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The validity of modulation equations for extended systems with cubic nonlinearities

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Synopsis

Modulation equations play an essential role in the understanding of complicated systems near the threshold of instability. Here we show that the modulation equation dominates the dynamics of the full problem locally, at least over a long time-scale. For systems with no quadratic interaction term, we develop a method which is much simpler than previous ones. It involves a careful bookkeeping of errors and an estimate of Gronwall type.

As an example for the dissipative case, we find that the Ginzburg–Landau equation is the modulation equation for the Swift–Hohenberg problem. Moreover, the method also enables us to handle hyperbolic problems: the nonlinear Schrödinger equation is shown to describe the modulation of wave packets in the Sine–Gordon equation

1. Introduction

We consider scalar evolutionary problems on the real line. In the parabolic case, we are interested in the behaviour of systems close to the threshold of instability. If a spatially homogeneous solution becomes unstable, typically a whole band of wave numbers turns unstable. In this situation, the bifurcating solutions can be approximately described by a so-called amplitude or modulation equation ([**3**, **8**, **9**]). As an example, we study the (scalar) *Swift-Hohenberg equation* ([**2**]):

$$\partial_t u(t,x) = L_\lambda(\partial_x)u(t,x) - u^3(t,x), \quad \text{with} \quad L_\lambda(\partial_x)u = -(1+\partial_x^2)^2 u + \lambda u. \quad (1.1)$$

The trivial solution $u \equiv 0$ is unstable for $\lambda > 0$ and, linearising at $u = 0$, we find solutions of the form $u(t,x) = e^{\mu t + ikx}$, where $\mu(k) = -(1-k^2)^2 + \lambda$ is positive for k close to ± 1. One expects that for small $\lambda > 0$ there are solutions which are slow modulations in time and space of the critical modes $e^{\pm ix}$. Using the scalings $\lambda = \varepsilon^2$, $T = \varepsilon^2 t$ and $X = \varepsilon x$, we introduce the formal approximation

$$u_A(t,x) = \varepsilon(A(T,X)e^{ix} + \bar{A}(T,X)e^{-ix}). \quad (1.2)$$

Substituting this ansatz into (1.1) and equating the coefficients of e^{ix} of order $\mathcal{O}(\varepsilon^3)$ to zero, we find that the amplitude A has to satisfy the *Ginzburg-Landau equation*

$$\partial_T A = 4\,\partial_X^2 A + A - 3\,|A|^2 A. \quad (1.3)$$

Now taking a solution A, the question arises as to how well u_A approximates the solution $u(t,x)$ of the original problem which has the same initial data.

This question was treated in [**2**] for the Swift-Hohenberg problem and in [**11**] for general scalar equations with quadratic nonlinearities. The result obtained there implies that u_A approximates the solution u of the original problem up to an error of order $\mathcal{O}(\varepsilon^2)$, uniformly for all $(t, x) \in [0, T_0/\varepsilon^2] \times \mathbb{R}$. Yet both studies lead to very involved analyses. Here we want to show that it is possible to obtain the same results by a much simpler method, which, however, is restricted to the case where the nonlinearity starts with cubic terms. This is, for instance, always the case for problems with odd nonlinearities. Our method consists in finding a good approximate solution v_A which is typically one order more accurate than u_A. Then we estimate the error $R = (u - v_A)/\varepsilon^2$ directly from the equation which is found by substituting $u = v_A + \varepsilon^2 R$ into the full problem. The essential ingredients of the method are the *boundedness* of the semigroup generated by the linearised problem (at criticality) and the *relative smallness* of the nonlinear terms. In particular, the perturbation to the linear part is of order ε^2, which allows estimates over the time scale $1/\varepsilon^2$.

In contrast to the previous work, our analysis does not rely on the smoothing properties of the linearised flow. Hence, we are also able to treat hyperbolic problems. The formal derivation of amplitude equations for hyperbolic problems is discussed extensively in [1], but rigorous approximation results in the sense mentioned above are rare, see e.g. [5], § 6. However, for problems on bounded x-domains, the theory of averaging was applied to show that finite mode (Galerkin) truncations lead to good approximations over long time-scales, see [7, 10]. In the case of an unbounded x-domain, we are interested in the evolution of slowly modulated wave packets. For the *Sine–Gordon equation* $\partial_t^2 u = \partial_x^2 u - \sin u$, we prove that solutions of the type

$$u_A(t, x) = \varepsilon(A(\varepsilon^2 t, \varepsilon(x - vt))e^{i(kx-\omega t)} + \text{c.c.}), \quad \text{where} \quad \omega^2 = k^2 + 1, \ v = k/\omega, \tag{1.4}$$

are $\mathcal{O}(\varepsilon^{3/2})$-approximations (in the $L_2(\mathbb{R})$-norm) over the time scale $1/\varepsilon^2$ of an exact solution u. Here $A = A(T, X)$ has to be a solution of the *nonlinear Schrödinger equation*

$$2i\omega\, \partial_T A = (v^2 - 1)\partial_X^2 A + \tfrac{1}{2} |A|^2 A. \tag{1.5}$$

The simplicity of the method makes the theory amenable to several generalisations discussed in Section 4. For instance, one can treat vector-valued u, even with values in an infinite-dimensional Banach space, allowing for applications to problems on cylindrical domains.

2. The parabolic case

We prove the following approximation result for the Swift-Hohenberg problem (SHE) (1.1) through the Ginzburg–Landau equation (GLE) (1.3). It is shown in [2, Lemma 3.1] that (1.3) has for each initial datum $A(0, X) \in C_b^4(\mathbb{R})$ a unique solution which is defined for all $T > 0$ and bounded in $C_b^4(\mathbb{R})$. The same statement holds for the SHE. (Here $C_b(\mathbb{R})$ denotes the space of bounded and uniformly continuous functions.)

THEOREM 2.1. *Let $A = A(T, X)$ be a solution of the GLE and u_A the formal approximation* (1.2). *Then, for each $T_0 > 0$ and $d > 0$, there exist ε_0, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following statement holds. Let $u = u(t, x)$ be a solution of* *the SHE such that* $|u(0,x)-u_A(0,x)|\leqq d\varepsilon^2$; *then the estimate*

$$|u(t,x)-u_A(t,x)|<C\varepsilon^2, \quad \textit{for all } (t,x)\in[0,T_0/\varepsilon^2]\times\mathbb{R}, \tag{2.1}$$

is satisfied.

Remarks 2.2. Note that u_A, and hence u, are $\mathcal{O}(\varepsilon)$ and that the error is one order smaller. The approximation occurs on the natural time-scale $\mathcal{O}(1/\varepsilon^2)$ for the modulations. Taking solutions u_1 and u_2 with two initial conditions $\mathcal{O}(\varepsilon^2)$-close to $u_A(0,\cdot)$ we find, by using (2.1) and the triangle inequality, that u_1 and u_2 stay $\mathcal{O}(\varepsilon^2)$-close over the time interval $[0,T_0/\varepsilon^2]$.

Proof of Theorem 2.1. We want to show that the error $u(t,x)-u_A(t,x)$ remains of order $\mathcal{O}(\varepsilon^2)$ over the time $t\leqq T_0/\varepsilon^2$. However, substituting u_A into (1.1) leaves the residual term $\varepsilon^3A^3e^{3ix}$ (cf. [**2**]) which, upon integration over $[0,T_0/\varepsilon^2]$, leads to an error $\mathcal{O}(\varepsilon)$. To avoid this difficulty, we use an improved approximation:

$$v_A(t,x)=\varepsilon A(T,X)e^{ix}-\varepsilon^3\tfrac{1}{64}A(T,X)^3e^{3ix}+\text{c.c.}$$

Using the relation

$$\begin{aligned}&-L_0(\partial_x)(B(\varepsilon x)e^{nix})\\&=[(1-n^2)^2B+\varepsilon 4in(1-n^2)B'+\varepsilon^2(2-6n^2)B''+\varepsilon^3 4inB'''+\varepsilon^4B'''']e^{nix},\end{aligned}$$

we find the residuum

$$\begin{aligned}\rho(\varepsilon,t,x)&=\partial_t v_A-L_0v_A-\varepsilon^2v_A+v_A^3\\&=\varepsilon^3[(\partial_TA-4\,\partial_X^2A-A)e^{ix}-(1-3^2)^2\tfrac{1}{64}A^3e^{3ix}+\text{c.c.}]\\&\quad+\varepsilon^3(Ae^{ix}+\bar{A}e^{-ix})^3+\mathcal{O}(\varepsilon^4).\end{aligned}$$

Since A solves the GLE, the whole $\mathcal{O}(\varepsilon^3)$-term vanishes.

We let $\varepsilon^2R(t,x)=u(t,x)-v_A(t,x)$ and obtain

$$\begin{aligned}\partial_tR&=L_0R+\varepsilon^2a(\varepsilon,t,x)R+\varepsilon^3N(\varepsilon,t,x,R)+\varepsilon^2r(\varepsilon,t,x),\\R(0,x)&=(u(0,x)-v_A(0,x))/\varepsilon^2,\end{aligned} \tag{2.2}$$

where $a(\varepsilon,t,x)=1-3(v_A(t,x)/\varepsilon)^2$, $N(\varepsilon,t,x,R)=-3(v_A/\varepsilon)R^2-\varepsilon R^3$, $r=\rho/\varepsilon^4$ is bounded over $(0,\varepsilon_0)\times[0,\infty)\times\mathbb{R}$, and $|R(0,x)|\leqq 2d$ for sufficiently small ε_0.

We solve this equation by turning it into an integral equation in $C_b(\mathbb{R})$ equipped with the standard supremum norm. Henceforth, we omit the dependence on the x-variable. The linear problem $\partial_tR=L_0R+f(t)$, $R(0)=g$, can be solved by the semigroup $G(t)=e^{L_0t}$. In Lemma 2.3 we prove that $G(t)$ is a uniformly bounded strongly continuous semigroup on $C_b(\mathbb{R})$. Thus, (2.2) transforms into

$$R(t)=G(t)R(0)+\varepsilon^2\int_0^t G(t-s)[a(\varepsilon,s)R(s)+\varepsilon N(\varepsilon,s,R(s))+r(s)]\,ds. \tag{2.3}$$

For each $D>0$, we have $\|N(\varepsilon,s,R)\|\leqq M$ for all R with $\|R\|\leqq D$ and $\varepsilon\in(0,\varepsilon_0)$. With $\|G(s)\|$, $\|r\|$, $|a(s)|\leqq C$, we estimate

$$\|R(t)\|\leqq 2Cd+\int_0^t\varepsilon^2C^2\,\|R(s)\|\,\mathrm{d}s+\varepsilon^2tC(\varepsilon M+C),$$

as long as $R(t)$ stays in the ball of radius D. With $\varepsilon^2 t \leqq T_0$, Gronwall's inequality yields

$$\|R(t)\| \leqq \tilde{C} e^{\varepsilon^2 C^2 t}, \qquad \tilde{C} = 2Cd + T_0 C(\varepsilon M + C), \tag{2.4}$$

for $t \leqq T_0/\varepsilon^2$. Let $\hat{C} = 2Cd + T_0 C(C + C)$ and $D = \hat{C} e^{C^2 T_0}$, and make ε_0 smaller, such that $\varepsilon M \leqq C$ (hence $\tilde{C} \leqq \hat{C}$). Then (2.4) shows that $\|R(t)\| \leqq D$ for all $t \leqq T_0/\varepsilon^2$. The desired result now follows from $u(t,x) - u_A(t,x) = \varepsilon^2 R(t,x) - \varepsilon^3 (A^3 e^{3ix} + \text{c.c.})/64$. □

Lemma 2.3. *The semigroup* $G(t) = e^{L_0 t}: C_b(\mathbb{R}) \to C_b(\mathbb{R})$, $t \geqq 0$, *is uniformly bounded.*

Proof. We use the representation of G as a Dunford integral over the resolvent

$$G(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (L_0 - \lambda)^{-1} \, d\lambda,$$

where $\Gamma \subset \mathbb{C}$ is a curve from $(-1+i)\infty$ to $(-1-i)\infty$ not intersecting $(-\infty, 0]$. The lemma is proved, if the resolvent $(L_0 - \lambda)^{-1}$ exists in the whole sector $\mathbb{C}_S = \{\lambda \in \mathbb{C}: \lambda \neq 0, |\arg \lambda| < 2\pi/3\}$ and satisfies

$$\|(L_0 - \lambda)^{-1}\| \leqq M/|\lambda| \quad \text{for all } \lambda \in \mathbb{C}_S \tag{2.5}$$

(cf. [6, chap. IX.1]). For $\lambda \in \mathbb{C}_S$, we define $\omega = (-\lambda)^{\frac{1}{2}}$ with $\operatorname{Im} \omega > 0$. Then, $L_0 u - \lambda u = -(1 + \partial_x^2 + \omega)(1 + \partial_x^2 - \omega) u = f$ has the solution

$$u = (L_0 - \lambda)^{-1} f = -K_{a_+} \circ K_{a_-} f, \quad \text{with} \quad K_a g(x) = \int_{\mathbb{R}} \frac{-1}{2a} e^{-a|x-\xi|} g(\xi) \, d\xi,$$

where $a_\pm = (-1 \pm \omega)^{\frac{1}{2}}$ with $\operatorname{Re} a_\pm > 0$. Hence, the resolvent satisfies

$$\|(L_0 - \lambda)^{-1}\| \leqq \frac{1}{|a_+| \operatorname{Re} a_+} \frac{1}{|a_-| \operatorname{Re} a_-} = \frac{1}{\sqrt{|1 + \lambda|}} \frac{1}{\operatorname{Re} a_+ \operatorname{Re} a_-}. \tag{2.6}$$

From $|\arg \lambda| < 2\pi/3$, we find with $\rho^2 = |\lambda|$ the estimates $|-1 \pm \omega| \geqq (1 - \sqrt{3}\rho + \rho^2)^{\frac{1}{2}}$ and $|\arg(-1 \pm \omega)| \geqq \alpha(\rho)$, where $\tan(\pi - \alpha) = \rho/(1 + \sqrt{3}\rho/2)$ and $\alpha \in (2\pi/3, \pi)$. This implies $\operatorname{Re} a_\pm \geqq (1 - \sqrt{3}\rho + \rho^2)^{\frac{1}{4}} \cos(\alpha(\rho)/2)$. In both limits, $\rho \to 0$ and $\rho \to \infty$, we find $\sqrt{|1 + \lambda|} \operatorname{Re} a_+ \operatorname{Re} a_- > c\rho^2 = c|\lambda|$, which is the desired estimate (2.5). Thus, the lemma is proved. □

3. The hyperbolic case

We now consider a hyperbolic problem where the interest lies in the time- and space-dependent modulations of oscillations around a trivial state. As a model, we consider the Sine–Gordon equation (SGE)

$$\partial_t^2 u = \partial_x^2 u - u + g(u), \quad \text{where} \quad g(u) = u - \sin u. \tag{3.1}$$

Here $e^{i(kx - \omega t)}$ are the solutions of the linearised problem, where k and ω are related by the *dispersion relation* $\omega^2 = k^2 + 1$. The group velocity is given by $v = \partial\omega/\partial k = k/\omega$ and satisfies $|v| < 1$.

We are looking for modulated travelling waves in the form u_A of (1.4), i.e. we define the slow time $T = \varepsilon^2 t$ and the slow space variable $X = \varepsilon(x - vt)$ (cf. [1]). Guided by the proof of Theorem 2.1, we use the improved approximation

$$v_A(t, x) = \varepsilon A(T, X)e^{i(kx-\omega t)} - \frac{1/6}{9k^2 - 9\omega^2 + 1}\varepsilon^3 A(T, X)^3 e^{3i(kx-\omega t)} + \text{c.c.}.$$

Due to the correction term, we find the residuum

$$\begin{aligned}\rho(\varepsilon, t, x) &= \partial_t^2 v_A - \partial_x^2 v_A + v_A - g(v_A)\\ &= \varepsilon^3(-2i\omega\, \partial_T A + (v^2 - 1)\, \partial_X^2 A + \tfrac{1}{2}|A|^2 A)e^{i(kx-\omega t)} + \text{c.c.} + \mathcal{O}(\varepsilon^4).\end{aligned}$$

Hence, if A satisfies the nonlinear Schrödinger equation (NLSE) (1.5), the residuum is $\mathcal{O}(\varepsilon^4)$.

We have to be aware of the fact that the above estimates are pointwise in $(t, x) \in [0, \infty) \times \mathbb{R}$. However, for wave problems it is more convenient to work in energy space, namely $(u, \partial_t u) \in Y = H^1(\mathbb{R}) \times L_2(\mathbb{R})$. It is well-known that, for any initial conditions in Y, the SGE has a unique solution $u = u(t, x)$ such that $t \in \mathbb{R} \to (u(t, \cdot), \partial_t u(t, \cdot)) \in Y$ is continuous and bounded. Similarly, the NLSE has a unique solution $A = A(T, X)$ such that $T \in \mathbb{R} \to A(T, \cdot) \in Z = H^1(\mathbb{R})$ is continuous and bounded.

In constructing the approximate solution v_A with a given solution A of the NLSE, we have to recall the spatial scaling $X = \varepsilon(x - vt)$. Hence we obtain

$$\|v_A(t, \cdot)\| \leqq 2\varepsilon^{1/2}\|A(\varepsilon^2 t, \cdot)\| + C\varepsilon^{5/2}\|A(\varepsilon^2 t, \cdot)\|_1^3.$$

Here and further on $\|\cdot\|$ denotes the L_2-norm and $\|\cdot\|_1$ the H^1-norm in the appropriate variable. For the residuum we have $\|\rho(\varepsilon, t, \cdot)\| = \mathcal{O}(\varepsilon^{7/2})$. Again we omit the x-dependence.

For the error we use the ansatz $\varepsilon^{3/2}R = u - v_A$ and obtain the equation

$$\partial_t^2 R = \partial_x^2 R - R + \varepsilon^2 a(\varepsilon, t)R + \varepsilon^{5/2}N(\varepsilon, t, R) + \varepsilon^2 r(\varepsilon, t), \tag{3.2}$$

where $a(\varepsilon, t) = (1 - \cos v_A(t))/\varepsilon^2$, $r = \rho/\varepsilon^{7/2}$. To show that N is $\mathcal{O}(1)$, we use

$$\begin{aligned}N(\varepsilon, t, R) &= \varepsilon^{-4}[g(\varepsilon^{3/2}R + v_A) - g(v_A) - g'(v_A)\varepsilon^{3/2}R]\\ &= \varepsilon^{-4}g''(v_A + \theta\varepsilon^{3/2}R)\varepsilon^3 R^2 = \frac{1}{\varepsilon}\sin(v_A + \theta\varepsilon^{3/2}R)R^2,\end{aligned}$$

where $\theta \in [0, 1]$ from the mean value theorem. Since $1/\varepsilon \sin(\ldots)$ as well as $a(\varepsilon, t)$ act as multiplication operators in $L_2(\mathbb{R})$, their operator norm is the supremum norm. Hence, using $\|v_A\|_\infty = \mathcal{O}(\varepsilon)$ and $\|R\|_\infty \leqq C\|R\|_1 = \mathcal{O}(1)$, we have the desired result.

Now, the analysis of Theorem 2.1 can be repeated by writing (3.2) as a first-order system for $(R, \partial_t R)$ in the Banach space Y. The linear part is $\partial_t(R, S) = L(R, S) = (S, \partial_x^2 R - R)$ and the associated semigroup $G(t) = e^{Lt}$ is an isometry for the norm $\|(R, S)\|_Y^2 = \int_{\mathbb{R}}[(\partial_x R)^2 + R^2 + S^2]\,dx$. Moreover, the nonlinear mapping N is well defined from Y into $L_2(\mathbb{R})$. Going through the estimates of the proof of Theorem 2.1 gives the following result:

THEOREM 3.1. *Let $A = A(T, X)$ be a solution of the NLSE such that the derivatives $\partial_T^n \partial_X^k A$ are in $C([0, T_1], L_2(\mathbb{R}))$ for $n + k \leqq 2$ and let u_A be the formal* approximation (1.4). *Then, for each* $T_0 \leqq T_1$ *and each* $d>0$, *there exist* ε_0, $C>0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *the following statement holds. Let* $u = u(t, x)$ *be a solution of the SGE such that* $\|(u(0), \partial_t u(0)) - (u_A(0), \partial_t u_A(0))\|_Y \leqq d\varepsilon^{\frac{3}{2}}$; *then the estimate*

$$\|(u(t), \partial_t u(t)) - (u_A(t), \partial_t u_A(t))\|_Y \leqq C\varepsilon^{\frac{3}{2}} \quad \text{for} \quad t \in [0, T_0/\varepsilon^2]$$

is satisfied.

We remark here that the theory cannot be carried out in spaces of functions which are bounded (e.g. quasi-periodic), since the semigroup $G(t)$ of the dispersive wave operator is not uniformly bounded on spaces like $C_b(\mathbb{R})$.

4. Discussion and generalisations

From the proof of Theorem 2.1 we see that the result can be generalised in several directions. Firstly, we may increase the order of accuracy of the formal solution v_A. This can be done as described in **[11]** or **[4]**. Having the residuum $\rho = \partial_t v_A - L_\varepsilon v_A + v_A^3$ of order $\mathcal{O}(\varepsilon^n)$ with $n>4$, we introduce the scaling $R = \varepsilon^{2-n}(u - v_A)$ and again obtain the same equation (2.2) but with ε^{n-1} as a factor of $N(\varepsilon, \ldots)$. Again we obtain $|R(t, x)| = \mathcal{O}(1)$ on the time interval $[0, T_0/\varepsilon^2]$, and thus $|u(t, x) - v_A(t, x)| \leqq C\varepsilon^{n-2}$. A similar result holds in the hyperbolic case.

Secondly, we can treat arbitrary nonlinearities g with $g(u) = \mathcal{O}(|u|^3)$ instead of u^3 or $u - \sin u$. Moreover, in the parabolic case g could also depend on the derivatives $\partial_x^k u$, $k = 1, 2, 3$. Then the smoothing properties of the semigroup have to be exploited, see **[2, 11]**. Moreover, an explicit t- and x-dependence for g can be allowed, as long as this dependence is uniform and we are able to construct approximate solutions which have a sufficiently small residuum.

The case of a quadratic leading term in the nonlinearity is explicitly excluded, since there the linear perturbation term $g'(v_A(t, x))R$ would only be of order $\mathcal{O}(\varepsilon)$. An error estimate would only be possible on the shorter time-scale $1/\varepsilon$, which is not the natural time-scale for the modulations. For the quadratic case, the only rigorous results over the correct time scale are given in **[11]**.

Thirdly, we note that the method is simple enough to allow for the case when u is vector-valued, in particular for studying partial differential equations on cylindrical domains, where $u(t, x)$ can be thought of as having values in a Banach space. We leave this for future research.

From the integral equation (2.3), we can also prove the existence of the solution R by the standard contraction mapping principle. Note that the Lipschitz constant is small even over time intervals of length T_0/ε^2, due to the factor ε^α with $\alpha>0$ in front of $N(\varepsilon, t, R)$. This observation is helpful in problems where the full system does not guarantee global existence for all initial data. Then the solutions with initial conditions of modulation type, i.e. as in (1.2), exist as long as predicted by the associated modulation equation, while general initial conditions might lead to blow up in finite time, independently of ε.

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