

The use of the Adomian decomposition method for solving multipoint boundary value problems

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Abstract

In this paper, a method for solving multipoint boundary value problems is presented. The main idea behind this work is the use of the well-known Adomian decomposition method. In this technique, the solution is found in the form of a rapid convergent series. Using this method, it is possible to obtain the solution of the general form of multipoint boundary value problems. The Adomian decomposition method is not affected by computation round off errors and one is not faced with the necessity of large computer memory and time. To show the efficiency of the developed method, numerical results are presented.

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1. Introduction

The Adomian decomposition method is useful for obtaining the closed form and numerical approximations of linear or nonlinear differential equations. This method has been applied to obtain formal solutions to a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, integro-differential, differential delay, integral and partial differential equations.

The method was proposed by the American mathematician, G Adomian (1923–96). It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using Adomian polynomials [1–3].

Generally this method is useful for problems that can be written in the following form which appears in a large number of problems in applied sciences

$$u - \Theta(u) = g, \quad (1.1)$$

where u is unknown, Θ usually is a nonlinear operator and g is given. Depending on the nonlinear form Θ , the Adomian decomposition method can be considered as an efficient method. This technique has many advantages over classical techniques. It avoids perturbation in order to find solutions of given nonlinear equations.

In recent years, a lot of attention has been paid to the study of the Adomian decomposition method to investigate various scientific models. This method is applied to solve various kinds of ordinary differential equations. In particular, this method is useful for nonlinear differential equations [3]. Furthermore, this method is used for finding the numerical solution of higher order differential equations in [4–9].

Another method for solving various types of problem is proposed by He which is known as the homotopy perturbation method [10]. It is shown that the Adomian decomposition method could not always satisfy all boundary conditions in solving partial differential equations [11]. Some new approaches for overcoming this difficulty have been proposed in [10, 12]. In recent years, the variational iteration method has been widely applied for solving different kinds of problems [13, 14]. This method does not need calculation of the Adomian polynomials which can be considered as the main advantage of the variational iteration method over the Adomian decomposition method.

The present work is aimed at producing approximate solutions which are obtained in rapidly convergent series with elegantly computable components by the Adomian decomposition technique. It is well known in the literature that the decomposition method provides the solution in a rapidly convergent series where the series may lead to the solution in a closed form if it exists. The rapid convergence

of the solution is guaranteed by the work conducted by Cherruault *et al* [15, 16].

Consider the following ordinary differential equation of order n

$$y^{(n)} = g(x, y, y', \dots, y^{(n-1)}). \tag{1.2}$$

Without loss of generality let $x \in [0, 1]$. Also, it is assumed that g has properties which guarantee the existence and uniqueness of the solution of the problem. If the solution or its derivatives are given at m points ($m \leq n$), then the problem is said an m -point boundary value problem (BVP).

Multipoint BVPs for ordinary differential equations appear in modelling of some physical problems. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint BVP. Also in [17] many problems in the theory of elastic stability are handled by multipoint problems.

As a small sample of some theoretical works on these kind of problems, see [18–20]. Despite of the large amount of works which are done on the theoretical aspects of these kind of equations, few works are available on the numerical analysis of multipoint BVPs. For some numerical methods for solving multipoint ordinary differential equations, refer to [21, 22].

In this paper, the application of the Adomian decomposition method [23, 24] for finding an approximate solution for multipoint BVPs has been investigated.

The organization of the rest of this paper is as follows; in section 2, the Adomian decomposition method is applied to some ordinary differential equations with given multipoint boundary conditions. To present a clear overview of the method, several examples have been shown in section 3. A conclusion is presented in section 4.

2. Solution using the Adomian decomposition method

Consider the operator form of an ordinary differential equation in the following form

$$L(y) - N(y) = f, \tag{2.1}$$

for $0 \leq x \leq 1$ where $L = d^n/dx^n$ is the n th-order derivative operator, N usually is a nonlinear operator which contains differential operators with order less than two and f is a given function. Assume that the inverse operator L^{-1} exists and it can conveniently be taken as the definite integral for a function in the following form

$$L^{-1}(\cdot) = \int_0^x \int_0^{t_n} \int_0^{t_{n-1}} \dots \int_0^{t_2} (\cdot) dt_1 \dots dt_{n-1} dt_n. \tag{2.2}$$

Applying the inverse operator L^{-1} to both sides of (2.1) yields

$$L^{-1}L(y) = L^{-1}N(y) + L^{-1}f. \tag{2.3}$$

Thus we have

$$y(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} y^{(k)}(0) = L^{-1}N(y) + L^{-1}f, \tag{2.4}$$

or equivalently

$$y(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} y^{(k)}(0) + L^{-1}f + L^{-1}N(y). \tag{2.5}$$

Set $A_k = y^{(k)}(0)$ for $0 \leq k \leq n - 1$. Now according to the decomposition procedure of Adomian, we construct the unknown function $y(x)$ by a sum of components defined [4, 5, 25] by the following decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{2.6}$$

where

$$y_0(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} A_k + L^{-1}f(x), \tag{2.7}$$

$$y_{n+1}(x) = L^{-1}N_n, \quad n \geq 0. \tag{2.8}$$

Based on the Adomian decomposition method, solution of the equation (2.1) is considered as the series (2.6) and take the nonlinear expressions $N(y)$ by the infinite series of the Adomian polynomials given by

$$N(y) = \sum_{n=0}^{\infty} N_n, \tag{2.9}$$

where the component N_n is an appropriate Adomian polynomial which is calculated using the method introduced in [1]. Adomian polynomials are found by calculating the nonlinear operator N_n in the following form:

$$N_n(y_0, y_1, \dots, y_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left[\sum_{k=0}^{\infty} \lambda^k y_k \right] \right]_{\lambda=0}, \quad n \geq 0. \tag{2.10}$$

Notice that if N is a linear operator then we have $N_n = y_n$. In some nonlinear problems, it is not easy to calculate the Adomian polynomials easily. But using a few number of Adomian polynomials results an accurate approximation of the problem [7, 26, 27].

The resulting solution converges [15] to the closed form of the problem. The most important work about convergence has been carried by Cherruault [15]. Other references about theoretical treatments of convergence of Adomian decomposition method are found in [2]. A new approach of convergence of the decomposition method is presented by Ngarhasta *et al* [28].

By calculating the terms y_0, y_1, y_2, \dots , the solution y can be obtained. Based on the Adomian decomposition method [8, 9, 29], we constructed the solution y as

$$y = \lim_{n \rightarrow \infty} \phi_n, \tag{2.11}$$

where the $(n + 1)$ -term approximation of the solution is defined in the following form

$$\phi_n = \sum_{k=0}^n y_k(x), \quad n > 0. \tag{2.12}$$

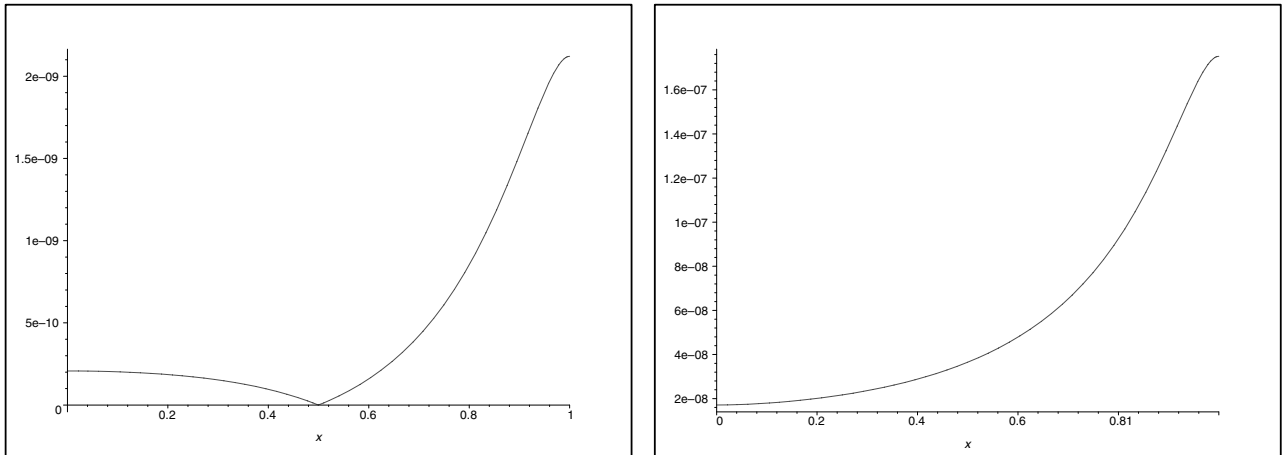


Figure 1. Plot of the $|\phi_{10} - y|$ (left) and $|\phi_{10} - y|/|y|$ (right).

The solution here is given in a series form that generally converges very rapidly in real physical problems.

Applying decomposition procedure of Adomian, it is found that the series solution of $y(x)$ follows with constants A_k for $0 \leq k \leq n$ which are unknown. To find these constants, the boundary conditions at other points are imposed. This enables us to obtain the approximation of the solution defined in (2.12) which results in a system of $n + 1$ equations with $n + 1$ unknowns A_k for $0 \leq k \leq n$. By solving this equation that usually is nonlinear, the values A_k and the solution of the multipoint BVP follow immediately.

3. Test examples

3.1. Example 1

As the first example consider the following third-order linear differential equation with its boundary conditions at three different points [21]

$$y''' - k^2y' + a = 0, \tag{3.1}$$

$$y'(0) = y'(1) = 0, \quad y(0.5) = 0, \tag{3.2}$$

where the physical constants are $k = 5$ and $a = 1$. The function $y(x)$ shows the shear deformation of sandwich beams. The analytic solution of this problem is written as

$$y(x) = \frac{a}{k^3} \left(\sinh \frac{k}{2} - \sinh kx \right) + \frac{a}{k^2} \left(x - \frac{1}{2} \right) + \frac{a}{k^3} \tanh \frac{k}{2} \left(\cosh kx - \cosh \frac{k}{2} \right). \tag{3.3}$$

Applying the Adomian decomposition method to the problem we have

$$y = y(0) + xy'(0) + \frac{1}{2}x^2y''(0) - L^{-1}(1) + L^{-1}(25y'(t_1)), \tag{3.4}$$

where

$$L^{-1}(\cdot) = \int_0^x \int_0^{t_3} \int_0^{t_2} dt_1 dt_2 dt_3. \tag{3.5}$$

In this problem, we have $A_1 = 0$. Thus we obtain

$$y_0 = A_0 + A_2 \frac{1}{2}x^2 - \frac{1}{6}x^3, \tag{3.6}$$

$$y_1 = \frac{25}{24}A_2x^4 - \frac{5}{24}x^5, \tag{3.7}$$

and so on. Using 11 terms of series solution we can write

$$A_0 = -0.1210, \quad A_2 = 0.1973.$$

In figure 1, the error function $|\phi_{10} - y|$ and $|\phi_{10} - y|/|y|$ are plotted. Other components of the series solution are found easily. Better approximations can be obtained using more components of the series solution.

3.2. Example 2

In this example, we consider the four-point fourth-order nonlinear ordinary differential equation [21]

$$y^{(4)} + yy' - 4x^7 - 24 = 0, \tag{3.8}$$

$$\begin{aligned} y(0) &= 0, & y^{(3)}(0.25) &= 6, \\ y''(0.5) &= 3, & y(1) &= 1. \end{aligned} \tag{3.9}$$

The exact solution of this problem is $y(x) = x^4$. In this problem, we have

$$y = y(0) + xy'(0) + \frac{1}{2}x^2y''(0) + \frac{1}{6}x^6y^{(3)}(0) + L^{-1}(4t_1^7 + 24) - L^{-1}(y(t_1)y'(t_1)), \tag{3.10}$$

where

$$L^{-1}(\cdot) = \int_0^x \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} (\cdot) dt_1 dt_2 dt_3 dt_4. \tag{3.11}$$

In this example, we have $A_0 = 0$. Now using the Adomian decomposition method we get

$$y_0 = A_1x + \frac{1}{2}A_2x^2 + \frac{1}{6}A_3x^3 + \frac{1}{1980}x^{11} + x^4, \tag{3.12}$$

$$y_n = -L^{-1}(N_n), \quad n \geq 1, \tag{3.13}$$

where $N_n(x) = N_n(y_0(x), y_1(x), \dots, y_n(x))$ are Adomian polynomials of nonlinear operator $N(y) = yy'$.

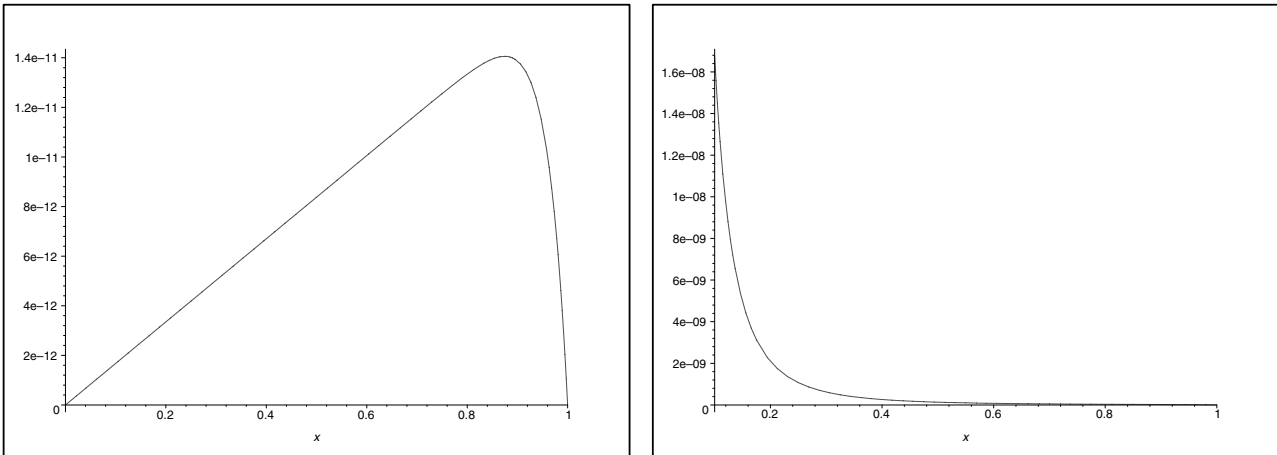


Figure 2. Plot of the $|\phi_2 - y|$ (left) and $|\phi_2 - y|/|y|$ (right).

In figure 2, the results are shown using only three terms of the Adomian decomposition method. In this case, we have

$$A_1 = -0.1674e - 10, \quad A_2 = -0.3665e - 13, \\ A_3 = -0.1635e - 13.$$

3.3. Example 3

Consider the following three-point second-order nonlinear ordinary differential equation [30]

$$y'' + \frac{3}{8}y + \frac{2}{1089}[y']^2 + 1 = 0, \quad 0 \leq x \leq 1, \quad (3.14)$$

$$y(0) = 0, \quad y(\frac{1}{3}) = y(1). \quad (3.15)$$

In this problem, we have

$$y = y(0) + xy'(0) - L^{-1}(1) - L^{-1}(\frac{3}{8}y(t_1) - \frac{2}{1089}y'(t_1)), \quad (3.16)$$

where

$$L^{-1}(\cdot) = \int_0^x \int_0^{t_2} (\cdot) dt_1 dt_2. \quad (3.17)$$

In this example, we have $A_0 = 0$. Now using the Adomian decomposition method we have

$$y_0 = A_1x - \frac{1}{3}x^2, \quad (3.18)$$

$$y_n = -L^{-1}(N_n), \quad n \geq 1, \quad (3.19)$$

where $N_n(x) = N_n(y_0(x), y_1(x), \dots, y_n(x))$ are Adomian polynomials of nonlinear operator $N(y) = [y']^2$.

In table 1, the results of the present method and the results of the successive iteration method introduced in [30] are shown using only three terms of both methods. In this case, we have $A_1 = 0.7065$.

4. Conclusion

The Adomian decomposition method is applied to multipoint BVPs successfully. This method provides an accurate approximation of the solution. As a main advantage

Table 1. The results of the Adomian decomposition method and the successive iteration method.

x	Adomian decomposition method	Successive iteration method
0.1	0.0656	0.0656
0.2	0.1209	0.1211
0.3	0.1658	0.1661
0.4	0.2001	0.2004
0.5	0.2236	0.2240
0.6	0.2363	0.2367
0.7	0.2382	0.2385
0.8	0.2291	0.2295
0.9	0.2091	0.2095

of this method over traditional numerical methods, the decomposition procedure of Adomian does not require discretization of the solution. Therefore, unlike other numerical methods, this method does not result in any large system of linear or nonlinear equations. Thus it is not affected by computation round off errors and the solution is found without taking a long time and a large amount of computer memory. The Adomian decomposition method provides a closed form of the solution.

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