sum_{s=0}^{n-1} F_{s-1}^2 = F_{n-1} F_{n-2} + 1,$$

and we get

$$c_1(C) = \begin{cases} \sqrt{5F_n F_{n-1} + 4}, & n \text{ odd}, \\ \sqrt{5F_n F_{n-1}}, & n \text{ even}. \end{cases}$$

From (4), we further infer

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1} + 4}, & n \text{ odd}, \\ \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1}}, & n \text{ even}. \end{cases}$$

□

Corollary 2.13 *Let $A = \mathrm{SCirc}_r(L_0, L_1, \ldots, L_{n-1})$ be a symmetric r-circulant matrix and $|r| \geq 1$, then we obtain*

$$\|A\|_2 \leq \begin{cases} \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1} + 4}, & n \text{ odd}, \\ \sqrt{(n-1)|r|^2 + 1} \times \sqrt{5F_n F_{n-1}}, & n \text{ even}, \end{cases}$$

where $r \in \mathbb{C}$, $\| \cdot \|_2$ is the spectral norm, and L_n and F_n denote the nth Lucas and Fibonacci numbers, respectively.

Corollary 2.14 *Let $A = \mathrm{Circ}_r(L_0^2, L_1^2, \ldots, L_{n-1}^2)$ be a r-circulant matrix and $|r| \geq 1$, then*

$$\|A\|_2 \leq \begin{cases} (5F_n F_{n-1} + 4)\sqrt{n[(n-1)|r|^2 + 1]}, & n \text{ odd}, \\ 5F_n F_{n-1}\sqrt{n[(n-1)|r|^2 + 1]}, & n \text{ even}, \end{cases}$$

where $r \in \mathbb{C}$, $\|\cdot\|_2$ is the spectral norm, and L_n and F_n denote the nth Lucas and Fibonacci numbers, respectively.

Proof Since $A = \mathrm{Circ}_r(L_0^2, L_1^2, \ldots, L_{n-1}^2)$ is a r-circulant matrix, if the matrices $B = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$ and $C = \mathrm{Circ}_r(L_0, L_1, \ldots, L_{n-1})$, we get $A = B \circ C$; thus from (5), Theorems 2.4, and 2.12, we obtain

$$\|A\|_2 \leq \begin{cases} (5F_nF_{n-1} + 4)\sqrt{n[(n-1)|r|^2 + 1]}, & n \text{ odd}, \\ 5F_nF_{n-1}\sqrt{n[(n-1)|r|^2 + 1]}, & n \text{ even}. \end{cases}$$ □

3 Examples

Example 1 Let $A = \mathrm{Circ}(F_0, F_1, \ldots, F_{n-1})$ be a circulant matrix, in which F_i $(i = 0, 1, \ldots, n-1)$ denotes the Fibonacci number.

From Table 1, it is easy to find that the upper bounds for the spectral norm, of Theorem 2.1 are more accurate than Theorem 1.3 when $n \geq 4$.

Example 2 Let $A = \mathrm{Circ}(L_0, L_1, \ldots, L_{n-1})$ be a circulant matrix, where L_i $(i = 0, 1, \ldots, n-1)$ denotes the Lucas sequence.

Let $n \geq 3$, and it is easy to find that the upper bounds for the spectral norm of Theorem 2.4 are more accurate than Theorem 1.4 (see Table 2).

Example 3 Let $A = \mathrm{Circ}_2(F_0, F_1, \ldots, F_{n-1})$ be a 2-circulant matrix, in which F_i $(i = 0, 1, \ldots, n-1)$ denotes the Fibonacci number.

Let $n \geq 4$, and it is easy to find that the upper bounds for the spectral norm of Theorem 2.8 are more precise than Theorem 1.5 (see Table 3).

Table 1 Numerical results of $a_i = F_i$, $r = 1$

n	Theorem 2.1	Theorem 1.3	$\frac{\text{Third column}}{\text{Second column}}$
2	1	1	$\frac{1}{1} = 1$
3	2	2	$\frac{2}{2} = 1$
4	$3\sqrt{2}$	6	$\frac{6}{3\sqrt{2}} = \sqrt{2}$
5	$\sqrt{60}$	15	$\frac{15}{\sqrt{60}} \approx 1.936$
6	$\sqrt{200}$	40	$\frac{40}{\sqrt{200}} = 2\sqrt{2}$
n	$\sqrt{(n-1)F_nF_{n-1}}$	F_nF_{n-1}	$\frac{F_nF_{n-1}}{\sqrt{(n-1)F_nF_{n-1}}} = \sqrt{\frac{F_nF_{n-1}}{n-1}}$

Table 2 Numerical results of $a_i = L_i$, $r = 1$

n	Theorem 2.4	Theorem 1.4	$\frac{\text{Third column}}{\text{Second column}}$
1	2	2	$\frac{2}{2} = 1$
2	$\sqrt{10}$	$\sqrt{10}$	$\frac{\sqrt{10}}{\sqrt{10}} = 1$
3	$\sqrt{42}$	$\sqrt{154}$	$\frac{\sqrt{154}}{\sqrt{42}} \approx 1.915$
4	$\sqrt{120}$	$\sqrt{810}$	$\frac{\sqrt{810}}{\sqrt{120}} \approx 2.598$
5	$\sqrt{395}$	$\sqrt{6{,}004}$	$\frac{\sqrt{6{,}004}}{\sqrt{395}} \approx 3.899$
6	$\sqrt{1{,}200}$	$\sqrt{39{,}400}$	$\frac{\sqrt{39{,}400}}{\sqrt{1{,}200}} \approx 5.730$
n			$\sqrt{n^{-1}(5F_nF_{n-1} + 1)}$ n odd, $\sqrt{n^{-1}(5F_nF_{n-1} - 3)}$ n even

Table 3 Numerical results of $a_i = F_i$, $r = 2$

n	Theorem 2.8	Theorem 1.5	$\frac{\text{Third column}}{\text{Second column}}$
2	2	2	$\frac{2}{2} = 1$
3	4	4	$\frac{4}{4} = 1$
4	$6\sqrt{2}$	12	$\frac{12}{6\sqrt{2}} = \sqrt{2}$
5	$4\sqrt{15}$	$\sqrt{30}$	$\frac{30}{4\sqrt{15}} \approx 1.936$
6	$20\sqrt{2}$	80	$\frac{80}{20\sqrt{2}} = 2\sqrt{2}$
n	$\sqrt{(n-1)\lvert r\rvert^2 F_n F_{n-1}}$	$\lvert r\rvert F_n F_{n-1}$	$\sqrt{(n-1)^{-1} F_n F_{n-1}}$

Table 4 Numerical results of $a_i = L_i$, $r = 2$

n	Theorem 2.12	Theorem 1.6	$\frac{\text{Third column}}{\text{Second column}}$
1	2	2	$\frac{2}{2} = 1$
2	5	4	$\frac{4}{5} = \frac{4}{5}$
3	$3\sqrt{14}$	$2\sqrt{231}$	$\frac{2\sqrt{231}}{3\sqrt{14}} \approx 2.708$
4	$\sqrt{390}$	54	$\frac{54}{\sqrt{390}} \approx 2.734$
5	$\sqrt{1{,}343}$	152	$\frac{152}{\sqrt{1{,}343}} \approx 4.418$
6	$10\sqrt{42}$	394	$\frac{394}{10\sqrt{342}} \approx 6.080$

Example 4 Let $A = \mathrm{Circ}_2(L_0, L_1, \ldots, L_{n-1})$ be a 2-circulant matrix where L_i ($i = 0, 1, \ldots, n-1$) denotes the Lucas sequence.

It can be seen from Table 4 that the upper bounds for the spectral norm of Theorem 2.12 are more precise than Theorem 1.6 when $n \geq 3$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

Project supported by Applied Fundamental Research Plan of Sichuan Province (No. 2013JY0178).

Received: 25 September 2014 Accepted: 12 February 2015 Published online: 28 February 2015

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