The validity of an operational solution of Laplace's equation

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THE VALIDITY OF AN OPERATIONAL SOLUTION OF LAPLACE'S EQUATION

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ABSTRACT

A physical problem that has received considerable attention is that of leakage in mine ventilation. Most underground methods of mining leave porous material between roadways, with the result that through leakage, the quantity of air reaching the working face is usually less than that entering the mine. Investigations into this problem have assumed that air flows through the barrier in a direction perpendicular to the roadways. This is clearly wrong unless flow is impervious in any other direction. In the present study, the more general consideration of air movement in any direction is made.

It is shown that the pressure of air at points in the barrier is governed by Laplace's equation and from other fluid flow relationships, certain non-linear boundary conditions apply. An operational method is used to find a solution but an approximation must be introduced for this to be achieved. The author wishes to establish the accuracy of this solution.

For the special case of streamlined flow in the airways, an analytical solution is able to be determined. However, to do so depends on the validity of a modified Dini expansion of a function. This is established so that the operational and analytical solutions can be compared.

Flow in the airways, however, is usually somewhat turbulent. A numerical method is developed for comparison purposes in this more general case, errors introduced by numerical approximation being reduced to a suitable level by checking with previous results from the analytical solution. Taking the numerical results as being 'exact', it appears that the operational solution is only a fair result.

CONTENTS

INTRODUCTION

THE NATURE OF MATHEMATICS

It is often not realised that the roots of all mathematics lie deep in problems arising in the physical world and that through abstraction and generalization various strands have been developed that may or may not bear some relationship to the initial events. The trend is constantly towards abstraction and generalization although continual new calls are made on mathematics by problems suggested by the outside world. In considering these new problems, all kinds of new methods may evolve. Some of these, as in the theory of operators, are used before their logical structure is completely understood.

This thesis follows these lines to a large extent. A physical problem, that of leakage in mine ventilation, is considered. Firstly, a model of an 'ideal' mine is constructed and with certain restrictions the pressure and quantity of air at points in the airways can be determined. One of the restrictions then is removed and it is found that there is no known technique for solving the equations that are formed. However, an approximate solution can be obtained by using an operational approach. The interest then moves from the physical problem to determining the accuracy of the operational solution. The aim of this thesis is to determine this accuracy.

FIGURE 1

LONG-WALL METHOD OF MINING

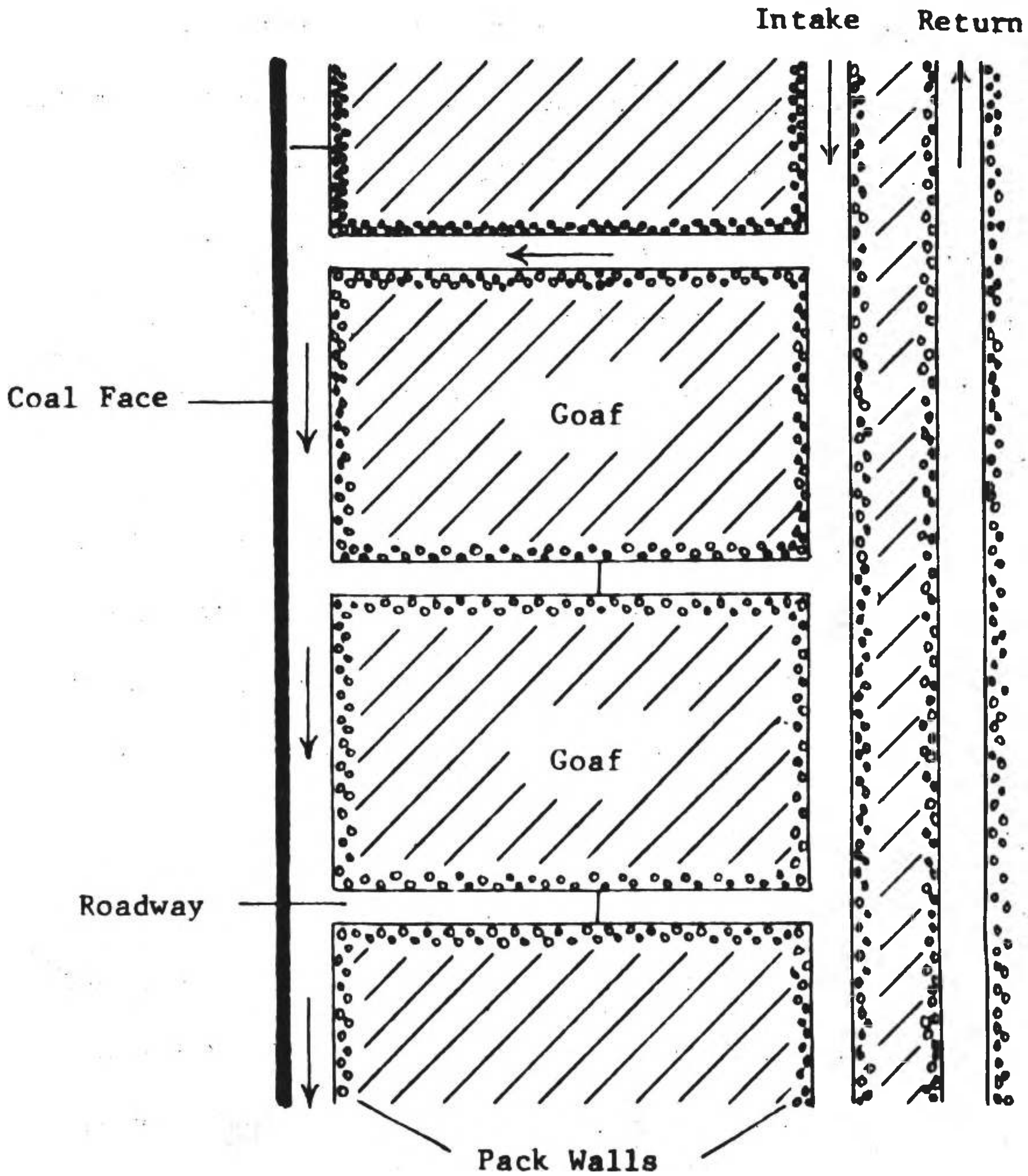


Figure 1

VENTILATION OF COAL MINES

In underground mining, coal is commonly extracted by what is called the long-wall method. As shown in figure 1, roadways are cut at right angles to the intake. These are lined with pack walls of stone and rubble which are obtained in the course of ripping down the roof above the roadways. The pack walls help support the roof and also aid in constraining the air to pass on to the coal face.

Coal is then extracted from the area between the roadways and when all is removed the roof is allowed to subside into the goaf. The roadways are then extended and the process is repeated as often as is necessary to complete the extraction of the coal in that direction.

To ventilate the roadways an exhaust fan is placed at the end of the return airway. This draws air out of the mine and thus causes fresh air to enter the intake and hence into the various roadways. However, air can leak through the porous goaf so that the quantity of air reaching the working face is usually less than the amount that enters the mine.

SCOPE OF THE THESIS

A considerable amount of work was done on the problem of leakage in mine ventilation by Keane et al. about fourteen year ago. By assuming air could only pass through the goaf at right angles to the roadways, they were able to obtain results from a mathematical model of a mine. A review of this work will firstly be given but then it is wished to remove the restriction concerning the passage of air through the goaf, for it is clearly wrong unless the goaf is impervious in the direction of the roadways.

It will be found in this more general consideration, where air can flow through the goaf in any direction, that pressure and quantity of air are related by Laplace's equation together with certain non-linear boundary conditions. An approximate solution is able to be obtained using operational methods but the accuracy of this solution needs to be checked.

For the special case where flow in the airways is taken to be streamlined, it is found that an analytical solution can be found. However, it depends on the validity of a modified Dini expansion of a function. This validity is obtained so that the operational and analytical solutions can be compared for the case of streamlined flow.

For turbulent flow, as is usual since the walls of airways are not smooth, a numerical method is needed for comparison purposes. Such a method is developed and is firstly checked by comparing results from it with previous results from streamlined flow considerations. With this being satisfactory, operational and numerical solutions can then be compared, to determine the accuracy of the former.

CHAPTER 1

THE THEORY OF MINE VENTILATION

FIGURE 2

MATHEMATICAL MODEL OF A MINE

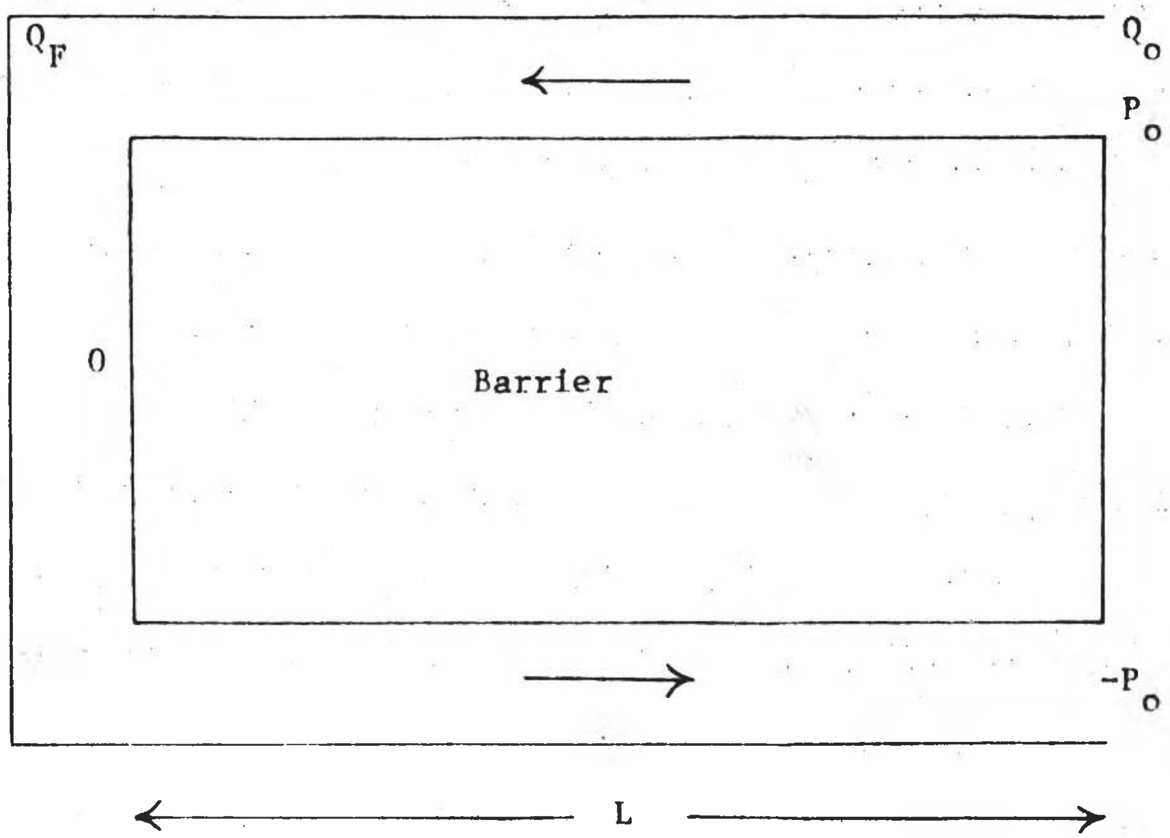


Figure 2

A MODEL OF A MINE

To determine the effect of leakage on mine ventilation, Peascod and Keane (1955) decided to consider a mine with two roadways, an intake and a return, that are of similar uniform construction and mathematically parallel. This ensures that the pressure drop will be the same in the intake and return so that a pressure P_o may be taken at the entrance, $-P_o$ at the fan and zero pressure at the working face. A quantity of air Q_o is taken to enter the mine while a quantity Q_F reaches the working face. Taking the length of the roadways as L, the mine then is as shown in figure 2.

The barrier is taken as being uniformly porous perpendicular to the airways and impervious in a direction parallel to them. Thus air leaking through the barrier will flow in a straight line from the intake to the return. The total conductance of the barrier is taken as G and the conductance per unit length of airway is g where $g = \frac{G}{L}$.

For the airways the total resistance to flow is given by R while the resistance per unit length is r where $r = \frac{R}{L}$.

For a relationship between pressure P and quantity of air Q at any point in the airway, we have from investigations into fluid flow that for streamlined flow

$$P = \rho Q \qquad (1.1)$$

and for fully turbulent flow

$$P = \rho Q^2 \tag{1.2}$$

where the constant ρ is a measure of the resistance to flow.

In the study made by Peascod and Keane it is taken that flow in the airways is fully turbulent while flow through the barrier is streamlined.

SOLUTION OF THE MODEL

Consider a section in the intake of length δl. In this section the pressure will drop by an amount δP and a quantity of air δQ is lost through the barrier under an influence of a pressure drop of 2P.

Since flow in the intake is taken as being fully turbulent, we have from equation (1.2) that

$$\delta P = r\ \delta l\ Q^2$$

and hence
$$\frac{dP}{dl} = rQ^2 \ . \tag{1.3}$$

Also from consideration of flow in the barrier, we have from equation (1.1) that

$$\delta Q = g\ \delta l . 2P$$

and thus
$$\frac{dQ}{dl} = 2gP \ . \tag{1.4}$$

Combining equations (1.3) and (1.4) we have the differential equation

$$\frac{dQ}{dP} = \frac{2g}{r}\frac{P}{Q^2}$$

or
$$Q^2\frac{dQ}{dP} - \frac{2G}{R}P = 0$$

remembering that $R = rL$ and $G = gL$.

On separating the variables and integrating we find that

$$Q^3 - \frac{3G}{R}P^2 = Q_F^3$$

or
$$P^2 = \frac{R}{3G}(Q^3 - Q_F^3) \quad . \tag{1.5}$$

Equation (1.5) is the general pressure - quantity relationship for flow in the mine airways.

As a particular case of equation (1.5) we have

$$P_o^2 = \frac{R}{3G}(Q_o^3 - Q_F^3) \tag{1.6}$$

since a quantity Q_o enters the mine under a pressure P_o.

It is also found that another relationship can be obtained. Q_F and Q_o can be related by certain hypergeometric functions by considering equations (1.4) and (1.5). From (1.4) we have

$$\int_{Q_F}^{Q_o}\frac{dQ}{P} = 2g\int_{o}^{L} dl \ .$$

Using (1.5) this becomes

$$\int_{Q_F}^{Q_o}\frac{dQ}{(Q^3-Q_F^3)^{\frac{1}{2}}} = 2\sqrt{\frac{RG}{3}} \ .$$

Expanding the integrand in a binomial series and then integrating term by term, we obtain the relationship

$$Q_F = \frac{3}{RG}\left[F\left\{\frac{1}{2},\frac{1}{6};\frac{7}{6};1\right\} - \left(\frac{Q_F}{Q_o}\right)^{\frac{1}{2}} F\left\{\frac{1}{2},\frac{1}{6};\frac{7}{6};\left(\frac{Q_F}{Q_o}\right)^3\right\}\right]^2 . \qquad (1.7)$$

Equations (1.5) and (1.7) then have been obtained as solutions of the mathematical model. They are relationships between the variable quantities involved.

IMPROVEMENTS TO THE MODEL

The preceding work is based on the assumption that flow in the airways is fully turbulent. It would be more realistic to assume that flow is somewhere between the two limits of fully turbulent and streamlined and to consider the relationship

$$P = \rho Q^n \quad , \ 1 \leqslant n \leqslant 2$$

for flow in the airways. Groden (1956) developed the theory for this more general case and obtained the results

$$P^2 = \frac{1}{n+1}\frac{R}{G}(Q^{n+1} - Q_F^{n+1}) \qquad (1.8)$$

as the relationship between pressure and quantity of air at points in the airways, and

$$Q_F = \left[\frac{n+1}{(n-1)^2}\cdot\frac{1}{RG}\left(F\left\{\frac{1}{2},\frac{n-1}{2n+2};\frac{3n+1}{2n+2};1\right\} - \left(\frac{Q_F}{Q_o}\right)^{\frac{n-1}{2}} F\left\{\frac{1}{2},\frac{n-1}{2n+2};\frac{3n+1}{2n+2};\left(\frac{Q_F}{Q_o}\right)^{n+1}\right\}\right)^2\right]^{\frac{1}{n-1}}$$

(1.9)

as the relationship between the quantity of air reaching the working face and the quantity of air entering the mine.

In a practical study of a mine in the Newcastle district, Rose (1960) found that a highly probable value of the flow index n was 1.85.

Another improvement in the model was made by Low (1956). His investigation takes into account the fact that resistances of the intake and return are not equal. This is the usual case since the intake is kept in good repair for the transport of men and materials while the return airway is neglected. His results indicate that the effects of the different resistances can be compensated by using the average resistance of the airways.

In the next chapter another modification will be made.

CHAPTER 2

FURTHER DEVELOPMENTS IN THE THEORY OF MINE VENTILATION

FIGURE 3

PLACEMENT OF AXES IN THE MODEL

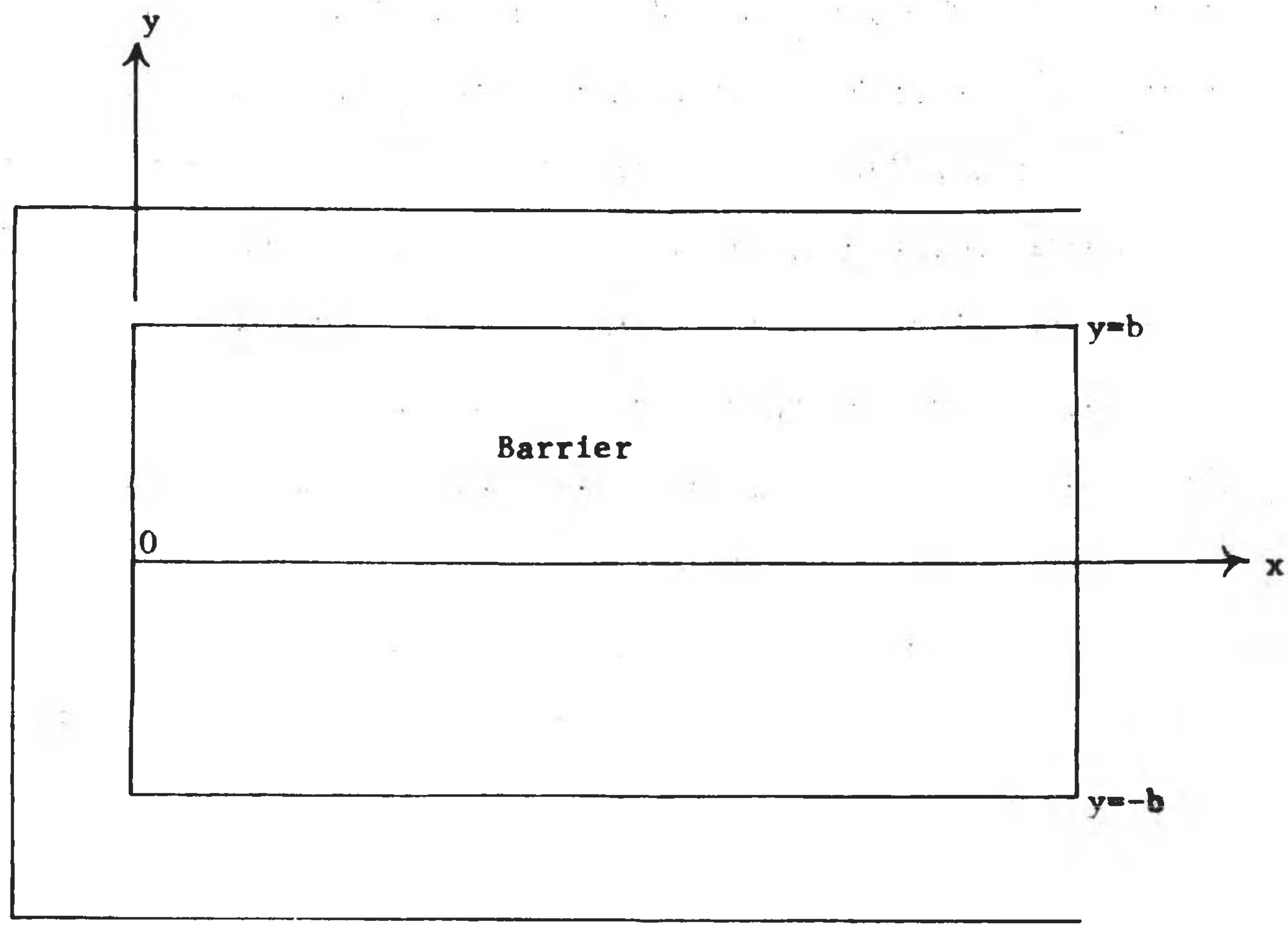


Figure 3

FURTHER MODIFICATION OF THE MODEL

Up to this stage it has been assumed that the air leaking through the goaf travels in a direction at right angles to the airways. We now wish to remove this restriction and place no limitation on the flow of air through the barrier. Flow through porous media is discussed by Schneidigger (1960) and Streeter (1961) but the approach to be used is suggested by Keane (1969).

Suppose in figure 3 that p is the pressure of air at any point (x,y) in the barrier and that u and v are the components of the velocity $\underset{\sim}{q}$ of the air at this point.

In the case of fluid seeping through a homogeneous isotropic porous medium, it is often assumed that the velocity vector $\underset{\sim}{q}$ of the fluid is given by

$$\underset{\sim}{q} = -\alpha \text{ grad } p$$

where α is the porosity of the medium. To allow more generality we shall take

$$u = -k^2 \alpha \frac{\partial p}{\partial x} \tag{2.1}$$

$$v = -\alpha \frac{\partial p}{\partial y} \tag{2.2}$$

so that if k = 0 we have the type of barrier assumed in the previous chapter, and if k = 1 we have the isotropic case where there is no preferable direction of motion.

Noting that the continuity equation for an incompressible fluid is

$$\operatorname{div} \underset{\sim}{q} = 0 \tag{2.3}$$

we have from (2.1), (2.2) and (2.3) that

$$k^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0. \tag{2.4}$$

This is the equation that is satisfied by the pressure in the barrier when there is no restriction on the direction of flow. Basset (1961), Binder (1958), and Lamb (1953) arrive at a similar result when considering two dimensional flow.

The problem now is to consider the solution of equation (2.4) subject to the boundary conditions

$$\left.\begin{aligned} P &= (p)_{y=b} \\ \frac{dQ}{dx} &= \left(2bg \frac{\partial p}{\partial y}\right)_{y=b} \\ \frac{dP}{dx} &= rQ^n \ . \end{aligned}\right\} \tag{2.5}$$

OPERATIONAL SOLUTION OF THE NEW EQUATION OF FLOW

In order to obtain an analytic solution an operational approach will be taken.

Putting $D \equiv \frac{\partial}{\partial x}$ equation (2.4) can be written

$$\frac{\partial^2 p}{\partial y^2} + k^2 D^2 p = 0.$$

Treating D like a constant, this differential equation has as a solution

$$p = \cos ky\, D.A(x) + \frac{\sin ky\, D}{kD}\,.\, B(x).$$

Since $p = 0$ when $y = 0$ we have that $A(x) = 0$ so that

$$p = \frac{\sin ky\, D}{kD}\,.\, B(x). \tag{2.6}$$

Applying the second boundary condition we have that

$$\frac{dQ}{dx} = 2bg \cos kbD\,.\, B(x).$$

But from (2.6)

$$P = \frac{\sin kbD}{kD} B(x)\,.$$

Thus
$$\frac{dQ}{dx} = 2bgkD \cot kbD\,.\, P.$$

Approximating the cotangent by the first two terms in its Laurent expansion, we have that

$$\frac{dQ}{dx} \doteq 2bgk\, D(\frac{1}{kbD} - \frac{kbD}{3})P.$$

That is

$$\frac{dQ}{dx} \doteq 2g(P - \frac{k^2b^2}{3} D^2P).$$

Remembering that we also have

$$\frac{dP}{dx} = rQ^n$$

then
$$\frac{dQ}{dx} \doteq 2gP - \frac{2gk^2b^2}{3}\frac{d}{dx}(rQ^n)\,. \tag{2.7}$$

Multiplying by rQ^n and rearranging terms gives

$$rQ^n\,dQ + \frac{2gk^2b^2}{3}\,nr^2Q^{2n-1}dQ \doteq 2gP\,dP.$$

On integration we obtain

$$\frac{rQ^{n+1}}{n+1} + \frac{gk^2b^2r^2}{3}\,Q^{2n} \doteq gP^2 + \text{constant}.$$

When $P = 0$ we have $Q = Q_F$. Hence

$$P^2 \doteq \frac{1}{n+1}\,\frac{r}{g}(Q^{n+1} - Q_F^{n+1}) + \frac{k^2b^2r^2}{3}(Q^{2n} - Q_F^{2n})$$

or

$$P^2 \doteq \frac{1}{n+1}\,\frac{R}{G}(Q^{n+1} - Q_F^{n+1}) + \frac{k^2b^2R^2}{3L^2}(Q^{2n} - Q_F^{2n}) \quad . \qquad (2.8)$$

Equation (2.8) is the pressure - quantity relationship for flow in the airways when there is no restriction to the flow through the barrier. It corresponds to equation (1.5) where $n = 2$ and $k = 0$, and to equation (1.8) where $k = 0$.

It is to be noted that from (2.8) the relationship

$$P_o^2 \doteq \frac{1}{n+1}\,\frac{R}{G}(Q_o^{n+1} - Q_F^{n+1}) + \frac{k^2b^2R^2}{3L^2}(Q_o^{2n} - Q_F^{2n}) \qquad (2.9)$$

exists between P_o, Q_o and Q_F.

In attempting to find a relationship between Q_F and Q_o, as was done in the previous chapter, we have from (2.7) that

$$\left(1 + \frac{2k^2b^2n\ RG}{3L^2}\,Q^{n-1}\right)\frac{dQ}{dx} \doteq 2gP.$$

Hence

$$\int_{Q_F}^{Q_o} \frac{1 + \dfrac{2k^2b^2nRG}{3L^2}\,Q^{n-1}}{P}\,dQ \doteq 2g\int_0^L dx.$$

Using (2.8) this becomes

$$\int_{Q_F}^{Q_o} \frac{1 + \frac{2k^2b^2n\ RG}{3L^2} Q^{n-1}}{\left[\frac{1}{n+1}\frac{R}{G}(Q^{n+1}-Q_F^{n+1}) + \frac{k^2b^2R^2}{3L^2}(Q^{2n}-Q_F^{2n})\right]^{\frac{1}{2}}} dQ \doteq 2G. \quad (2.10)$$

As we have already seen, equation (2.10) yields a relationship between Q_F and Q_o for the $k = 0$ case. However, in this more general consideration, when k can have values other than zero, the integrand has become too involved to proceed further, except, as we will later see, for the streamlined case where $n = 1$.

EXAMINATION OF THE OPERATIONAL SOLUTION

A solution for this more general treatment of mine ventilation has been determined but in so doing an approximation had to be introduced. We now wish to determine how accurate the solution actually is. To do this it will be shown that an analytical solution can be found for the case of streamlined flow. The operational solution, with $n = 1$, can then be checked against results from this. For general values of n, however, a numerical approach will be required to obtain accurate results. The errors introduced by numerical approximation can be reduced to a suitable level by comparison with the analytical solution for $n = 1$. The range of validity of the operational solution can then be checked by comparing results from it with those from the numerical solution.

CHAPTER 3

A SPECIAL FOURIER-BESSEL EXPANSION

THE ROLE OF THIS EXPANSION

In order that an analytic solution may be obtained it will be necessary to make use of a series expansion of the form

$$f(x) = \sum_{m=1}^{\infty} b_m \sin(\lambda_m x) \qquad (3.1)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of

$$\lambda \sin(kb\lambda) - 2bkgr \cos(kb\lambda) = 0 . \qquad (3.2)$$

A similar series occurs as a special case of a Dini expansion, namely,

$$F(x) = \sum_{m=1}^{\infty} B_m \frac{\cos(\lambda_m x)}{x} \qquad (3.3)$$

where, for some constant H, $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of

$$z \sin z + H \cos z = 0. \qquad (3.4)$$

Equations (3.2) and (3.4) are of the same form but (3.1) and (3.3) differ.

Dini series are discussed by Lebedev (1965) and Calif. Inst. of Tech. - Bateman Mss. Project (1953), and their general theory is given by Watson (1944). It is now intended to employ similar methods to those used by Watson to establish the validity of the expansion (3.1).

A slightly modified form of a special case of a Fourier-Bessel expansion will firstly be analysed. This series is

$$F(x) = \sum_{m=1}^{\infty} A_m \cos(j_m x) \qquad (3.5)$$

where $j_1, j_2, j_3, \ldots$ are the positive roots of

$$\cos z = 0. \tag{3.6}$$

An investigation of the more general expansion

$$F(x) = \sum_{m=1}^{\infty} B_m \cos(\lambda_m x) \tag{3.7}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of (3.4) will then follow. Integration of (3.7) would yield an expansion of the form (3.1) but this step, of course, will need careful justification.

INVESTIGATION OF THE FOURIER-BESSEL SERIES

Suppose a function $F(x)$ can be expanded in the form (3.5). Multiplying by $\cos(j_n x)$ and integrating between the limits 0 and 1 we have

$$\int_0^1 F(x)\cos(j_n x)dx = \sum_{m=1}^{\infty} A_m \int_0^1 \cos(j_m x)\cos(j_n x)dx \ .$$

Now

$$\begin{aligned}\int_0^1 \cos(j_m x)\cos(j_n x)dx &= 0 \quad \text{for } m \neq n \\ &= \tfrac{1}{2} \quad \text{for } m = n.\end{aligned}$$

Hence,

$$A_m = 2\int_0^1 F(t)\cos(j_m t)dt \ . \tag{3.8}$$

To investigate the validity of the expansion we need to consider, therefore, the limit

$$\lim_{n\to\infty} \sum_{m=1}^{n} 2\cos(j_m x) \int_0^1 F(t)\cos(j_m t)dt \quad .$$

To do this we shall consider the sum

$$\sum_{m=1}^{n} 2\cos(j_m x)\cos(j_m t) \quad 0 < x \leq 1,\ 0 \leq t \leq 1 \tag{3.9}$$

which we will denote by $T_n(t,x)$.

If we can express the m^{th} term of $T_n(t,x)$ as the residue at j_m of a function of the complex variable w, which has poles at $j_1, j_2, j_3, \ldots j_n$, then $T_n(t,x)$ can be expressed as an integral of this function around some suitable path.

Consider the function

$$\phi(w) = \frac{2[t\cos(xw)\sin(tw) - x\cos(tw)\sin(xw)]}{(t^2-x^2)\cos^2 w} . \tag{3.10}$$

If we put

$$g(w) = 2[t\cos(xw)\sin(tw) - x\cos(tw)\sin(xw)]$$

and then put $w = j_m + \theta$, where θ is small, Taylor's theorem gives that

$$\phi(w) = \frac{g(j_m) + \theta g'(j_m) + \tfrac{1}{2}\theta^2 g''(j_m) + \ldots}{(t^2-x^2)[\theta^2\sin^2 j_m + o(\theta^3)]}$$

$$= \left(\frac{g(j_m)}{t^2-x^2}\right)\frac{1}{\theta^2} + \left(\frac{g'(j_m)}{t^2-x^2}\right)\frac{1}{\theta} + \ldots$$

Thus the residue of $\phi(w)$ at j_m is

$$\frac{g'(j_m)}{t^2-x^2}$$

or $\qquad 2\cos(xj_m)\cos(tj_m)$.

Hence $T_n(t,x) = \frac{1}{2\pi i}\int_c \phi(w)\, dw$

where we shall take c as a rectangle with vertices at $\pm Bi$ and $A_n \pm Bi$ where B will tend to ∞ and A_n is chosen so that $j_n < A_n < j_{n+1}$, n being sufficiently large. In fact we may take

$$A_n = n\pi \ .$$

From Bi to -Bi the integral is zero since $\phi(w)$ is odd. To consider the integral along either the upper or lower side, put $w = u + iv$ and let $|v| \to \infty$. Then $\cos w = O(e^{|v|})$ so that $\phi(w) = O(e^{-(2-t-x)|v|})$ and hence the integrals along these sides tend to zero as $B \to \infty$.

Thus we have that

$$T_n(t,x) = \frac{1}{2\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \phi(w)dw \quad . \tag{3.11}$$

We can now determine an upper bound for $|T_n(t,x)|$. For values of w on the line joining $A_n - i\infty$ to $A_n + i\infty$ there exist positive constants c_1 and c_2 such that

$$|\cos(tw)| \leq c_1 e^{|tv|}, \ |\cos w| \geq c_2 e^{|v|} \ . \tag{3.12}$$

Hence, from (3.10), (3.11) and (3.12)

$$|T_n(t,x)| = \left|\frac{1}{2\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{2[t\ \cos(xw)\sin(tw) - x\ \cos(tw)\sin(xw)]}{(t^2-x^2)\ \cos^2 w}\, dw\right|$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|t-x|\, c_1^2 e^{|tv|+|xv|}}{|t^2-x^2|\, c_2^2 e^{|2v|}}\, dv$$

$$\leq \frac{4c_1^2}{\pi c_2^2 (2-x-t)\,|t^2-x^2|} \ . \tag{3.13}$$

This gives the upper bound of $|T_n(t,x)|$.

It is also noted that

$$\int_0^t (t^2-x^2)T_n(t,x)dt = \frac{1}{\pi i}\int_0^t\int_{A_n-i\infty}^{A_n+i\infty} \frac{t\ \cos(xw)\sin(tw)\ -\ x\ \cos(tw)\sin(xw)}{\cos^2 w}\,dw\,dt$$

$$= \frac{1}{\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{dw}{\cos^2 w}\int_0^t \Big\{t\ \cos(xw)\sin(tw)-x\ \cos(tw)\sin(xw)\Big\}dt$$

$$= \frac{1}{\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{\frac{1}{w}\cos(xw)\sin(tw)-t\ \cos(tw)\cos(xw)-x\sin(xw)\sin(tw)}{w\ \cos^2 w}\,dw$$

Thus,

$$\left|\int_0^t (t^2-x^2)T_n(t,x)dt\right| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|t+x|\,c_1^2 e^{(t+x)|v|}}{A_n c_2^2 e^{|2v|}}\,dv$$

$$\leqslant \frac{4c_1^2}{\pi c_2^2\ A_n(2-x-t)}\ . \qquad (3.14)$$

To continue further into the investigation of $T_n(t,x)$ we now wish to consider certain limiting cases of integrals in which it is involved. Firstly we will consider

$$\lim_{n\to\infty}\int_0^1 T_n(t,x)\ dt.$$

We have from (3.9), that

$$\int_0^1 T_n(t,x)dx = \int_0^1\sum_{m=1}^{n} 2\cos(j_m x)\cos(j_m t)dt$$

$$= \sum_{m=1}^{n} 2\ \cos(j_m x)\int_0^1 \cos(j_m t)dt$$

$$= \sum_{m=1}^{n} \frac{2\ \cos(j_m x)\sin(j_m)}{j_m}$$

$$= \sum_{m=1}^{n} \frac{2\cos(j_m x)}{j_m \sin(j_m)}$$

since $\sin(j_m)$ is either 1 or -1.

Putting $w = j_m+\theta$ in the function

$$\frac{2\cos(xw)}{w\cos w}$$

and applying Taylor's theorem, we find that the residue of the function at j_m is

$$\frac{-2\cos(xj_m)}{j_m \sin(j_m)}$$

so that we have

$$\int_0^1 T_n(t,x)\,dx = \frac{1}{2\pi i}\int_c \frac{2\cos(xw)}{w\cos w}\,dw$$

where c is the same path as before except that it is in the opposite direction and the origin is avoided by an indentation to the right of the imaginary axis.

As before, the integrals along the upper and lower sides of the rectangle tend to zero as $B\to\infty$. However, there is now a pole at the origin. Applying Taylor's theorem again we have

$$\frac{2\cos(xw)}{w\cos w} = \frac{2(1-\frac{1}{2}x^2w^2+\ldots)}{w(1-\frac{1}{2}w^2+\ldots)}$$

$$= 2\,\frac{1}{w} - x^2 w + \ldots$$

so that the residue at the origin is 2.

Since the integrand is odd, the contribution of the pole at the origin is 1 and hence

$$\int_0^1 T_n(t,x)\,dx = 1 - \frac{1}{2\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{2\cos(xw)}{w\cos w}\,dw. \qquad (3.15)$$

Now, for values of w on the line $A_n-i\infty$ to $A_n+i\infty$,

$$\left|\int_{A_n-i\infty}^{A_n+i\infty} \frac{2\cos(xw)}{w\cos w}\,dw\right| \leqslant \int_{-\infty}^{\infty} \frac{2c_1e^{x|v|}}{A_nc_2e^{|v|}}\,dv$$

$$\leqslant \frac{4c_1}{A_nc_2}\int_0^1 e^{-(1-x)|v|}\,dv$$

$$\leqslant \frac{4c_1}{A_nc_2(1-x)} \quad .$$

Thus $\int_{A_n-i\infty}^{A_n+i\infty} \frac{2\cos(xw)}{w\cos w}\,dw$ tends to zero as $n \to \infty$, for $0 \leqslant x < 1$, so that from (3.15)

$$\lim_{n\to\infty}\int T_n(t,x)\,dt = 1. \qquad (3.16)$$

Another limiting case we need to consider is

$$\lim_{n\to\infty}\int_0^x T_n(t,x)\,dt.$$

Following the same procedure as in the previous case, we have

$$\int_0^x T_n(t,x)dt = \sum_{m=1}^{n} 2\cos(j_m x)\int_0^x \cos(j_m t)dt$$

$$= \sum_{m=1}^{n} \frac{2\cos(j_m x)\sin(j_m x)}{j_m}$$

$$= \frac{1}{2\pi i}\int_c \frac{2\sin(xw)[\cos(w)\sin(xw)-\cos(xw)\sin w]}{w\cos w}\,dw$$

$$= \lim_{B\to\infty}\frac{1}{2\pi i}\int_{A_n-Bi}^{A_n+Bi}\frac{-2\sin(xw)\sin(w-wx)}{w\cos w}\,dw$$

$$= \lim_{B\to\infty}\frac{1}{2\pi i}\int_{A_n-Bi}^{A_n+Bi}\frac{\cos w-\cos(2xw-w)}{w\cos w}\,dw$$

$$= \tfrac{1}{2} - \lim_{B\to\infty}\int_{A_n-Bi}^{A_n+Bi}\frac{\cos(2x-1)w}{w\cos w}\,dw\,. \qquad (3.17)$$

As a consequence of a lemma proved by Watson (1944, p.587) we have that

$$\int_{A_n-Bi}^{A_n+Bi}\frac{\cos(2x-1)w}{w\cos w}\,dw \to 0 \qquad \text{as } B\to\infty$$

and that the integral is bounded for $\tfrac{1}{2} \leq x \leq 1$.

Hence we have the result, from (3.17), that

$$\lim_{n\to\infty}\int_0^x T_n(t,x)dt = \tfrac{1}{2} \qquad (3.18)$$

and if we close the range of values of x on the right so that $0 < x \leq 1$, we can infer that

$$\left| \int_0^x T_n(t,x)dt \right| < U \quad . \tag{3.19}$$

That is, the integral is bounded for $0 < x \leq 1$. It can be similarly shown that the integral

$$\int_0^t T_n(t,x)dt$$

is bounded for $0 \leq t \leq 1$.

As a final consideration we note from (3.16) and (3.18) that

$$\lim_{n\to\infty} \int_x^1 T_n(t,x)dt = \tfrac{1}{2} \; . \tag{3.20}$$

ANALOGUE OF THE RIEMANN-LEBESGUE LEMMA

Lemma: If (a,b) is any part of the closed interval (0,1) such that x is not an internal point nor an end point of (a,b), then the existence and absolute convergence of

$$\int_a^b F(t)dt$$

are sufficient to ensure that as $n \to \infty$,

$$\int_a^b F(t)\; T_n(t,x)dt$$

tends to zero for $0 < x \leq 1$.

To prove this lemma we shall firstly suppose $F(t)$ is bounded and that the origin is not an end point of (a,b). These restrictions will then be removed.

(I) Let $F(t) = (t^2-x^2)G(t)$ where $|G(t)|$ has an upper bound K in (a,b). Divide (a,b) into p equal parts by the points $t_1, t_2, \dots t_{p-1}$, where p is such that for any arbitrary positive number ε

$$\sum_{m=1}^{p} (U_m - L_m)(t_m - t_{m-1}) < \varepsilon$$

where U_m and L_m are the upper and lower bounds of $G(t)$ in (t_{m-1}, t_m).

Let $G(t) = G(t_{m-1}) + w_m(t)$

where $|w_m(t)| \leq U_m - L_m$ in (t_{m-1}, t_m). Then

$$\left|\int_a^b F(t)T_n(t,x)\,dt\right| = \left|\sum_{m=1}^{p} G(t_{m-1})\int_{t_{m-1}}^{t_m}(t^2-x^2)T_n(t,x)\,dt + \sum_{m=1}^{p}\int_{t_{m-1}}^{t_m}(t^2-x^2)T_n(t,x)w_m(t)\,dt\right|$$

$$\leq \sum_{m=1}^{p}\left|G(t_{m-1})\right| \cdot \left|\int_{t_{m-1}}^{t_m}(t^2-x^2)T_n(t,x)\,dt\right| + \sum_{m=1}^{p}\left|\int_{t_{m-1}}^{t_m}(t^2-x^2)T_n(t,x)w_m(t)\,dt\right|$$

$$\leq \frac{8\,c_1^2\,pK}{\pi c_2^2 A_n(2-x-b)} + \frac{4c_1^2}{\pi c_2^2(2-x-b)}\sum_{m=1}^{p}(U_m-L_m)\int_{t_{m-1}}^{t_m} dt$$

from the inequalities (3.13) and (3.14).

Hence,

$$\left|\int_a^b F(t)T_n(t,x)dt\right| \leqslant \frac{4c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{2pK}{A_n}+\varepsilon\right).$$

Now, the choice of ε fixes p so that we can choose A_n such that $A_n > \frac{2pK}{\varepsilon}$.

Hence $$\left|\int_a^b F(t)T_n(t,x)dt\right| \to 0 \text{ as } n \to \infty \text{ for } 0 < x \leqslant 1.$$

(II) Now suppose G(t) is bounded only on intervals $\gamma_1, \gamma_2, \ldots, \gamma_{r+1}$ in (a,b) but not on the remaining intervals $\mu_1, \mu_2, \ldots, \mu_r$ so that

$$\sum_{i=1}^{r} \int_{\mu_i} |G(t)|dt < \varepsilon.$$

When t lies in one of the intervals μ_i we have

$$|(t^2-x^2)T_n(t,x)| \leqslant \frac{4c_1^2}{\pi c_2^2(2-x-b)} .$$

Hence, if K is the upper bound of $|G(t)|$ in the intervals $\gamma_1, \gamma_2, \ldots \gamma_{r+1}$, then

$$\left|\int_a^b F(t)T_n(t,x)dt\right| = \left|\sum_{i=1}^{r+1}\int_{\gamma_i} G(t)(t^2-x^2)T_n(t,x)dt + \sum_{i=1}^{r}\int_{\mu_i} G(t)(t^2-x^2)T_n(t,x)dt\right|$$

$$\leqslant \left|\sum_{i=1}^{r+1}\int_{\gamma_i} G(t)(t^2-x^2)T_n(t,x)dt\right| + \sum_{i=1}^{r}\int_{\mu_i} |G(t)(t^2-x^2)T_n(t,x)|dt$$

$$\leqslant \frac{4c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{2(r+1)pK}{A_n}+\varepsilon\right) + \frac{4c_1^2}{\pi c_2^2(2-x-b)}\varepsilon$$

$$\leqslant \frac{8c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{(r+1)pK}{A_n}+\varepsilon\right).$$

If we take ε sufficiently small and then take A_n sufficiently large, then the expression on the right is arbitrarily small and we have the required result.

(III) Finally, suppose $\int_0^b F(t)dt$ exists and is absolutely convergent. We can then choose a η so small that

$$\int_0^\eta \left|\frac{F(t)}{t^2-x^2}\right| dt < \varepsilon.$$

Then,

$$\left|\int_0^\eta F(t)T_n(t,x)dt\right| \leq \int_0^\eta |F(t)|\,|T_n(t,x)|\,dt$$

$$\leq \frac{4c_1^2}{\pi c_2^2(2-x-b)}\int_0^\eta \left|\frac{F(t)}{t^2-x^2}\right| dt$$

$$< \frac{4c_1^2\,\varepsilon}{\pi c_2^2(2-x-b)}\ .$$

From part (II), if K is the upper bound of $|G(t)|$ in (η,b) when the intervals μ_i are omitted, then

$$\left|\int_0^b F(t)T_n(t,x)dt\right| < \frac{8c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{(r+1)pK}{A_n} + \frac{3\varepsilon}{2}\right)$$

and once again the right hand side and hence the left, tends to zero, thus completing the proof.

VALIDITY OF THE FOURIER-BESSEL EXPANSION

We are now able to examine the conditions under which the expansion (3.5) is valid.

To do this we can firstly consider the following theorem.

Theorem: If $F(t)$ is defined on an interval $(0,1)$, and has limited total fluctuation in any interval (a,b) where $0 < a < b < 1$, and if $\int_0^1 F(t)dt$ exists and is absolutely convergent, then for $x \in (a,b)$ the series

$$\sum_{m=1}^{\infty} A_m \cos(j_m x)$$

is convergent and its sum is

$$\tfrac{1}{2}[F(x-0) + F(x+0)].$$

To prove this, we note that from (3.8) and (3.9)

$$\sum_{m=1}^{n} A_m \cos(j_m x) = \int_0^1 F(t)T_n(t,x)dt$$

and since

$$\lim_{n\to\infty} \int_0^x T_n(t,x)dt = \lim_{n\to\infty} \int_x^1 T_n(t,x)dt = \tfrac{1}{2}$$

as shown in (3.18) and (3.20), we have

$$\tfrac{1}{2}[F(x-0)+F(x+0)] = \lim_{n\to\infty} F(x-0)\int_0^x T_n(t,x)dt + \lim_{n\to\infty} F(x+0)\int_x^1 T_n(t,x)dt.$$

Put

$$S_n(x) = \int_0^x \{F(t)-F(x-0)\}T_n(t,x)dt + \int_x^1 \{F(t)-F(x+0)\}T_n(t,x)dt.$$

We then wish to show that $S_n(x) \to 0$ as $n \to \infty$.

Consider the integral

$$I = \int_x^1 \{F(t)-F(x+0)\}T_n(t,x)dt. \qquad (3.21)$$

The function F(t)-F(x+0) has limited total fluctuation in (x,b) and so we may write

$$F(t)-F(x+0) = \chi_1(t)-\chi_2(t)$$

where $\chi_1(t)$, $\chi_2(t)$ are bounded positive increasing functions in (x,b) such that

$$\chi_1(x+0) = \chi_2(x+0) = 0.$$

Hence, for any arbitrary positive number ε there exists a positive number $\delta \leqslant b - x$ such that for $x \leqslant t \leqslant x + \delta$ we have

$$0 \leqslant \chi_1(t) < \varepsilon, \quad 0 \leqslant \chi_2(t) < \varepsilon. \tag{3.22}$$

Now, (3.21) can be expressed in the form

$$I = \int_{x+\delta}^{1} \{F(t)-F(x+0)\}T_n(t,x)\,dt+\int_{x}^{x+\delta}\chi_1(t)T_n(t,x)\,dt-\int_{x}^{x+\delta}\chi_2(t)T_n(t,x)\,dt.$$

From the analogue of the Riemann-Lebesgue lemma it follows that for n sufficiently large

$$\left|\int_{x+\delta}^{1}\{F(t)-F(x+0)\}T_n(t,x)\,dt\right| < \varepsilon$$

and from the second mean-value theorem, there is a number ζ between 0 and δ such that

$$\int_{x}^{x+\delta}\chi_1(t)T_n(t,x)\,dt = \chi_1(x+\delta)\int_{x+\zeta}^{x+\delta} T_n(t,x)\,dt .$$

But from (3.14) we have that

$$\left|\int_0^x T_n(t,x)\,dt\right| < U$$

and from (3.22) $\chi_1(x+\delta) < \varepsilon$, so that

$$\left|\int_x^{x+\delta} \chi_1(t)T_n(t,x)\,dt\right| < 2\varepsilon U \ .$$

Similarly,

$$\left|\int_x^{x+\delta} \chi_2(t)T_n(t,x)\,dt\right| < 2\varepsilon U \ .$$

Hence $I \to 0$ as $n \to \infty$ for ε sufficiently small.

The other part of $S_n(x)$ can be shown to tend to zero in a similar manner. Thus $S_n(x) \to 0$ as $n \to \infty$. This means that the difference between $\sum_{m=1}^{n} A_m \cos(j_m x)$ and $\frac{1}{2}[F(x-0)+F(x+0)]$ is arbitrarily small and hence the required result is obtained.

Finally we can place one more condition on $F(t)$. If $F(t)$ has all the properties of the previous result and is also continuous on $(0,1)$, then we can show that $\sum_{m=1}^{\infty} A_m \cos(j_m x)$ is uniformly convergent to $F(x)$ throughout the interval (a,b) where $0 < a < b < 1$.

To prove this, the integral

$$\int_{x+\delta}^{1} \{F(t)-F(x)\}T_n(t,x)\,dt$$

would have to tend to zero uniformly as $n \to \infty$. This means, from the analogue of the Riemann-Lebesgue lemma that

$$\int_{x+\delta}^{1}\{F(t)-F(x)\}dt$$

is bounded.

Now,
$$\left|\int_{x+\delta}^{1}\{F(t)-F(x)\}dt\right| \leq \int_{0}^{1}|F(t)|dt + |F(x)|\int_{0}^{1}dt$$

and this is bounded since F(x) is continuous and therefore bounded in (a,b).

Similarly, the other integrals introduced in the previous proof tend to zero uniformly. Hence, as $n \to \infty, \sum_{m=1}^{n} A_m \cos(j_m x) \to F(x)$ uniformly.

Thus, in summary, if F(t) is defined and continuous on (0,1) and $\int_0^1 F(t)dt$ exists and is absolutely convergent, then for $x \in (a,b)$ where $0 < a < b < 1$, we have that

$$F(x) = \sum_{m=1}^{\infty} A_m \cos(j_m x)$$

where the coefficients A_m are given by

$$A_m = 2\int_0^1 F(t)\cos(j_m t)dt$$

and the j_m are the positive roots of

$$\cos z = 0.$$

CHAPTER 4

A SPECIAL DINI EXPANSION

INVESTIGATION OF THE DINI SERIES

Suppose now that a function F(x) can be expanded in the form

$$F(x) = \sum_{m=1}^{\infty} B_m \cos(\lambda_m x) \tag{4.1}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of the equation

$$z \sin z + H \cos z = 0, \tag{4.2}$$

H being some constant. This is basically the Dini expansion (3.3), although the theory must be modified slightly.

We note that for the case $m \neq n$

$$\int_0^1 \cos(\lambda_m x)\cos(\lambda_n x)\,dx = \frac{\lambda_m \sin\lambda_m \cos\lambda_n - \lambda_n \sin\lambda_n \cos\lambda_m}{\lambda_m^2 - \lambda_n^2}. \tag{4.3}$$

From (4.2)

$$\lambda_n \sin\lambda_n = -H\cos\lambda_n$$

and

$$\lambda_m \sin\lambda_m = -H\cos\lambda_m$$

so that the right hand side of (4.3) is zero.

For the case m = n we have

$$\int_0^1 \cos^2(\lambda_m x)\,dx = \frac{\lambda_m + \sin\lambda_m \cos\lambda_m}{2\lambda_m}. \tag{4.4}$$

On multiplying (4.1) by $\cos(\lambda_n x)$ and then integrating between the limits 0 and 1, we obtain as a consequence of (4.3) and (4.4) that

$$B_m = \frac{2\lambda_m}{\lambda_m+\sin\lambda_m\cos\lambda_m}\int_0^1 F(t)\cos(\lambda_m t)dt \ . \qquad (4.5)$$

Thus, to examine the validity of expansion (4.1) we need to investigate the sum

$$\sum_{m=1}^{n} \frac{2\lambda_m \cos(\lambda_m x) \cos(\lambda_m t)}{\lambda_m+\sin\lambda_m\cos\lambda_m} \ .$$

To do this we will proceed in a similar manner as before.

Consider the function

$$\psi(w) = \frac{2w \cos(xw)\cos(tw)}{\cos w(w\sin w + H\cos w)} \qquad (4.6)$$

which has poles at $j_1, j_2, j_3, \ldots$ and $\lambda_1, \lambda_2, \lambda_3, \ldots$, where the j_m are as previously defined in (3.6).

Putting $w = j_m+\theta$ and applying Taylor's theorem we have

$$\psi(w) = \left(\frac{2j_m \cos(j_m x)\cos(j_m t)}{-j_m \sin^2 j_m}\right) \frac{1}{\theta} + \ldots$$

so that the residue of $\psi(w)$ at j_m is

$$-2 \cos(j_m x)\cos(j_m t).$$

Putting $w = \lambda_m+\theta$, we have that

$$\psi(w) = \left(\frac{2\lambda_m \cos(\lambda_m x) \cos(\lambda_m t)}{\cos\lambda_m(\sin\lambda_m+\lambda_m\cos\lambda_m-H\sin\lambda_m)}\right) \frac{1}{\theta} + \ldots$$

$$= \left(\frac{2\lambda_m \cos(\lambda_m x) \cos(\lambda_m t)}{\lambda_m-\sin\lambda_m(\lambda_m\sin\lambda_m+H\cos\lambda_m)+\sin\lambda_m\cos\lambda_m}\right) \frac{1}{\theta} + \ldots$$

thus giving the residue of $\psi(w)$ at λ_m as

$$\frac{2\lambda_m\cos(\lambda_m x)\cos(\lambda_m t)}{\lambda_m+\sin\lambda_m\cos\lambda_m}$$

since $\lambda_m\sin\lambda_m+H\cos\lambda_m = 0$ from (4.2).

Now if we let D_n be a number between λ_n and λ_{n+1} such that it is not equal to any of the j_m, and if j_N is the greatest of the j_m that is less than D_n, then we can define

$$S_n(t,x;H) = \sum_{m=1}^{n} \frac{2\lambda_m\cos(\lambda_m x)\cos(\lambda_m t)}{\lambda_m+\sin\lambda_m\cos\lambda_m} - \sum_{m=1}^{N} 2\cos(j_m x)\cos(j_m t) \tag{4.7}$$

and we will wish to prove that

$$\sum_{m=1}^{n} B_m\cos(\lambda_m x) - \sum_{m=1}^{N} A_m\cos(j_m x) = \int_0^1 F(t)S_n(t,x;H)dt. \tag{4.8}$$

We have from (4.7) and (4.6) that

$$S_n(t,x;H) = \frac{1}{2\pi i}\int_c \psi(w)\ dw$$

where c is the rectangle with vertices $\pm Bi$, $D_n \pm Bi$ where $B \to \infty$. The integral tends to zero along the upper and lower sides as before when $B \to \infty$ and so we have

$$S_n(t,x;H) = \frac{1}{2\pi i}\int_{D_n-i\infty}^{D_n+i\infty} \psi(w)dw - \frac{1}{2\pi i}P\int_{-i\infty}^{i\infty} \psi(w)dw$$

where P denotes Cauchy's principal value.

Since $\psi(w)$ is odd, the second integral vanishes and thus

$$S_n(t,x;H) = \frac{1}{2\pi i}\int_{D_n-i\infty}^{D_n+i\infty} \frac{2w\ \cos(wx)\cos(wt)}{\cos w(w\sin w+H\cos w)}\,dw\ . \qquad (4.9)$$

By considering values of w on the line $D_n-i\infty$ to $D_n+i\infty$ we have from (4.9) and from similar considerations to (3.12) that

$$\left|S_n(t,x;H)\right| = \left|\frac{1}{2\pi i}\int_{D_n-i\infty}^{D_n+i\infty} \frac{2w\ \cos(wx)\cos(wt)}{\cos w(w\sin w+H\cos w)}\,dw\right|$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(D_n+iv)c_1^2e^{(x+t)|v|}}{c_2^2e^{|v|}\{(D_n+iv)e^{|v|}+He^{|v|}\}}\,dv$$

$$\leq \frac{2c_1^2}{\pi c_2^2}\int_0^{\infty} e^{-(2-x-t)|v|}dv$$

$$\leq \frac{2c_1^2}{\pi c_2^2(2-x-t)}\ . \qquad (4.10)$$

We also have that

$$\int_0^t S_n(t,x;H)dt = \frac{1}{2\pi i}\int_0^t\int_{D_n-i\infty}^{D_n+i\infty} \frac{2w\ \cos(xw)\cos(tw)}{\cos w(w\sin w+H\cos w)}\,dw\ dt$$

$$= \frac{1}{\pi i}\int_{D_n-i\infty}^{D_n+i\infty} \frac{\cos(xw)\sin(tw)}{\cos w(w\sin w+H\cos w)}\,dw\ .$$

Hence,

$$\left|\int_0^t S_n(t,x;H)dt\right| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{c_1^2e^{(t+x)|v|}}{c_2^2e^{|2v|}D_n}\,dv$$

$$\leq \frac{2c_1^2}{\pi c_2^2D_n(2-x-t)}\ . \qquad (4.11)$$

ANALOGUE OF THE RIEMANN-LEBESGUE LEMMA

Lemma: If (a,b) is any part of the interval (0,1) then the existence and absolute convergence of

$$\int_a^b F(t)dt$$

are sufficient to ensure that as $n \to \infty$,

$$\int_a^b F(t)S_n(t,x;H)dt$$

tends to zero for $0 < x \leqslant 1$.

The proof is similar to that given before and so much of the same terminology can apply again.

(I) Suppose F(t) is bounded and let K be the upper bound of $|F(t)|$ in (a,b) where $a > 0$. Then by dividing (a,b) into p parts as before and putting

$$F(t) = F(t_{m-1}) + w_m(t)$$

we have that

$$\left|\int_a^b F(t)S_n(t,x;H)dt\right| = \left|\sum_{m=1}^{p} F(t_{m-1})\int_{t_{m-1}}^{t_m} S_n(t,x;H)dt + \sum_{m=1}^{p}\int_{t_{m-1}}^{t_m} w_m(t)S_n(t,x;H)dt\right|$$

$$\leqslant \frac{2c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{2pK}{D_n} + \varepsilon\right)$$

using (4.10) and (4.11). The expression on the right can be made as small as we like so that as $n \to \infty$,

$$\left|\int_a^b F(t)S_n(t,x;H)dt\right| \to 0.$$

(II) Now suppose F(t) is only bounded in intervals $\gamma_1, \gamma_2, \ldots, \gamma_{r+1}$ in (a,b) but not in the remaining intervals $\mu_1, \mu_2, \ldots, \mu_r$ and that

$$\sum_{i=1}^{r} \int_{\mu_i} |F(t)|dt < \varepsilon.$$

Then

$$\left|\int_a^b F(t)S_n(t,x;H)dt\right| \leq \left|\sum_{i=1}^{r+1}\int_{\gamma_i} F(t)S_n(t,x;H)dt\right| + \sum_{i=1}^{r}\int_{\mu_i}|F(t)S_n(t,x;H)|dt$$

$$\leq \frac{4c_1^2}{\pi c_2^2(2-x-b)}\left(\frac{(r+1)pK}{D_n} + \varepsilon\right)$$

which again tends to zero as $n \to \infty$.

(III) Finally, suppose $\int_0^b F(t)dt$ exists and is absolutely convergent so that for an arbitrarily small η

$$\int_0^\eta |F(t)|dt < \varepsilon.$$

Then since

$$\left|\int_0^\eta F(t)S_n(t,x;H)dt\right| \leq \frac{2c_1^2}{\pi c_2^2(2-x-b)}\int_0^\eta |F(t)|dt$$

we have

$$\left|\int_0^b F(t)S_n(t,x;H)\,dt\right| \leq \frac{4c_1^2}{\pi c_2^2(2-x-b)}\left\{\frac{(r+1)pK}{D_n} + \frac{3\varepsilon}{2}\right\}$$

which tends to zero as $n \to \infty$ as required.

VALIDITY OF THE DINI EXPANSION

As a result of the last theorem, if $\int_0^1 F(t)dt$ exists and is absolutely convergent, then for $0 < x \leq 1$ we have from (4.8) that

$$\sum_{m=1}^{n} B_m \cos(\lambda_m x) - \sum_{m=1}^{N} A_m \cos(j_m x) \to 0 \qquad (4.12)$$

as $n \to \infty$.

Now, in a monotonic function the positive zeros are interlaced with the positive poles. If we consider the function

$$g(z) = \frac{z \sin z + H \cos z}{\cos z}$$

$$= z \tan z + H$$

we note that this is monotonic so that the numbers λ_m and j_m are interlaced. Therefore, D_n may be chosen so that after a certain stage, $n - N$ has the same value for all values of n.

Then, since

$$\sum_{m=N+1}^{n} A_m \cos(j_m x) \to 0$$

throughout $(0,1)$ we have from (4.12) that

$$\sum_{m=1}^{n} B_m \cos(\lambda_m x) - \sum_{m=1}^{n} A_m \cos(j_m x) \to 0 \qquad (4.13)$$

as $n \to \infty$.

Further, as a result of work done in the previous chapter, if $F(x)$ has limited total fluctuation in (a,b) where $0 < a < b < 1$, then from (4.13) the series

$$\sum_{m=1}^{\infty} B_m \cos(\lambda_m x)$$

converges to the sum

$$\tfrac{1}{2}[F(x-0) + F(x+0)]$$

for all $x \in (a,b)$.

If, also, $F(x)$ is continuous in (a,b) then the series converges uniformly to $F(x)$. That is, if $F(x)$ satisfied all these conditions, then

$$F(x) = \sum_{m=1}^{\infty} B_m \cos(\lambda_m x)$$

where B_m is given by

$$B_m = \frac{2\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 F(t)\cos(\lambda_m t)\,dt$$

and the λ_m are the positive roots of

$$z \sin z + H \cos z = 0.$$

CHAPTER 5

A MODIFIED DINI EXPANSION

FURTHER CONSIDERATION OF THE DINI SERIES

We have shown that under certain conditions a function F(x) can be expanded as a series of the form

$$\sum_{m=1}^{\infty} B_m \cos(\lambda_m x) \tag{5.1}$$

where B_m and λ_m are determined as indicated in (4.5) and (4.2). We have, however, only established the validity of the expansion for values of x within the interval (a,b) where $0 < a < b < 1$. We have not considered the behaviour of the series for x values in the intervals (0,b) and (a,1). However, we can infer from Watson (1944, pp.601-605) that provided the same conditions hold in these two intervals, the expansion will converge as before.

Our interest now moves to a consideration of the series

$$\sum_{m=1}^{\infty} b_m \sin(\lambda_m x) \tag{5.2}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are as before. This series can be obtained by either differentiation or integration of (5.1) but the use of these operations will need to be carefully considered.

DIFFERENTIATION AND INTEGRATION OF THE SERIES

Suppose that $\sum_{m=1}^{\infty} B_m \cos(\lambda_m x)$ converges uniformly to F(x) in (0,1). That is,

$$F(x) = \sum_{m=1}^{\infty} B_m \cos(\lambda_m x). \tag{5.3}$$

Integration of (5.3) yields

$$f(x) = \sum_{m=1}^{\infty} C_m \sin(\lambda_m x) \tag{5.4}$$

where $f(x) = \int_0^x F(t)dt$, $x \leq 1$ and $f(0) = 0$.

From (5.4) and (4.5) we have that

$$C_m = \frac{B_m}{\lambda_m}$$

$$= \frac{2}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f'(x) \cos(\lambda_m x) dx. \tag{5.5}$$

That is,

$$C_m = \frac{2}{\lambda_m + \sin\lambda_m \cos\lambda_m} \left\{ f(1)\cos\lambda_m + \lambda_m \int_0^1 f(x) \sin(\lambda_m x) dx \right\}$$

$$= \frac{2\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f(x) \sin(\lambda_m x) dx + \frac{2f(1)\cos\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m}. \tag{5.6}$$

Now suppose $G(x)$ is a continuous function on $(0,1)$ and is such that

$$G(x) = \sum_{m=1}^{\infty} D_m \cos(\lambda_m x) \tag{5.7}$$

where

$$D_m = \frac{2\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 G(x) \cos(\lambda_m x) dx. \tag{5.8}$$

Differentiation of (5.7) yields

$$f(x) = \sum_{m=1}^{\infty} C_m \sin(\lambda_m x) \tag{5.9}$$

where it is taken that $G'(x) = f(x)$ and $C_m = -\lambda_m D_m$.

Thus, from (5.8) we now have that C_m is given by

$$\begin{aligned}
C_m &= \frac{-2\lambda_m^2}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 \cos(\lambda_m x)\,dx \int_0^x f(t)\,dt \\
&= \frac{-2\lambda_m^2}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f(t)\,dt \int_t^1 \cos(\lambda_m x)\,dx \\
&= \frac{-2\lambda_m^2}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 \left\{ \frac{\sin\lambda_m}{\lambda_m} - \frac{\sin(\lambda_m t)}{\lambda_m} \right\} f(t)\,dt \\
&= \frac{2\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f(x) \sin(\lambda_m x)\,dx - \frac{2\lambda_m \sin\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f(x)\,dx.
\end{aligned} \tag{5.10}$$

Comparing (5.6) and (5.10) we therefore require for consistent results that

$$\sum_{m=1}^{\infty} \frac{-2\lambda_m \sin(\lambda_m) \sin(\lambda_m x)}{\lambda_m + \sin\lambda_m \cos\lambda_m} \int_0^1 f(x)\,dx = \sum_{m=1}^{\infty} \frac{2f(1)\cos\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \sin(\lambda_m x) \; .$$

Now from (4.2) we have that

$$-\lambda_m \sin\lambda_m = H \cos\lambda_m .$$

Therefore, we require

$$\sum_{m=1}^{\infty} \frac{2H\cos\lambda_m \sin(\lambda_m x)}{\lambda_m+\sin\lambda_m\cos\lambda_m}\int_0^1 f(x)\,dx = \sum_{m=1}^{\infty} \frac{2f(1)\cos\lambda_m \sin(\lambda_m x)}{\lambda_m+\sin\lambda_m\cos\lambda_m}\ .$$

This is so if

$$H\int_0^1 f(x)\,dx = f(1),$$

which is not likely, or if

$$\sum_{m=1}^{\infty} \frac{2\cos\lambda_m \sin(\lambda_m x)}{\lambda_m+\sin\lambda_m\cos\lambda_m} = 0. \qquad (5.11)$$

To prove this result, consider the function

$$h(w) = \frac{2\sin(xw)}{w\sin w + H\cos w}$$

which has poles at $\lambda_1, \lambda_2, \lambda_3, \ldots$.

To find the residue at λ_m, put $w = \lambda_m + \theta$.

Then,

$$h(w) = \left(\frac{2\sin(\lambda_m x)}{\lambda_m\cos\lambda_m+\sin\lambda_m-H\sin\lambda_m}\right)\frac{1}{\theta} + \ldots$$

$$= \left(\frac{2\sin(\lambda_m x)\cos\lambda_m}{\lambda_m-\sin\lambda_m(\lambda_m\sin\lambda_m+H\cos\lambda_m)+\sin\lambda_m\cos\lambda_m}\right)\frac{1}{\theta} + \ldots$$

from which we see that the residue at λ_m is

$$\frac{2\sin(\lambda_m x)\cos\lambda_m}{\lambda_m+\sin\lambda_m\cos\lambda_m}\ .$$

Therefore,

$$\sum_{m=1}^{n} \frac{2\cos\lambda_m \sin(\lambda_m x)}{\lambda_m + \sin\lambda_m \cos\lambda_m} = \frac{1}{2\pi i}\int_c \frac{2\sin(xw)}{w\sin w + H\cos w}\, dw$$

where c is the same rectangular path as in the previous chapter. Now, since h(w) is odd, the integral from Bi to -Bi is zero and the integrals along the upper and lower sides vanish as $B \to \infty$. Then, for values of w on the line joining $D_n - i\infty$ to $D_n + i\infty$, we have

$$\left|\sum_{m=1}^{n} \frac{2\cos\lambda_m \sin(\lambda_m x)}{\lambda_m + \sin\lambda_m \cos\lambda_m}\right| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{c_1 e^{x|v|}}{D_n c_2 e^{|v|}}\, dv$$

$$\leq \frac{2c_1}{\pi c_2 D_n (1-x)} .$$

The right hand side vanishes as $n \to \infty$ so that result (5.11) is proved.

This means that the expansion (5.2) can be obtained from (5.1) either by integration or differentiation.

VALIDITY OF THE MODIFIED DINI EXPANSION

We have now established an expansion

$$f(x) = \sum_{m=1}^{\infty} b_m \sin(\lambda_m x)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of

$$z \sin z + H \cos z = 0$$

and b_m is given by

$$b_m = \frac{2}{\lambda_m + \sin\lambda_m \cos\lambda_m}\int_0^1 f'(x)\cos(\lambda_m x)\, dx$$

or

$$b_m = \frac{2\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m}\int_0^1 f(x)\sin(\lambda_m x)\,dx.$$

We note, however, that such an expansion is only valid if f(x) is such that f'(x) satisfies the conditions in the previous chapter.

CHAPTER 6

COMPARISON OF THE OPERATIONAL AND ANALYTICAL SOLUTIONS FOR STREAMLINED FLOW

EQUATION OF FLOW FOR THE STREAMLINED CASE

As previously mentioned, the value of the flow index n may be taken as 1 for streamlined flow. Then, from (2.4) and (2.5), for flow in the airways and barrier,we have that pressure and quantity are connected by the differential equation

$$k^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

with the associated boundary conditions

$$P = (p)_{y=b}$$

$$\frac{dQ}{dx} = \left(2bg \frac{\partial p}{\partial y}\right)_{y=b}$$

$$\frac{dP}{dx} = rQ.$$

It is noted that now the last two equations in these boundary conditions can be combined to give the relationship

$$\frac{\partial^2 p}{\partial x^2} = 2bgr \frac{\partial p}{\partial y} \qquad (6.1)$$

for y = b.

Also it will be found that further consideration can be given to the operational solution obtained in chapter 2 and that an analytical solution may be determined. These two solutions can then be compared to establish the degree of accuracy of the operational solution for n = 1.

OPERATIONAL SOLUTION FOR STREAMLINED FLOW

Putting n = 1 in equations (2.8), (2.9) and (2.10) we have the solutions

$$P^2 \doteqdot \left(\frac{R}{2G} + \frac{k^2b^2R^2}{3L^2}\right)(Q^2-Q_F^2), \tag{6.2}$$

being the general relationship between pressure and quantity in the airways,

$$P_0^2 \doteqdot \left(\frac{R}{2G} + \frac{k^2b^2R^2}{3L^2}\right)(Q_0^2-Q_F^2), \tag{6.3}$$

giving the relationship between P_0, Q_0, Q_F, and

$$\int_{Q_F}^{Q_o} \frac{dQ}{\sqrt{Q^2-Q_F^2}} \doteqdot \frac{2G\sqrt{\frac{R}{2G} + \frac{k^2b^2R^2}{3L^2}}}{1 + \frac{2k^2b^2RG}{3L^2}} . \tag{6.4}$$

The integral in (6.4) can now be evaluated to yield the relationship between Q_F and Q_0,

$$Q_o = Q_F \cosh\left(\sqrt{\frac{6RGL^2}{3L^2+2k^2b^2RG}}\right) . \tag{6.5}$$

This case lends itself to still more relationships between the variables P_o, Q_o, and Q_F. Substituting Q_o from (6.5) into (6.3) we have that

$$P_0^2 \doteqdot Q_F^2\left(\frac{R}{2G} + \frac{k^2b^2R^2}{3L^2}\right)\sinh^2\left(\sqrt{\frac{6RGL^2}{3L^2+2k^2b^2RG}}\right)$$

or

$$Q_F \doteqdot \frac{P_o}{R}\sqrt{\frac{6RGL^2}{3L^2+2k^2b^2RG}}\ \operatorname{cosech}\left(\sqrt{\frac{6RGL^2}{3L^2+2k^2b^2RG}}\right). \tag{6.6}$$

Thus for streamlined flow we have several results that may be used to determine the pressure and quantity of air at points in the airways.

ANALYTICAL SOLUTION FOR STREAMLINED FLOW

To solve the differential equation analytically we can separate the variables by putting

$$p = X(x)Y(y). \tag{6.7}$$

This gives the equation

$$\frac{k^2}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

from which we have that

$$\frac{d^2X}{dx^2} = \lambda^2 X \tag{6.8}$$

and

$$\frac{d^2Y}{dy^2} = -\lambda^2 k^2 Y \tag{6.9}$$

where λ is some constant.

Now, equations (6.8) and (6.9) have as solutions

$$X = A\cosh(\lambda x) + B\sinh(\lambda x)$$

and

$$Y = C\cos(\lambda ky) + D\sin(\lambda ky)$$

where A,B,C,D are arbitrary constants.

Applying the first boundary condition, which implies that $p = 0$ when $x = 0$ or $y = 0$, and using (6.7) we have that

$$p = BD\sinh(\lambda x)\sin(\lambda ky).$$

The other boundary condition, (6.1), gives us that λ is a root of the equation

$$\lambda \sin(\lambda kb) = 2bgrk \cos(\lambda kb). \tag{6.10}$$

Thus we have as solution

$$p = \sum_{m=1}^{\infty} c_m \sinh(\lambda_m x)\sin(\lambda_m ky) \tag{6.11}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the roots of (6.10) and the c_m are constants.

Now, if a function f(y) is such that it can be expanded in the form

$$f(y) = \sum_{m=1}^{\infty} b_m \sin(\lambda_m ky) \tag{6.12}$$

where the λ_m are roots of

$$z \sin(kbz) + H \cos(kbz) = 0$$

then the coefficients b_m are given by

$$b_m = \frac{2}{kb\lambda_m + \sin(kb\lambda_m)\cos(kb\lambda_m)} \int_0^b f'(t)\cos(k\lambda_m t)\,dt. \tag{6.13}$$

Thus, if we assume that no air leaks into the barrier between the mine entrances so that

$$\frac{\partial p}{\partial y} = \frac{P_o}{b} \tag{6.14}$$

when x = L, then from (6.11), (6.12), (6.13) and (6.14)

$$c_m \sinh\lambda_m L = \frac{2}{kb\lambda_m + \sin(kb\lambda_m)\cos(kb\lambda_m)} \int_o^b \frac{P_o}{b}\cos(k\lambda_m t)\,dt$$

$$= \frac{P_o}{bk\lambda_m}\left(\frac{2\sin(kb\lambda_m)}{kb\lambda_m + \sin(kb\lambda_m)\cos(kb\lambda_m)}\right).$$

Hence the pressure of the air at points in the roadways is given by

$$P = \frac{P_o}{b}\sum_{m=1}^{\infty} \frac{2\sin^2(kb\lambda_m)}{k\lambda_m[kb\lambda_m+\sin(kb\lambda_m)\cos(kb\lambda_m)]}\,\frac{\sinh(\lambda_m x)}{\sinh(\lambda_m L)} \qquad (6.15)$$

and since

$$\frac{dP}{dx} = rQ,$$

the quantity of air entering the mine is given by

$$Q_o = \frac{P_o}{rb}\sum_{m=1}^{\infty} \frac{2\sin^2(kb\lambda_m)}{k^2b\lambda_m+k\sin(kb\lambda_m)\cos(kb\lambda_m)}\coth(\lambda_m L) \qquad (6.16)$$

while the quantity of air reaching the working face is

$$Q_F = \frac{P_o}{rb}\sum_{m=1}^{\infty} \frac{2\sin^2(kb\lambda_m)}{k^2b\lambda_m+k\sin(kb\lambda_m)\cos(kb\lambda_m)}\ \mathrm{cosech}(\lambda_m L). \qquad (6.17)$$

COMPARISON OF THE TWO SOLUTIONS

So that results from the operational solution may be compared with those from the analytical solution, values will need to be given to the variables concerned. As a guide to the values that might be taken, it is noted that Rose (1960,p.21) considers a problem of determining R and G for a mine of efficiency 0.4, efficiency being determined by the ratio Q_F to Q_o. With a unit quantity of air entering the mine under a pressure of one unit, it is calculated that for a flow index of 1.4, R = 2.35 and G = 1.54.

A more efficient mine would have a smaller resistance and conductance while a less efficient one would have larger values. To take this into account, R will be given values 1.0, 2.0, 3.0 while G will have corresponding values of 0.5, 1.5, 2.5. By letting k = 1 the isotropic case will be considered so that there is no preferable direction of flow through the barrier. The width of the barrier can be set by putting b = 1 and then the effect of various lengths of airways can be seen by letting L be 1, 3 and 5. It is noted, however, that the longer the airway, the more realistic is the case. Then, by putting $P_O = 1$, values of Q_O and Q_F can be determined from (6.5), (6.6) and (6.16), (6.17).

Results from these equations are listed in table 1 and are shown diagramatically in graphs 1 to 6. It appears that the solutions obtained by the two methods agree reasonably well for small values of R and G but as the resistance and conductance increase, so does the differance between the results. However, this difference does seem to decrease as the length of airway increases so that for streamlined flow and for long airways, the operational solution might be reasonably accurate.

TABLE 1

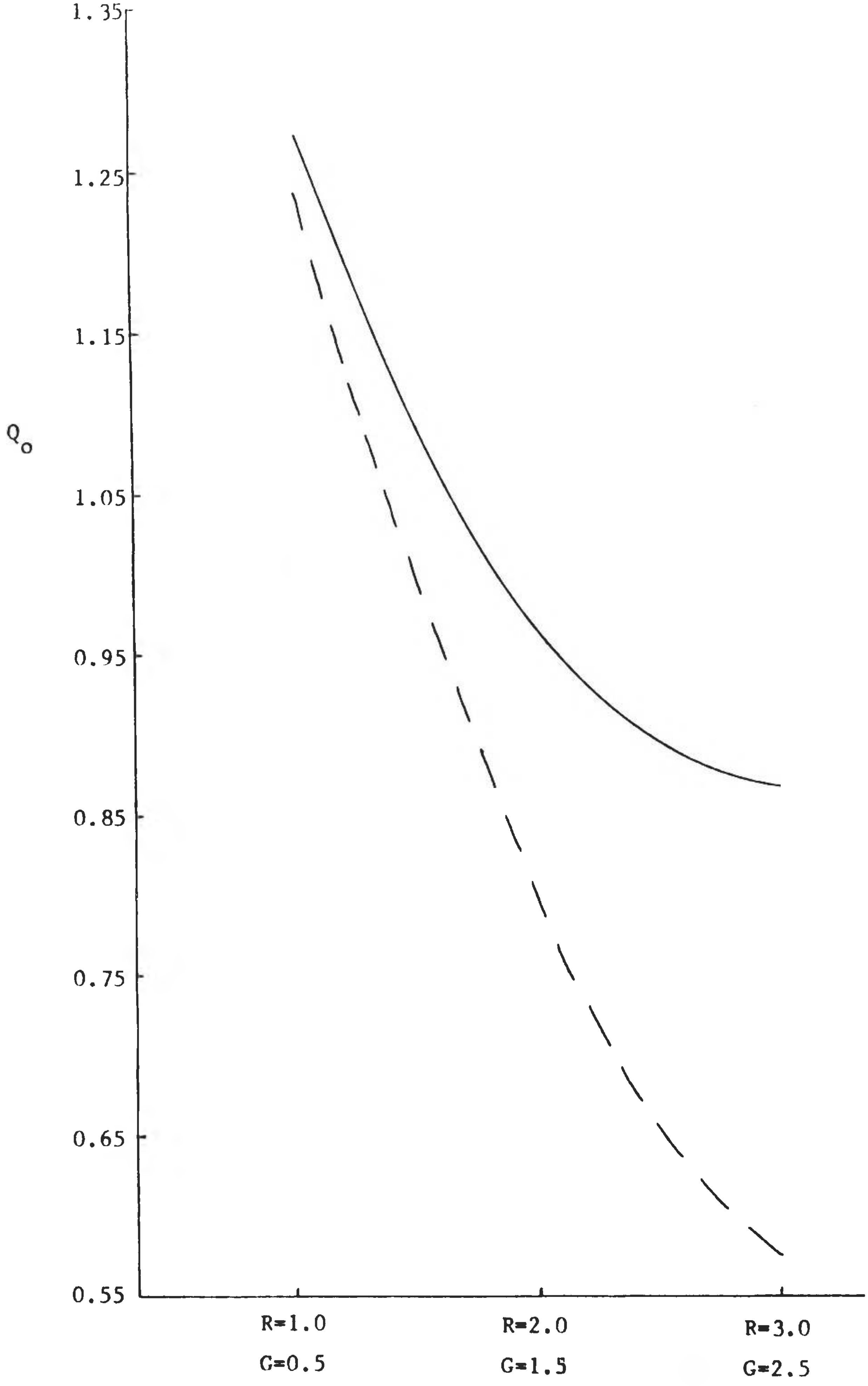
RESULTS FROM THE OPERATIONAL AND ANALYTICAL SOLUTIONS

			Operational Solution	Analytical Solution
R = 1.0, G = 0.5	L = 1	Q_o	1.238	1.273
		Q_F	0.885	0.877
	L = 3	Q_o	1.303	1.305
		Q_F	0.856	0.856
	L = 5	Q_o	1.310	1.310
		Q_F	0.853	0.853
R = 2.0, G = 1.5	L = 1	Q_o	0.796	0.965
		Q_F	0.365	0.344
	L = 3	Q_o	1.135	1.164
		Q_F	0.245	0.244
	L = 5	Q_o	1.200	1.208
		Q_F	0.225	0.225
R = 3.0, G = 2.5	L = 1	Q_o	0.574	0.868
		Q_F	0.227	0.210
	L = 3	Q_o	1.039	1.117
		Q_F	0.093	0.093
	L = 5	Q_o	1.181	1.207
		Q_F	0.069	0.069

GRAPH 1

QUANTITY OF AIR ENTERING A MINE OF LENGTH

L = 1 FOR VARIOUS RESISTANCES TO FLOW.

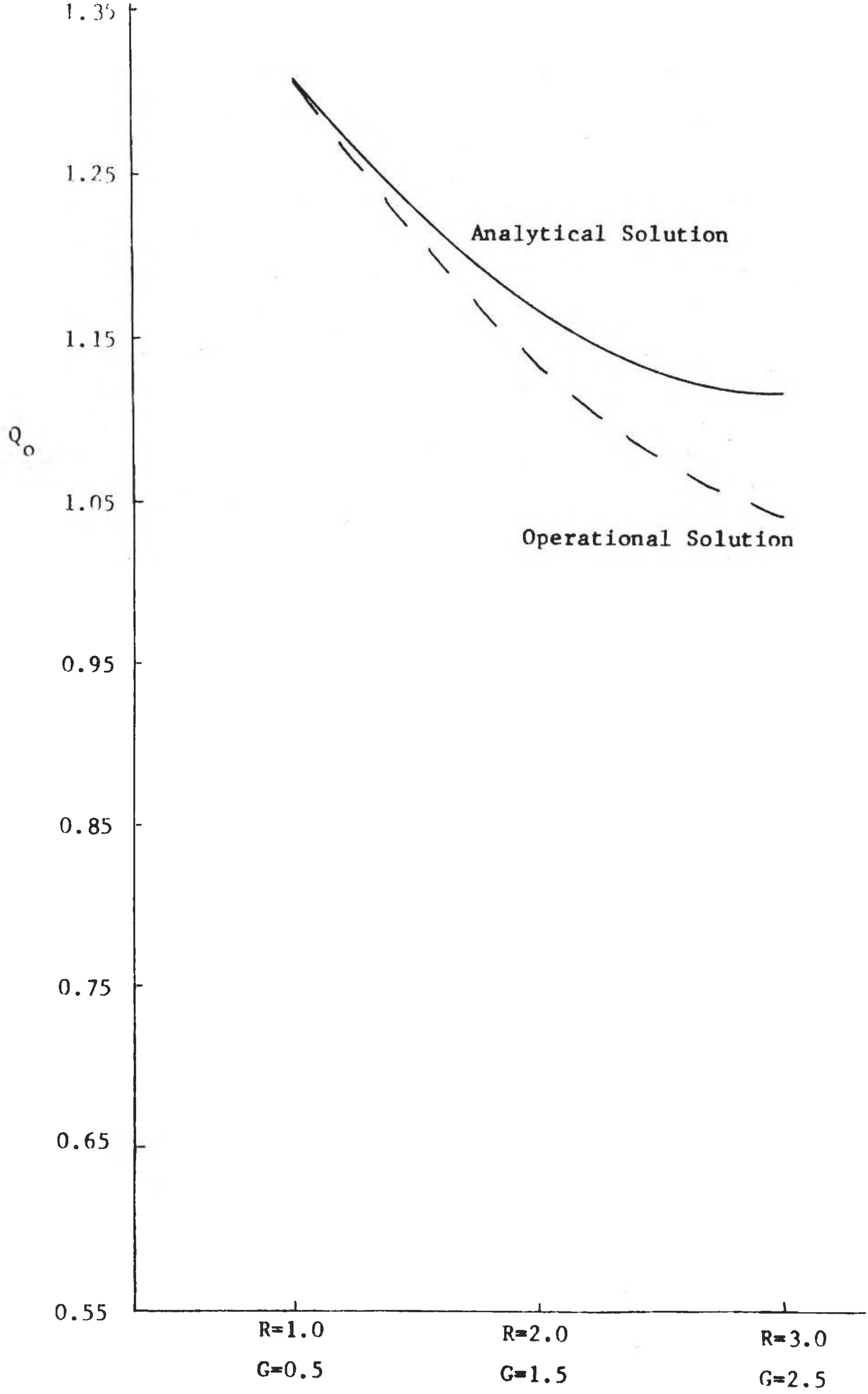


Graph 1

GRAPH 2

QUANTITY OF AIR ENTERING A MINE OF LENGTH

L = 3 FOR VARIOUS RESISTANCES TO FLOW.

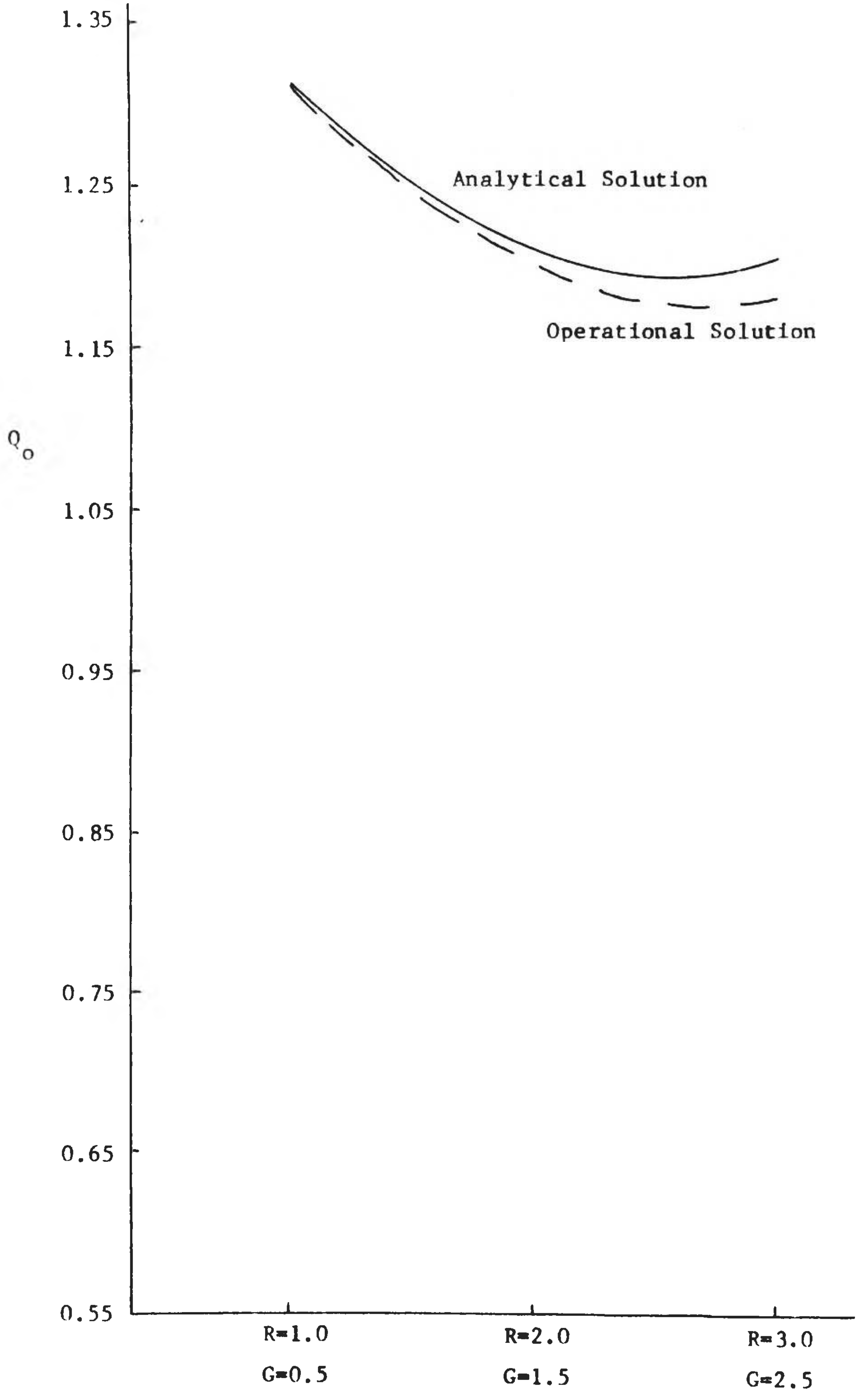

1.35
1.25
1.15
1.05
0.95
0.85
0.75
0.65
0.55
Q_o
Analytical Solution
Operational Solution
R=1.0
G=0.5
R=2.0
G=1.5
R=3.0
G=2.5

Graph 2

GRAPH 3

QUANTITY OF AIR ENTERING A MINE OF LENGTH

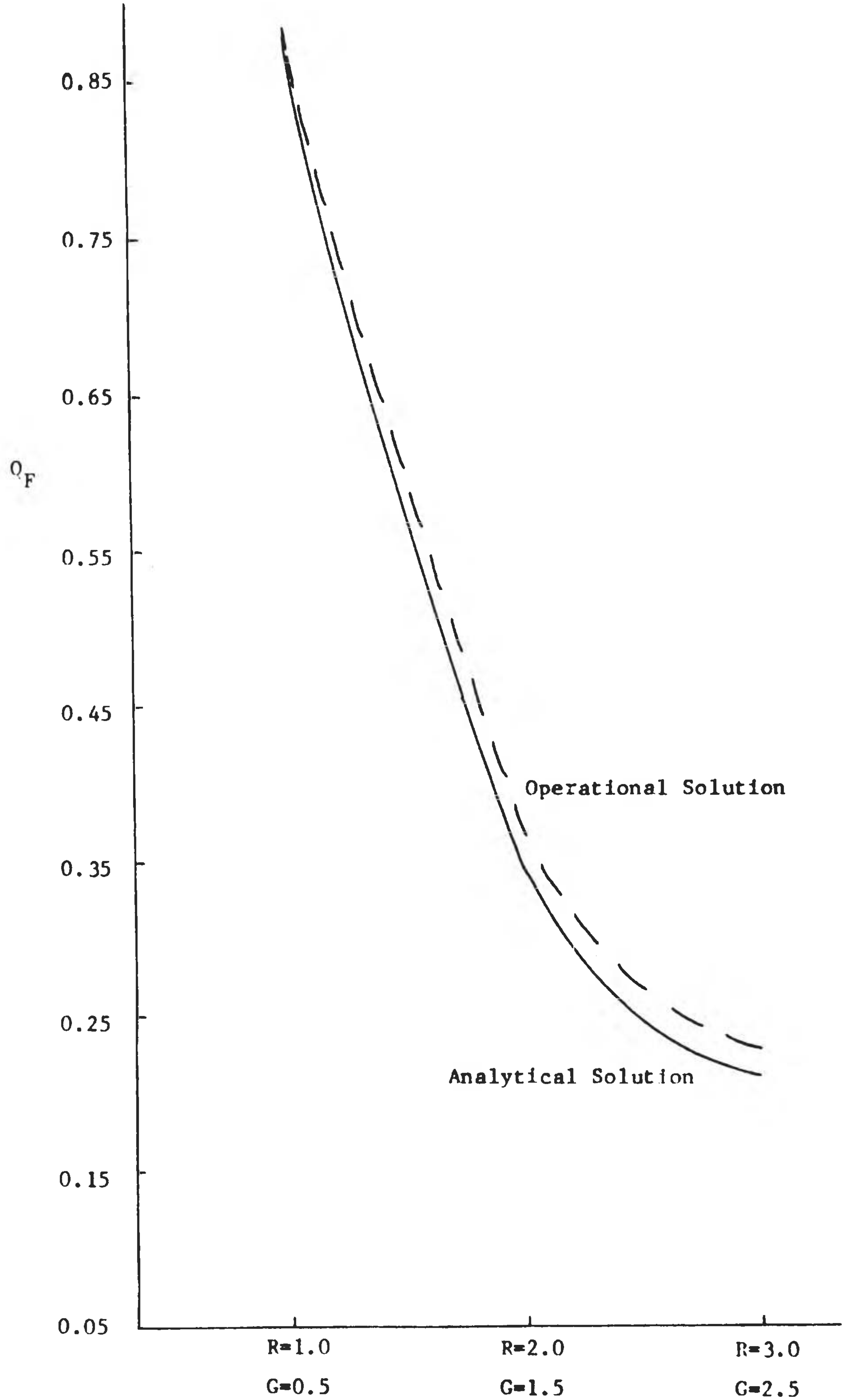
L = 5 FOR VARIOUS RESISTANCES TO FLOW.


1.35
1.25
1.15
1.05
0.95
0.85
0.75
0.65
0.55
Q_o
Analytical Solution
Operational Solution
R=1.0
G=0.5
R=2.0
G=1.5
R=3.0
G=2.5

<u>Graph 3</u>

GRAPH 4

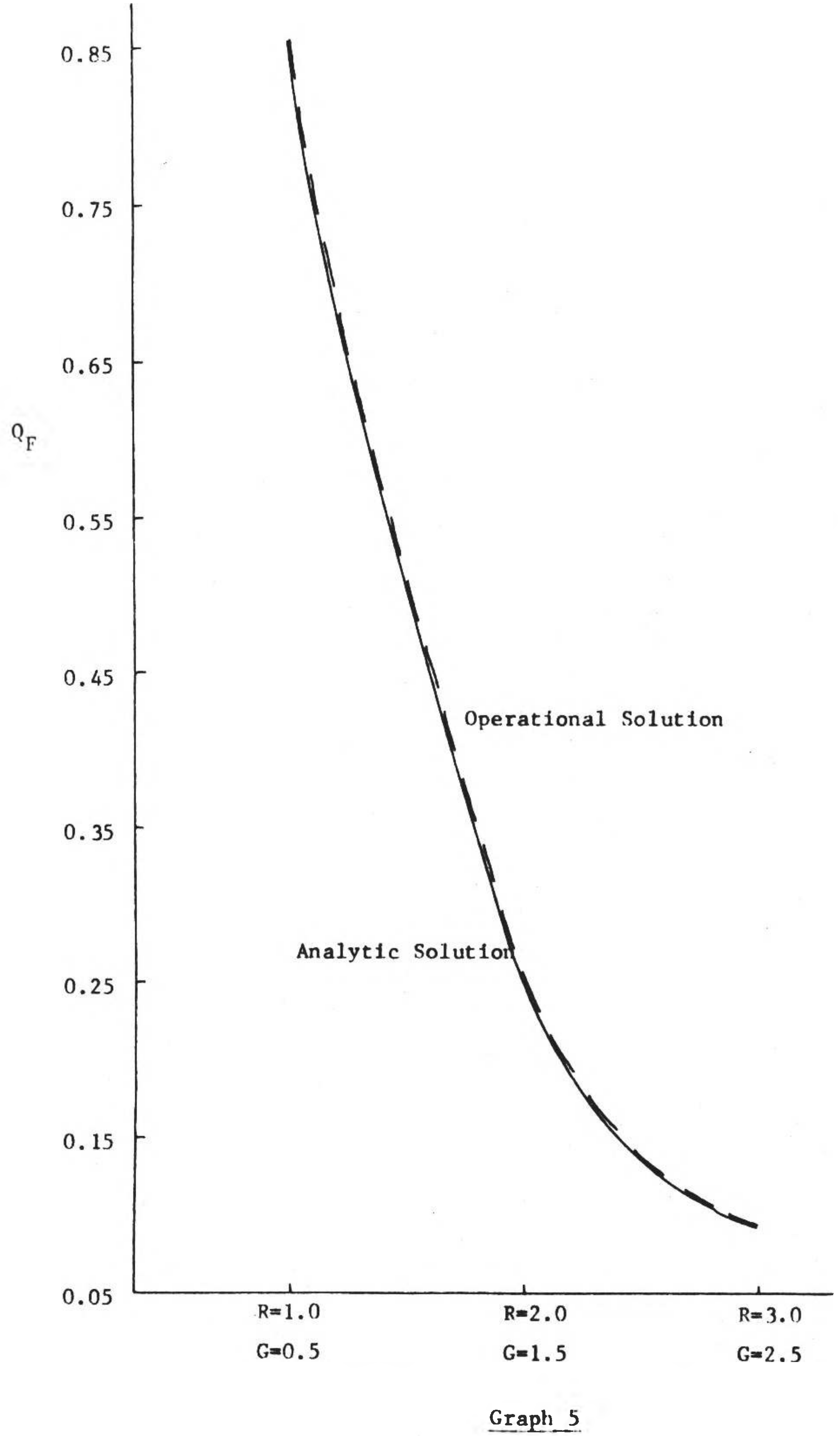
QUANTITY OF AIR REACHING THE WORKING FACE OF A MINE OF LENGTH $L = 1$ FOR VARIOUS RESISTANCES TO FLOW.


0.85
0.75
0.65
Q_F
0.55
0.45
0.35
0.25
0.15
0.05
Operational Solution
Analytical Solution
R=1.0
G=0.5
R=2.0
G=1.5
R=3.0
G=2.5

Graph 4

GRAPH 5

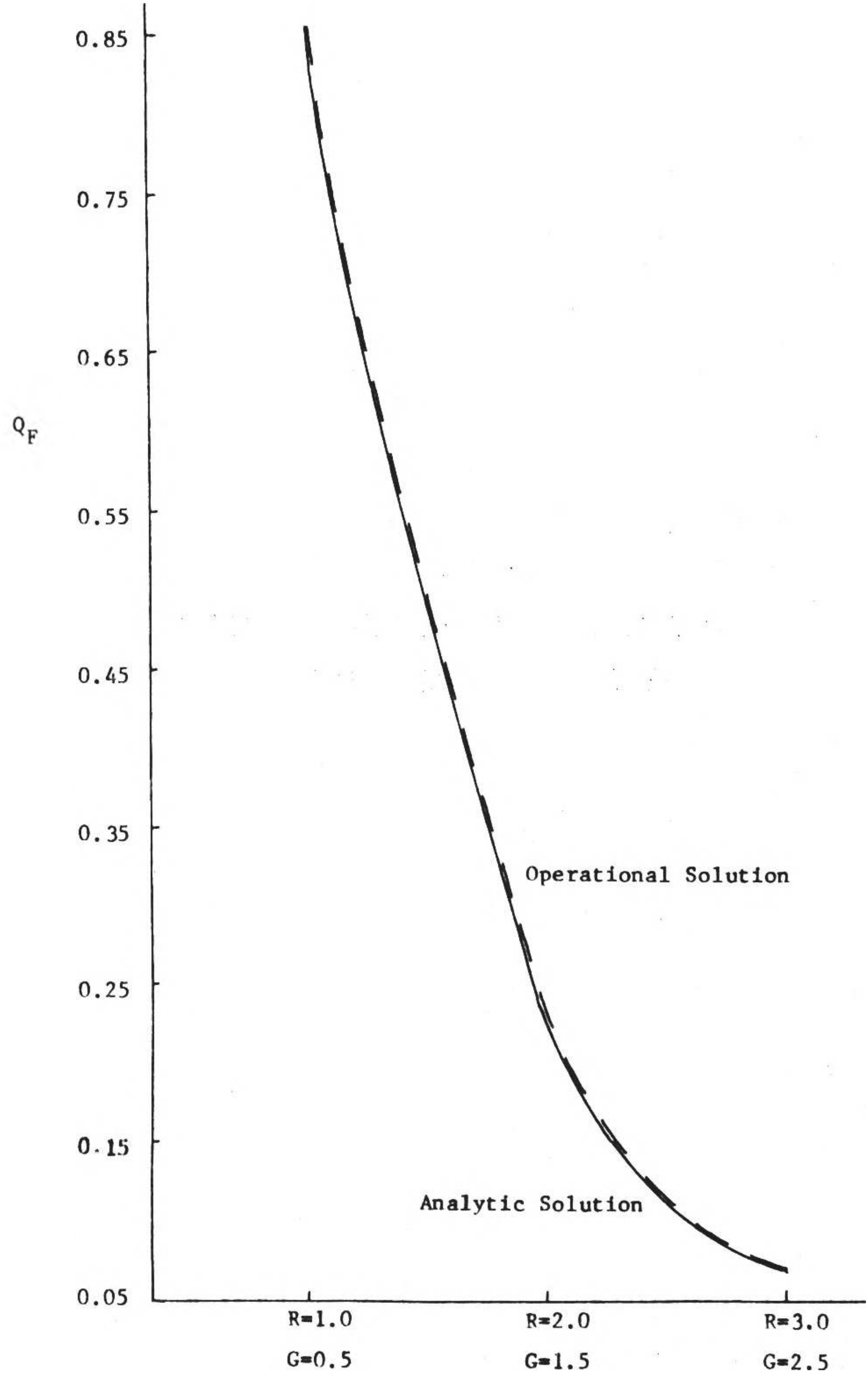
QUANTITY OF AIR REACHING THE WORKING FACE OF A MINE OF LENGTH L = 3 FOR VARIOUS RESISTANCES TO FLOW.


0.85
0.75
0.65
Q_F
0.55
0.45
Operational Solution
0.35
Analytic Solution
0.25
0.15
0.05
R=1.0
G=0.5
R=2.0
G=1.5
R=3.0
G=2.5

Graph 5

GRAPH 6

QUANTITY OF AIR REACHING THE WORKING FACE OF A MINE OF LENGTH L = 5 FOR VARIOUS RESISTANCES TO FLOW.


0.85
0.75
0.65
Q_F
0.55
0.45
0.35
Operational Solution
0.25
0.15
Analytic Solution
0.05
R=1.0
G=0.5
R=2.0
G=1.5
R=3.0
G=2.5

Graph 6

CHAPTER 7

COMPARISON OF THE OPERATIONAL AND NUMERICAL SOLUTIONS FOR TURBULENT FLOW

REVIEW OF THE OPERATIONAL SOLUTION FOR THE TURBULENT CASE

The streamlined case was considered because analytic results could be obtained. However, because of the method of construction of airways, flow through them is usually somewhat turbulent. This means that the value of the flow index is in the range $1 \leq n \leq 2$. We now wish to consider this more general situation.

It has been shown that the pressure and quantity of the air in the mine are related by equation (2.4) together with its boundary conditions (2.5). An approximate solution, given by (2.8), has been obtained but its accuracy needs to be determined. To do this it will suffice if we consider (2.9), that is,

$$P_o^2 \doteqdot \frac{1}{n+1}\,\frac{R}{G}(Q_o^{n+1} - Q_F^{n+1}) + \frac{k^2b^2R^2}{3L^2}(Q_o^{2n} - Q_F^{2n}).$$

P_o, k, b, R, G and L can be given values as before and n can take values in the above range. If, then, values can be obtained for Q_o and Q_F by some other means, they can be compared with corresponding values from (2.9). This can be done by solving (2.4) numerically to obtain values of Q_o and Q_F. These values of Q_F can then be used in (2.9) and corresponding values of Q_o can be determined by using the Newton-Raphson procedure. Corresponding values of Q_o, as obtained by the numerical and operational solutions, can then be compared.

NUMERICAL SOLUTION FOR TURBULENT FLOW

It is required to find a numerical solution of the differential equation

$$k^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

with boundary conditions

$$P = (p)_{y=b}$$

$$\frac{dQ}{dx} = (2bg\frac{\partial p}{\partial y})_{y=b}$$

$$\frac{dP}{dx} = rQ^n.$$

There is no lack of information concerning a numerical solution of Laplace's equation. Fox (1962), Milne (1953) and Southwell (1946) all discuss the topic in detail and it is the approach outlined in Shaw (1953) that will be largely followed. However, most problems discussed have simpler boundary conditions. In our problem only values along three sides are known, those along the remaining boundary need to be determined. To do this values will firstly be estimated, and using these, the partial differential equation will be solved. From the other boundary conditions values of Q can be determined as well as new boundary values of P. The process can then be repeated as many times as is necessary to obtain the desired accuracy.

FIGURE 4

SUBDIVISION OF THE BARRIER INTO A MESH

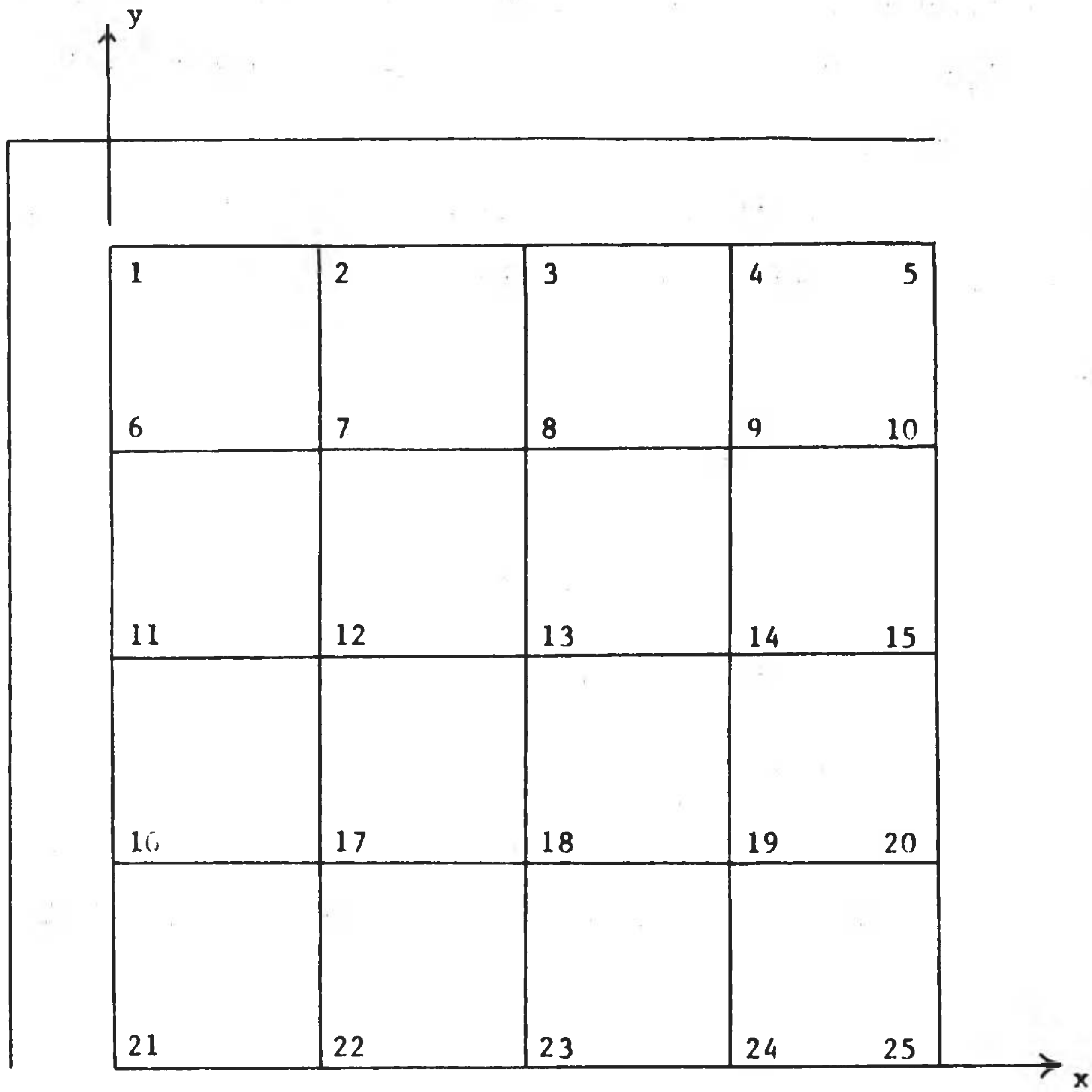


Figure 4

To explain the numerical method more fully, the part of the barrier in the first quadrant is divided into a mesh as show in figure 4. This part is chosen because of symmetric relationships of the barrier.

For the partial differential equation, (2.4), we can set up the finite difference equations

$$\left.\begin{array}{l} p_8 + p_2 + p_6 + p_{12} - 4p_7 = 0 \\ p_9 + p_3 + p_7 + p_{13} - 4p_8 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p_{20} + p_{14} + p_{18} + p_{24} - 4p_{19} = 0 \end{array}\right\} \qquad (7.1)$$

where, for any internal point ω, we have

$$p_\alpha + p_\beta + p_\gamma + p_\delta - 4p_\omega = 0$$

according to the point pattern in figure 5.

To solve (7.1) we rewrite the equations in residual form. That is,

$$\left.\begin{array}{l} R_1 = p_8 + p_2 + p_6 + p_{12} - 4p_7 \\ R_2 = p_9 + p_3 + p_7 + p_{13} - 4p_8 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ R_9 = p_{20} + p_{14} + p_{18} + p_{24} - 4p_{19} \end{array}\right\} \qquad (7.2)$$

Then, for correct values of p everywhere, all the R_i will be zero. For values of p that are approximately correct they will be small, whilst for values of p far from correct, one or more of the

FIGURE 5

POINT PATTERN FOR FINITE DIFFERENCE EQUATIONS

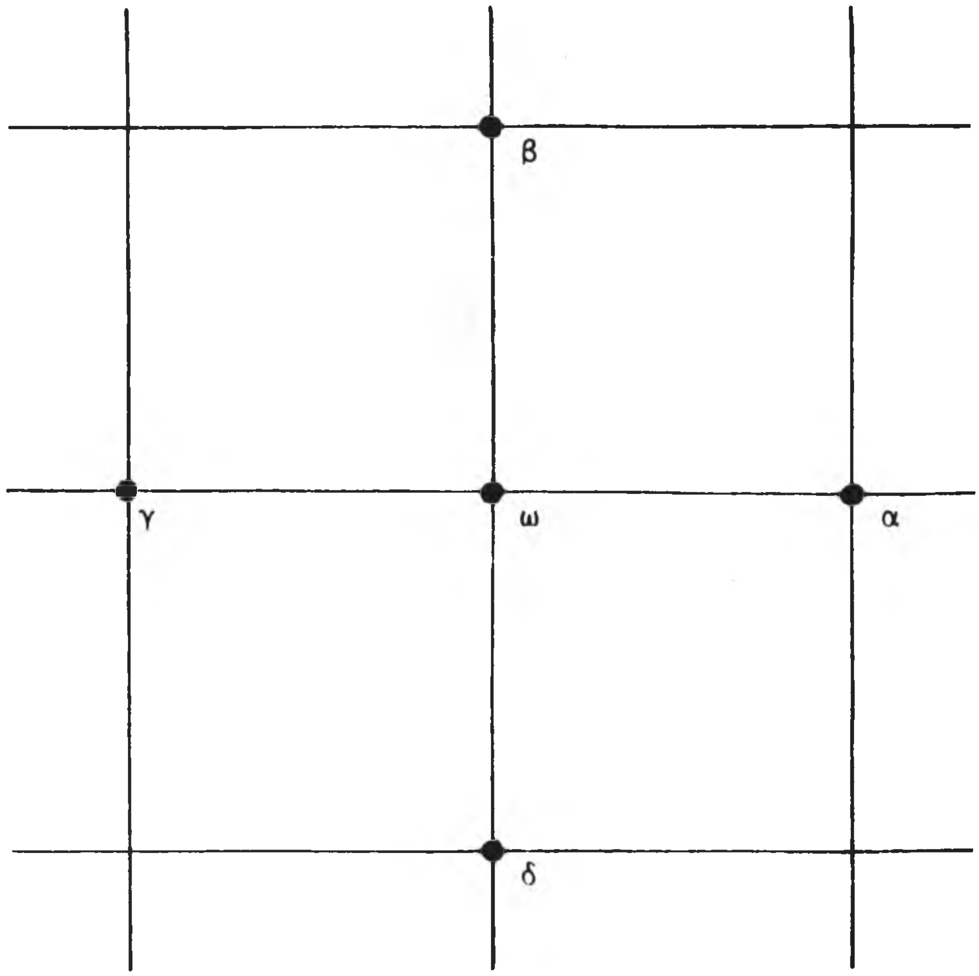


Figure 5

FIGURE 6

DIAGRAMETRIC REPRESENTATION OF THE RESIDUAL OPERATOR

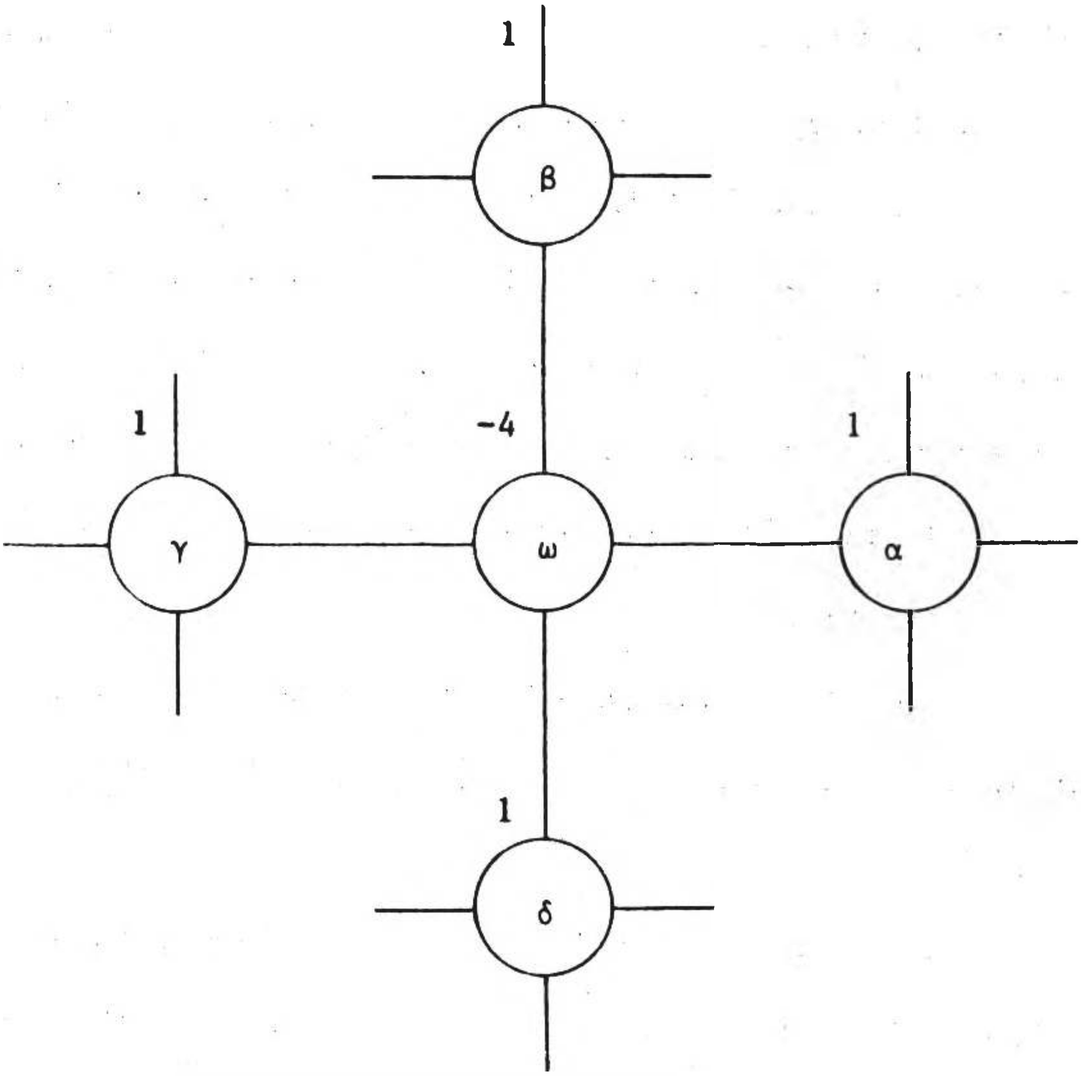


Figure 6

residuals will be large. Thus we can assign values to p at all internal points, calculate the residuals at these points from (7.2) and then by relaxation methods reduce these residuals.

The residual operators indicated in (7.2) can be more conveniently expressed diagramatically by figure 6. Thus to compute the value of the residual R_ω at a point ω, we multiply the values of the p's at the five points ω, α, β, γ, δ by the multipliers shown on the left of the respective points and then add the products. The residual at ω may then be placed on the right of the point.

The residuals can then be systematically reduced to values as small as desired by applying the relaxation operator given in figure 7.

This operator is applied so as to approximately liquidate the largest residual at each step. By a unit alteration to the value of p at point ω, this operator alters the residual R_ω by -4 and at the same time alters the residuals at the four surrounding points each by 1. The largest residual then is again sought and reduced. This process is repeated until the desired accuracy is achieved.

To speed up the calculations a block operator may be used initially to reduce the sum of the residuals to zero. The

FIGURE 7

THE RELAXATION OPERATOR AT A POINT ω

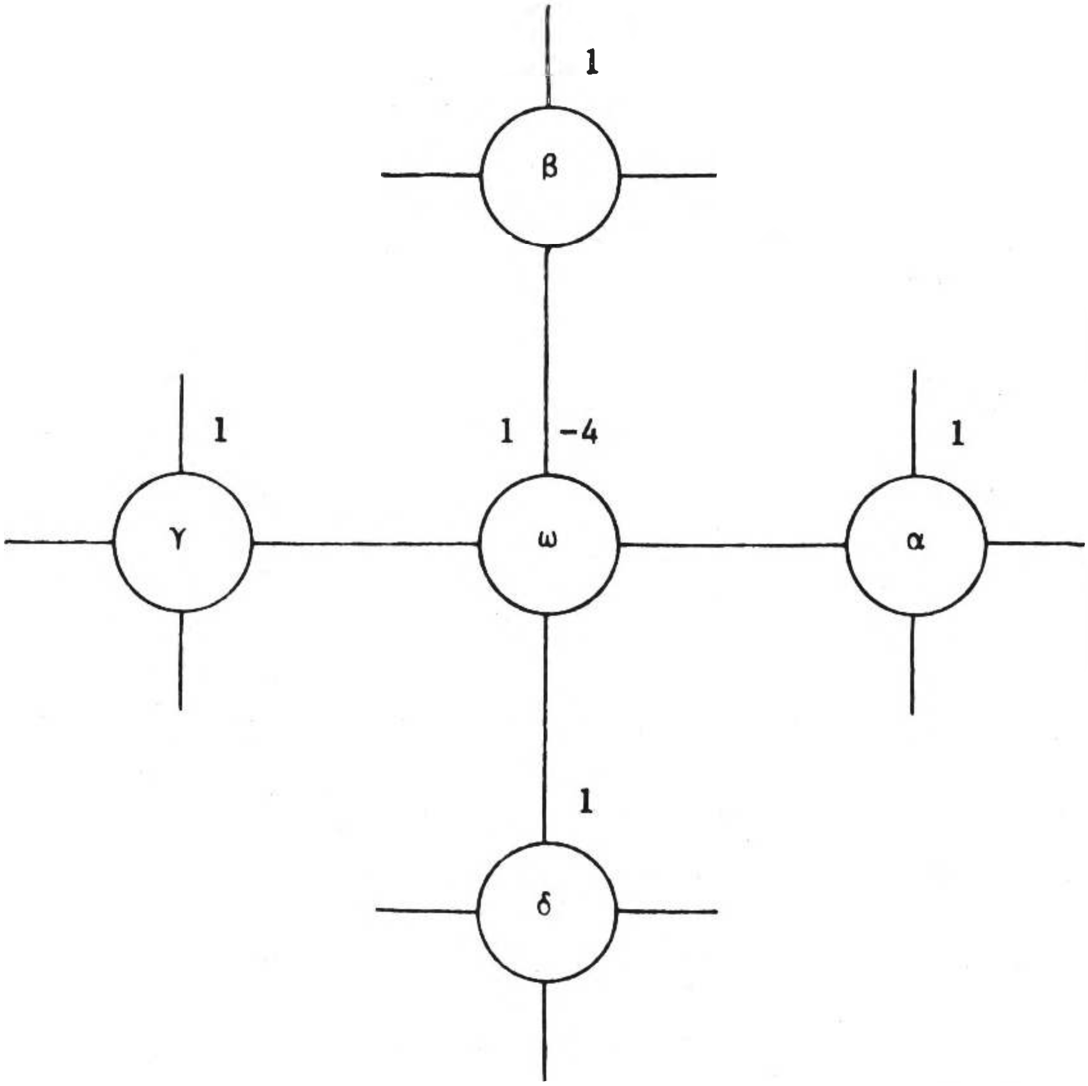


Figure 7

FIGURE 8

A BLOCK OPERATOR

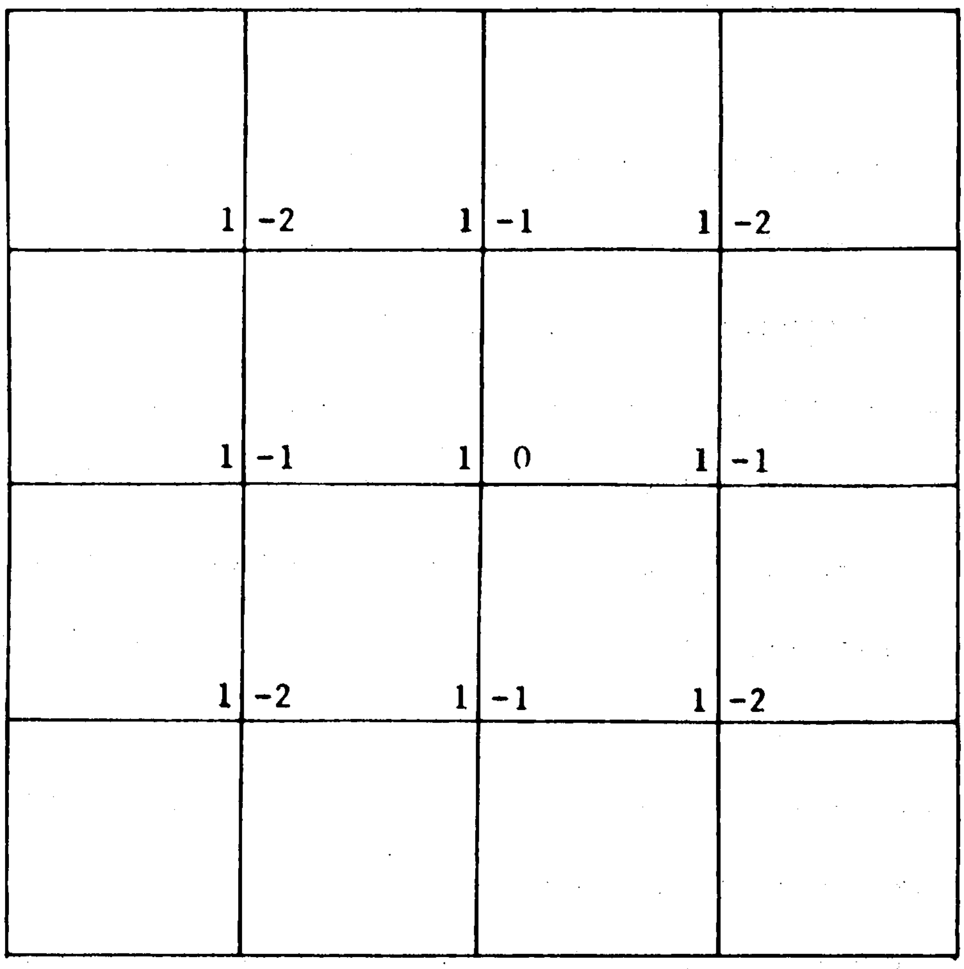


Figure 8

block operator is formed by considering unit alterations at the various points in the block. Doing this at each internal point in figure 4, for example, results in the block operator shown in figure 8.

The sum of the residuals of this block operator is -12. If then the sum of the residuals at the internal points of the mesh can be determined, a suitable multiple of the block operator can be used to reduce the sum of residuals to zero. Point relaxation can then be used to reduce the residuals to be required level.

The method to be used for the numerical solution of equation (2.4) has just been outlined. We now wish to consider the role of the boundary conditions (2.5).

Referring to figure 9 and to initial considerations of the model we have that there is zero pressure along the working face, so that $p = 0$ along BC, and because of symmetric relationship of the model, p will be zero along CD.

Along AD we have assumed that

$$\frac{\partial p}{\partial y} = \frac{P_o}{b}$$

so that the pressure drops linearly from 1 to 0.

Values of P along AB, however, will need to be approximated. In an article by Peascod (1955) it is suggested that the pressure distribution along the intake might be depicted as in figure 9.

FIGURE 9

BOUNDARY VALUES FOR THE NUMERICAL SOLUTION

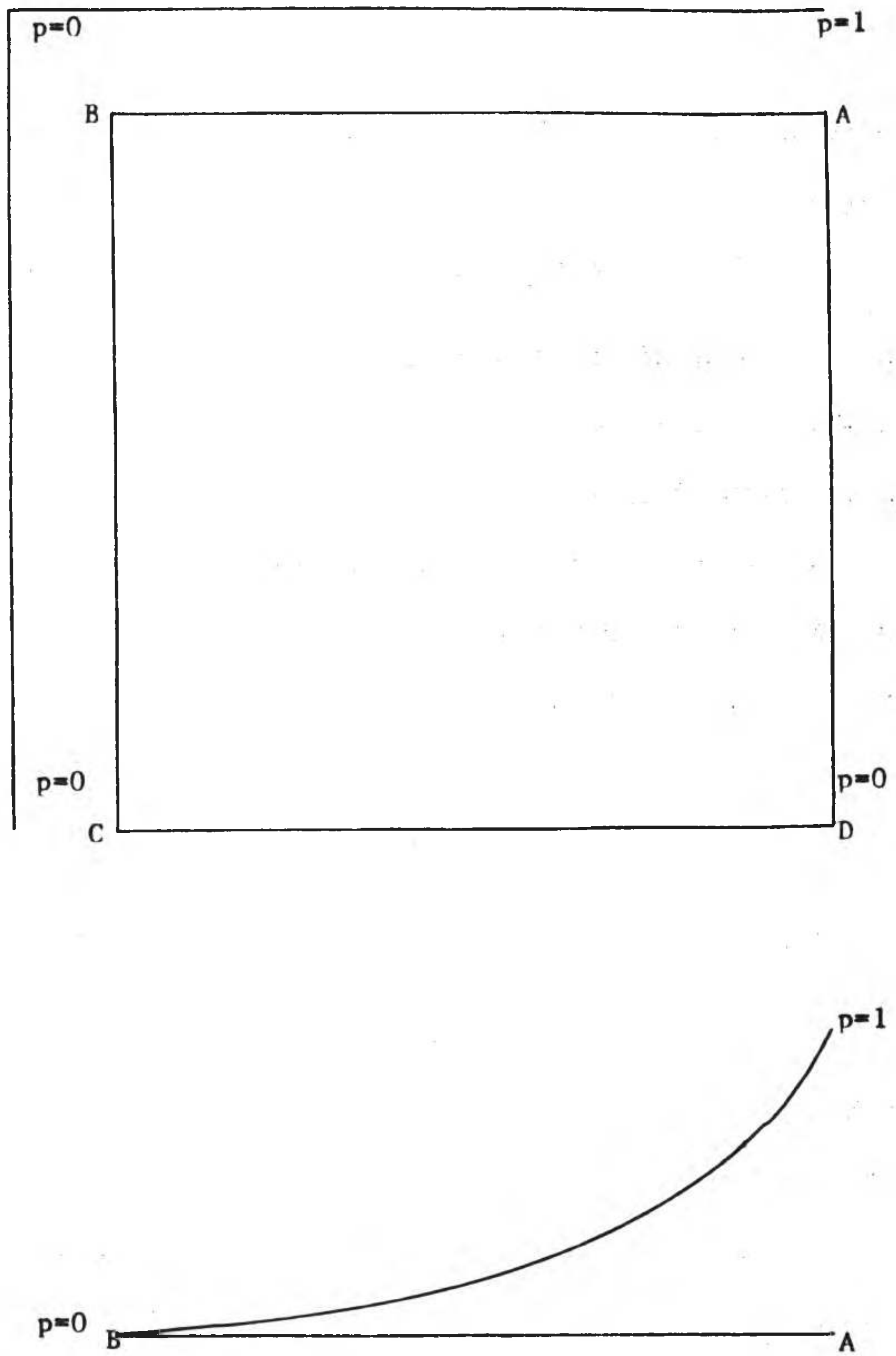


Figure 9

For purposes of numerical calculation then, we might approximate P along AB from

$$P = \left(\frac{2-4m}{L^2}\right)x^2 - \left(\frac{1-4m}{L}\right)x \qquad (7.3)$$

which is a parabolic relationship such that p = 0 when x = 0, P = 1 when x = L and P = m when $x = \frac{L}{2}$. The value of m is estimated from (6.15).

Using these boundary values, equation (2.4) can be solved numerically by relaxation methods to obtain values of p at the internal points of the mesh.

Then from

$$\frac{dP}{dx} = rQ^n$$

we have that

$$\frac{1}{2h}(P_{i+1} - P_{i-1}) = rQ_i^n \qquad (7.4)$$

where h is the width of the mesh subdivisions and i = 1,2,3,... j, j being the number of points in each line of the mesh.

Using fictitious values P_o and P_{j+1}, calculated from (7.3), values of Q_1, Q_2, Q_3, ... Q_j can be found from (7.4).

However, values of Q_i can also be determined from

$$\frac{dQ}{dx} = 2bg\,\frac{\partial p}{\partial y}$$

since this can be considered in the form

$$Q_i - Q_{i-2} = 2bg(p_{i-1} - p_{i+2j}), \qquad (7.5)$$

the i and j having the same meanings as before, but $i = 3,4,5,\cdots j$.

Setting Q_1 the same, equation (7.5) can be used to determine Q_3. Then, since the pressure drop at that end of the airway is small, Q_2 may be approximated linearly from

$$Q_2 = \frac{Q_1 + Q_3}{2} .$$

Thereafter, the remaining Q_4, Q_5, ... Q_j can be found by repeated application of (7.5).

For correct initial values of P in the intake it would be expected that the values of the Q_i, as obtained from (7.4) and (7.5) would agree reasonably well. On the other hand, the correlation would probably be poor if the values of P were not correct. In that case it is thought that the averages might be better approximations for the values of Q at the various points of the airway. Using these new values of Q_i, equation (7.4) could then be used to determine another set of values of P in the intake.

This cycle can be continued until the differences between corresponding values of Q are small enough to give the accuracy required.

COMPARISON OF THE TWO SOLUTIONS

So that results from the operational and numerical solutions can be compared for various degrees of turbulence, only the values

R = 1.0 and G = 0.5 will be considered for the resistance and conductance, but n will be given the values 1.0, 1.2, 1.4, 1.6, 1.8 and 2.0. The other variables will have the same values as were taken for previous calculations.

For numerical results a mesh size of one ninth is taken. Values of Q_O and Q_F are determined and as was mentioned earlier in the chapter, the Q_F values are then used in the operational solution to obtain corresponding values of Q_O. The results from the two approaches are given in table 2, and are depicted graphically in graphs 7, 8 and 9.

Figures from the numerical solution for the case n = 1 agree quite favourably with those from the series solution in table 1. Thus, taking the numerical results as being 'exact', it seems that the difference between the operational and numerical solutions increases as n increases.

TABLE 2

RESULTS FROM THE OPERATIONAL AND NUMERICAL SOLUTIONS

$R = 1.0$, $G = 0.5$

			Operational Solution	Numerical Solution
$n = 1.0$	$L = 1$	Q_o	1.223	1.262
		Q_F	0.863	0.863
	$L = 3$	Q_o	1.300	1.295
		Q_F	0.852	0.852
	$L = 5$	Q_o	1.308	1.302
		Q_F	0.851	0.851
$n = 1.2$	$L = 1$	Q_o	1.137	1.184
		Q_F	0.754	0.754
	$L = 3$	Q_o	1.227	1.220
		Q_F	0.741	0.741
	$L = 5$	Q_o	1.237	1.227
		Q_F	0.740	0.740
$n = 1.4$	$L = 1$	Q_o	1.087	1.095
		Q_F	0.678	0.678
	$L = 3$	Q_o	1.185	1.167
		Q_F	0.653	0.653
	$L = 5$	Q_o	1.196	1.176
		Q_F	0.652	0.652

TABLE 2 (Continued)

RESULTS FROM THE OPERATIONAL AND NUMERICAL SOLUTIONS (Continued)

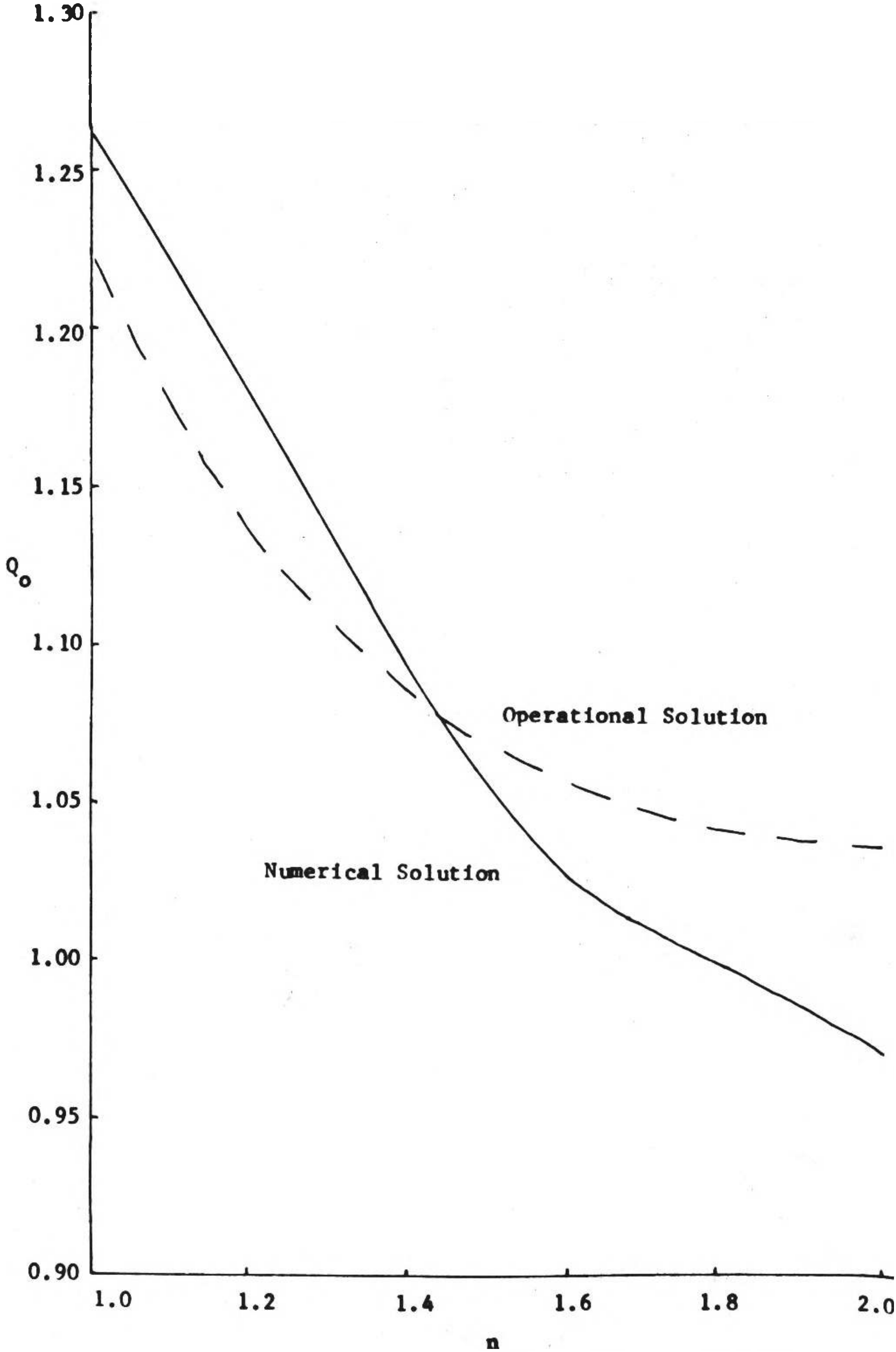
$R = 1.0$, $G = 0.5$

			Operational Solution	Numerical Solution
$n = 1.6$	$L = 1$	Q_o	1.059	1.025
		Q_F	0.619	0.619
	$L = 3$	Q_o	1.160	1.126
		Q_F	0.579	0.579
	$L = 5$	Q_o	1.173	1.140
		Q_F	0.578	0.578
$n = 1.8$	$L = 1$	Q_o	1.044	1.006
		Q_F	0.576	0.576
	$L = 3$	Q_o	1.149	1.103
		Q_F	0.516	0.516
	$L = 5$	Q_o	1.162	1.107
		Q_F	0.513	0.513
$n = 2.0$	$L = 1$	Q_o	1.037	0.971
		Q_F	0.536	0.536
	$L = 3$	Q_o	1.145	1.082
		Q_F	0.456	0.456
	$L = 5$	Q_o	1.159	1.086
		Q_F	0.454	0.454

GRAPH 7

QUANTITY OF AIR ENTERING A MINE OF LENGTH L = 1

FOR VARIOUS FLOW INDICES.

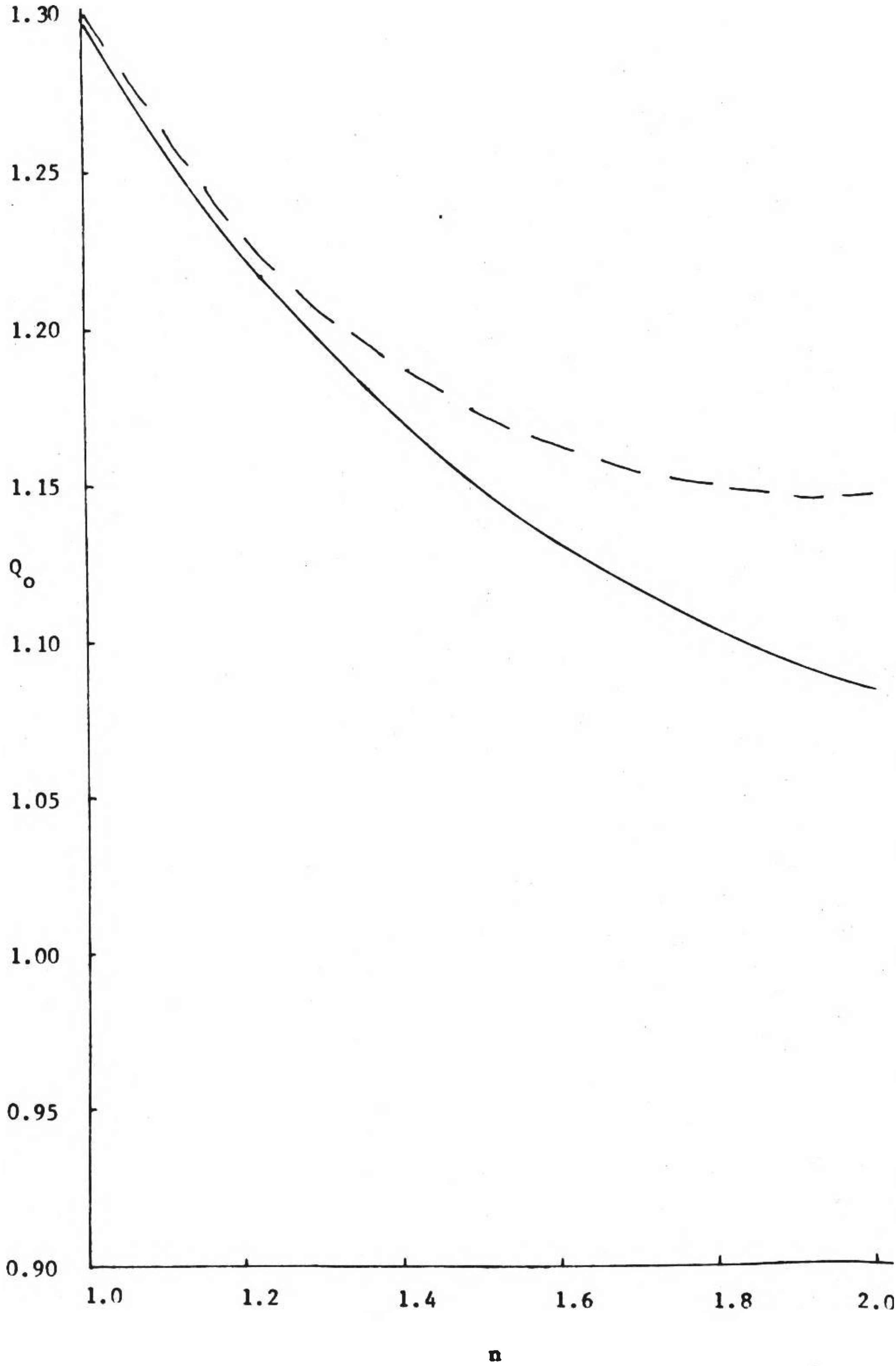

1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
0.90
Q_o
Operational Solution
Numerical Solution
1.0
1.2
1.4
1.6
1.8
2.0
n

Graph 7

GRAPH 8

QUANTITY OF AIR ENTERING A MINE OF LENGTH L = 3

FOR VARIOUS FLOW INDICES.

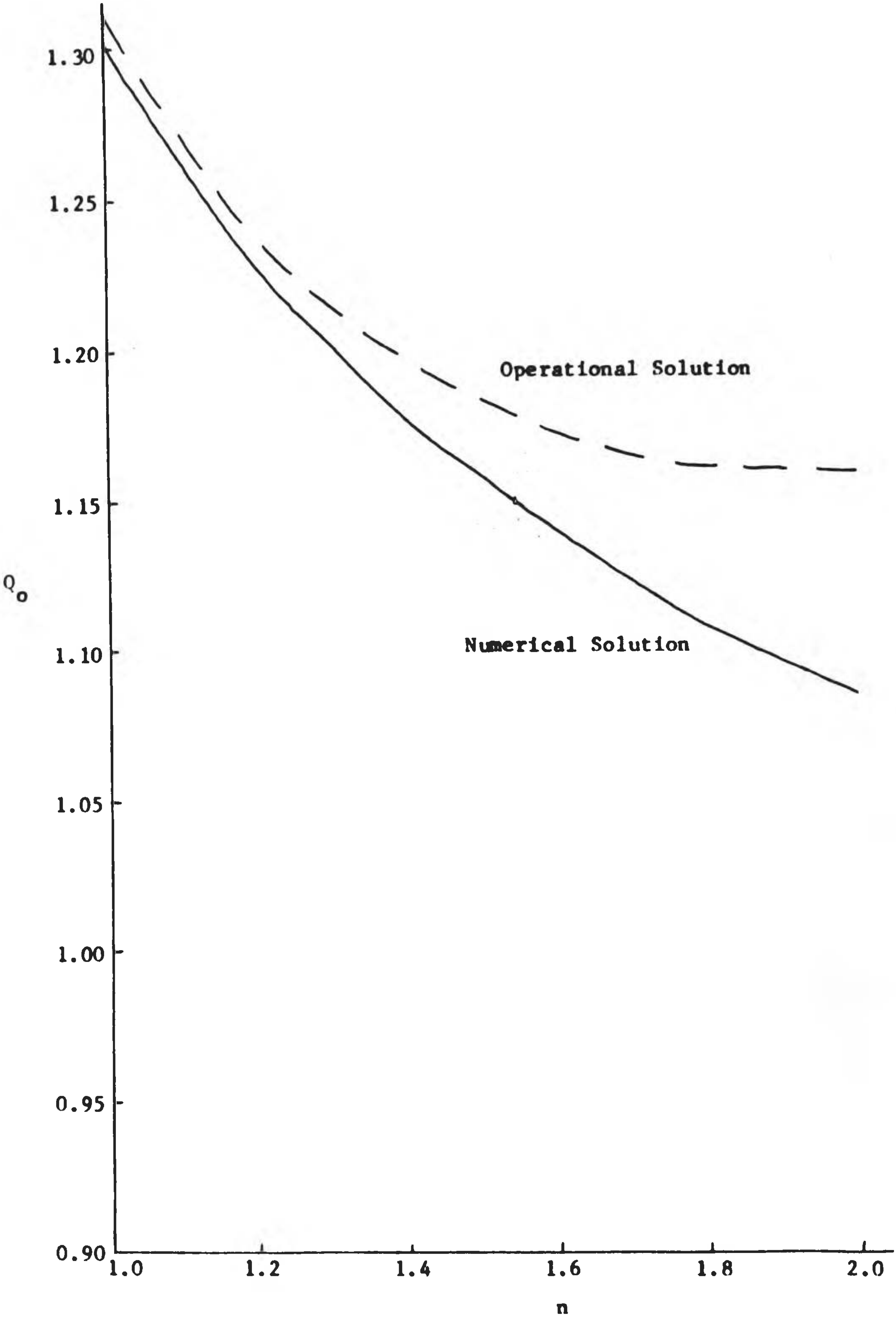


Graph 8

GRAPH 9

QUANTITY OF AIR ENTERING A MINE OF LENGTH L = 5

FOR VARIOUS FLOW INDICES.



1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
0.90
Q_o
Operational Solution
Numerical Solution
1.0
1.2
1.4
1.6
1.8
2.0
n

Graph 9

CHAPTER 8

CONCLUSION

In considering the problem of leakage in mine ventilation, it has been shown that pressure and quantity of air at points in the mine are governed by Laplace's equation together with associated boundary conditions. An approximate solution pertaining to flow in the airways was obtained using operational methods, but its accuracy needed to be checked.

The situation of streamlined flow in airways was firstly investigated to see if an analytic solution could be determined. It was found that provided the pressure at the entrance could be expanded in a modified Dini series, such a solution existed. Chapters 3, 4 and 5 were devoted to establishing the validity of this expansion. The theory of a special case of a Fourier-Bessel series was firstly modified, this then being extended to Dini series considerations. Integration was shown to be permissible so that the expansion was established and its use verified.

In chapter 6 series solutions were obtained analytically giving, in terms of the pressure at the entrance, the pressure at points in the airway, the quantity of air entering the mine and the quantity of air reaching the working face. The opportunity was taken at this juncture to compare results from the operational solution with those from the series. It was found the results agreed quite favourably for reasonably long airways.

However, flow in the airways is usually turbulent. For comparison purposes it was necessary to obtain 'exact' results from a numerical solution. The numerical approach was developed in chapter 7 and using relaxation methods the quantities of air at the entrance and at the working face were determined for various degrees of turbulence. Compared with these results it appears that the discrepancy of the operational solution increases as the turbulence increases, and hence it is only a fair result.

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