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THE USE OF WAVELET TRANSFORMS IN THE SOLUTION OF

TWO-PHASE FLOW PROBLEMS

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The Use of Wavelet Transforms in the Solution
of Two-Phase Flow Problems

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Abstract

In this paper we present the use of wavelets to solve the
non-linear Partial Differential Equation (PDE) of two-phase
flow in one dimension. The wavelet transforms allow a
drastically different approach in the discretization of space.
In contrast to the traditional trigonometric basis functions,
wavelets approximate a function not by cancellation but by
placement of wavelets at appropriate locations. When an
abrupt change, such as a shock wave or a spike, occurs in
a function, only local coefficients in a wavelet approxima-
tion will be affected. The unique feature of wavelets is their
Multi-Resolution Analysis (MRA) property, which allows
seamless investigation at any spatial resolution. The use
of wavelets is tested in the solution of the one-dimensional
Buckley-Leverett problem against analytical solutions and
solutions obtained from standard numerical models. Two
classes of wavelet bases (Daubechies and Chui-Wang) and
two methods (Galerkin and collocation) are investigated.
We determine that the Chui-Wang wavelets and a collo-
cation method provide the optimum wavelet solution for
this type of problem. Increasing the resolution level im-
proves the accuracy of the solution, but the order of the
basis function seems to be far less important. Our results
indicate that wavelet transforms are an effective and ac-
curate method which does not suffer from oscillations or
numerical smearing in the presence of steep fronts.

Introduction

Wavelets constitute unconditional bases for a variety of
function spaces, such as $L^2(\mathcal{R})$ spaces, Sobolev spaces, and
Besov spaces. Thus, they can provide accurate approxima-
tions of functions in such spaces. With multi-resolution
analysis (MRA) properties, compact support, and (semi-
)orthogonality, wavelets provide an attractive alternative as
bases for numerical solution of differential equations.

In contrast to the traditional trigonometric sine and cosine
basis functions, which have infinite support and are not in
$L^2(\mathcal{R})$, wavelet bases are $L^2(\mathcal{R})$ functions and may have
compact support. Unlike trigonometric approximation, ap-
proximation with wavelet bases does not rely on cancel-
lation. When an abrupt change, such as a shock wave or
a spike, occurs in a function, only local coefficients in a
wavelet approximation will be affected. No global cancel-
lation is needed. Due to the MRA properties, wavelets are
inherent multi-level bases. Mallat's decomposition and re-
construction algorithm[1] provides a fast transition between
approximations at different levels.

Due to the above properties and advantages over
trigonometric series, wavelets have been used in a vari-
ety of areas, including numerical solution of differential
equations. Glowinski et al.[2], Jaffard[3], Qian and Weiss[4],
Nikolaou and You[5] used wavelets or/and scaling functions
directly as bases with Galerkin's method or other methods;
Lorentz and Madych[6], Liandrat et al.[7], Jaffard[3], Xu and
Shann[8], Dahlke and Weinreich[9], Jawerth and Sweldens[10]
adapted (modified) wavelet bases for some specific differ-
ential operators to solve PDEs. Galerkin's method was
used with the adapted bases to obtain diagonal matrices of
unknown coefficients. Beylkin[11] used wavelets for alge-
braic manipulations to simplify the results from traditional
discretization methods.

In this work, we discuss the use of wavelets in the
solution of the Buckley-Leverett[12] problem of nonlinear
one-dimensional two-phase flow in porous media. In the
following sections, we first introduce some basic concepts
in wavelet analysis, including MRA and wavelet bases
(Daubechies and Chui-Wang), which we will use in later
sections. Then we discuss the use of wavelets for the
numerical solution of the Buckley-Leverett problem, and
present numerical examples.

Multi-Resolution Analysis

Scaling Functions and Multiresolution Analysis. A function, $\phi(x) \in L^2(\mathcal{R})$, is called a scaling function that generates a multiresolution analysis (MRA) in the subspaces, $\cdots$, V_{n-1}, V_n, V_{n+1}, $\ldots$, if the following conditions are satisfied[1]:

(i) $V_j \subset V_{j+1}$, $\forall j$;

(ii) $f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$;

(iii) $f(x) \in V_n \Leftrightarrow f(x + 2^{-n}) \in V_n$;

(iv) $lim_{n \to \infty} V_n = \bigcup_n V_n$ is dense in $L^2(\mathcal{R})$;

(v) $lim_{n \to -\infty} \bigcap_n V_n = \{0\}$;

(vi) The set $\{\phi(x-k)\}_{k \in \mathcal{Z}}$ forms a Riesz or unconditional basis for V_0, i.e. there exist constants A and B, with $0 < A \le B < \infty$, such that,

$$A \sum_{k \in \mathcal{Z}} |c_k|^2 \le \left\| \sum_{k \in \mathcal{Z}} c_k \phi(x - k) \right\|_2^2 \le B \sum_{k \in \mathcal{Z}} |c_k|^2 \tag{1}$$

for any sequence $\{c_k\} \in l_2$, the space of all square summable sequences ($A = B = 1$ for an orthonormal basis).

A scaling function, $\phi(x)$, and a set of related coefficients, $\{p(k)\}_{k \in \mathcal{Z}}$, are constructed such that they satisfy the so-called two-scale relation or refinement equation,

$$\phi(x) = \sum_k p(k) \phi(2x - k) \tag{2}$$

and some additional conditions. We say that the scaling function $\phi(x)$ has compact support if and only if finitely many coefficients $p(k)$ are non-zero.

Translations of the scaling function, $\{\phi(x-k)\}$, form a Riesz or unconditional basis of a subspace $V_0 \subset L^2(\mathcal{R})$. Furthermore, through translation of ϕ by a factor of 2^n and dilation by a factor of $k \cdot 2^{-n}$, a Riesz basis, $\{\phi_{n,k}(x)\}_{k \in \mathcal{Z}}$, is obtained for the subspace $V_n \subset L^2(\mathcal{R})$, where

$$\phi_{n,k}(x) = 2^{n/2} \phi(2^n x - k) \tag{3}$$

corresponding to resolution level n. Thus, the scaling function, $\phi(x)$, generates a set of bases for a sequence of nested subspaces of $L^2(\mathcal{R})$, and tends to $L^2(\mathcal{R})$ as the resolution level, n, goes to infinity.

Wavelets and the Detail Signals. A wavelet function is defined by

$$\psi(x) = \sum_k q(k) \phi(2x - k) \tag{4}$$

where $q(k)$ are a set of coefficients for the two-scale relationship of wavelet basis. The wavelet function, $\psi(x)$, is also called the basic (mother, analyzing) wavelet. If we define

$$\psi_{n,k}(x) = 2^{n/2} \psi(2^n x - k) \tag{5}$$

then, a Riesz basis for the subspace W_n is given by $\{\psi_{n,k}(x)\}_{k \in \mathcal{Z}}$, where W_n turns out to be the orthogonal complement[1,13] of V_n to V_{n+1}. The signals contained in the subspaces W_n are called detail signals or difference signals, because they are the difference between the subspaces V_{n+1} and V_n. The subspaces W_n have the following properties,

(i) $V_{n+1} = V_n \oplus W_n$, i.e. W_n is the orthogonal complement of V_n to V_{n+1};

(ii) W_{n_1} is orthogonal to W_{n_2} if $n_1 \ne n_2$;

(iii) $\oplus_{n=-\infty}^{\infty} W_n = L^2(\mathcal{R})$, i.e. the set $\{\psi_{n,k}(x)\}_{n,k \in \mathcal{Z}}$ is a Riesz basis for $L^2(\mathcal{R})$.

A wavelet basis is orthonormal if any two translated and/or dilated wavelets satisfy the orthogonality condition:

$$\int_{-\infty}^{\infty} \psi_{n,k}(x) \psi_{m,l}(x) dx = \delta_{n,m} \delta_{k,l}, \qquad n, m, l, k \in \mathcal{Z} \tag{6}$$

A wavelet basis is semi-orthogonal if wavelets at different resolution levels only satisfy the semi-orthogonality condition

$$\int_{-\infty}^{\infty} \psi_{n,k}(x) \psi_{m,l}(x) dx = 0, \quad n \ne m, \ n, m, l, k \in \mathcal{Z} \tag{7}$$

For semi-orthogonal wavelet ψ, a dual basis, $\tilde{\psi}_{n,k}$, exists[13], which satisfies the bi-orthogonality condition,

$$\int_{-\infty}^{\infty} \psi_{n,k}(x) \tilde{\psi}_{m,l}(x) dx = \delta_{n,m} \delta_{k,l} \qquad n, m, k, l \in \mathcal{Z}, \tag{8}$$

and must be used in place of $\psi_{n,k}(x)$ to evaluate the approximation coefficient $b_{n,k}$ below. For an orthonormal basis, the dual is the basis itself.

Multi-Level Approximation with Wavelets. For an orthonormal basis, the best approximation of an $L^2(\mathcal{R})$ function, $f(x)$, by a function $\mathcal{A}_n f(x)$ in the subspace V_n of $L^2(\mathcal{R})$ (i.e. at resolution level n) is given by the orthogonal projection of f on V_n, as follows:

$$f(x) \simeq \mathcal{A}_n f(x) = \sum_k a_{n,k} \phi_{n,k}(x) \tag{9}$$

where $a_{n,k}$ is given by the inner product of $f(x)$ and $\phi_{n,k}(x)$,

$$a_{n,k} = \langle f, \phi_{n,k} \rangle = \int_{-\infty}^{\infty} f(x) \phi_{n,k}(x) dx \tag{10}$$

The operator $\mathcal{A}_n$: $L^2(\mathcal{R}) \ni f(x) \to \mathcal{A}_n f(x) \in V_n$, is a linear orthogonal projection operator[1] resulting in the best approximation of $f(x)$ in V_n.

Similarly, we can define an operator $\mathcal{D}_n : L^2(\mathcal{R}) \ni f(x) \to \mathcal{D}_n f(x) \in W_n$ similar to $\mathcal{A}_n$. Then,

$$\mathcal{D}_n f(x) = \sum_k b_{n,k} \psi_{n,k}(x) \qquad (11)$$

where $\{b_{n,k}\}$ are coefficients that can be determined through an equation analogous to (5), i.e.

$$b_{n,k} = \langle f, \psi_{n,k} \rangle = \int_{-\infty}^{\infty} f(x) \psi_{n,k}(x) dx \qquad (12)$$

Approximation of a function, $f(x)$, can be conducted at different resolution levels (corresponding to different n's), and the approximations in the subspaces are recursive. Thus, for a function, $f(x)$, we have, at resolution level n,

$$\mathcal{A}_n f(x) = \sum_k a_{n,k} \phi_{n,k}(x)$$

$$= \underbrace{\sum_k a_{n-1,k} \phi_{n-1,k}(x)}_{\mathcal{A}_{n-1} f(x) \in V_{n-1}} + \underbrace{\sum_k b_{n-1,k} \psi_{n-1,k}(x)}_{\mathcal{D}_{n-1} f(x) \in W_{n-1}}$$

$$(13)$$

$\mathcal{A}_{n-1} f(x)$ can be decomposed again and this decomposition can be conducted recursively to a desired level.

For semi-orthogonal bases, the dual bases should be used in Eqs. 10 and 11 to evaluate the approximation coefficients, $a_{n,k}$, and the difference coefficients, $b_{n,k}$.

Mallat's Decomposition and Reconstruction Algorithm. Mallat[1] developed a very efficient algorithm for multi-resolution analysis using wavelets. Using this algorithm, the relationships between the approximation coefficients at different levels, e.g. $\{a_{n,k}\}$ at level n, and $\{a_{n-1,k}\}$ and $\{b_{n-1,k}\}$ at level $n-1$, are determined through some sequences of constants which depend only on the wavelet basis being used.

Wavelet Bases

A classical example of a compactly supported orthonormal wavelet basis of $L^2(\mathcal{R})$ is the Haar[14] basis,

$$\psi_H(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases} \qquad (14)$$

The corresponding scaling function is,

$$\phi_H(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases} \qquad (15)$$

However, approximation with the Haar basis is not very accurate and lacks smoothness.

In recent years, some more powerful wavelet bases have been developed[15,16,17]. In this section we will focus on the Daubechies[18] and the Chui-Wang's B-spline wavelets[13,19,20] which we use in our simulation. The Daubechies wavelets have compact support and are orthonormal. Daubechies wavelets do not have an explicit or analytic mathematical expression, but are constructed through a recursive algorithm. Chui-Wang's B-spline wavelets have compact support and are semi-orthogonal. Both Chui-Wang wavelets and their duals have analytic expressions. However, the dual wavelets do not have compact support.

Daubechies' Wavelet Bases. The construction of Daubechies wavelets starts from finding a sequence with finite non-zero terms, $p(k)$, $k = 0, \cdots, 2m-1$, for the two-scale relation of the scaling function (see Eq. 2), where m is an index of the wavelet functions. m determines the number of non-zero coefficients, the support, and the regularity of the wavelet functions. $m = 1$ leads to the Haar wavelet given by Eq. 14. Daubechies[18,21] gave the numerical values of the reconstruction and decomposition sequences for different m (not unique). For low order bases, the reconstruction sequences have explicit expressions.

Since Daubechies wavelets are orthonormal, the non-zero two-scale coefficients $\{p(k)\}$ for the wavelet function ψ (Eq. 2.3) are given by,

$$q(k) = (-1)^k p(1-k), \qquad k = 2 - 2m, \cdots, 1 \quad (16)$$

Note that these two sequences, $\{p(k)\}$ and $\{q(k)\}$, have only a finite number of non-zero terms. The decomposition sequences are given by,

$$g(k) = p(-k), \quad k = 0, \cdots, 2m - 1 \qquad (17)$$

$$h(k) = q(-k), \quad k = 2 - 2m, \cdots, 1 \qquad (18)$$

The support for the scaling and the wavelet functions are

$$supp \ \phi_{n,k} = [2^{-n}k, 2^{-n}(k + 2m - 1)]$$

$$supp \ \psi_{n,k} = [2^{-n}(k + 1 - m), 2^{-n}(k + m)]$$

These bases also have the continuity property, $\phi_{n,k}(x)$, $\psi_{n,k}(x) \in C^{\lambda_m}$, i.e they belong to the space of Hölder continuous functions with index λ_N, which satisfy

$$\left| f^{(k)}(x) - f^{(k)}(x + h) \right| \le K |h|^{\alpha} \qquad (19)$$

where $k = [\lambda_m]$; $0 \le \alpha = \lambda_N - k < 1$; and K a positive constant. The value of λ_m is determined by the wavelet order m. The differentiability of these wavelet functions increases with the index m. When $m = 2$, the Daubechies wavelet is continuous everywhere, left-differentiable at dyadic points (i.e. at $x = k \cdot 2^{-l}$), and nowhere right-differentiable at dyadic points.

Daubechies wavelet bases also have up to m zero moments, i.e.

$$\mathcal{M}_\psi = \int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \qquad k = 0, \cdots, m-1$$

(20)

which means that the scaling bases can represent polynomials of degrees up to $m-1$ exactly.

Figs. 1 and 2 present Daubechies' scaling and wavelet functions of order $m = 4$. The regularity of the basis functions increases with the order m. However, the support increases as well.

Chui-Wang Wavelet Bases. Chui and Wang[19,20] and Chui[13] developed semi-orthogonal wavelets based on B-spline functions, which are defined by the recursive convolution formula,

$$N_m(x) = \int_{-\infty}^{\infty} N_{m-1}(x-t)N_1(t)dt = \int_0^1 N_{m-1}(x-t)dt$$

(21)

starting from the box function,

$$N_1(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(22)

It can be shown that $N_m(x)$ is a scaling function that generates an MRA. The two-scale relation for this scaling function is,

$$\phi(x) = N_m(x) = \sum_{k=0}^{m} 2^{-m+1} \binom{m}{k} \phi(2x-k)$$
$$= \sum_{k=0}^{m} p(k)\phi(2x-k).$$

(23)

The wavelet basis is given by

$$\psi(x) = 2^{-m+1} \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1)N_{2m}^{(m)}(2x-k)$$

(24)

where

$$N_{2m}^{(m)}(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \phi(x-k)$$

(25)

Chui-Wang wavelets are semi-orthogonal. The basis and its dual satisfy the bi-orthogonality condition in Eq. 8. Chui-Wang wavelets are compactly supported, smooth (in C^k), and symmetric. Their drawback is that the dual wavelet and scaling functions do not have compact supports, although they have good decay far from their centers. The reconstruction sequences are thus infinite, but

have fast (exponential) decay. The support for the scaling and the wavelet functions are

$$supp \; \phi_{n,k} = [2^{-n}k, 2^{-n}(k+m)]$$
$$supp \; \psi_{n,k} = [2^{-n}k, 2^{-n}(k+2m-1)]$$

One of the major advantages of Chui-Wang bases is that there exist analytic expressions for scaling functions, wavelet functions and their duals (piece-wise polynomials). Figs. 3 and 4 give the scaling and wavelet bases of order $m = 4$.

For MRA, the reconstruction sequences are the coefficients in the two-scale relations of scaling functions and wavelet functions, i.e.,

$$p(k) = 2^{-m+1} \binom{m}{k}, \; k = 0, \cdots, m$$

(26)

$$q(k) = (-1)^k 2^{-m+1} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k+1-l), \quad (27)$$

$k = 0, \cdots, 3m-2$ for the Chui-Wang basis of mth order. Only a finite number of terms are non-zero. However, the decomposition sequences have infinite non-zero terms, which, however, have fast (exponential) decay. The numerical values of these sequences for Chui-Wang wavelets can be found in Chui[13].

Wavelet Solution of the Buckley-Leverett Problem

Neglecting gravitational and capillary effects, the two-phase flow of immiscible incompressible liquids through a one-dimensional homogeneous porous medium is given by the Buckley-Leverett[12] PDE:

$$R(x,t) = \frac{\partial S}{\partial t} + u \frac{\partial f(S)}{\partial x} = 0,$$

(28)

where S is the water saturation, x is the distance along the flow path, $u = q/(\phi A)$, ϕ is the porosity, A is the cross-sectional area of flow, and q is the constant volumetric injection/production rate. The term

$$f(S) = \begin{cases} 0, & \text{if } 0 < S \le S_{wr} \\ \dfrac{\lambda_w}{\lambda_w + \lambda_o}, & \text{if } S_{wr} < S \le 1-S_{or} \\ 1, & \text{if } 1-S_{or} < S \le 1, \end{cases}$$

(29)

where λ_w and λ_o denote the water and oil mobilities, and S_{wr} and S_{or} are respectively the water and oil irreducible saturations. For the Buckley-Leverett problem

$$\lambda_w = S - S_{wr}, \quad \text{and} \quad \lambda_o = 1 - S - S_{or}.$$

(30)

The initial and boundary conditions are

$$S(0,t) = 1 - S_{or}, \quad \text{and} \quad S(x,0) = S_{wr}.$$

(31)

We use the Method of Weighted Residuals (MWR) to solve this problem. In MWR the solution of a differential equation, $S(x,t)$, is approximated by a finite series of functions, $\phi_{n,k}(x)$, as follows:

$$S(x,t) \approx \sum_{k=1}^{m} a_{n,k}(t)\, \phi_{n,k}(x) \qquad (32)$$

where $\phi_{n,k}(x)$ are called basis or trial functions, $a_{n,k}(t)$ are coefficients to be determined to satisfy the PDE in Eq. 28, and m is the number of basis functions. In general, the approximate solution cannot satisfy the original equation exactly, and substitution of the approximate solution into the original PDE results in a residual. Thus, the coefficients $a_{n,k}(t)$ in Eq. 32 should minimize the residual.

The method of weighted residuals minimizes $R(x)$ by forcing it to zero in the domain Ω using weighting (test) functions, $w_j(x)$, such that, for every weighting function,

$$\int_{\Omega} R(x)\, w_j(x) dx = 0, \quad \text{for} \quad j = 1, \cdots, \nu \qquad (33)$$

where ν is the number of weighting functions to be determined by the type of boundary conditions and the number of basis functions, m.

In this work, we will use scaling functions and wavelet functions as the Galerkin bases. Because of their properties (orthonormality or semi-orthogonality, compact support, multi-level structure, unconditional bases), they can provide accurate approximation of solutions of PDEs[2].

Wavelet-Galerkin Method. In Galerkin's method, the weighting functions are chosen to be the basis functions, i.e., $w_j(x) = \phi_{n,j}(x)$. Thus we have,

$$\int_{\Omega} R(x,t)\phi_{n,j}(x) dx = 0, \quad \text{for} \quad j = 1, \cdots, \nu. \qquad (34)$$

Boundary conditions may impose some constraints on the solution and result in additional equations for the unknown coefficients $\{a_{n,k}\}$[22].

Substitution of Eq. 32 into Eq. 34 results in the following residual:

$$R_n(x,t) = \sum_k \frac{da_{n,k}(t)}{dt}\phi_{n,k}(x) + q\frac{\partial f}{\partial x}(S). \qquad (35)$$

Using Galerkin's method with $\phi_{n,j}(x)$ as weighting functions, we have

$$\sum_k \frac{da_{n,k}(t)}{dt} \int_0^1 \phi_{n,j}(x)\phi_{n,k}(x) dx$$
$$\qquad (36)$$
$$= -q\phi_{n,j}(x)f(S)\Big|_0^1 + q \int_0^1 \phi'_{n,j}(x)f(S) dx.$$

By solving this nonlinear ordinary differential equation, we can get the approximate solution at a desired time t.

It has been previously reported[23,24,25] that the direct application of Galerkin's method to the two-phase immiscible displacement problem without capillary pressure cannot yield satisfactory results. We encountered the same difficulties in this work. Such difficulties can be alleviated by introducing some measure to smear the sharp front, such as using upstream weighting techniques, or adding an artificial capillary term.

Upstream weighting did not produce satisfactory results in Galerkin-wavelet formulation. The addition of a small artificial capillary pressure produced better results. The saturation equation with capillary pressure can be found in Spivak et al.[26], and is given by:

$$\phi\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left[u\, f(S) + \lambda_o\, f(S)\frac{\partial P_c}{\partial S}\frac{\partial S}{\partial x}\right] = 0, \qquad (37)$$

where P_c is the capillary pressure.

Wavelet-Collocation Method. When using the collocation method, we minimize the residual function by using the standard delta function at some points, x_j, i.e.

$$\int_{\Omega} R(x,t)\delta(x_j) dx = 0, \quad j = 1, \cdots, \nu \qquad (38)$$

which is equivalent to

$$R(x_j,t) = 0, \quad j = 1, \cdots, \nu \qquad (39)$$

where x_j's are collocation points. The collocation points can be evenly distributed in the domain of interest. Starting from Eq. 35, we use an upstream weighting scheme between adjacent collocation points for the nonlinear term $f(S)$. The initial condition is approximated by the collocation method, i.e. by solving the linear system

$$\sum_k a_{n,k}(0)\phi_{n,k}(x_j) = S(0,x_j), \qquad (40)$$

where x_j's are collocation points. For evenly distributed collocation points, $x_j = (j-1)/(n_p-1)$, $j = 1, \cdots, n_p$, where n_p is the total number of coefficients.

Results and Discussion

The Buckley-Leverett problem was solved using both the Galerkin-wavelet and the collocation-wavelet formulation. In all cases $S_{wr} = 0.16$, $S_{or} = 0.2$, and $u = 2.134 \times 10^{-4}\ m/s$. In the Galerkin-wavelet formulation the slope of the artificial capillary pressure $dP_c/dS = -2\ Pa$. The Daubechies wavelets are not continuously differentiable, require special treatment to handle the derivatives, result in unsatisfactory solutions, and will not be further discussed.

The Chui-Wang wavelets are much smoother and continuously differentiable, and produced significantly better results. In the following discussion Chui-Wand wavelets are used exclusively.

Galerkin-Wavelet Method. The results of the simulation for $m = 4$ and $n = 4$ are shown in **Fig.** 5 at the following times: 300, 600, 900, 1200, and 1500 days. We also show the analytical and numerical[27] solutions at $t = 1500$ days. The numerical solution uses a uniform grid of 40 grid-blocks. It is evident that although there is a general agreement with the numerical solution, the Galerkin-wavelet solutions are inaccurate and exhibit oscillations ahead of the front. Without exploring the matter in detail, we believe that this behavior may be due to the treatment of the boundary conditions. We also believe that increasing the level of resolution n will improve the solution. We do not show solutions at times less than $t = 1500$ days because the results are analogous and the differences less pronounced.

Collocation-Wavelet Method. The results from six sets of simulations for $m = 2, 4$ and $n = 4, 5, 6$ are shown in **Figs.** 6 through 11. The analytical and numerical[27] solutions at $t = 1500$ days are also shown. It is evident that an increasing resolution level results in an increasing accuracy of the solution. For $n = 6$ we obtain a solution which is in excellent agreement with the analytical solution, better than the corresponding numerical[27] solution, and capable of tracing accurately the steep front. All the collocation-wavelet solutions are free from the oscillations observed in the Galerkin-wavelet solutions. Increasing the order m of the basis function from 2 to 4 seems to improve the accuracy of the solution. This improvement is more pronounced at lower resolution levels n, and seems to have a minimal (if any) effect on the solution accuracy for $n = 6$.

From these results, the conclusion to be drawn is that under the described conditions collocation-wavelet methods are superior to Galerkin-wavelet solutions. However, caution needs to be exercised in the interpretation of these results. The subject of wavelet use in the solution of PDE is a new and unexplored area, about which the existing information base is very limited. Inevitably, we had to use techniques derived from traditional space discretization schemes. We would like to approach our results only as a first indication of (and not as the definitive authority in) the performance of wavelets in the solution of PDEs. We are convinced that while wavelets may have be a very promising technique, significant effort will be needed before they can realize their full potential and begin rivaling traditional numerical simulation schemes in commercial simulators.

Acknowledgements

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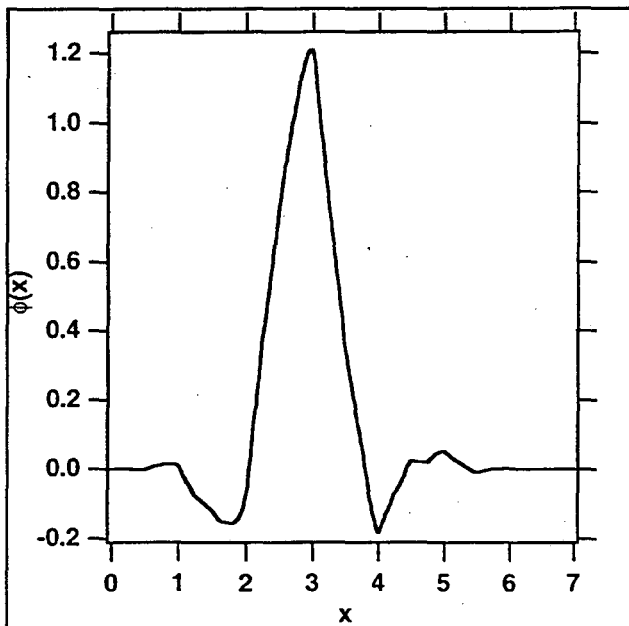
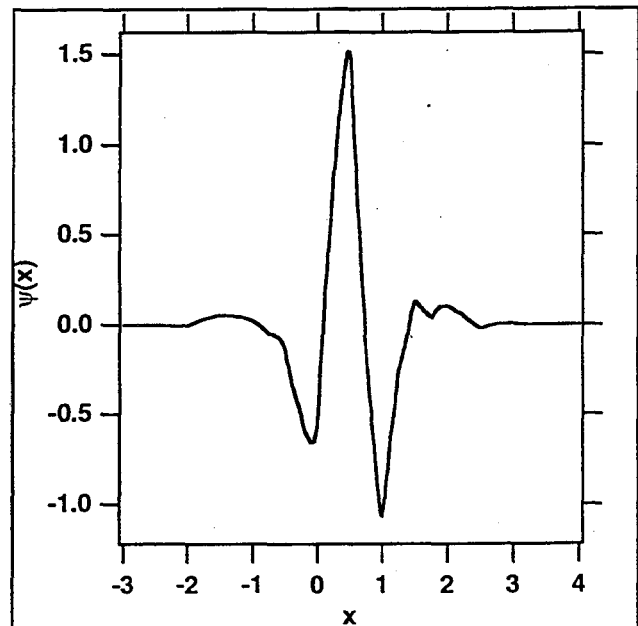Fig. 1 - Daubechies scaling function $\phi(x)$ of order m = 4.

Fig. 2 - Daubechies wavelet function $\psi(x)$ of order m = 4.
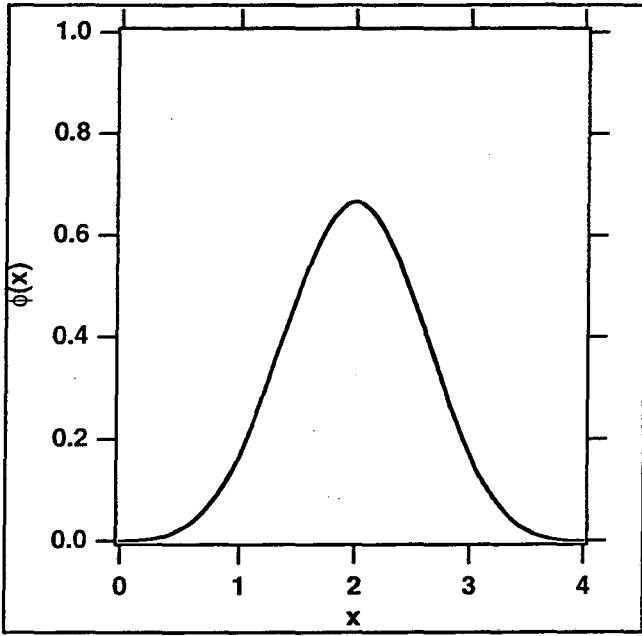
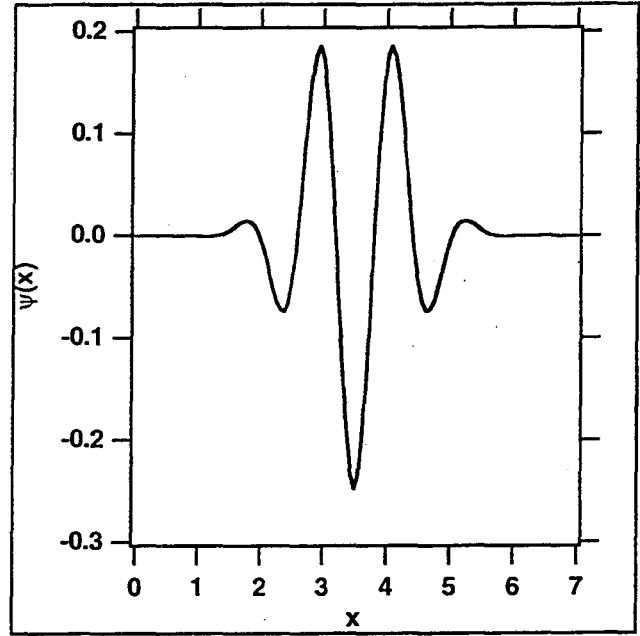Fig. 3 - Chui-Wang scaling function $\phi(x)$ of order m = 4.

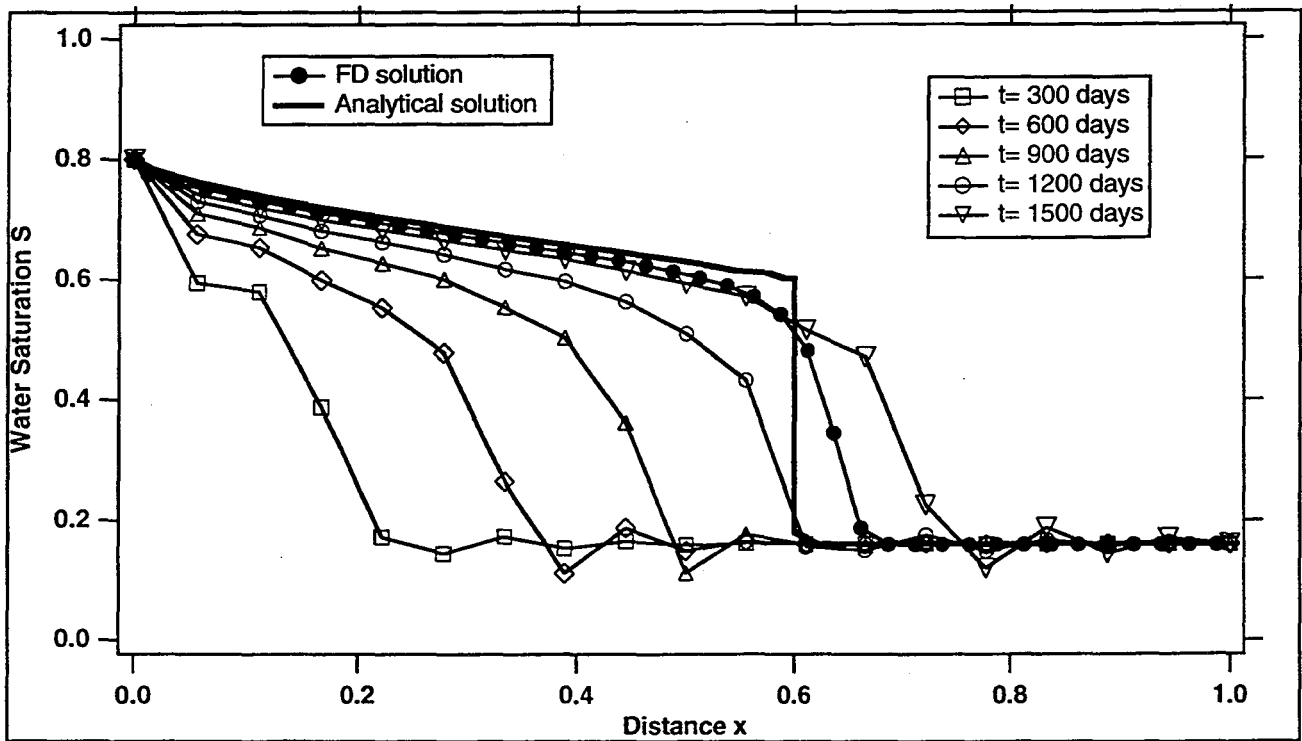Fig. 4 - Chui-Wang wavelet function $\psi(x)$ of order m = 4.

Fig. 5 - Galerkin-wavelet solution of the Buckley-Leverett problem with m = 4 and n = 4 (n_p = 37).

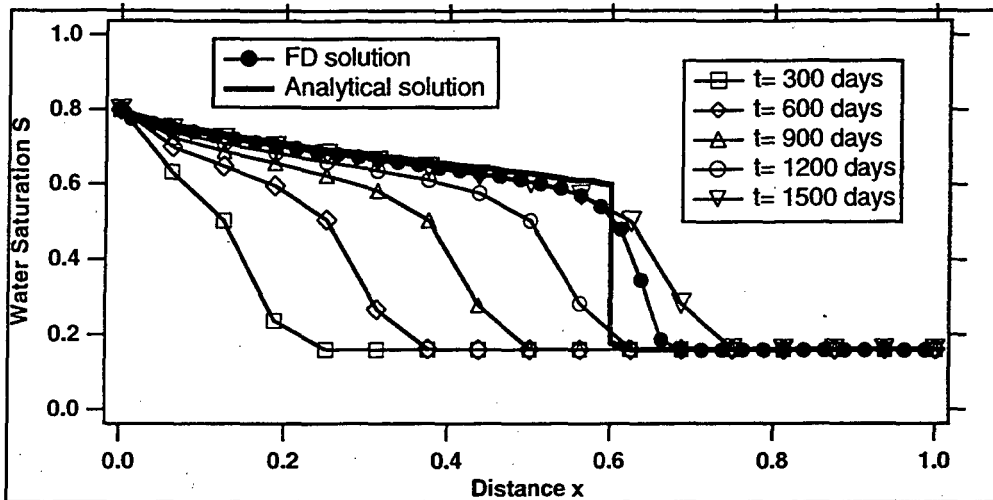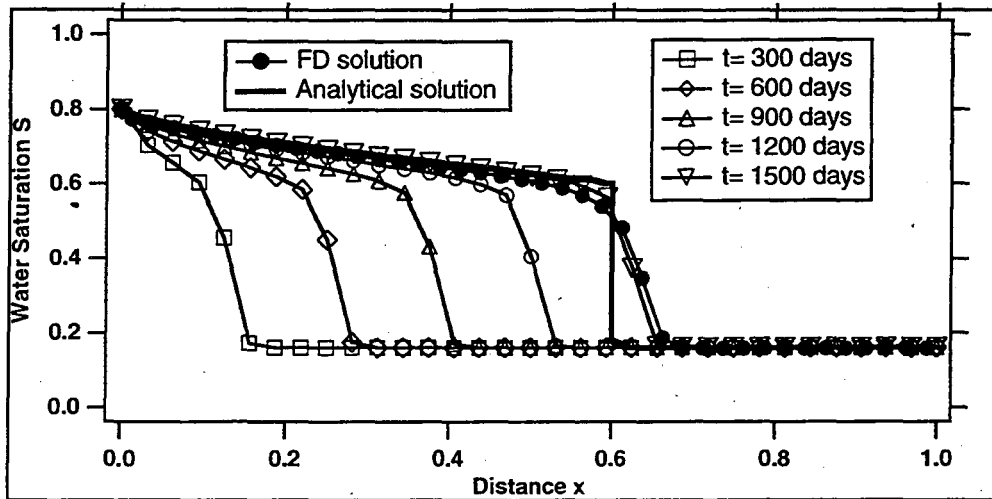**Fig. 6 - Collocation-wavelet solution of the Buckley-Leverett problem with m = 2 and n = 4 (n_p = 17).**

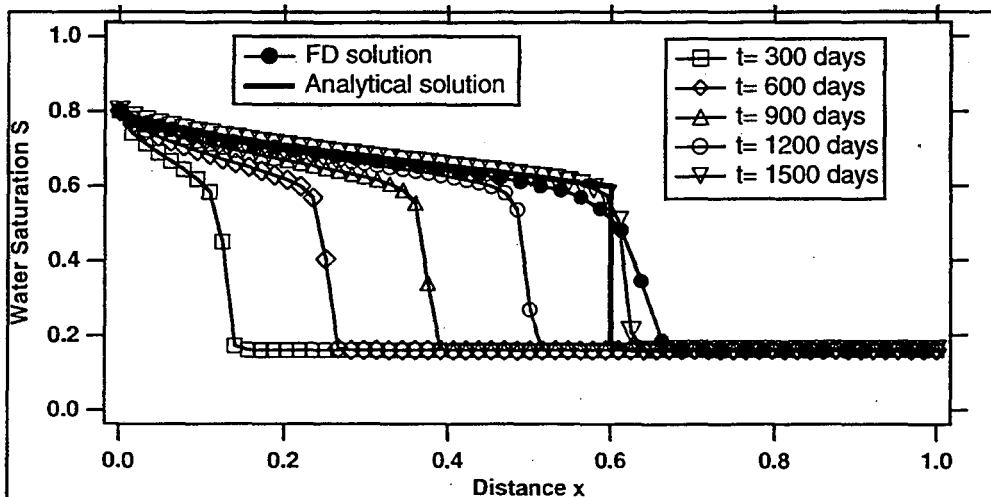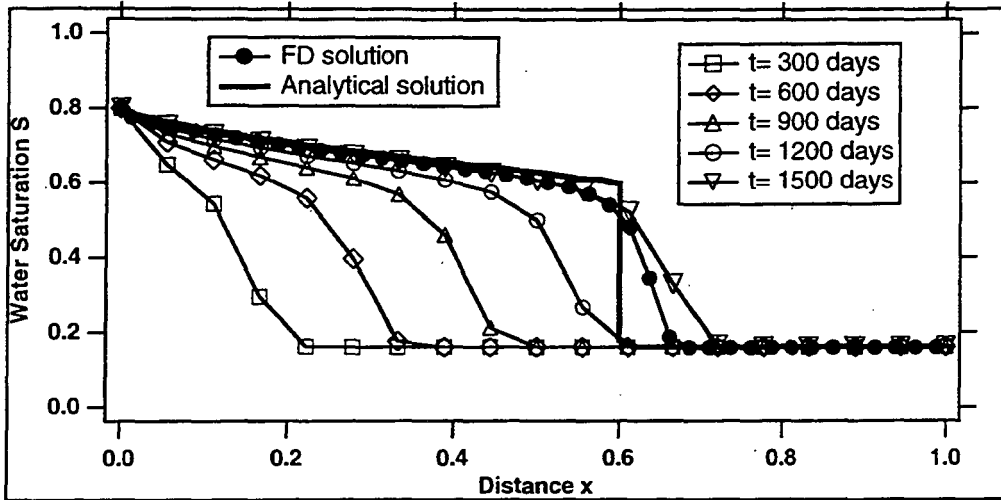**Fig. 7 - Collocation-wavelet solution (m = 2, n = 5, n_p = 33).**

Fig. 8 - Collocation-wavelet solution (m = 2, n = 6, n_p = 65).

Fig. 9 - Collocation-wavelet solution of the Buckley-Leverett problem with m = 4 and n = 4 (n_p = 19).
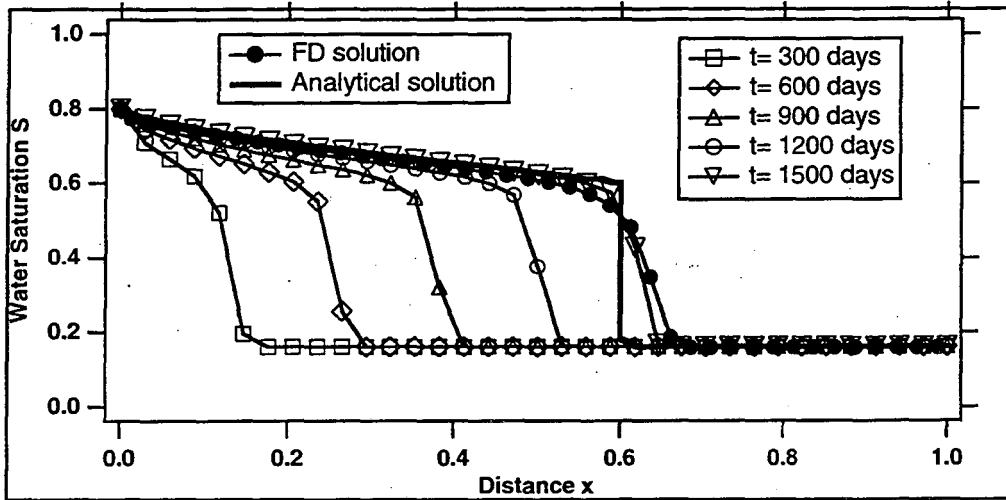

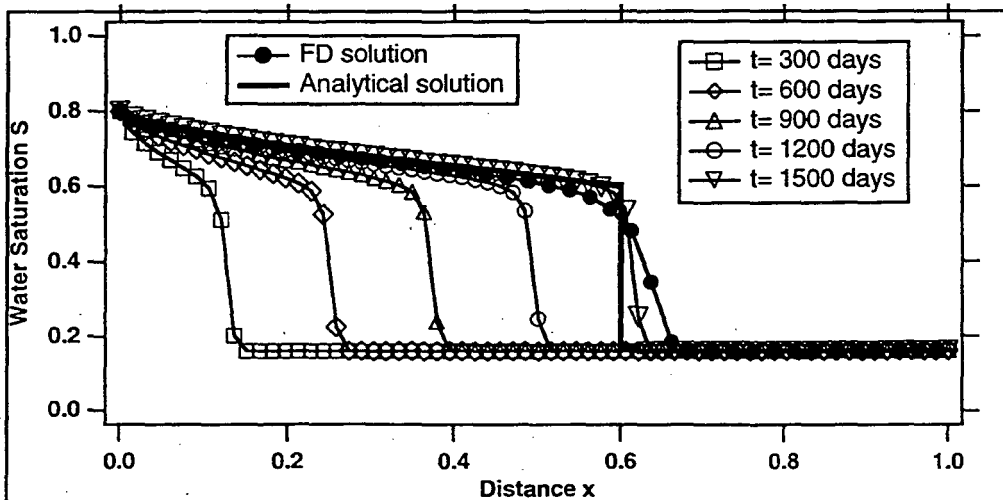**Fig. 10 - Collocation-wavelet solution (m = 4, n = 5, n_p = 35).**

Fig. 11 - Collocation-wavelet solution (m = 4, n = 6, n_p = 67).