THE USE OF THE MELLIN TRANSFORM IN FINDING THE STRESS DISTRIBUTION IN AN INFINITE WEDGE

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SUMMARY

By making use of the Mellin transform a formal solution is obtained for the stress distribution in an infinite wedge under fairly general conditions of surface loading. The results for the particular case in which each surface is subjected to unit pressure for a finite distance measured from the vertex of the wedge are reduced to infinite integrals. These can be evaluated exactly when the wedge is a semi-infinite solid and are in a form suitable for numerical computation for other wedge angles.

1. FORMAL solutions for fairly general conditions of surface loading have recently been obtained by Tranter and Craggs (1) and Sneddon (2) in the case of axially symmetrical stress. If the axis of symmetry is the z-axis and the loading is on the curved surfaces $r =$ constant, Tranter and Craggs make use of the complex form of the Fourier integral transform, while for loading on the plane surfaces $z =$ constant, Sneddon employs the Hankel transform. A formal solution for the stress distribution in an infinite wedge under similar fairly general surface tractions can be obtained by using the Mellin transform. Such a solution would appear to cover cases (e.g. discontinuities) which are excluded from the solution given by Timoshenko (3) for a polynomial distribution of load and to give a more direct result than could be obtained from an extension of Shepherd's work (4) for isolated forces acting at points of the faces of the wedge.

2. Taking cylindrical polar coordinates (r, θ, z) and taking the faces of the wedge as $\theta = \pm\alpha$, we have to find a stress function ϕ satisfying†

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2\phi = 0 \quad (0 < r < \infty,\ -\alpha < \theta < \alpha), \tag{1}$$

and the boundary conditions

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = f_1(r) \quad (\theta = \alpha)$$
$$= f_2(r) \quad (\theta = -\alpha), \tag{2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = s_1(r) \quad (\theta = \alpha)$$
$$= s_2(r) \quad (\theta = -\alpha), \tag{3}$$

† Timoshenko (3), pp. 53, 55.

where $f_1(r)$, $f_2(r)$ are the normal stresses and $s_1(r)$, $s_2(r)$ the shear stresses applied to the faces of the wedge, all supposed given functions of r. Once ϕ has been found, the stresses σ_θ, $\tau_{r\theta}$ are given by the expressions shown in (2) and (3), while the third stress is given by

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}. \tag{4}$$

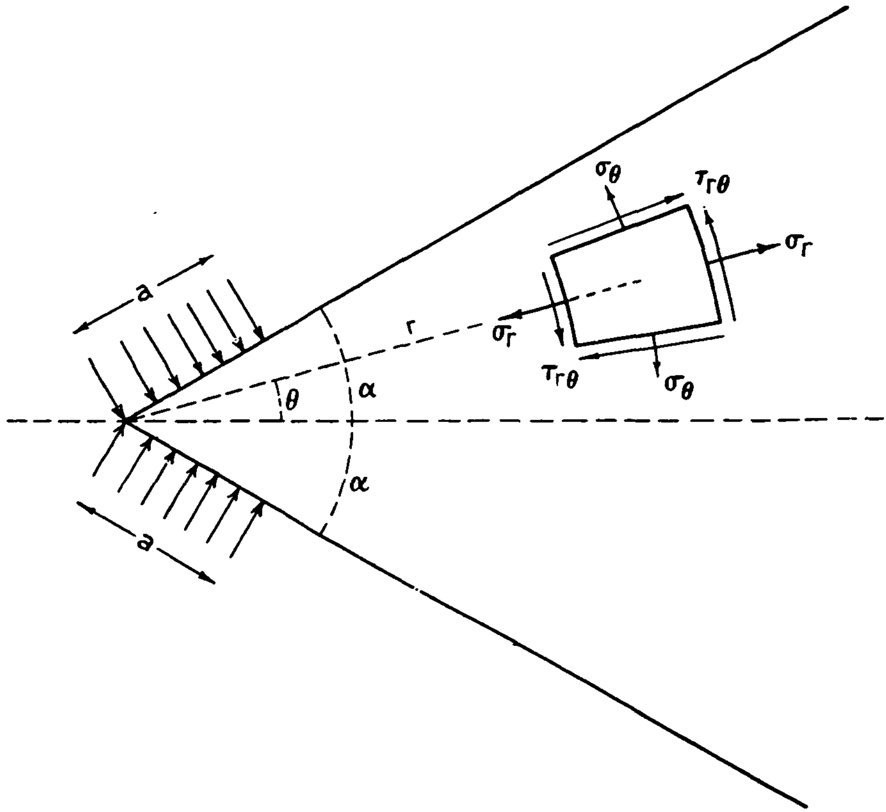

FIG. 1. The surface loading shown is for the particular case of § 3.

Assuming ϕ is such that $r^{p+n}\dfrac{\partial^n\phi}{\partial r^n}$ $(n = 0,1,2,3)$, $r^p\dfrac{\partial^n\phi}{\partial\theta^n}$ $(n = 1,2)$, and $r^{p+1}\dfrac{\partial^3\phi}{\partial r\partial\theta^2}$ all tend to zero as $r \to \infty$, and writing $\bar{\phi}$ for the Mellin transform of ϕ, i.e.

$$\bar{\phi} = \int\limits_0^\infty \phi r^{p-1}\,dr, \tag{5}$$

integration by parts gives:

$$\left.\begin{aligned} \int_0^\infty r\frac{\partial^{n+1}\phi}{\partial r\partial\theta^n}r^{p-1}\,dr &= -p\frac{d^n\bar{\phi}}{d\theta^n} \quad (n = 0,1,2),\\ \int_0^\infty r^2\frac{\partial^{n+2}\phi}{\partial r^2\partial\theta^n}r^{p-1}\,dr &= p(p+1)\frac{d^n\bar{\phi}}{d\theta^n} \quad (n = 0,2),\\ \int_0^\infty r^3\frac{\partial^3\phi}{\partial r^3}r^{p-1}\,dr &= -p(p+1)(p+2)\bar{\phi},\\ \int_0^\infty r^4\frac{\partial^4\phi}{\partial r^4}r^{p-1}\,dr &= p(p+1)(p+2)(p+3)\bar{\phi}. \end{aligned}\right\} \tag{6}$$

Multiplying (1) by r^{p+3}, integrating with respect to r from 0 to ∞ and using (6), we find

$$\frac{d^4\bar{\phi}}{d\theta^4}+[(p+2)^2+p^2]\frac{d^2\bar{\phi}}{d\theta^2}+p^2(p+2)^2\bar{\phi} = 0. \tag{7}$$

Equations (2) and (3), after multiplication by r^{p+1} and integration with respect to r from 0 to ∞, give

$$p(p+1)\bar{\phi} = F_1(p), \qquad (p+1)\frac{d\bar{\phi}}{d\theta} = S_1(p) \quad (\theta = \alpha), \tag{8}$$

$$p(p+1)\bar{\phi} = F_2(p), \qquad (p+1)\frac{d\bar{\phi}}{d\theta} = S_2(p) \quad (\theta = -\alpha), \tag{9}$$

where

$$F_n(p) = \int_0^\infty r^{p+1}f_n(r)\,dr, \qquad S_n(p) = \int_0^\infty r^{p+1}s_n(r)\,dr \quad (n = 1,2). \tag{10}$$

The solution of (7) is

$$\bar{\phi} = A\sin p\theta+B\cos p\theta+C\sin(p+2)\theta+D\cos(p+2)\theta, \tag{11}$$

where A, B, C, D depend on p and α. Substitution in (8) and (9) and some reduction yields

$$2p(p+1)G(\alpha,p)A = \{S_1(p)+S_2(p)\}p\sin(p+2)\alpha-\{F_1(p)-F_2(p)\}(p+2)\cos(p+2)\alpha, \tag{12}$$

$$-2(p+1)G(\alpha,p)C = \{S_1(p)+S_2(p)\}\sin p\alpha-\{F_1(p)-F_2(p)\}\cos p\alpha, \tag{13}$$

$$2p(p+1)H(\alpha,p)B = \{S_1(p)-S_2(p)\}p\cos(p+2)\alpha+\{F_1(p)+F_2(p)\}(p+2)\sin(p+2)\alpha, \tag{14}$$

$$-2(p+1)H(\alpha,p)D = \{S_1(p)-S_2(p)\}\cos p\alpha+\{F_1(p)+F_2(p)\}\sin p\alpha, \tag{15}$$

where

$$\left.\begin{aligned} G(\alpha,p) &= (p+1)\sin 2\alpha-\sin 2(p+1)\alpha,\\ H(\alpha,p) &= (p+1)\sin 2\alpha+\sin 2(p+1)\alpha. \end{aligned}\right\} \tag{16}$$

The Mellin transform $\bar{\phi}$ of the stress function ϕ is thus completely known. More interest, however, lies in the stresses themselves, and if we denote the Mellin transform by a 'bar', we have

$$\left.\begin{aligned}
\overline{(r^2\sigma_\theta)} &= \int\limits_0^\infty r^2\frac{\partial^2\phi}{\partial r^2}r^{p-1}\,dr = p(p+1)\bar{\phi},\\
\overline{(r^2\sigma_r)} &= \int\limits_0^\infty \left(r\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial\theta^2}\right)r^{p-1}\,dr = \frac{d^2\bar{\phi}}{d\theta^2}-p\bar{\phi},\\
\overline{(r^2\tau_{r\theta})} &= -\int\limits_0^\infty r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)r^{p-1}\,dr = (p+1)\frac{d\bar{\phi}}{d\theta}.
\end{aligned}\right\} \tag{17}$$

The stresses can now be found from Mellin's inversion formula (5) and we have

$$\left.\begin{aligned}
\sigma_\theta &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} p(p+1)\bar{\phi}r^{-p-2}\,dp,\\
\sigma_r &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \left[\frac{d^2\bar{\phi}}{d\theta^2}-p\bar{\phi}\right]r^{-p-2}\,dp,\\
\tau_{r\theta} &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} (p+1)\frac{d\bar{\phi}}{d\theta}r^{-p-2}\,dp.
\end{aligned}\right\} \tag{18}$$

The formal solution is now complete, $\bar{\phi}$ and its derivatives with respect to θ being found from equations (11) to (16). The line integrals in (18) can be evaluated in terms of infinite integrals from which numerical computation is possible. The conversion is straightforward but somewhat tedious, and the process is illustrated in § 3 for a particular case of the loading.

3. The particular case considered is that in which the faces of the wedge are each subjected to unit pressure for a distance a measured from the vertex, the rest of these faces being free from normal stress and the whole of both faces being free from shear stress.

For this case (10) gives

$$\left.\begin{aligned}
F_1(p) &= F_2(p) = -\int\limits_0^a r^{p+1}\,dr = -a^{p+2}/(p+2),\\
S_1(p) &= S_2(p) = 0.
\end{aligned}\right\} \tag{19}$$

Equations (12) to (15) lead to

$$\left.\begin{aligned} A = C &= 0, \\ p(p+1)H(\alpha,p)B &= -a^{p+2}\sin(p+2)\alpha, \\ (p+2)(p+1)H(\alpha,p)D &= a^{p+2}\sin p\alpha. \end{aligned}\right\} \tag{20}$$

Substitution in (11) and (18) then gives:

$$\left.\begin{aligned} \sigma_\theta &= -\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{p+2}\left[\sin(p+2)\alpha\cos p\theta-\frac{p}{p+2}\sin p\alpha\cos(p+2)\theta\right]\frac{dp}{H(\alpha,p)}, \\ \sigma_r &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{p+2}\left[\sin(p+2)\alpha\cos p\theta-\frac{p+4}{p+2}\sin p\alpha\cos(p+2)\theta\right]\frac{dp}{H(\alpha,p)}, \\ \tau_{r\theta} &= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{p+2}[\sin(p+2)\alpha\sin p\theta-\sin p\alpha\sin(p+2)\theta]\frac{dp}{H(\alpha,p)}. \end{aligned}\right\} \tag{21}$$

For values of α between 0 and $\frac{1}{2}\pi$ it is easy to show that the only zero of $H(\alpha,p)$ in the strip for which the real part of p lies between -2 and 0 is $p=-1$. The line integrals in (21) can be replaced by integrals from $-\infty$ to 0 and from 0 to ∞ along the line for which the real part of p is -1, less πi times the residue at $p=-1$. Omitting details of the algebra, we find

$$\left.\begin{aligned} \frac{\pi r}{2a}(\sigma_\theta-\sigma_r) &= \frac{\pi\sin\alpha\cos\theta}{2\alpha+\sin 2\alpha}-\int\limits_0^\infty P(\xi)\sin\{\xi\log(a/r)\}\,d\xi, \\ \frac{\pi r}{2a}(\sigma_\theta+\sigma_r) &= -\frac{\pi\sin\alpha\cos\theta}{2\alpha+\sin 2\alpha}+\int\limits_0^\infty [P(\xi)-\xi Q(\xi)]\frac{\sin\{\xi\log(a/r)\}}{1+\xi^2}\,d\xi- \\ &\qquad -\int\limits_0^\infty [Q(\xi)+\xi P(\xi)]\frac{\cos\{\xi\log(a/r)\}}{1+\xi^2}\,d\xi, \\ \frac{\pi r}{a}\tau_{r\theta} &= \int\limits_0^\infty R(\xi)\cos\{\xi\log(a/r)\}\,d\xi, \end{aligned}\right\} \tag{22}$$

where $P(\xi)$, $Q(\xi)$, $R(\xi)$ are given by

$$\left.\begin{aligned} (\xi\sin 2\alpha+\sinh 2\alpha\xi)P(\xi) &= \sin(\alpha-\theta)\cosh(\alpha+\theta)\xi+\sin(\alpha+\theta)\cosh(\alpha-\theta)\xi, \\ (\xi\sin 2\alpha+\sinh 2\alpha\xi)Q(\xi) &= \cos(\alpha-\theta)\sinh(\alpha+\theta)\xi+\cos(\alpha+\theta)\sinh(\alpha-\theta)\xi, \\ (\xi\sin 2\alpha+\sinh 2\alpha\xi)R(\xi) &= \sin(\alpha-\theta)\sinh(\alpha+\theta)\xi-\sin(\alpha+\theta)\sinh(\alpha-\theta)\xi. \end{aligned}\right\} \tag{23}$$

For the particular case ($\alpha = \frac{1}{2}\pi$), when the wedge becomes a semi-infinite solid, the integrals in (22) can be evaluated exactly by making use of the two results (6)

$$\left.\begin{aligned} \int_0^\infty \frac{\cosh px}{\sinh \frac{1}{2}\pi x}\sin mx\,dx &= \frac{\sinh 2m}{\cos 2p+\cosh 2m}, \\ \int_0^\infty \frac{\sinh px}{\sinh \frac{1}{2}\pi x}\cos mx\,dx &= \frac{\sin 2p}{\cos 2p+\cosh 2m} \end{aligned}\right\} \quad (4p^2 \not> \pi^2).$$

The results are

$$\left.\begin{aligned} \sigma_\theta-\sigma_r &= \frac{4ar\cos\theta}{\pi}\left(\frac{r^2+a^2\cos 2\theta}{r^4+2a^2r^2\cos 2\theta+a^4}\right), \\ \sigma_\theta+\sigma_r &= -\frac{2}{\pi}\left[\pi-\tan^{-1}\left(\frac{2ar\cos\theta}{a^2-r^2}\right)\right] \quad (r < a) \\ &= -\frac{2}{\pi}\tan^{-1}\left(\frac{2ar\cos\theta}{r^2-a^2}\right) \quad (r > a), \\ \tau_{r\theta} &= \frac{2ar\cos\theta}{\pi}\left(\frac{a^2\sin 2\theta}{r^4+2a^2r^2\cos 2\theta+a^4}\right), \end{aligned}\right\} \quad (24)$$

which can be shown to agree with those given by Love (7), who treats this particular case by an entirely different method. For other values of α it would appear that the stresses can only be found by evaluating the integrals in (22) numerically. The method developed by Filon (8) for trigonometric integrals of this type has proved very convenient for this purpose.

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