The validity of weighted automata

Jacques Sakarovitch

CNRS / Telecom ParisTech

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Sylvain Lombardy (Univ. Bordeaux)

and have been published in

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Outline

This work addresses, and proposes a solution to,
the problem of ε-transition removal in weighted automata.

The problem lies in effectivity.

The solution is based on a new, and more constrained,
definition of the validity of weighted automata.

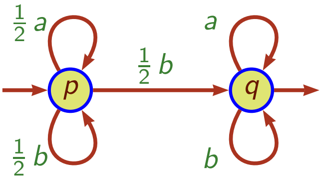
The definition insures that
algorithms are successful on valid automata.

In some (interesting) cases, we are able to establish that
success of algorithms implies validity of automata.

This solution provides a sound theoretical framework for
the algorithms implemented in VAUCANSON.

The weighted automaton model

$\mathcal{C}_1$



- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

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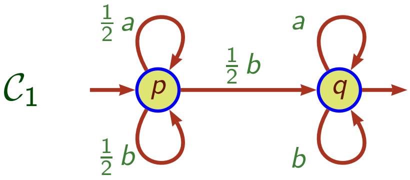

- Weight of a path c: *product* of the weights of transitions in c
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$$\xrightarrow{1} p \xrightarrow{\frac{1}{2}\,b} p \xrightarrow{\frac{1}{2}\,a} p \xrightarrow{\frac{1}{2}\,b} q \xrightarrow{1}$$

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$$b\,a\,b \quad \longmapsto \quad \frac{1}{2} + \frac{1}{8} = \frac{5}{8} = \langle 0.101 \rangle_2 \qquad |\mathcal{C}_1| \colon A^* \longrightarrow \mathbb{Q}$$

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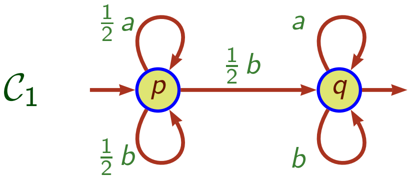

$$|\mathcal{C}_1| \in \mathbb{Q}\langle\langle A^* \rangle\rangle$$

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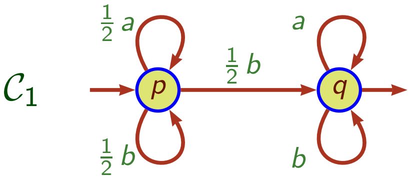
$$|\mathcal{C}_1| = \frac{1}{2}\,b + \frac{1}{4}\,a\,b + \frac{1}{2}\,b\,a + \frac{3}{4}\,b\,b + \frac{1}{8}\,a\,a\,b + \frac{1}{4}\,a\,b\,a + \frac{3}{8}\,a\,b\,b + \frac{1}{2}\,b\,a\,a + \ldots$$

The weighted automaton model



$$\mathcal{C}_1 = \left\langle\, I_1, \underline{E_1}, T_1 \,\right\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\,a + \frac{1}{2}\,b & \frac{1}{2}\,b \\ 0 & a + b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

The weighted automaton model

$$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle \qquad\qquad \underline{E} = \text{incidence matrix}$$

$$\underline{E}_{p,q} = \sum \{\mathbf{wl}(e) \,|\, e \quad \text{transition from } p \text{ to } q\}$$

$$\underline{E}^{n}_{p,q} = \sum \{\mathbf{wl}(c) \,|\, c \quad \text{computation from } p \text{ to } q \text{ of length } n\}$$

$$\underline{E}^{*} = \sum_{n \in \mathbb{N}} \underline{E}^{n}$$

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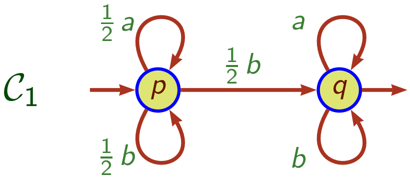

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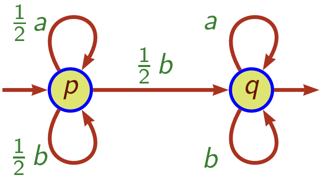

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Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$

whose coefficients are effectively computable

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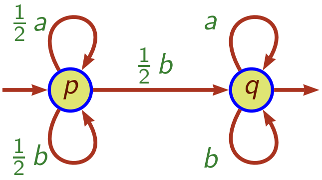

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Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$ whose coefficients are effectively computable

Where is the problem ?

We want to deal with automata whose transitions may be labelled by the empty word ε

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

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Every ε-NFA is equivalent to an NFA

Usefulness of ε-transitions:

Preliminary step for many constructions on NFA's:

- Product and star of position automata
- Thompson construction
- Construction of the universal automaton
- Computation of the image of a transducer
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Removal of ε-transitions is implemented in all automata software

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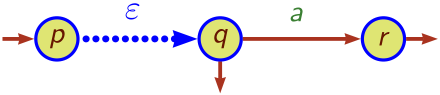

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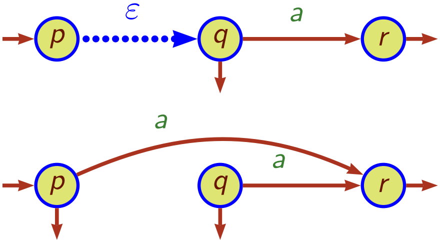

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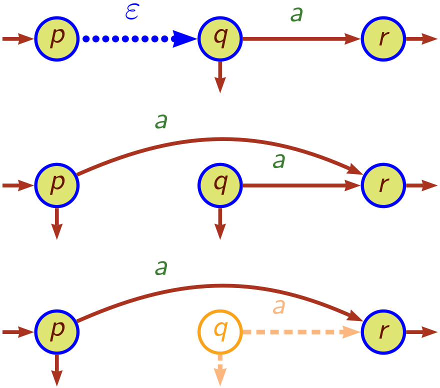

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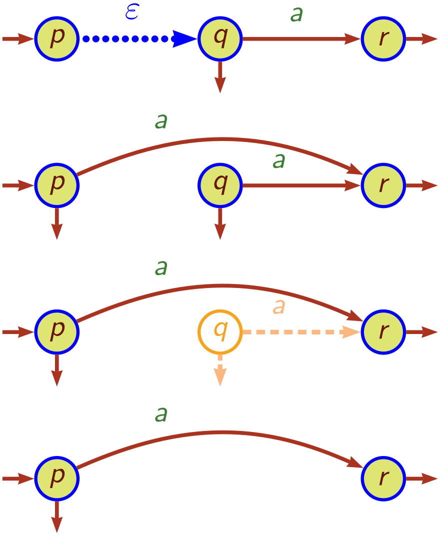

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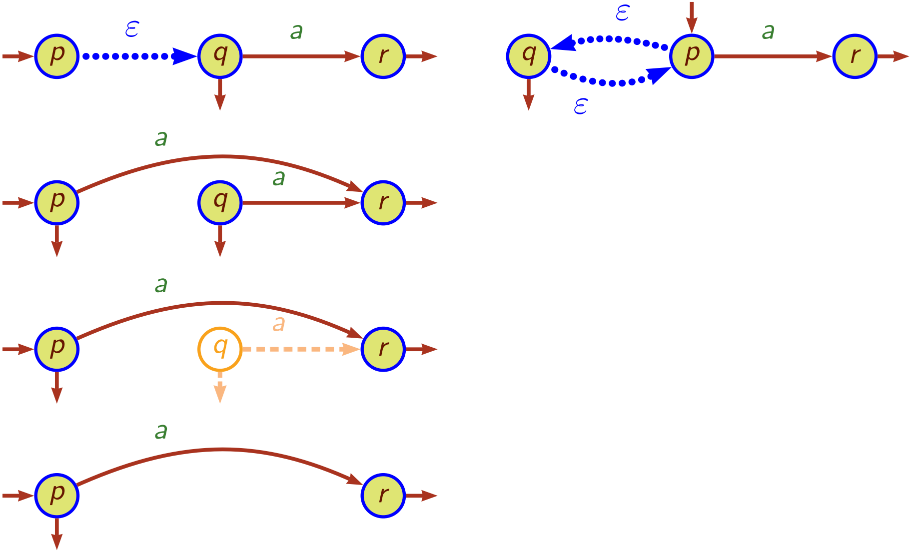

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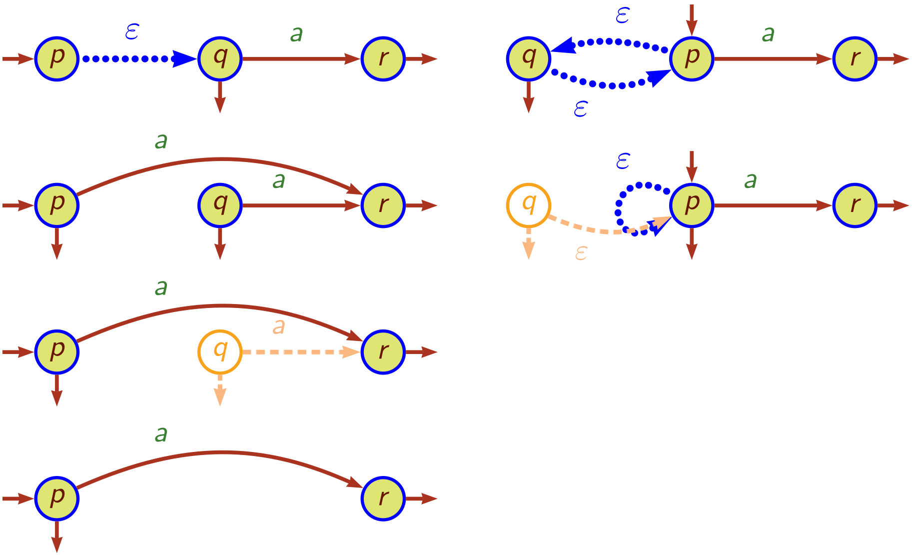

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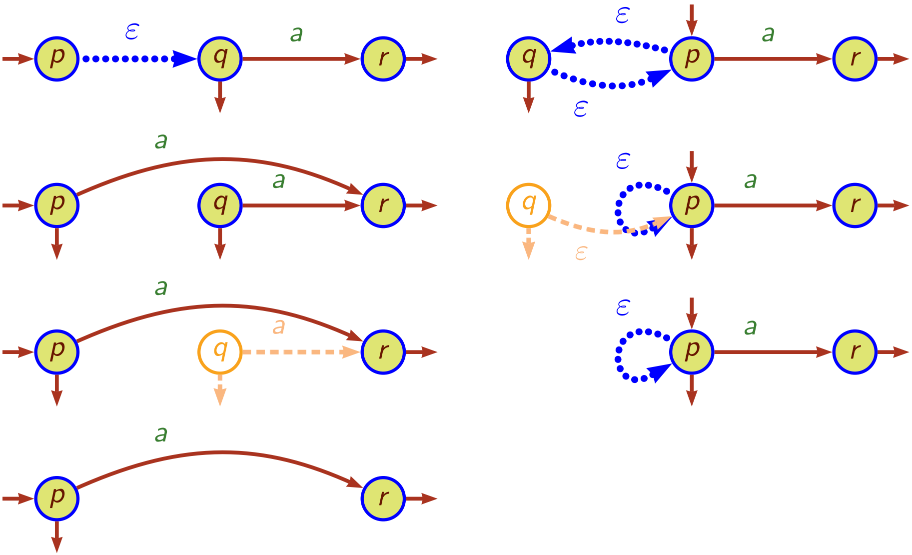

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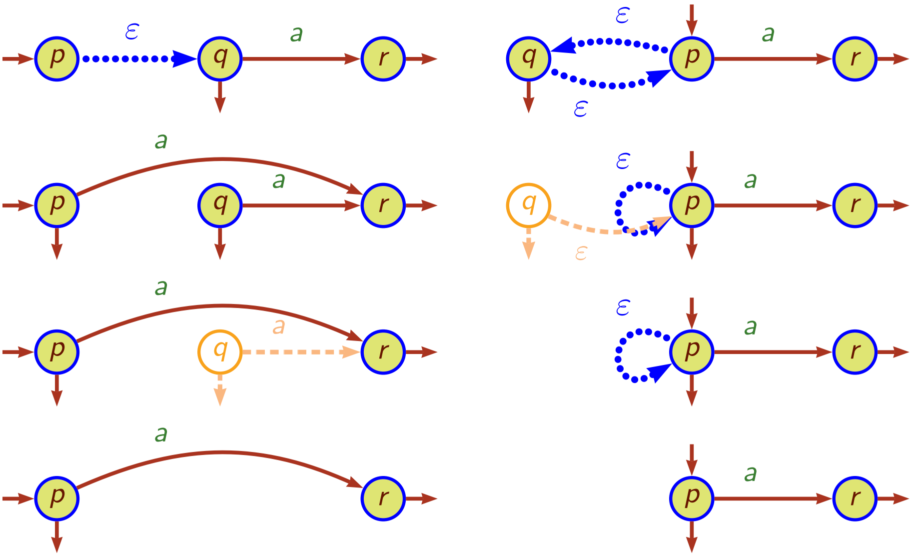

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

A proof

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ $\qquad\qquad$ $\underline{E}$ transition matrix of $\mathcal{A}$

Entries of $\underline{E}$ = subsets of $A \cup \{\varepsilon\}$

$$L(\mathcal{A}) = I \cdot \underline{E}^* \cdot T$$

$$\underline{E} = \underline{E}_0 + \underline{E}_{\mathsf{p}}$$

$$L(\mathcal{A}) = I \cdot (\underline{E}_0 + \underline{E}_{\mathsf{p}})^* \cdot T = I \cdot (\underline{E}_0^* \cdot \underline{E}_{\mathsf{p}})^* \cdot \underline{E}_0^* \cdot T$$

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ equivalent to $\mathcal{B} = \left\langle\, I, \underline{E}_0^* \cdot \underline{E}_{\mathsf{p}}, \underline{E}_0^* \cdot T \,\right\rangle$

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One *proof* = several *algorithms* for *computing* $\underline{E}_0^{*}$ or $\underline{E}_0^{*} \cdot \underline{E}_{\mathrm{p}}$

A basic question in weighted automata theory

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Is every ε-WFA is equivalent to a WFA?

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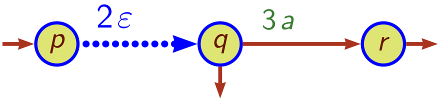

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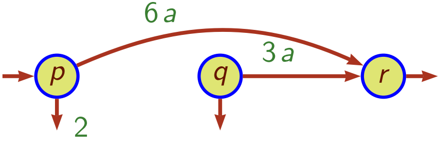

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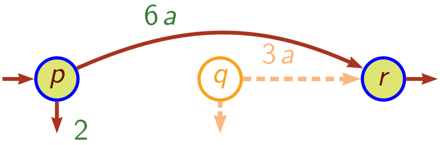

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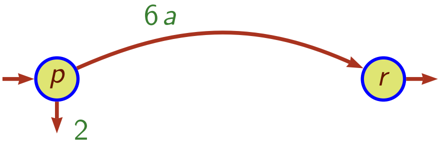

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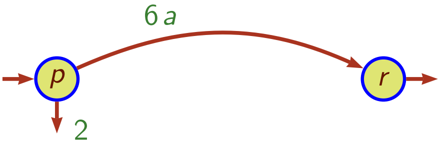

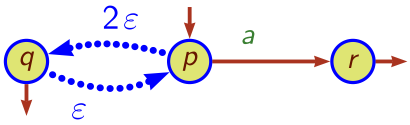

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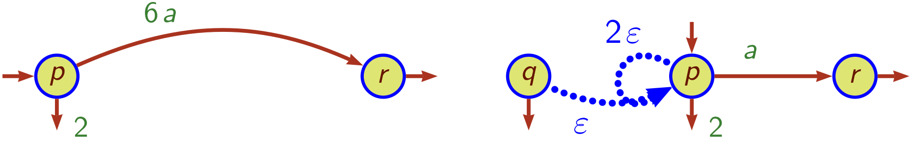

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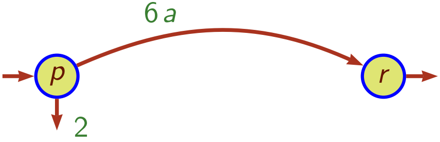

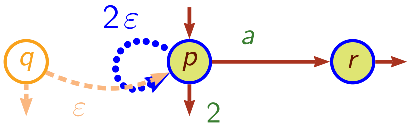

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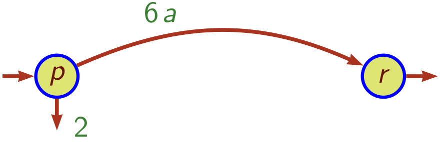

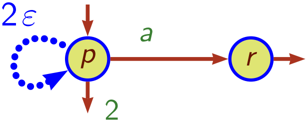

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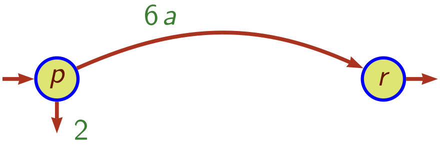

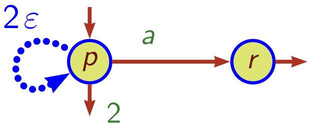

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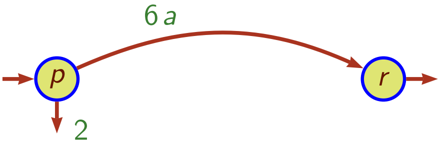

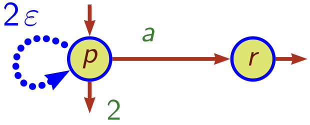

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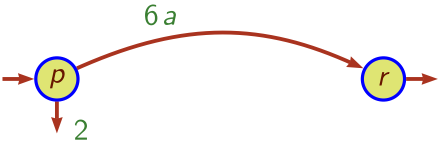

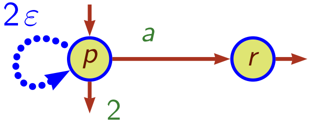

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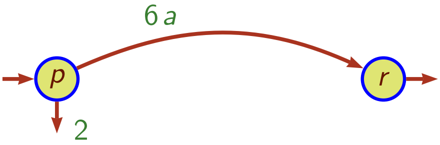

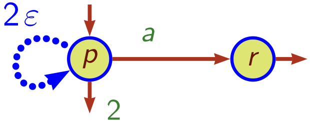

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$$a \quad \longmapsto \quad 1 + 2 + 4 + \cdots$$

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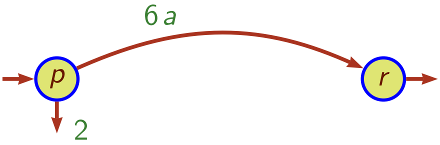

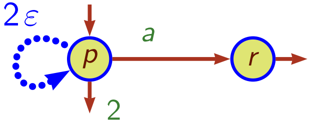

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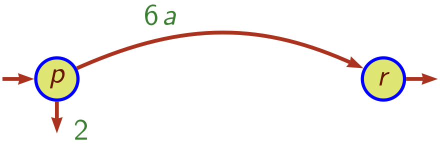

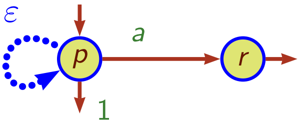

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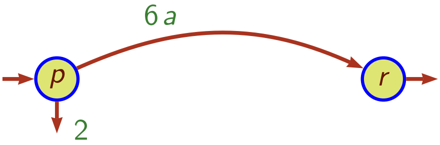

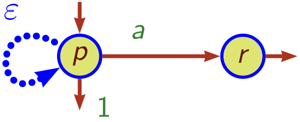

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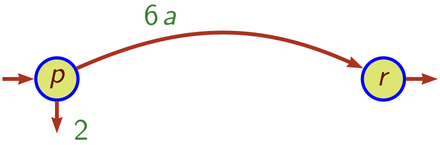

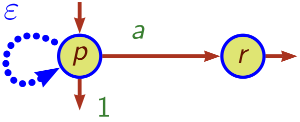

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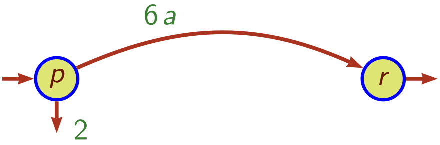

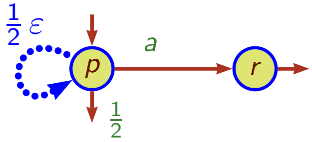

A basic question in weighted automata theory

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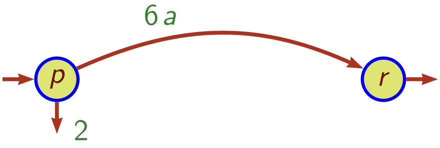

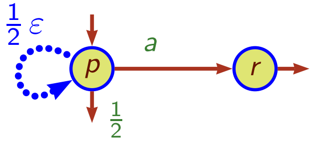

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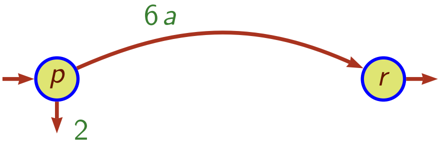

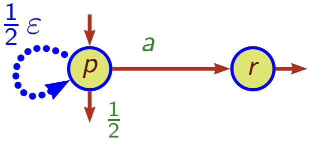

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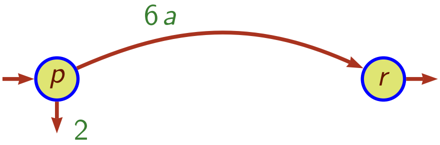

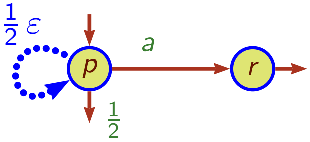

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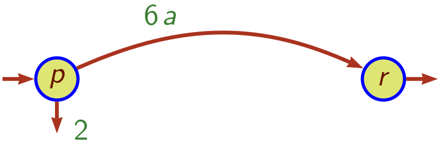

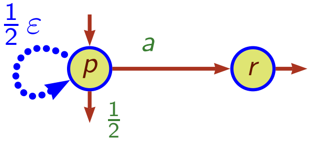

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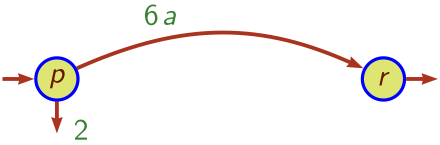

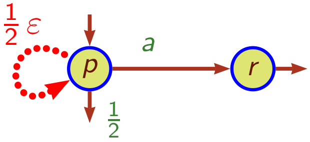

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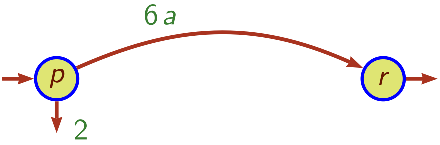

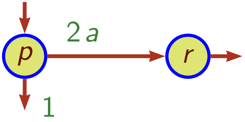

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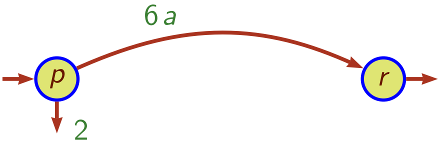

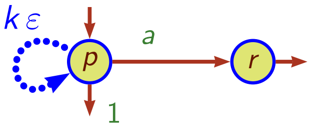

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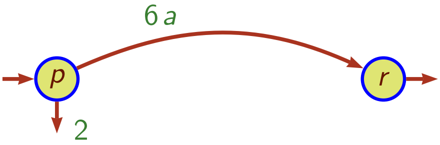

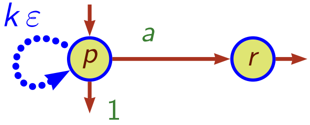

$a \quad \longmapsto \quad k^*$ undefined? defined?

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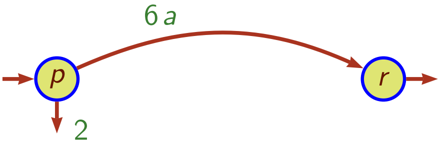

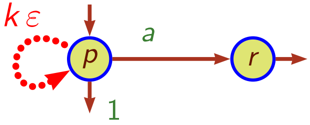

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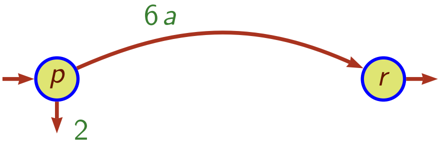

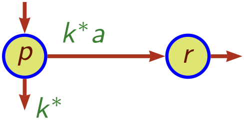

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Is every ε-WFA is equivalent to a WFA?

certainly not !

A basic question in weighted automata theory

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Is every ε-WFA is equivalent to a WFA?

certainly not !

New questions

Which ε-WFAs have a *well-defined* behaviour? i.e. are *valid* ?

How to compute the behaviour of a *valid* ε-WFA ?

How to decide if an ε-WFA is *valid*?

A chicken and egg problem

automaton	algorithm
$\mathcal{A}$	A
valid ?	success ?

A chicken and egg problem

automaton		**algorithm**
$\mathcal{A}$		A
valid ?		**success ?**
valid	$\Longrightarrow$	**success**

A chicken and egg problem

automaton		algorithm
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valid	$\Longrightarrow$	success
valid	$\overset{?}{\Longleftarrow}$	success

Behaviour of weighted automata

$\mathcal{A} = \langle\, \mathbb{K}, Q, A, E, I, T \,\rangle$ possibly with ε-transitions

$u \in A^*$ paths labelled by u in $\mathcal{A}$ possibly infinitely many

$\langle|\mathcal{A}|, u\rangle$ sum of weights of computations labelled by u in $\mathcal{A}$

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Every u in A^* is the label of a finite number of paths

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First solution

behaviour well-defined $\Longleftrightarrow$ acyclic

(Kuich–Salomaa 86, Berstel–Reutenauer 84-88; 11)

Behaviour of weighted automata

$\mathcal{A}$ not acyclic $\Rightarrow$ weight of u in $\mathcal{A}$ may be an infinite sum.

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Accepting the idea of infinite sums

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Second solution

Accepting the idea of infinite sums

Topological point of view

Infinite sums are given a meaning via a topology on $\mathbb{K}$

Topology on $\mathbb{K}$ defines a topology on $\mathbb{K}\langle\langle A^* \rangle\rangle$

Topology allows to define summable families in $\mathbb{K}\langle\langle A^* \rangle\rangle$

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$\mathcal{A} = \langle\, \mathbb{K}, Q, A, E, I, T \,\rangle$ possibly with ε-transitions

$\mathsf{P}_{\mathcal{A}}$ set of all paths in $\mathcal{A}$

$|\mathcal{A}|$ well-defined $\iff$ $\mathbf{WL}(\mathsf{P}_{\mathcal{A}})$ summable

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Definition taken in previous works (Lombardy, S. 03 –)

- Yields a consistent theory
- Two pitfalls for effectivity
  - *effective computation* of a summable family may not be possible
  - *effective computation* may give values to non summable families

Valid weighted automata

$\mathcal{A} = \langle \mathbb{K}, Q, A, E, I, T \rangle$ possibly with ε-transitions

E^* *free monoid* generated by E

$\mathsf{P}_{\mathcal{A}}$ *set of paths* in $\mathcal{A}$ (local) rational subset of E^*

Definition
R rational family of paths of $\mathcal{A}$ $R \in \mathrm{Rat}\mathrm{E}^* \wedge R \subseteq \mathsf{P}_{\mathcal{A}}$

Definition
$\mathcal{A}$ is valid iff
$\forall R$ rational family of paths of $\mathcal{A}$, $\mathbf{WL}(R)$ is summable

Valid weighted automata

Validity implies well-definition of behaviour

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Theorem

$\mathcal{A}$ is valid iff the behaviour of every covering of $\mathcal{A}$ is well-defined

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Eg. $\mathbb{R}$, $\mathbb{Q}$.

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Reminder

We do not know yet how to decide whether a $\mathbb{Q}$ - or an $\mathbb{R}$ -automaton is valid.

Deciding validity

Straightforward cases

- Non starable semirings (eg. $\mathbb{N}$, $\mathbb{Z}$)

  $\mathcal{A}$ valid $\iff$ $\mathcal{A}$ acyclic
- Complete topological semirings (eg. $\mathcal{N}$) every $\mathcal{A}$ valid
- Rationally additive semirings (eg. $\mathrm{Rat}\, A^*$) every $\mathcal{A}$ valid
- Locally closed commutative semirings every $\mathcal{A}$ valid

Deciding validity

Definition

$\mathbb{K}$ topological, ordered, positive semiring (TOPS)

is star-domain downward closed (SDC) if

$$\forall k, h \in \mathbb{K}\,,\ k < h \qquad h \text{ starable} \implies k \text{ starable}$$

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$\mathbb{N}$, $\mathcal{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, $\mathbb{Z}\mathrm{min}$, $\mathrm{Rat}\,A^*$,... are TOPS SDC

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Theorem

$\mathbb{K}$ topological, ordered, positive, star-domain downward closed

A $\mathbb{K}$-automaton is valid if, and only if,

the state-by-state ε-removal algorithm succeeds

Deciding validity

Definition

If $\mathcal{A}$ is a $\mathbb{Q}$- or $\mathbb{R}$-automaton,

then $\mathsf{abs}(\mathcal{A})$ is a $\mathbb{Q}_+$- or $\mathbb{R}_+$-automaton

Theorem

A $\mathbb{Q}$- or $\mathbb{R}$-automaton $\mathcal{A}$ is valid if and only if $\mathsf{abs}(\mathcal{A})$ is valid.

Hidden parts

- The problematic examples
- The removal algorithm itself:
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues
- Automata and expressions validity
- 'Infinitary' axioms : *strong*, *star-strong* semirings
- Links with the 'axiomatic' approach (Bloom–Ésik–Kuich) :
  Th: A starable star-strong semiring is an iteration semiring.
- References to previous work (on removal algorithm):
  - *locally closed* srgs (Ésik–Kuich), *k-closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)

Conclusion

- Semiring structure is weak, topology does not help so much.
- This weakness imposes a restricted definition of validity,
  in order to guarantee success of validity algorithms.
- Axiomatic approach does not allow
  to deal wit most common numerical semirings: $\mathbb{Z}$min, $\mathbb{Q}$
- On 'usual' semirings,
  the new definition of validity coincides with the former one.

Conclusion (2)

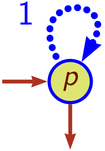
- Apart the trivial cases, and the TOPS SCD case, decision of validity is never granted, and has to be established.
- On ‘usual’ semirings, validity is decidable.
- The new definition of validity fills the ‘effectivity gap’ left open by the former one.

Hidden parts

- The problematic examples

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_1$



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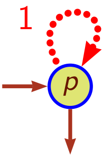

$(1)^* = \text{undefined}$

$\mathbb{N}$ natural integers $|\mathcal{A}_1|$ not defined

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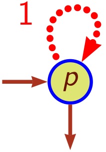

$$(1)^* = +\infty$$

$\mathbb{N}$		natural integers	$\lvert\mathcal{A}_1\rvert$	not defined
$\mathcal{N}$	$\mathbb{N} \cup +\infty$	compact topology	$\lvert\mathcal{A}_1\rvert$	defined

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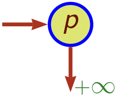

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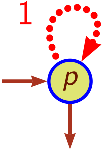

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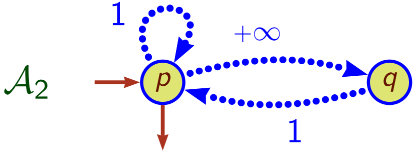
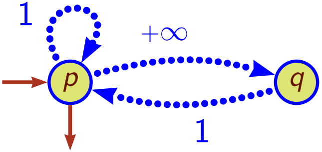
$\mathbb{N}_\infty$	$\mathbb{N} \cup +\infty$	discrete topology	$\lvert\mathcal{A}_1\rvert$	not defined

Problems in computing the behaviour of a weighted automaton



Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_2$



$\mathcal{N}$ $\quad$ $\mathbb{N} \cup +\infty$ $\quad$ compact topology $\quad$ $|\mathcal{A}_1|$ $\quad$ defined

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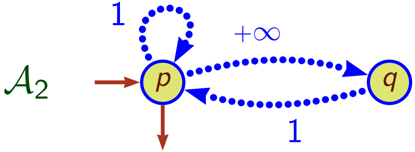

$\mathcal{N}$	$\mathbb{N} \cup +\infty$	compact topology	$\lvert\mathcal{A}_1\rvert$	defined
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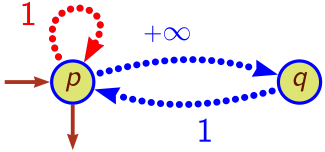

$(1)^* =$ undefined

$\mathcal{N}$	$\mathbb{N} \cup +\infty$	compact topology	$\lvert\mathcal{A}_1\rvert$	defined
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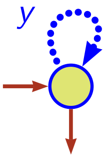


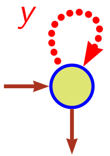
$$\mathbb{S} \subset \mathbb{N}^{2\times 2}, \quad x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{\mathbb{S}}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x+y = \infty_{\mathbb{S}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\mathbb{S}$ equipped with the discrete topology

$0_{\mathbb{S}}$, y, and $\infty_{\mathbb{S}}$ starable $\qquad$ $x = y^2$ $\qquad$ x not starable

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_3$



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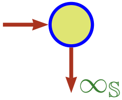

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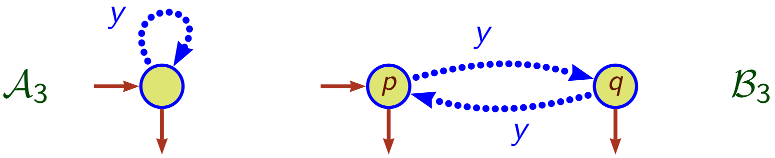

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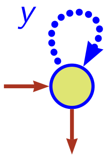

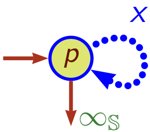

$\mathcal{B}_3$

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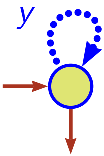
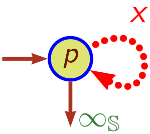
$0_{\mathbb{S}}$, y, and $\infty_{\mathbb{S}}$ starable $\qquad x = y^2 \qquad$ x not starable

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_3$





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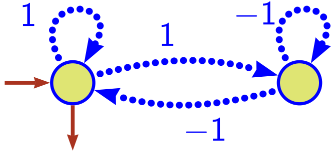
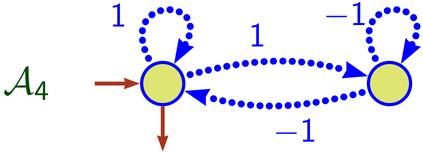
Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_4$



Problems in computing the behaviour of a weighted automaton

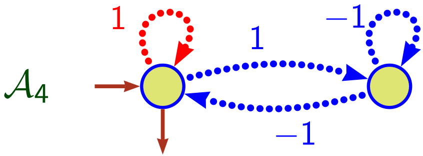


$$\mathcal{A}_4 = \langle\, I_4, \underline{E_4}, T_4 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$|\mathcal{A}_4| = I_4 \cdot \underline{E_4}^* \cdot T_4$$

$$\underline{E_4}^2 = 0 \quad \Longrightarrow \quad \underline{E_4}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad |\mathcal{A}_4| = 2$$

Problems in computing the behaviour of a weighted automaton



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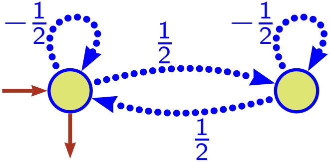
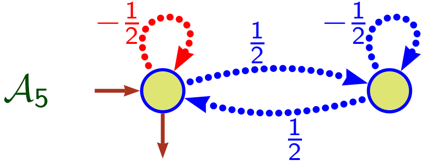
Problems in computing the behaviour

$\mathcal{A}_5$



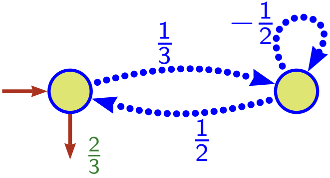
Problems in computing the behaviour



$$\left(-\frac{1}{2}\right)^* = \frac{2}{3}$$

Problems in computing the behaviour

$\mathcal{A}_5$



Problems in computing the behaviour

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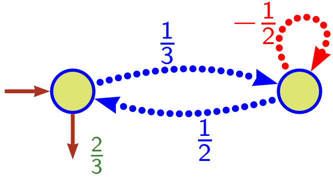

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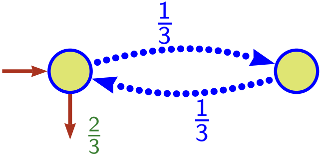

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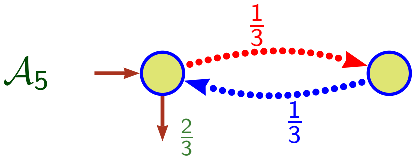

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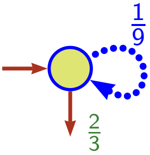

Problems in computing the behaviour

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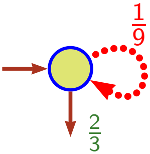
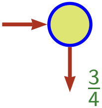


$$\left(\frac{1}{9}\right)^* = \frac{9}{8}$$

Problems in computing the behaviour

$\mathcal{A}_5$



Problems in computing the behaviour

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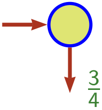


$$\mathcal{A}_5 = \langle\, I_5, \underline{E_5}, T_5 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

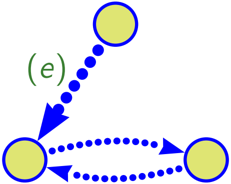
$$|\mathcal{A}_5| = I_5 \cdot \underline{E_5}^* \cdot T_5$$

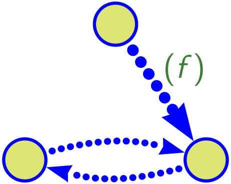
$$\underline{E_5}^3 = \underline{E_5} \quad \Longrightarrow \quad \underline{E_5}^* \quad \text{undefined} \quad \Longrightarrow \quad |\mathcal{A}_5| \quad \text{undefined}$$

Hidden parts

- The problematic examples
- The removal algorithm itself:
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues

Termination issues

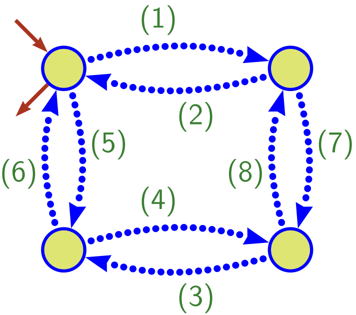




weighted ε-removal procedure does not terminate

if newly created ε-transitions are stored in a stack

Termination issues



weighted ε-removal procedure does not terminate

if newly created ε-transitions are stored in a queue

Hidden parts

- The problematic examples
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- Automata and expressions validity

Automata and expressions validity

'Kleene' theorem

Automata	$\Longleftrightarrow$	Expressions
$\mathcal{A}$	$\Longleftrightarrow$	E
Weighted automata	$\Longleftrightarrow$	Weighted expressions

Automata and expressions validity

'Kleene' theorem

$$\text{Automata} \iff \text{Expressions}$$

$$\mathcal{A} \iff \mathsf{E}$$

$$\text{Weighted automata} \iff \text{Weighted expressions}$$

Notion of a valid expression

$$\mathsf{E} \ \textit{valid} \iff \mathsf{c}(\mathsf{E}) \ \text{well-defined}$$

$\mathsf{c}(\mathsf{E})$ computed by a bottom-up traversal of the syntactic tree of E

Automata and expressions validity

Valid $\mathcal{A}$ yields valid E

Valid E yields valid $\mathcal{A}$ with Glushkov construction

Valid E may yield non valid $\mathcal{A}$ with Thompson construction

Automata and expressions validity

Valid $\mathcal{A}$ yields valid E

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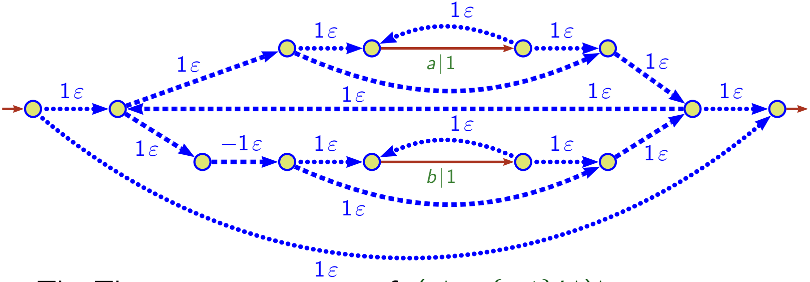

The Thompson automaton of $(a^* + \{-1\}b^*)^*$

Hidden parts

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- Automata and expressions validity
- ‘Infinitary’ axioms : *strong*, *star-strong* semirings

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Definition

A topological semiring is a *strong* semiring

if the product of two summable families is a summable family

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Definition

A topological semiring is a *strong* semiring
if the product of two summable families is a summable family

Theorem

$\mathbb{K}$ *strong semiring* $\quad$ $s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ *starable iff* $s_0 \in \mathbb{K}$ *starable*

Hidden parts

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Definition

A topological semiring is a *strong* semiring
if the product of two summable families is a summable family

Definition

A topological semiring is a *star-strong* semiring if
the star of a summable family, whose sum is starable, is summable

Hidden parts

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- ‘Infinitary’ axioms : *strong*, *star-strong* semirings
- Links with the ‘axiomatic’ approach (Bloom–Ésik–Kuich):

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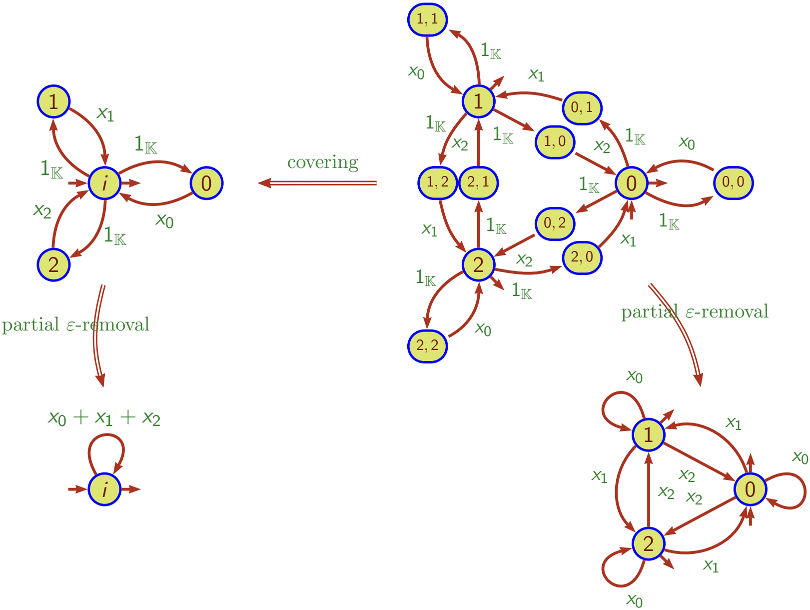
Theorem

A starable star-strong semiring is an iteration semiring

Group identities



Hidden parts

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- Automata and expressions validity
- ‘Infinitary’ axioms : *strong*, *star-strong* semirings
- Links with the ‘axiomatic’ approach (Bloom–Ésik–Kuich):
- References to previous work (on removal algorithm):
  - *locally closed* srgs (Ésik–Kuich), k-*closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)