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The Upper Envelope of Piecewise Linear Functions: Tight Bounds on the Number of Faces*

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Abstract. This note proves that the maximum number of faces (of any dimension) of the upper envelope of a set of n possibly intersecting d-simplices in $d+1$ dimensions is $\Theta(n^d\alpha(n))$. This is an extension of a result of Pach and Sharir [PS] who prove the same bound for the number of d-dimensional faces of the upper envelope.

1. Introduction

This note considers the combinatorial complexity[1] of the upper envelope of a finite set of (possibly intersecting) d-dimensional simplices[2] in $(d+1)$-dimensional Euclidean space. In order to define the notion of an envelope we think of each d-simplex as the graph of a real-valued, linear d-variate function. This function, f, is defined so that $x_{d+1}=f(x_1, x_2, \ldots, x_d)$ whenever $(x_1, x_2, \ldots, x_d, x_{d+1})$ is in the simplex. If no such x_{d+1} exists we conveniently set $f(x_1, x_2, \ldots, x_d)=-\infty$. The (*upper*) *envelope* of the set of simplices is now the pointwise maximum of all corresponding d-variate functions. The (upper) envelope of more general piecewise linear d-variate functions is implicitly defined since the graph of every such function is a collection of d-dimensional polyhedra

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[1] By the *combinatorial complexity* we mean the number of faces of any dimension $k<d$. In our analysis we assume that d, the number of dimensions, is a fixed constant.

[2] A *d-dimensional simplex* (or *d-simplex*) in $d+1$ dimensions is the intersection of a hyperplane with $d+1$ half-spaces, where a *half-space* is defined as the set of points on and to one side of a hyperplane.

which can be decomposed into d-simplices. To prove an upper bound on the combinatorial complexity of the envelope of n d-simplices we assume without loss of generality that the d-simplices are in general position. Among other things this means that the hyperplanes that contain the d-simplices are nonvertical.[3] Other implications of the general position assumption are implicitly used whenever it is convenient.

Let S be such a set of n d-simplices in $d+1$ dimensions and let M_S be its envelope. If we project every face of M_S vertically onto the hyperplane $x_{d+1}=0$ we get a cell complex,[4] M_S^*, and we denote the number of k-faces[5] of M_S^* by $\psi_k(S)$ for $0 \le k \le d$. Formally, we consider the sum of the $\psi_k(S)$ as the combinatorial complexity of M_S. This note proves tight upper bounds for $\psi_k^{(d+1)}(n)$, where

$$\psi_k^{(d+1)}(n) = \max\{\psi_k(S) | S \text{ a set of } n \text{ } d\text{-simplices in } d+1 \text{ dimensions}\},$$

for all $0 \le k \le d$ and constant values of d. Prior to this note, tight bounds were known for all k only if $d+1=2, 3$ and for $k=d$ if $d>3$. In two dimensions $(d+1=2)$, S is a set of (possibly intersecting) line segments in the plane. Using so-called Davenport–Schinzel sequences of order 3 [HS] and [WS] prove that $\psi_k^{(2)}(n) = \Theta(n\alpha(n))$, for $k=0,1$, where $\alpha(n)$ is the extremely slowly growing inverse of Ackermann's function. [PS] proves $\psi_d^{(d+1)}(n) = O(n^d\alpha(n))$ using a divide-and-conquer argument and shows that this upper bound is tight by extending the two-dimensional lower bound construction of [WS] to three and higher dimensions. In $d+1=3$ dimensions the Euler characteristic can be used to extend the upper bound for 2-faces to 0-faces (vertices) and 1-faces (edges). In this note we prove the following result.

Theorem. *$\psi_k^{(d+1)}(n) = \Theta(n^d\alpha(n))$ for $0 \le k \le d$.*

In other words, the combinatorial complexity of the envelope of n d-simplices in $d+1$ dimensions is proportional to $n^d\alpha(n)$ in the worst case. It is easy to verify the lower bound of the theorem. [PS] shows that there is a collection of n d-simplices in $d+1$ dimensions such that the number of d-faces of the envelope is $\Omega(n^d\alpha(n))$. The lower bound for $0 \le k < d$ follows since every d-face has at least one k-face in its boundary and every k-face belongs to the boundary of at most some constant number of d-faces, if we assume general position of the d-simplices. The constant is linear in d. The proof of the upper bound is presented in Section 2 of this note. It is an extension of the divide-and-conquer proof of

[3] A hyperplane is *nonvertical* if it intersects the $(d+1)$st coordinate axis in a unique point.

[4] A *cell complex* is a collection of closed convex sets (called *faces*) of various dimensions such that the relative interiors of the faces partition the space and the intersection of any two faces is again a face.

[5] A maximal connected component, f, of the intersection of M_S^* with a k-dimensional affine subspace is a k-face of M_S^* if the interior of f relative to the subspace is nonempty and f is not contained in the relative interior of a $(k+1)$-face of M_S^*.

[PS]. Combinatorial extensions and algorithmic applications of the theorem can be found in [EGS].

2. Proof of the Theorem

We first review the main steps of the proof and then describe each step in appropriate detail. Most of the arguments are concerned with a refinement, $\bar{M}_S$, of the cell complex M_S^* in d dimensions. $\bar{M}_S$ has the nice property that every face is convex. Being a refinement of M_S^* the number of faces of $\bar{M}_S$ is certainly an upper bound on the number of faces of M_S^*. The overall structure of the proof is inductive over the number of dimensions. In a specific dimension, $d+1$, we use a divide-and-conquer argument, that is, we form subsets of S, the set of d-simplices, consider the envelopes of these subsets and combine them to get the envelope of S. More precisely, we consider the cell complexes $\bar{M}$ of the subsets and combine those to get $\bar{M}_S$. The combination makes use of the convexity of $\bar{M}_S$'s faces and the inductively available upper bounds on the combinatorial complexity of envelopes in d dimensions. A careful choice of the subsets of S allows us to prove the upper bound of the theorem for $2 \le k \le d$. Finally, we use the Euler characteristic for cell complexes to extend the upper bound to $k = 0, 1$. The order in which we present the various steps of the proof is different from the order used in this outline.

Definition of $\bar{M}_S$. As mentioned above, $\bar{M}_S$ is a refinement of M_S^* which is a cell complex in d dimensions. (The d-dimensional space is identified with the hyperplane $x_{d+1} = 0$ in $d+1$ dimensions.) Recall that M_S^* is obtained by projecting every face of M_S vertically onto $x_{d+1} = 0$. To obtain $\bar{M}_S$ from M_S^* we also project each d-simplex in S vertically onto $x_{d+1} = 0$ and, in addition, extend each $(d-1)$-face of each projected d-simplex to the full hyperplane in $x_{d+1} = 0$ that contains it. Thus, $\bar{M}_S$ is M_S^* after superimposing an arrangement[6] of $(d+1)n$ hyperplanes; the arrangement is denoted by A_S.

It is convenient to think of $\bar{M}_S$ as a refinement of A_S: every cell (i.e., d-face) of A_S is further decomposed by projections of intersections between d-simplices. Consider the vertical slab, V_c, in $d+1$ dimensions whose points project vertically to points of some cell c of A_S. Restricted to V_c, a d-simplex in S cannot be distinguished from the (d-dimensional) hyperplane that contains the d-simplex. It follows that M_S, the envelope of S, restricted to V_c is the boundary of the convex polyhedron that is the intersection of the half-spaces bounded from below by the hyperplanes containing the d-simplices cutting through V_c. This implies that in $\bar{M}_S$ every cell of A_S is further decomposed into convex faces. Consequently, every face of $\bar{M}_S$ is convex. We let $\bar{\psi}_k(S)$ denote the number of k-faces of $\bar{M}_S$.

[6] An *arrangement* in d dimensions is the cell complex obtained by dissecting the space with a finite number of hyperplanes. If n is the number of hyperplanes then the number of faces of the arrangement is $O(n^d)$ (see [Grü] and [E]).

Use of the Euler Characteristic

The Euler characteristic of a cell complex in d dimensions is a linear relation for the numbers of k-faces, $0 \le k \le d$. For $\bar{M}_S$ it has the simple form

$$\sum_{k=0}^{d} (-1)^k \bar{\psi}_k(S) = 1 + (-1)^d$$

since all faces of $\bar{M}_S$ are convex and therefore simply connected (see [Gre]). Assuming $\bar{\psi}_k(S) = O(n^d\alpha(n))$ for $2 \le k \le d$ we get

$$|\bar{\psi}_0(S) - \bar{\psi}_1(S)| = O(n^d\alpha(n)).$$

Thus, the number of vertices and edges of $\bar{M}_S$ can be asymptotically more than $n^d\alpha(n)$ only if their difference is small, that is, $O(n^d\alpha(n))$. However, by assumption of general position every vertex of $\bar{M}_S$ is incident upon $d+1$ edges if it lies inside a cell of A_S, and between $d+2$ and $2d$ if it lies on the boundary of a cell of A_S. In any case, we have

$$\bar{\psi}_1(S) \ge \frac{d+1}{2} \bar{\psi}_0(S)$$

which implies that both $\bar{\psi}_0(S)$ and $\bar{\psi}_1(S)$ can be at most proportional to their difference, as long as $d \ge 2$. This proves $\bar{\psi}_k^{(d+1)}(n) = O(n^d\alpha(n))$ for $k = 0, 1$ if the same upper bound holds for $2 \le k \le d$.

An Exercise in Solving Recurrence Relations

Later we prove that indeed $\bar{\psi}_k^{(d+1)}(n) = O(n^d\alpha(n))$ for $2 \le k \le d$. The type of recurrence relation that we have to deal with is of the form

$$T(n) = \binom{m}{d+1-k} \cdot T\left(\frac{d+1-k}{m} \cdot n\right) + O(n^d\alpha(n)),$$

where $m > d+1-k$ is an integer constant independent of n. The solution to this recurrence relation is $O(n^d\alpha(n))$ if the homogeneous solution is $O(n^{d-\varepsilon})$ for some $\varepsilon > 0$. We show that m can always be chosen such that this is true.

The homogeneous solution of the above recurrence relation is n^β, with

$$\beta = \log_2\binom{m}{d+1-k} \Big/ \log_2 \frac{m}{d+1-k}.$$

The requirement $\beta < d$ can be rewritten as

$$\binom{m}{d+1-k} < \left(\frac{m}{d+1-k}\right)^d$$

which is equivalent to

$$\frac{(d+1-k)^d}{(d+1-k)!} < \frac{m^d}{m \cdot (m-1) \cdot \cdots \cdot (m-d+k)}.$$

The ratio on the right side has d factors in the numerator and $d+1-k$ factors in the denominator which implies that

$$\frac{(d+1-k)^d}{(d+1-k)!} < m$$

is sufficient to guarantee $\beta < d$ as long as $d+1-k < d$ which is equivalent to $k \geq 2$. Thus, the recurrence relation solves to $O(n^d \alpha(n))$ if $k \geq 2$ and m is chosen appropriately. The above calculation shows that choosing m exponentially in d is sufficient.

Adding Hyperplanes

The final step of the proof (described later) takes the envelopes of a constant number of subsets of S and obtains the envelope of S by combining those envelopes. Let $S_1, S_2, \ldots, S_\mu$ be the subsets of S and consider the cell complexes $\bar{M}_{S_i}$ for $1 \leq i \leq \mu$. When we combine those cell complexes it is important that they are refinements of the same arrangement as $\bar{M}_S$, namely of A_S. To satisfy this need, we superimpose A_S on $\bar{M}_{S_i}$, for every $1 \leq i \leq \mu$, and call the resulting cell complex $\bar{\bar{M}}_{S_i}$. Adding hyperplanes to $\bar{M}_{S_i}$ clearly increases the number of faces. We now show that the effect of adding hyperplanes on the number of faces is surprisingly small.

When we add a hyperplane we create new k-faces that lie in the hyperplane and we cut old k-faces into pairs of new k-faces; in the latter case the hyperplane contains a $(k-1)$-face that splits the old k-face. Thus, we can estimate the increase in combinatorial complexity from $\bar{M}_{S_i}$ to $\bar{\bar{M}}_{S_i}$ by counting the faces in the hyperplanes added to $\bar{M}_{S_i}$. The number of hyperplanes added to $\bar{M}_{S_i}$ is at most $(d+1)n$ and thus linear in the size of S.[7]

Consider now the decomposition of a hyperplane, h, in $\bar{\bar{M}}_{S_i}$. In order to bound the number of faces in h we use the following auxiliary claim, which we also establish using induction over the number of dimensions. The claim considers cell complexes that are slightly more general than the cell complexes $\bar{M}$.

Claim. *Let S be a finite set of d-simplices in $d+1$ dimensions, let $\bar{M}_S$ be the cell complex in d dimensions as defined earlier, and let $\bar{M}$ be $\bar{M}_S$ after adding a finite number of hyperplanes (in d dimensions). The number of faces of $\bar{M}$ is $O(N^d \alpha(N))$, where N is the number of d-simplices in S plus the number of hyperplanes added to $\bar{M}_S$.*

[7] Some of the hyperplanes of A_S are already present in $\bar{M}_{S_i}$ and do not have to be added.

If $d = 1$, S is a finite set of line segments in the plane. The vertical projection of the upper envelope of S is a decomposition of the x_1-axis into intervals. [WS] establishes that the number of intervals is $O(n\alpha(n))$ if $n = |S|$. If we add $N - n$ points to the subdivision of the x_1-axis we get at most $O(n\alpha(n) + N)$ intervals which is smaller than $O(N\alpha(N))$ and thus the claim is correct for $d = 1$.

We now come back to hyperplane h which intersects the other hyperplanes in a $(d-1)$-dimensional arrangement consisting of $O(n^{d-1})$ faces. The decomposition of h in $\bar{M}_S$ is a refinement of this arrangement which can be obtained from a cross-section of M_S as follows. Let h' be the vertical hyperplane in $d+1$ dimensions whose intersection with $x_{d+1} = 0$ is h. The cross-section $M_S \cap h'$ is the envelope of $O(n)$ $(d-1)$-simplices[8] in h' which has $O(n^{d-1}\alpha(n))$ faces by inductive assumption (the above claim for $(d-1)$-simplices in d dimensions). Inductively, we can also assume that the decomposition of h in $\bar{\bar{M}}_{S_i}$ (which we obtain by superimposing the vertical projection of the cross-section with the arrangement in h described earlier) has at most $O(n^{d-1}\alpha(n))$ faces. Thus, the total number of faces in the cell complexes $\bar{\bar{M}}$ (taken over all sets S_i for $1 \leq i \leq \mu$) is at most $O(n^d\alpha(n))$ larger than the total number of faces of the cell complexes $\bar{M}$ (taken over the same collection of sets).

Notice that the argument makes no use of the fact that every hyperplane added to $\bar{M}_{S_i}$ contains a $(d-1)$-face of the vertical projection of a d-simplex in S_i. It can therefore be applied to any odd hyperplane that we like to add. This is important for proving the claim for $d+1$ dimensions which can thus be done along the same lines.

Combining Envelopes

For this step of the proof it is important that M_S, the envelope of S, restricted to a vertical slab defined by a cell of A_S, is the lower boundary of a convex polyhedron. Thus, every face is convex and every intersection of $d+1-k$ d-simplices (for $0 \leq k \leq d$) contains at most one k-face within this slab. Let us now fix k to some integer between 2 and d including the limits. We partition S into $m > d+1-k$ subsets of approximately equal sizes[9] and then form

$$\mu = \binom{m}{d+1-k}$$

sets of size approximately $n \cdot (d+1-k)/m$ by merging every combination of $d+1-k$ subsets. For example, if $k = d$ then the new sets are the original m subsets, and if $k = d-1$ the sets are the unions of any two original subsets. It is important to see that any $(d+1-k)$-tuple of d-simplices is contained in at least one of the μ sets.

[8] h' intersects a d-simplex in a $(d-1)$-dimensional convex polytope which can be decomposed into a constant number of $(d-1)$-simplices.

[9] S can be partitioned such that the sizes of any two subsets differ by at most 1.

We now come back to M_S, the envelope of S, restricted to the vertical slab, V_c, defined by cell c of the arrangement A_S in $x_{d+1}=0$. This restricted part of M_S corresponds to the decomposition of c induced by $\bar{M}_S$. We consider the μ sets formed above and denote them by $S_1, S_2, \ldots, S_\mu$. If a k-face f of $\bar{M}_S$ lies inside c, then it is contained in the projection of the intersection of some $d+1-k$ d-simplices $s_1, s_2, \ldots, s_{d+1-k}$. There is at least one index j, $1 \le j \le \mu$, such that S_j contains all those simplices. By convexity, $\bar{M}_{S_j}$ restricted to c has a k-face g that contains f; g is also contained in the projection of $s_1 \cap s_2 \cap \cdots \cap s_{d+1-k}$. It follows that the number of k-faces of $\bar{M}_S$ within c is at most the total number of k-faces of $\bar{M}_{S_1}, \bar{M}_{S_2}, \ldots, \bar{M}_{S_\mu}$ in c. The total number of k-faces of $\bar{M}_S$ is thus at most the sum of the numbers of k-faces of $\bar{M}_{S_1}$ through $\bar{M}_{S_\mu}$. By the argument in the previous step of the proof we therefore get

$$T(n) = \binom{m}{d+1-k} T\left(\frac{d+1-k}{m} \cdot n\right) + O(n^d \alpha(n)),$$

where $T(n)$ is the maximum number of k-faces of $\bar{M}_S$, that is, $T(n) = \bar{\psi}_k^{(d+1)}(n)$. The analysis of this recurrence relation presented earlier implies that the constant m can be chosen so that the solution is $O(n^d \alpha(n))$. This implies

$$\bar{\psi}_k^{(d+1)}(n) = O(n^d \alpha(n)) \qquad \text{for} \quad 2 \le k \le d.$$

The same bound for $k = 0, 1$ is now implied by our considerations of the Euler characteristic of $\bar{M}_S$. This completes the proof of the theorem. □

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