# The Use of Sliding Spectral Windows for Parameter Estimation in Power System Disturbance Monitoring

Peter O'Shea

***Abstract*—The monitoring of power systems after faults or disturbances is an important problem. These disturbances generally give rise to oscillating modal components, which in a worst case scenario, can be exponentially growing sinusoids. The latter, if not detected and damped out, can pose a serious threat to system reliability. It is thus necessary to monitor whether any of these modes do exhibit exponential growth (rather than the more acceptable scenario of exponential decay). There are currently a number of approaches to predicting/monitoring disturbances in power system networks. One approach is eigenanalysis, based on a linearized modeling of the power system [1]. A more direct approach is spectral analysis of the signals recorded immediately after a fault or disruption. For this latter approach both Prony's method [2] and conventional Fourier techniques have been used [5].**

**This paper presents a Fourier based algorithm for estimating the parameters of the oscillating modes which arise after a system disruption. The algorithm is based on the sliding window method discussed in [5], but has a number of innovations.**

***Index Terms*—Fourier, modal analysis, optimization, signal analysis, stability.**

## I. Introduction

There are some distinct advantages to using Fourier methods rather than Prony's method for estimating the parameters of the post disturbance modes. Fourier methods are: 1) known to be robust to noise [4], 2) able to be implemented efficiently due to the availability of the Fast Fourier transform (FFT), and 3) not as susceptible to problems with model mismatches, as is Prony's method [6]. Noise performance is a particularly important practical issue, as disturbance records are often plagued by measurement noise and "ghost modes."

A Fourier based method for determining the parameters of a damped oscillating mode was given in [5]. It relies on using two different windows applied at different starting points in the signal. The Fourier transform is applied to the two different windows of the signal, and from these two transforms, the parameters can be estimated. The formulae given in [5] for determining the parameters, however, assume that the windows are "rectangular." This paper proposes that "smoothly tapering" windows be used. These windows help to eliminate unwanted interference, with leads to simplified parameter estimation formulae. Moreover, some rather limiting conditions which were imposed on the window lengths in [5] can be lifted. This relaxation of the restrictions on the window lengths paves the way for procedures to either optimize performance in noise, or more effectively resolve multiple modes.

The parameter estimation method proposed in this paper is discussed in Section II, while Section III considers optimization of the length of the sliding windows in noise. Section IV discusses how the window length should be modified in order to ensure that multiple modes can be properly resolved. Simulations are provided in Section V to show how well the new algorithm works on: a) a single mode, b) a closely spaced modal pair benchmark signal [3], [2], and c) a real power system example. Simulations are also used to validate the optimization formulae given in Section V.

## II. Estimating Modal Parameters in a Decaying Sinusoid

Consider the noiseless real signal, $x(n)$, corresponding to a single oscillating mode which arises after a system disturbance. Assume that it can be modeled by:

$$x(n) = A\cos(\omega_0 n + \phi)e^{-\sigma n}, \qquad n = 0, \cdots, N-1. \quad (1)$$

where

- $A$ is the amplitude,
- $\omega_0$ is the angular frequency,
- $\phi$ is the initial phase, and
- $\sigma$ is the damping factor.

The cyclic sampling frequency is assumed without loss of generality to be unity, and hence the angular sampling frequency is $2\pi$.

Poon and Lee [5] showed that if one takes a window of the signal and then calculates the Fourier transform, there will be an energy concentration around the angular frequency, $\omega_0$, and around $-\omega_0$. They also showed that the Fourier amplitude at $\omega_0$ will be a measure of the signal energy at this frequency. If the Fourier transform is then calculated in another window of the same width, but applied at some time later, the Fourier spectrum would have decreased in magnitude because of the decay of the signal with time. The amount of this decrease is related to the damping factor of the mode, a fact which Poon and Lee were able to use to derive a formula for the damping factor.

An alternative way of looking at the sliding window technique is to consider that an exponentially decaying window is sliding across a fixed, infinite length sinusoidal signal. This is illustrated graphically in Fig. 1 for a mode with $\sigma = 0.003$ s$^{-1}$, $\omega_0 = \pi/16$ rad/s, $\phi = \pi/10$ rad and $A = 1$. The infinite length sinusoid is shown in Fig. 1(a), while Fig. 1(b) shows several snapshots of the exponentially decaying 1024 sample window sliding across the sinusoid. The magnitude spectrum

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The author is with the Department of Communication and Electronic Engineering, Royal Melbourne Institute of Technology, GPO Box 2476V, Melbourne, 3001, Australia.

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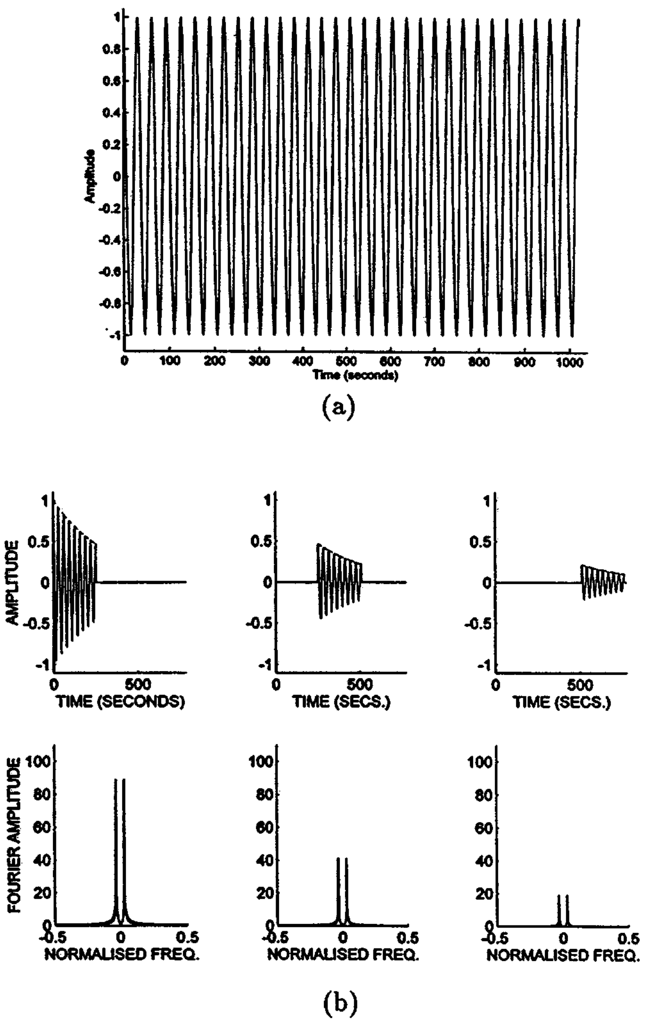

Fig. 1. (a) A sinusoidal signal. (b) Three snapshots of a sliding window of the sinusoid and the corresponding spectra.

of the signal corresponding to each snapshot is also shown in Fig. 1(b). It is seen that there is a clear energy concentration in the low frequency region [in fact it is at $f = \omega/(2\pi) = \pm 1/32$ Hz] in all spectra. The amplitude spectrum maintains an almost constant shape as the window slides, but decreases in magnitude (due to the decay of the sliding window). The rate of decay of the Fourier transform as the window slides was used in [5] to determine the damping factor of the mode.

An important point must be made at this juncture. The estimation of the damping factor is complicated by the fact that there are two separate components in the spectrum of the windowed mode. The first is the "positive frequency" component around $\omega_0$ and the second is the "negative frequency" component around $-\omega_0$. These two components are seen in Fig. 2(a) for the same signal that was used in Fig. 1, but with a window length of 32 samples; the solid line represents the positive frequency component, while the negative frequency component corresponds to the broken line.

It is clear in the figure that these components interfere with, or "leak" into one another. The superposition of these two components is shown in Fig. 2(b). Now as the exponentially decaying window moves along the signal, the spectrum of the positive frequency component retains its shape exactly, but decreases

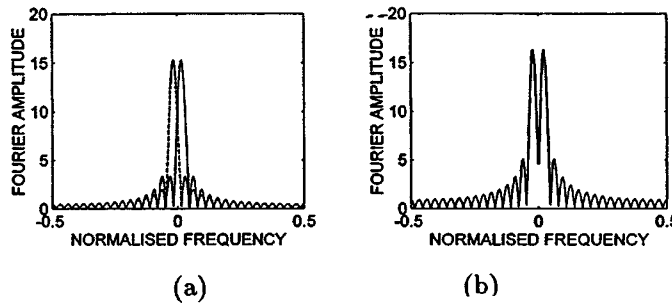

Fig. 2. (a) The positive frequency component (full line) and negative frequency component (dashed line) of a decaying sinusoid. (b) The superposition of the positive and negative frequency components.

in magnitude. The same holds true for the negative frequency component. The shape of the *superposition* of the positive and negative frequency components, however, varies as the window slides, due to the interference between them. Thus, it is not a straightforward matter to determine the damping factor by just monitoring the amplitude change in the spectrum as the window slides along.

This problem of interference has led to tight restrictions on the application of the damping factor estimation formula given in [5]. In particular, Poon and Lee specified a requirement that the window length only have certain discrete values, at which the interference turns out to be zero. This restriction on the window length is disadvantageous for two reasons. First, the allowable window lengths are dependent on frequency and hence a different analysis must be performed for each mode present in the signal.

Second, there is no chance of optimizing the window length for noise performance or for resolution of multiple modes. Realizing that the restrictions on the window lengths arise because of interference, it becomes evident that one can remedy the situation by using "smooth" sliding windows. Smooth windows do not abruptly onset or terminate as do rectangular ones, but rather gradually rise to a peak and gradually fall. Another way of saying this is that for a smooth window, the first (and often second and third) derivative with respect to time has no discontinuities.

A standard rectangular window is shown as a dotted line in Fig. 3(a). A decaying sinusoid ($A = 1$, $\sigma = 0.015$ s$^{-1}$, $\phi = 0$, and $\omega_0 = \pi/2$ rad/s) as seen through this rectangular window is shown as a full line in the same figure. Its spectrum is shown in Fig. 3(c). A smooth "Kaiser" window with a $\beta$ parameter of 7 [4] is shown as a dotted line in Fig. 3(b). The full line curve in Fig. 3(b) shows the decaying sinusoid described above, as seen through the Kaiser window. The spectrum of the smooth windowed mode is shown in Fig. 3(d). Comparison of Fig. 3(c) and (d) shows that the smooth window has much less leakage, and as a result, has separated the positive and negative frequency components more effectively.

Note that there is a significant amplitude difference between Fig. 3(c) and (d). This is because the Fourier transform does not display the amplitude of a signal directly, but rather reveals the *square root of the energy* of the signal as a function of frequency. The window function (both length and shape) has a strong bearing on the spectral magnitude, as it determines what part of the signal energy the Fourier transform "sees." A smooth

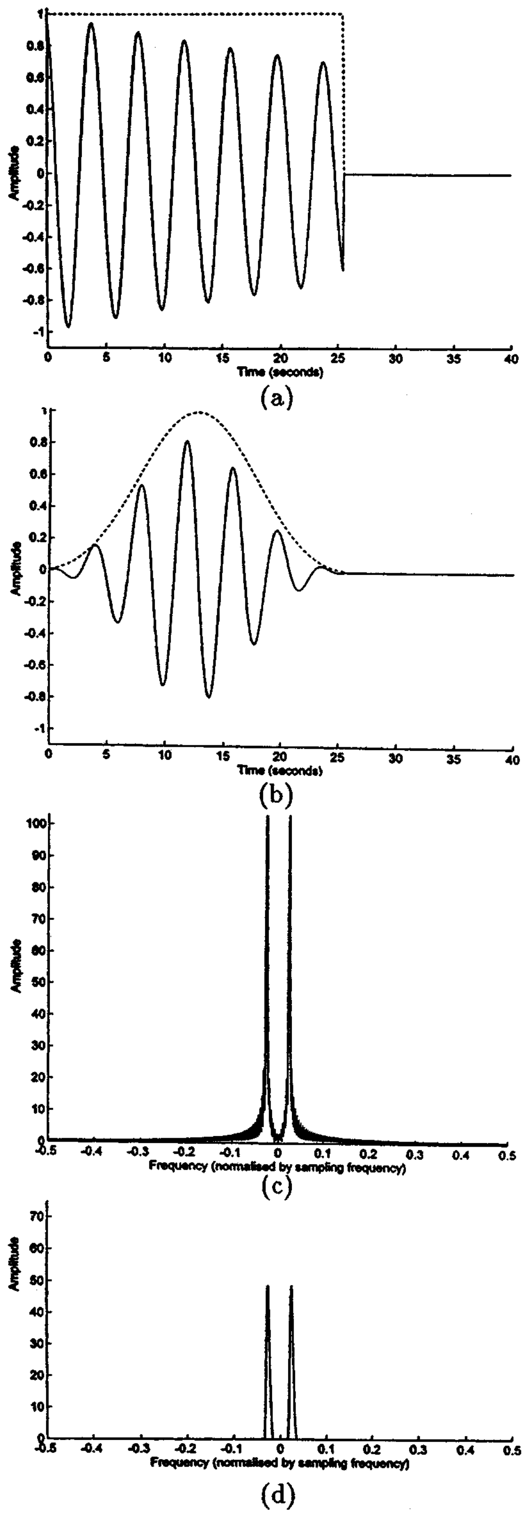


Fig. 3. (a) A mode windowed by a rectangular window, (b) the same mode windowed by a smooth window, (c) the spectrum of the signal in (a), (d) the spectrum of the signal in (b).

window tapers on and off gradually, as seen in Fig. 3(b); because of this, it "lets through" different amounts of signal energy to what a rectangular one does. The precise relationship between the amplitude in the time domain signal and the spectral peak values observed in Fig. 3 is given by equation (19) in Appendix A.

If one applies two smooth windows to a decaying sinusoid at different points in time, the relative amplitudes in the Fourier transforms of the two windows can reliably be used to determine the damping factor. This is explained more fully in Appendix A, (18) of which gives the estimation formula for the damping factor. Formulae for determining the frequency and (complex) amplitude are also given respectively in (11) and (19) of Appendix A. These formulae are similar to those in [5], but are simpler. Unlike those in [5], however, they are valid for all window lengths.

One practical issue needs to be mentioned in using the parameter estimation formulae in (18), (11), and (19). The removal of the interference between positive and negative frequency components will not be effective if the modal frequency is less than a couple of multiples of $1/N_w$ from 0 Hz, where $N_w$ is the window length used. This can be checked, and if necessary, an alternative technique such as Prony's method can be used.

## III. Window Length Optimization in the Presence of Noise

The relaxation in this paper of the window length constraint imposed in [5] brings a number of practical advantages to the sliding window method. First, since the window length no longer depends on $\omega_0$, multiple modes at different frequencies can be processed with the one set of windows. Second, if desired, one can "optimize" the window length for any given mode to enhance noise performance, or to facilitate good resolution of multiple modes. This section considers optimization from a noise perspective.

The optimal window length, $N_{w_{\text{opt}}}$, is found in Appendix B for the case of a decaying sinusoid in additive noise, for a given ratio, $k_3$, of window separation to window length. The optimization is done so as to yield the minimum mean-square error estimate of $\sigma$, this parameter often being the most crucial to estimate in practice. The optimal window length, $N_{w_{\text{opt}}}$ is given by:

$$N_{w_{\text{opt}}} \approx \underset{N_w}{\operatorname{argmax}} \left( \frac{e^{2k_3\sigma N_w} + k_{22}e^{k_3\sigma N_w} + k_{11}}{N_w \left( \sum_{n=0}^{n=N_w} w(n)e^{-\sigma n} \right)^2} \right) \tag{2}$$

where

$$k_{11} = \frac{\text{var}\{q_1\}}{\text{var}\{q_2\}}, \quad k_{22} = \frac{\text{cov}\{q_1 q_2\}}{\text{var}\{q_2\}}, \tag{3}$$

$$k_3 = \frac{N_g}{N_w} \tag{4}$$

where

| | |
|---|---|
| $q_1$ | is the noise contribution in the 1st window at $\omega_0$, |
| $q_2$ | is the noise contribution in the 2nd window at $\omega_0$, |
| var$\{.\}$ | denotes the variance, and |
| cov$\{.\}$ | signifies the covariance. |

$N_g$ is the number of samples between windows, and $N_w + 1$ is the number of samples in each window.

Equation (2) can be conveniently evaluated numerically. Simulations in Section V will verify the results in (2). It should be noted at this point that (2) has meaning only for modes where $\sigma > 0$. Otherwise the optimal window length is infinite. Note also that (2) indicates that the optimal window length can be obtained very simply as long as the damping factor is known. Since this is one of the parameters to be estimated it will not be *known a priori,* but can be *estimated* via a preliminary analysis with arbitrary length windows. The recommended algorithmic procedure is summarized below.

### *A. The Algorithm for Optimizing Noise Performance*

Step 1: Take an FFT of the data and determine the frequency of the mode by extracting the peak in the frequency domain, according to (11). The FFT length should be relatively long (i.e. it should cover most of the data) to give a reasonable estimate of $\omega_0$ in noise.

Step 2: To a *pair* of arbitrary length smooth windows, apply FFT's and use (18) and (19) to determine preliminary estimates for the damping factor and complex amplitude parameters for the mode.

Step 3: Use (2) to determine the (near) optimal window length for the mode. Then recalculate the damping factor and complex amplitude parameters via (18) and (19). (FFTs do not need to be re-calculated; the frequency domain values only need to be calculated at $\hat{\omega}_0$, requiring order $N_{w_{\rm opt}}$ operations.)

*1) The Method of Minimizing the Residual Energy:* In many practical situations, it may be difficult to know *a priori* or estimate the statistical parameters, $k_{11}$ and $k_{22}$. Hence the use of the optimization formula in (2) is problematical. An alternative procedure for setting the window length is to use the "method of minimizing the residual energy." In this method, it is assumed that there is an observed signal, consisting of a mode of the form of (1), plus some additive noise. The smooth sliding window algorithm is used to estimate the signal parameters (amplitude, frequency, phase, and damping factor), and a "signal estimate" is reconstructed from these estimated parameters. This signal estimate is then subtracted from the observed signal, and the result of the subtraction is referred to as the "residual." If the parameter estimates were error free, the residual would consist solely of the additive noise which was on the observed signal. More commonly, the parameter estimates will contain some error, and the residual will contain not only the additive noise but an additional component due to the mismatch between the noise free signal and the signal estimate. The energy of the residual effectively provides a measure of the quality of the parameter estimation process.

One can try to obtain the "best" parameter estimates by varying the window length until the residual energy is minimized. Doing so is equivalent to finding the window length which gives rise to parameter estimates with the least square error. It is also possible to use the method of minimizing the residual to optimize other variables such as the amount of overlap used.

## IV. Extension to Multiple Modes

The techniques presented for single modes are readily extendable to multiple modes. When multiple modes are present, their frequency contributions will not interfere substantially, providing that smooth windows are used and that the frequencies are adequately resolved. In practice, this means the modal frequencies must be separated by more than about 3 times $1/N_w$. The precise separation required will depend on the window type used.

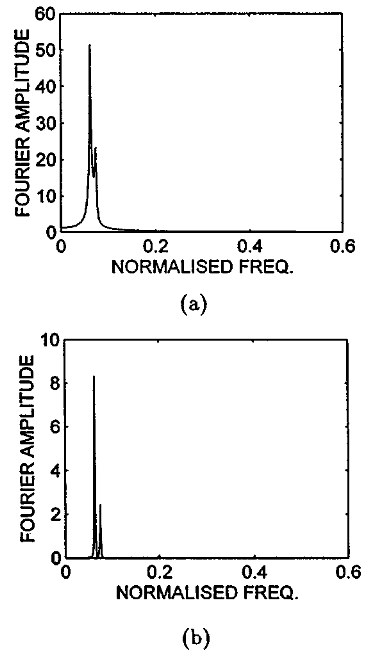


Fig. 4. The spectrum of a modal pair windowed with (a) a rectangular window, and (b) a smooth window.

The effectiveness of the sliding smooth window method for multiple modes is now shown with a graphical example. A signal is chosen with two modes whose frequencies are sufficiently separated so that they do not interfere. The two modes have different damping factors and amplitudes. The signal is specified by:

$$\begin{aligned} x(n) = &0.8 \cos\left(\frac{\pi}{8}n + \frac{\pi}{2}\right)e^{-0.008n} \\ &+ 0.4 \cos\left(\frac{\pi}{6.75}n - \frac{\pi}{4}\right)e^{-0.01n} \end{aligned} \tag{5}$$

Fig. 4(a) shows the spectrum of the signal after being windowed by a 1024 point rectangular window, while Fig. 4(b) shows the spectrum of the signal which has been windowed with a 1024 point Kaiser window (with $\beta = 15$). The two different modes in Fig. 4(b) are clearly separated, permitting easy processing of the two modes.

It is worth noting that the spectral peak values in Fig. 4(b) are not proportional to their amplitude values in the time domain signal. This is because the window has a large bearing on the amount of energy "let through" into the Fourier transform. In particular, because a smooth window turns on more slowly than does a rectangular one, it tends to give rise to relatively diminished spectral peaks for more heavily damped modes. This is why the more heavily damped mode appears proportionally smaller in Fig. 4(b) than it does in Fig. 4(a).

## V. Simulations

### *A. Parameter Estimation for a Single Mode, "Test1"*

The first signal tested was "Test1," a decaying sinusoid immersed in some white additive noise, with a signal to noise

TABLE I
TRUE AND ESTIMATED VALUES FOR A SINGLE MODE

| Modal parameter | True value | Estimated value (10dB noise) |
|---|---|---|
| Damping factor | -0.030000 sec$^{-1}$ | -0.030000 sec$^{-1}$ |
| Frequency | 0.714285 Hz | 0.714285 Hz |
| Amplitude | 1.000000 | 0.992592 |
| Phase | 0.000000 radians | 0.000575 radians |

TABLE II
TRUE AND ESTIMATED VALUES FOR A TWO COMPONENT BENCHMARK SIGNAL [3]

| Modal parameter | True value | Estimated value |
|---|---|---|
| Damping factor of 1st mode | 0.020000 sec$^{-1}$ | 0.020041 sec$^{-1}$ |
| Frequency of 1st mode | 0.021484 Hz | 0.021484 Hz |
| Amplitude of 1st mode | 1.000000 | 0.992592 |
| Phase of 1st mode | 0.0 radians | 0.000575 radians |
| Damping factor of 2nd mode | -0.005 sec$^{-1}$ | -0.004878 sec$^{-1}$ |
| Frequency of 2nd mode | 0.666666 Hz | 0.667114 Hz |
| Amplitude of 2nd mode | 1.000000 | 1.002685 |
| Phase of 2nd mode | 0.000000 radians | -0.041812 radians |

power ratio (SNR) of 10 dB. The parameters of "Test1" are $A = 1, \omega_0 = 4.488$ rad/s, $\phi = 0$, and $\sigma = 0.03\,\mathrm{s}^{-1}$. The smooth sliding window algorithm was applied to the signal (using a window length of 180 samples, a delay between windows of 20 samples, and a Kaiser window with $\beta$ parameter of 15). A sampling frequency of 1 was used, in this and all subsequent simulations. These true parameter values are listed in the first column of Table I, while the second column shows the parameter estimates actually obtained. Because the estimation technique is Fourier based, it is robust to additive noise [6], and the estimate errors are seen to be very small.

### B. Parameter Estimation for Benchmark Signal, "Test2"

The second signal analyzed was the benchmark signal proposed in [3] and [2] for damping factor estimators. This signal has two closely spaced modes, with the parameters of the two modes being shown in the first column of Table II. The estimates obtained for the parameters (using a Kaiser window with a $\beta$ factor of 5) are shown in the second column of Table II. It must be said that the example was a difficult one because of the closeness of the modal frequencies. The signal was analyzed by estimating the parameters of the highest energy mode first, then subtracting an estimate of this mode from the time domain, and finally analyzing the second mode. The estimates were all found to be quite close to the true values.

### C. Verification of Window Length Optimization Formula

The first signal used for this section was of the form of (1), with $\omega_0 = \pi/2$, $\sigma = 0.03$, $\phi = 0$, and $A = 1$. Stationary, white noise of power, 1/500, was added to the signal. The two windows were 1024 sample Kaiser windows (with a $\beta$ parameter of 15), and they were consecutive in time. The optimal window length may be determined theoretically from (2). Since the noise is white and stationary, $k_{11} = 1$, $k_{22} = 0$, and since there is no window overlap, $k_3 = 1$. $N_{w_{\rm opt}}$ can be found numerically from (2) to be:

$$N_{w_{\rm opt}} = 36 \quad (6)$$

1000 simulations were run with the mean square error in the damping factor estimate being plotted in Fig. 5 as a function of window length. The error is seen to be minimized at a window length of 36, compared with the theoretical prediction of 36.

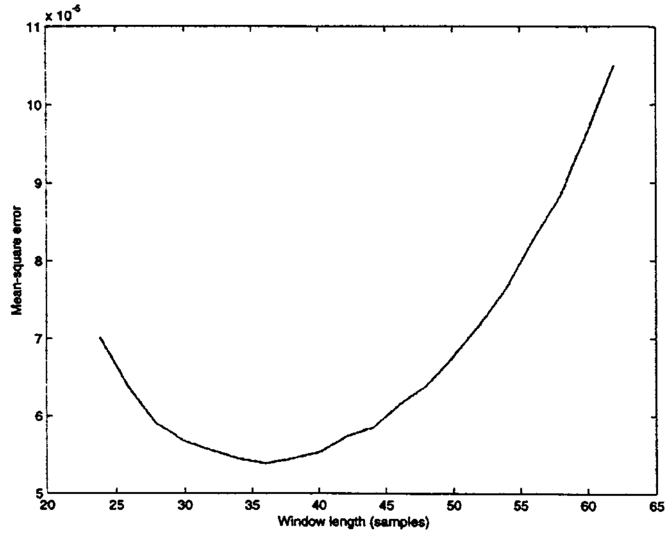


Fig. 5. Mean-square error of $\hat{\sigma}$ vs. window length.

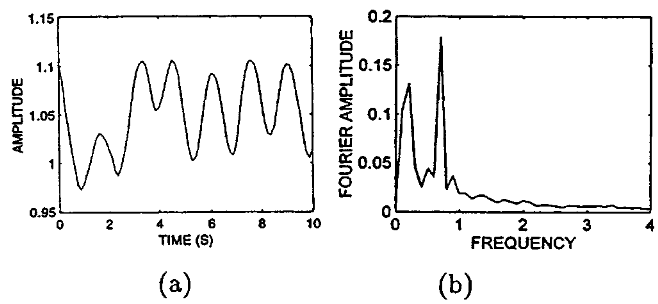


Fig. 6. (a) The swing curve from Pacific Gas and Electric Co. [5], (b) its spectrum.

A second simulation was run in which the noise was colored with 25 percent window overlap. (i.e. $k_3 = 0.75$). The statistical parameters, $k_{11}$ and $k_{22}$ were given by $k_{11} = 1$, $k_{22} = 0.55$. The simulation showed the minimum error to occur at 49, compared with the theoretically predicted value of 46.

### D. Parameter Estimation for a Real Power System Example

The real power system data used was the voltage swing curve of a test case by the Pacific Gas and Electric Co. first presented in [5]. The swing curve consisted of two modes, one of which was a component at around 0.67 Hz with potentially problematical damping, (its existence has been known in the Western Systems Coordinating Council interconnected system). The signal and the spectrum of the two modes are shown respectively in Fig. 6(a) and (b). (The mean was removed before forming the spectrum). The two modes are seen clearly, one at 0.22 Hz, and the other at 0.67 Hz.

In [5] a sliding window of length 2.2445 seconds was used in order to determine the damping factor for the mode at 0.67 Hz. This length was chosen so as to satisfy the window length constraints given in (5) (i.e. so as to eliminate interference between positive and negative frequency components). For the mode at $+0.67$ Hz, however, the use of such a window length is highly questionable; the interference from the positive frequency component at $+0.22$ Hz, being relatively close, would be expected to be much greater than that from the component at $-0.67$ Hz. The interference between the components at $+0.67$ and $+0.22$ Hz is also one of the probable reasons why a strong frequency drift of the $+0.67$ Hz component was reported to have occurred in (5). When a longer smoother window was used so that interference between the components at $+0.22$ and $+0.67$ Hz was effectively eliminated, the drift in frequency between windows was observed to be minimal.

It should be noted that the restriction on the window lengths probably contributed significant error to the estimates of frequency, damping factor and amplitude in [5]. This could be surmised, despite the true parameter values being unknown, by examining the energy of the residual. The parameter estimates found in [5] gave rise to a comparatively high energy residual. The parameter estimates found using the techniques proposed in this paper yielded a much lower energy residual, i.e. the estimates found in this paper had statistically higher likelihood of being the true values. This is explained further in the next paragraph.

The smooth sliding window method was used to first determine the frequency and damping factor of the mode around 0.67 Hz. The frequency estimate was found to be 0.672 Hz. The window length, starting time and $\beta$ parameter (for a Kaiser taper function) were selected by a certain amount of trial and error so that a low energy residual was obtained. The starting time, length, and $\beta$ parameter respectively of the first window were selected to be 0.86 s, 3.45 s, and 2. The second window was applied at 6.42 s, with the length and $\beta$ parameter being kept the same as in the first window. The damping factor was found to be $-0.0211$ s$^{-1}$, which is very different to the value of $+0.0137$ s$^{-1}$ obtained in [5]. The amplitude and phase for the mode were found to be 0.0376 and $-0.6131$ rad respectively. It should be noted that since the energy of the residual corresponding to these values was lower (by 34 percent) than the residual corresponding to the estimates given in [5], the statistical likelihood of the values in this paper being correct, is substantially higher.

The mode at 0.22 Hz was analyzed after subtracting an estimate of the modal component at 0.67 Hz in the time domain, as in [5]. A pair of Kaiser windows of length, 4.2 s, and $\beta$ parameter, 0, were used. The first window was applied at 0 seconds and the second window was applied at 4.82 s. All of these values were obtained again by using the method of minimizing residual energy. The frequency, damping factor, amplitude and phase subsequently found were 0.219 Hz, 0.265 s$^{-1}$, 0.041, and 1.293 rad, respectively.

## VI. Conclusion

A Fourier based algorithm for estimating the parameters of a damped oscillating mode has been presented. The algorithm uses the spectra of a pair of possibly overlapping smooth windows, with the length of the windows able to be chosen so as to provide good frequency resolution or optimal noise performance. A theoretical analysis has been presented and simulations have verified the effectiveness of the theoretical results.

## Appendix A

Consider the real signal model specified in (1). Assume the window function used, $w(n)$, is smooth and has $N_w+1$ samples. If the signal is windowed starting at $n_1$ and ending at $n_1+N_w$ the Fourier transform will be given by:

$$F_{n_1}(\omega) = \mathcal{F}[w(n-n_1)A\cos(j\omega_0 n+\phi)e^{-\sigma n}] \tag{7}$$
$$= \mathcal{F}[w(n-n_1)e^{\sigma n}] * \mathcal{F}[A\cos(j\omega_0 n+\phi)] \tag{8}$$
$$= \mathcal{F}[w(n-n_1)e^{\sigma n}] * \frac{Ae^{j\phi}}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \tag{9}$$
$$= \frac{Ae^{h\phi}}{2}[SW_{n_1}(\omega-\omega_0)+SW_{n_1}(\omega+\omega_0)] \tag{10}$$

where

$\mathcal{F}$ denotes the (discrete-time) Fourier transform [4],
$*$ signifies convolution,
$\delta(.)$ is the impulse function, and
$SW_{n_1}(\omega)$ $= \mathcal{F}[w(n-n_1)e^{-\sigma n}]$.

If the window, $w(n)$, smoothly tapers and $\sigma$ is finite, then $SW(\omega)$ will peak either at, or extremely close to the origin. The modal frequency estimate may therefore be determined from the relation:

$$\hat{\omega}_0 = \underset{\omega}{\text{argmax}}\, F_{n_1}(\omega) \tag{11}$$

The value of the Fourier transform at $\omega_0$ is given by:

$$F_{n_1}(\omega_0) = \frac{Ae^{j\phi}}{2}(SW_{n_1}(0)+SW_{n_1}(2\omega_0)) \tag{12}$$
$$\approx \frac{Ae^{j\phi}}{2}SW_{n_1}(0) \tag{13}$$

[since $SW_{n_1}(\omega)$ will have dropped very close to 0 at $\omega = 2\omega_0$]. It is assumed that the second window starts at time, $n_2 = n_1 + N_g$, and ends at $n_2 + N_w$. Then the Fourier transform at $\omega_0$ in this second window is:

$$F_{n_2}(\omega_0) \approx \frac{Ae^{j\phi}}{2}SW_{n_2}(0) \tag{14}$$

The ratio of the two windowed Fourier transform amplitudes at $\omega_0$ is:

$$\frac{F_{n_2}(\omega_0)}{F_{n_1}(\omega_0)} = \frac{SW_{n_2}(0)}{SW_{n_1}(0)}$$
$$= \frac{\sum_{n=n_1+N_g}^{n_1+N_g+N_w} w(n)e^{-\sigma n}}{\sum_{n=n_1}^{n_1+N_w} w(n)e^{\sigma n}} \tag{15}$$
$$= \frac{e^{-\sigma(n_1+N_g)}\sum_{n=0}^{N_w} w(n)e^{-\sigma n}}{e^{-\sigma(n_1+N_g)}\sum_{n=0}^{N_w} w(n)e^{-\sigma n}} \tag{16}$$
$$= e^{-\sigma N_g} \tag{17}$$

Rearranging (17) yields the following estimation formula for the damping factor:

$$\hat{\sigma} = \frac{1}{N_g} \log \left[ \frac{F_{n_2}(\omega_0)}{F_{n_1}(\omega_0)} \right] \tag{18}$$

Eqn. (13) can be rearranged to give the following estimation formula for the "complex amplitude":

$$A\hat{e}^{j\phi} = 2F_{n_1}(\omega_0)/SW_{n_1}(0) \tag{19}$$

## APPENDIX B

Assume that the signal of interest is of the form of (1), but with some additive noise present. The noise may be colored. Using (13), the Fourier transform in the first window, evaluated at $\omega_0$, is:

$$F_{n_1}(\omega_1) \approx \frac{Ae^{j\phi}SW_{n_1}(0)}{2} + q_1 \tag{20}$$

$$= \frac{Ae^{j\phi}SW_{n_1}(0)}{2} \left(1 + \frac{2q_1}{Ae^{j\phi}SW_{n_1}(0)}\right) \tag{21}$$

where $q_1$ is a random variable corresponding to the noise contribution at $\omega_0$. Similarly, the complex Fourier amplitude at $\omega_0$ in the second window is given by:

$$F_{n_2}(\omega_0) \approx \frac{Ae^{j\phi}SW_{n_2}(0)}{2} \left(1 + \frac{2q_2}{Ae^{j\phi}SW_{n_2}(0)}\right) \tag{22}$$

where $q_2$ is a random variable corresponding to the noise contribution at $\omega_0$ in this second window.

The ratio of the two Fourier amplitudes, $F_{n_1}(\omega_0)$ and $F_{n_2}(\omega_0)$, will be:

$$\frac{F_{n_2}(\omega_0)}{F_{n_1}(\omega_0)} \approx \frac{SW_{n_2}(0)}{SW_{n_1}(0)} \frac{\left(1 + \frac{2q_2}{Ae^{j\phi}SW_{N_g}(0)}\right)}{\left(1 + \frac{2q_1}{Ae^{j\phi}SW_0(0)}\right)} \tag{23}$$

$$= e^{-\sigma N_g} \frac{\left(1 + \frac{2q_2 e^{\sigma N_g}}{Ae^{j\phi}SW_{n_1}(0)}\right)}{\left(1 + \frac{2q_1}{Ae^{j\phi}SW_{n_1}(0)}\right)} \tag{24}$$

[using (17) from Appendix A].

It is assumed that the noise to signal ratio at $\omega_0$ in the Fourier transform of both windows is small. Then the following approximation can be made:

$$\frac{F_{n_2}(\omega_0)}{F_{n_1}(\omega_0)} \approx e^{-\sigma N_g} \left(1 + \frac{2q_2 e^{\sigma N_g} - 2q_1}{Ae^{j\phi}SW_{n_1}(0)}\right) \tag{25}$$

[since $1/(1+x) \approx (1-x)$ if $x$ is small]. Substituting (25) into (18) gives:

$$\hat{\sigma} \approx \sigma - \frac{1}{N_g} \log \left(1 + \frac{2q_2 e^{\sigma N_g} - 2q_1}{Ae^{j\phi}SW_{n_1}(0)}\right) \tag{26}$$

$$\approx \sigma - \left(\frac{2(q_2 e^{\sigma N_g} - q_1)}{N_g Ae^{j\phi}SW_{n_1}(0)}\right) \tag{27}$$

[since $\log(1+x) \approx x$ if $x$ is small].

The variance of $\hat{\sigma}$ as given in (27) is:

$$\mathrm{var}(\hat{\sigma}) \approx (e^{2\sigma N_g}\mathrm{var}(q_2) + \mathrm{var}(q_1) + e^{\sigma N_g}\mathrm{cov}(q_1 q_2)) \cdot \left(\frac{4}{N_g^2 A^2 e^{j2\phi} SW_{n_1}(0)^2}\right) \tag{28}$$

where var(.) and cov(.) denote the variance and covariance functions respectively.

Now $\mathrm{var}(q_1) = k_1 N_w$ and $\mathrm{var}(q_2) = k_2 N_w$ for a smooth sliding window of arbitrary length, where $k_1$ and $k_2$ are constants which are dependent on the shape of the window and the spectral character of the noise in each window [6]. The covariance between $q_1$ and $q_2$ will be given by $\mathrm{cov}(q_1 q_2) = k_{12}$, where again, $k_{12}$ is a constant which is dependent on the window shape, the spectral character of the noise *and* the percentage window overlap.

Using these variance expressions, one can expand the expression for the variance of the $\sigma$ estimate as:

$$\mathrm{var}(\hat{\sigma}) \approx \frac{4N_w(e^{2\sigma N_g}k_2 + k_{12}e^{\sigma N_g} + k_1)}{N_g^2 N_w A^2 e^{j2\phi} SW_{n_1}(0)^2} \tag{29}$$

Substitution of $SW_{n_1}(0) = \sum_{n=0}^{n=N_w} w(n)e^{-\sigma n}$, $N_g/N_w = k_3$, $k_{11} = k_1/k_2$, and $k_{22} = k_{12}/k_2$ into (29) yields:

$$\mathrm{var}(\hat{\sigma}) \approx \frac{4k_2(e^{2k_3\sigma N_w} + k_{22}e^{k_3\sigma N_w} + k_{11})}{k_3^2 N_w A^2 e^{j2\phi} \left(\sum_{n=0}^{n=N_w} w(n)w^{-\sigma n}\right)^2} \tag{30}$$

In minimizing the above expression with respect to $N_w$ (while keeping $k_3$, the ratio of the window separation to window length constant), multiplicative constants can be neglected. Thus the optimal window length is given by:

$$N_{w_{\mathrm{opt}}} \approx \underset{N_w}{\mathrm{argmax}} \left( \frac{(e^{2k_3\sigma N_w} + k_{22}e^{k_3\sigma N_w} + k_{11})}{N_w \left(\sum_{n=0}^{n=N_w} w(n)e^{-\sigma n}\right)^2} \right) \tag{31}$$

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**Peter O'Shea** received the B.E., Dip.Ed., and Ph.D. from the University of Queensland in 1978, 1983 and 1991 respectively. He has worked as an Engineer at the Overseas Telecommunications Commission for three years, and at the School of Electrical and Electronic Systems Engineering, Queensland University of Technology for three years in a postdoctoral and lecturing capacity. He is currently lecturing in the Department of Communication and Electronic Engineering at RMIT, Melbourne, Australia.