Mathematical Finance, Vol. 7, No. 3 (July 1997), 241–286

THE VALUATION OF AMERICAN OPTIONS ON MULTIPLE ASSETS*

MARK BROADIE

Graduate School of Business, Columbia University

JÉRÔME DETEMPLE

Faculty of Management, McGill University; and CIRANO

In this paper we provide valuation formulas for several types of American options on two or more assets. Our contribution is twofold. First, we characterize the optimal exercise regions and provide valuation formulas for a number of American option contracts on multiple underlying assets with convex payoff functions. Examples include options on the maximum of two assets, dual strike options, spread options, exchange options, options on the product and powers of the product, and options on the arithmetic average of two assets. Second, we derive results for American option contracts with nonconvex payoffs, such as American capped exchange options. For this option we explicitly identify the optimal exercise boundary and provide a decomposition of the price in terms of a capped exchange option with automatic exercise at the cap and an early exercise premium involving the benefits of exercising prior to reaching the cap. Besides generalizing the current literature on American option valuation our analysis has implications for the theory of investment under uncertainty. A specialization of one of our models also provides a new representation formula for an American capped option on a single underlying asset.

KEY WORDS: option pricing, early exercise policy, free boundary, security valuation, multiple assets, caps, investment under uncertainty

1. INTRODUCTION

In this paper we analyze several types of American options on two or more assets. We study options on the maximum of two assets, dual strike options, spread options, and others. For each of these contracts we characterize the optimal exercise regions and develop valuation formulas.

Our analysis provides new insights since many contracts that are traded in modern financial markets, or that are issued by firms, involve *American* options on several underlying assets. A standard example is the case of an index option that is based on the value of a portfolio of assets. In this case the option payoff upon exercise depends on an arithmetic or geometric average of the values of several assets. For example, options on the S&P 100, which have traded on the Chicago Board of Options Exchange (CBOE) since March 1983, are American options on a value weighted index of 100 stocks. Other contracts pay the maximum of two or more asset prices upon exercise. Examples include option bonds and incentive contracts. Embedded American options on the maximum of two or more assets

*This paper was presented at CIRANO, Queen's University, MIT, Wharton, the 1994 Derivative Securities Conference at Cornell University, and at the 1994 Conference on Recent Developments in Asset Pricing at Rutgers University. We thank the seminar participants and an anonymous referee for their comments.

Manuscript received July 1994; final revision received June 1995.

Address correspondence to Mark Broadie at Columbia University, New York, NY 10027.

can also be found in firms choosing among mutually exclusive investment alternatives, or in employment switching decisions by agents. American spread options and options to exchange one asset for another also arise in several contexts. Gasoline crack spread options, traded on the NYMEX (New York Mercantile Exchange), are American options written on the spread between the NYMEX New York Harbor unleaded gasoline futures and the NYMEX crude oil futures. Likewise, heating oil crack spread options, also traded on the NYMEX, are American options on the spread between the NYMEX New York Harbor heating oil futures and the NYMEX crude oil futures. Options on foreign indices with exercise prices quoted in the foreign currency can now be bought by American investors (one example is the option on the Nikkei index warrants traded on the AMEX; another is the option on the CAC40 on the MONEF). Stock tender offers, which are American options to exchange the stock of one company for the stock of another, are also common in financial markets.

In most cases the underlying assets in these contracts pay dividends or have other cash outflows. It is well known that standard American options written on a single dividend paying underlying asset may be optimally exercised before maturity. The same is true for options on multiple dividend paying assets: The American feature is valuable and exercise prior to maturity may be optimal. However, when several asset prices determine the exercise payoff, the shape of the exercise region often cannot be determined by simple arguments or by appealing to the intuition known for the single asset case. Furthermore, the structure of the exercise region may differ significantly among the various contracts under investigation. As a result it is important to identify optimal exercise boundaries in order to provide a thorough understanding of these contracts.

In the last few years there has been much progress in the valuation of standard American options written on a single underlying asset (see, e.g., Karatzas 1988, Kim 1990, Jacka 1991, and Carr, Jarrow, and Myneni 1992). The optimal exercise boundary and the corresponding valuation formula have also been identified for American call options with constant and growing caps, which are contracts with nonconvex payoffs (see Broadie and Detemple 1995). European options on multiple assets have been studied previously. European options to exchange one asset for another were analyzed by Margrabe (1978). Johnson (1981) and Stulz (1982) provide valuation formulas for European put and call options on the maximum or minimum of two assets. Their results are extended to the case of several assets by Johnson (1987).

The case of American options on multiple dividend-paying underlying assets, however, has received little attention in the literature. In recent independent work, Tan and Vetzal (1994) perform numerical simulations to identify the immediate exercise region for some types of exotic options. Independent work by Geltner, Riddiough, Stojanovic (1994) also provides insights about the exercise region for a perpetual option on the best of two assets in the context of land use choice.

We start with an analysis of a prototypical contract with multiple underlying assets and a convex payoff: an American option on the maximum of two assets. One of the surprising results obtained is that it is never optimal to exercise this option prior to maturity when the underlying asset prices are equal, even if the option is deep in the money and dividend rates are very large. This counterintuitive result rests on the fact that delaying exercise enables the investor to capture the gains associated with the event that one asset price exceeds the other in the future. This gain is sufficiently important to offset the benefits of immediate exercise even when the underlying asset prices substantially exceed the exercise price of the option. Beyond its implications for the valuation of financial options, this result is also of importance for the theory of investment under uncertainty (e.g., Dixit and Pindyck 1994). In this context our analysis provides a new motive for waiting to invest—namely the benefits associated with the possibility of future dominance of one project over the other investments available to the firm. In a global economy in which firms are constantly confronted with multiple investment opportunities this motive may well be at work in decisions to delay certain investments. We also derive an interesting divergence property of the exercise region: For equal underlying asset prices, the distance to the exercise boundary is increasing in the prices.

Another contribution of the paper is a new representation formula for a class of contracts with nonconvex payoffs, such as capped exchange options. We show that the optimal exercise policy consists in exercising at the first time at which the ratio of the two underlying asset prices reaches the minimum of the cap and the exercise boundary of an uncapped exchange option. A valuation formula, in terms of the uncapped exchange option and the payoff when the cap is reached, follows. We also provide an alternative representation of the price of this option which involves the value of a capped exchange option with automatic exercise at the cap and an early exercise premium involving the benefits of exercising prior to reaching the cap. The optimal exercise boundary, in turn, is shown to satisfy a recursive integral equation based on this decomposition. When one of the two underlying asset prices is a constant our formulas provide the value of an American capped option on a single underlying asset (Broadie and Detemple 1995). Hence, beside generalizing the literature on American capped call options we also produce a new decomposition of the price of such contracts.

American max-options are analyzed in Section 2. Section 3 focuses on American spread options and the special case of exchange options. In Section 4 we build on the results of Section 3 in order to value American capped exchange options which have a nonconvex payoff function. American options based on the product of underlying asset prices, such as options on a geometric average, are analyzed in Section 5. In Section 6 American options on arithmetic averages are examined. Generalizations to the case of n underlying assets are given in Section 7 and proofs of the propositions are relegated to the appendices.

2. AMERICAN OPTIONS ON THE MAXIMUM OF TWO ASSETS

We consider derivative securities written on a pair of underlying assets which may be interpreted as stocks, indices, futures prices, or exchange rates. The prices of the underlying assets at time t, S_t^1, and S_t^2, satisfy the stochastic differential equations

$$dS_t^1 = S_t^1[(r-\delta_1)dt + \sigma_1 dz_t^1] \tag{2.1}$$

$$dS_t^2 = S_t^2[(r-\delta_2)dt + \sigma_2 dz_t^2] \tag{2.2}$$

where z^1 and z^2 are standard Brownian motion processes with a constant correlation ρ. To avoid trivial cases, we assume throughout that $|\rho| < 1$. Here r is the constant rate of interest, $\delta_i \geq 0$ is the dividend rate of asset i, and σ_i is the volatility of the price of asset i, $i = 1, 2$. The price processes (2.1) and (2.2) are represented in their risk neutral form. Throughout the paper, E_t^* denotes the expectation at time t under the risk neutral measure.

Let $C_t(S_t)$ denote the theoretical value of an American call option at time t on a single asset (e.g., asset 1 above) that matures at time T and has a strike price of K. Throughout the paper, this option is referred to as the *standard* option. Let $C_t^X(S_t^1, S_t^2)$ denote the

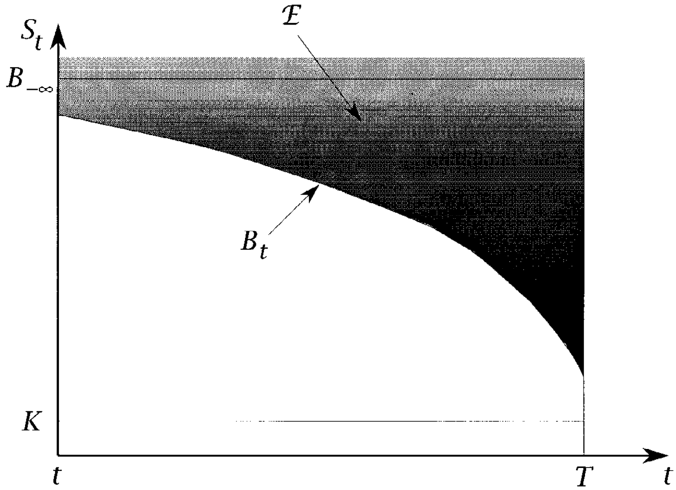

FIGURE 2.1. Illustration of B_t for a standard American call option.

theoretical value of an American call option on the maximum of two assets, or max-option for short. The payoff of the max-option, if exercised at some time t before maturity T, is $(\max(S_t^1, S_t^2) - K)^+$. The notation x^+ is short for $\max(x, 0)$. The optimal or immediate exercise region of an American call on a single underlying asset is $\mathcal{E} \equiv \{(S_t, t) : C_t(S_t) = (S_t - K)^+\}$. Similarly, for an American call option on the maximum of two assets, the immediate exercise region is $\mathcal{E}^X \equiv \{(S_t^1, S_t^2, t) : C_t^X(S_t^1, S_t^2) = (\max(S_t^1, S_t^2) - K)^+\}$.

Standard American Options

Before proceeding further, we review some essential results for standard American options (i.e., on a single underlying asset). Let B_t denote the immediate exercise boundary for a standard option on a single underlying asset. That is, $B_t = \inf\{S_t : (S_t, t) \in \mathcal{E}\}$. An illustration of B_t is given in Figure 2.1.

Van Moerbeke (1976) and Jacka (1991) show that B_t is continuous. Kim (1990) and Jacka (1991) show that B_t is decreasing in t. Kim (1990) shows that $B_{T^-} \equiv \lim_{t \to T} B_t = \max((r/\delta)K, K)$. Merton (1973) shows that B_t is bounded above and derives a formula for $B_{-\infty} \equiv \lim_{t \to -\infty} B_t$. Jacka (1991) shows that the option value $C_t(S_t)$ is continuous and the immediate exercise region $\mathcal{E}$ is closed.

Exercise Region of American Max-Options

How do the properties of the exercise region for a standard option compare to those for a max-option? For a standard American option, $(S_t, t) \in \mathcal{E}$ implies $(\lambda S_t, t) \in \mathcal{E}$ for all $\lambda \geq 1$.[1] By analogy, an apparently reasonable conjecture for $\mathcal{E}^X$ is

[1] See Proposition A.1 in Appendix A for a proof.

CONJECTURE 2.1. $(S_t^1, S_t^2, t) \in \mathcal{E}^X$ implies $(\lambda_1 S_t^1, \lambda_2 S_t^2, t) \in \mathcal{E}^X$ for all $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$.

For a call option on a single asset with a positive dividend rate, immediate exercise is optimal for all sufficiently large asset values. That is, there exists a constant M such that $(S_t, t) \in \mathcal{E}$ for all $S_t \geq M$. Hence a reasonable conjecture for $\mathcal{E}^X$ is

CONJECTURE 2.2. If $\delta_1 > 0$ and $\delta_2 > 0$ then there exist constants M_1 and M_2 such that $(S_t^1, S_t^2, t) \in \mathcal{E}^X$ for all $S_t^1 \geq M_1$ and all $S_t^2 \geq M_2$.

For standard options the exercise region $\mathcal{E}$ is convex with respect to the asset price. The analogous conjecture for $\mathcal{E}^X$ is

CONJECTURE 2.3. $(S_t^1, S_t^2, t) \in \mathcal{E}^X$ and $(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}^X$ implies $\lambda(S_t^1, S_t^2, t) + (1 - \lambda)(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}^X$ for all $0 \leq \lambda \leq 1$.

Surprisingly, all three conjectures concerning $\mathcal{E}^X$ turn out to be *false*.

However, by focusing on certain subregions of $\mathcal{E}^X$, properties similar to those for $\mathcal{E}$ do hold. Define the subregion $\mathcal{E}_i^X$ of the immediate exercise region $\mathcal{E}^X$ by $\mathcal{E}_i^X = \mathcal{E}^X \cap \mathcal{G}_i$ where $\mathcal{G}_i \equiv \{(S_t^1, S_t^2, t) : S_t^i = \max(S_t^1, S_t^2)\}$ for $i = 1, 2$. Proposition 2.1 below states that, prior to maturity, exercise is suboptimal when the prices of the underlying assets are equal. This result holds no matter how large the prices are and no matter how large the dividend rates are. In particular, $(S, S, t) \notin \mathcal{E}^X$ for all $S > 0$ and $t < T$. Proposition 2.1 is the reason for focusing attention on the subregions $\mathcal{E}_i^X$.

PROPOSITION 2.1. *If $S_t^1 = S_t^2 > 0$ and $t < T$ then $(S_t^1, S_t^1, t) \notin \mathcal{E}^X$. That is, prior to maturity exercise is not optimal when the prices of the underlying assets are equal.*

This proposition is proved in Appendix B. The intuition for the suboptimality of immediate exercise follows. Delaying exercise up to some fixed time $s > t$ provides at least

$$PV(s - t) = S_t^1 e^{-\delta_1(s-t)} - K e^{-r(s-t)}$$

plus a European option to exchange asset 2 for asset 1 with a maturity date s which has value $E_t^*[e^{-r(s-t)}(S_s^2 - S_s^1)^+]$. As s converges to t, the present value $PV(s - t)$ converges to $S_t^1 - K$ at a finite rate. The exchange option value, however, decreases to zero at an increasing rate which approaches infinity in the limit. Hence there is some time $s > t$ such that delaying exercise until s provides a strictly positive premium relative to immediate exercise.

The next proposition shows that subregions of the exercise region are convex.

PROPOSITION 2.2 (Subregion Convexity). *Let $S = (S^1, S^2)$ and $\tilde{S} = (\tilde{S}^1, \tilde{S}^2)$. Suppose $(S, t) \in \mathcal{E}_i^X$ and $(\tilde{S}, t) \in \mathcal{E}_i^X$ for a fixed $i = 1$ or 2. Given λ, with $0 \leq \lambda \leq 1$, define $S(\lambda) = \lambda S + (1 - \lambda)\tilde{S}$. Then $(S(\lambda), t) \in \mathcal{E}_i^X$. That is, if immediate exercise is optimal at S and $\tilde{S}$ and if $(S, t) \in \mathcal{G}_i$ and $(\tilde{S}, t) \in \mathcal{G}_i$ then immediate exercise is optimal at $S(\lambda)$.*

The convexity of the exercise region is a consequence of the convexity of the payoff function with respect to the pair (S^1, S^2) and a consequence of the multiplicative structure of the uncertainty in (2.1) and (2.2). Additional properties of the exercise region $\mathcal{E}^X$ are summarized in Proposition 2.3. In this proposition, B_t^i represents the exercise boundary for a standard American option on the single underlying asset i.

PROPOSITION 2.3. *Let $\mathcal{E}^X$ represent the immediate exercise region for a max-option. Then $\mathcal{E}^X$ satisfies the following properties.*

(i) $(S_t^1, S_t^2, t) \in \mathcal{E}^X$ *implies* $(S_t^1, S_t^2, s) \in \mathcal{E}^X$ *for all* $t \leq s \leq T$.
(ii) $(S_t^1, S_t^2, t) \in \mathcal{E}_1^X$ *implies* $(\lambda S_t^1, S_t^2, t) \in \mathcal{E}_1^X$ *for all* $\lambda \geq 1$.
(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}_1^X$ *implies* $(S_t^1, \lambda S_t^2, t) \in \mathcal{E}_1^X$ *for all* $0 \leq \lambda \leq 1$.
(iv) $(S_t^1, 0, t) \in \mathcal{E}_1^X$ *implies* $S_t^1 \geq B_t^1$.

In (ii), (iii)*, and* (iv)*, analogous results hold for the subregion* $\mathcal{E}_2^X$.

Property (i) says that the continuation region shrinks as time moves forward. Property (i) holds since a short maturity option cannot be worth more than the longer maturity option and it can attain the value of the longer maturity option if it is exercised immediately. Property (ii) states that the exercise subregion is connected in the direction of increasing S^1 (right connectedness). This follows since the option value at $(\lambda S_t^1, S_t^2, t)$ is bounded above by the option value at (S_t^1, S_t^2, t) plus the difference in the asset prices $\lambda S_t^1 - S_t^1$. Since immediate exercise is optimal by assumption at (S_t^1, S_t^2, t), the option value at $(\lambda S_t^1, S_t^2, t)$ is bounded above by its immediate exercise value (which can be attained by exercising immediately). Property (iii) is similar and states that the exercise subregion is connected in the direction of decreasing S^2 (down connectedness). Finally, since zero is an absorbing barrier for S^2, the max-option becomes an option on asset 1 only when $S^2 = 0$. In this case the optimal exercise region is delimited by the exercise boundary corresponding to an option on asset 1 alone.

Let $\mathcal{E}^X(t) = \{(S_t^1, S_t^2): (S_t^1, S_t^2, t) \in \mathcal{E}^X\}$ denote the t-section of $\mathcal{E}^X$ and similarly define $\mathcal{E}_i^X(t)$ by $\{(S_t^1, S_t^2): (S_t^1, S_t^2, t) \in \mathcal{E}_i^X\}$. Convexity of $\mathcal{E}_i^X(t)$ is assured by Proposition 2.2. This implies that the boundary of $\mathcal{E}_i^X(t)$ is continuous, except possibly at the endpoints where S_t^1 or S_t^2 is zero. However, continuity is assured at these points by part (iii) of Proposition 2.3.

The next proposition states that the immediate exercise region diverges from the diagonal (i.e., equal asset prices) as the asset prices become large. To state the result, let

$$R(\lambda_1, \lambda_2) \equiv \{(S^1, S^2) \in \mathbb{R}_+^2 : \lambda_2 S^1 < S^2 < \lambda_1 S^1\}$$

for $\lambda_2 < \lambda_1$ denote the open cone defined by the price ratios λ_1 and λ_2.

PROPOSITION 2.4 (Divergence of the exercise region). *Fix $t < T$. There exists λ_1 and λ_2 with $\lambda_2 < 1 < \lambda_1$ such that*

$$\mathcal{E}^X(t) \cap R(\lambda_1, \lambda_2) = \emptyset.$$

From the results in this section, we can plot the shape of a typical exercise region $\mathcal{E}^X$. An example is shown in Figures 2.2–2.4. Note in Figure 2.4 that $B_{T^-}^1 = \max((r/\delta_1)K, K)$

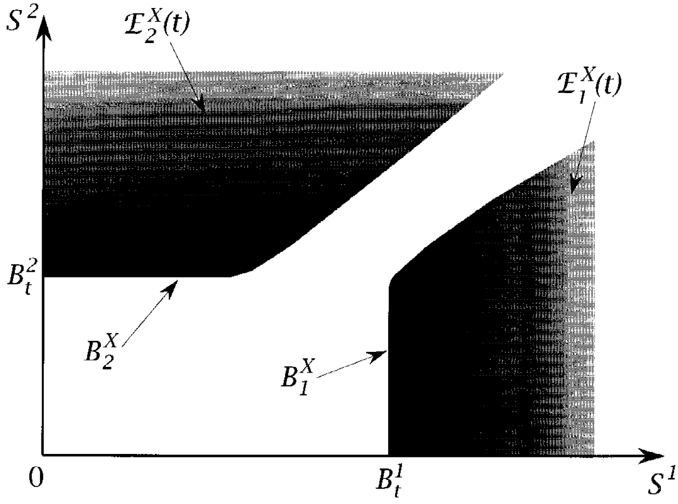

FIGURE 2.2. Illustration of $\mathcal{E}^X(t)$ for a max-option at time t with $t < T$.

and $B_{T^-}^2 = \max((r/\delta_2)K, K)$. The figures also show that $\max(S_t^1, S_t^2)$ is *not* a sufficient statistic for determining whether immediate exercise is optimal.

Valuation of American Max-Options

Recall $C_t^X(S_t^1, S_t^2)$ is the value of an American option on the maximum of two assets at time t with asset prices (S_t^1, S_t^2). In some cases, we will use $C^X(S^1, S^2, t)$ to denote $C_t^X(S_t^1, S_t^2)$.

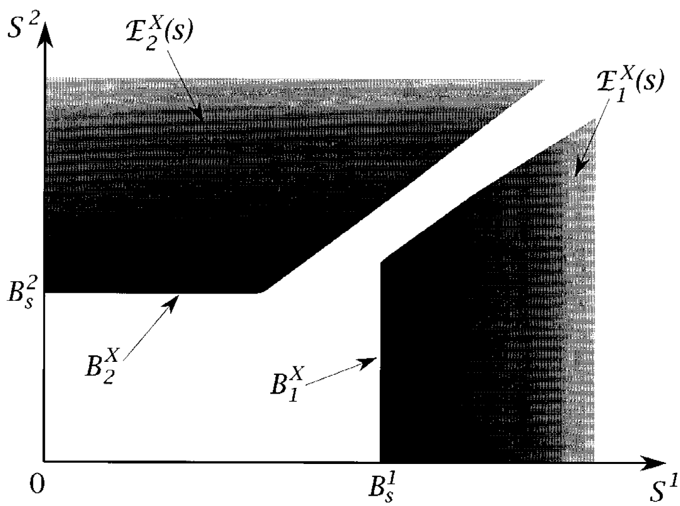

FIGURE 2.3. Illustration of $\mathcal{E}^X(s)$ for a max-option at time s with $t < s < T$.

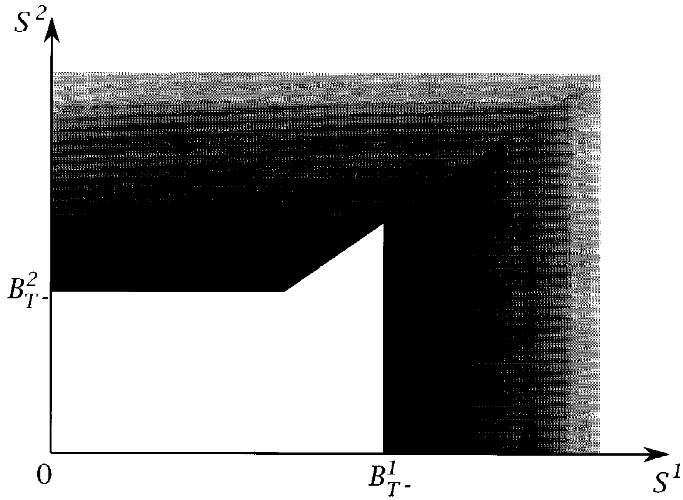

FIGURE 2.4. Illustration of $\mathcal{E}^X(T^-)$ for a max-option at time T^-.

PROPOSITION 2.5.

(i) *The value of the American max-option, $C^X(S^1, S^2, t)$, is continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times [0, T]$.*
(ii) *$C^X(\cdot, S^2, t)$ and $C^X(S^1, \cdot, t)$ are nondecreasing on $\mathbb{R}^+$ for all S^1, S^2 in $\mathbb{R}^+$ and all t in $[0, T]$.*
(iii) *$C^X(S^1, S^2, \cdot)$ is nonincreasing on $[0, T]$ for all S^1 and S^2 in $\mathbb{R}^+$.*
(iv) *$C^X(\cdot, \cdot, t)$ is convex on $\mathbb{R}^+ \times \mathbb{R}^+$ for all t in $[0, T]$.*

The continuity of $C^X(S^1, S^2, t)$ on $\mathbb{R}^+ \times \mathbb{R}^+ \times [0, T]$ follows from the continuity of the payoff function $(\max(S^1_t, S^2_t) - K)^+$ and the continuity of the flow of the stochastic differential equations (2.1) and (2.2). The monotonicity of $C^X(S^1, S^2, t)$ follows since $(\max(S^1_t, S^2_t) - K)^+$ is nondecreasing in S^1 and S^2. Property (iii) holds since a shorter maturity option cannot be more valuable. Convexity is implied by the convexity of the payoff function. The next proposition characterizes the option price in terms of variational inequalities (see Bensoussan and Lions 1978 and Jaillet, Lamberton, and Lapeyre 1990).

PROPOSITION 2.6 (Variational inequality characterization for max-options). *C^X has partial derivatives $\partial C^X/\partial S^i$, $i = 1, 2$, which are uniformly bounded and $\partial C^X/\partial t$ and $\partial^2 C^X/\partial S^i \partial S^j$, $i, j = 1, 2$, which are locally bounded on $[0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$. Define the operator $\mathcal{L}$ on the value function C^X by*

$$
\begin{aligned}
(2.3)\quad \mathcal{L}C^X &= (r - \delta_1)S^1 \frac{\partial C^X}{\partial S^1} + (r - \delta_2)S^2 \frac{\partial C^X}{\partial S^2} \\
&\quad + \frac{1}{2}\left[\sigma_1^2 (S^1)^2 \frac{\partial^2 C^X}{(\partial S^1)^2} + 2\rho\sigma_1\sigma_2 S^1 S^2 \frac{\partial^2 C^X}{\partial S^1 \partial S^2} + \sigma_2^2 (S^2)^2 \frac{\partial^2 C^X}{(\partial S^2)^2}\right] - rC^X.
\end{aligned}
$$

Then $C_t^X(S_t^1, S_t^2)$ *satisfies*

$$C_t^X \geq (\max(S_t^1, S_t^2) - K)^+; \qquad \frac{\partial C^X}{\partial t} + \mathcal{L}C^X \leq 0; \tag{2.4}$$

$$\left(\frac{\partial C^X}{\partial t} + \mathcal{L}C^X\right)((\max(S_t^1, S_t^2) - K)^+ - C_t^X) = 0$$

almost everywhere on $[0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$.

COROLLARY 2.1. *The spatial derivatives* $\partial C^X/\partial S^i$, $i = 1, 2$, *are continuous on* $[0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$.

Proposition 2.6 establishes the local boundedness of the partial derivatives of the value function $C^X(S_t^1, S_t^2, t)$. The continuity of the spatial derivatives follows from the convexity of $C^X(S^1, S^2, t)$ and the variational inequality $\frac{\partial C^X}{\partial t} + \mathcal{L}C^X \leq 0$. Although Proposition 2.6 provides a complete characterization of the value of the max-option, it is of interest to derive an alternative representation which provides additional insights about the determinants of the option value. This representation expresses the value of the American max-option as the value of the corresponding European option plus the gains from early exercise. Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992) provide such a representation for the standard American option when the underlying asset price follows a geometric Brownian motion process. The early exercise premium representation is the Riesz decomposition of the Snell envelope which arises in the stopping time problem associated with the valuation of the American option (see El Karoui and Karatzas 1991, Myneni 1992, and Rutkowski 1994).

Define the continuation region $\mathcal{C}$ to be the complement of $\mathcal{E}^X$, i.e., $\mathcal{C} \equiv \{(S_t^1, S_t^2, t) : C_t^X(S_t^1, S_t^2) > (\max(S_t^1, S_t^2) - K)^+\}$. The properties in Proposition 2.5 imply that the continuation region $\mathcal{C}$ is open and the immediate exercise region $\mathcal{E}^X$ is closed. Now define $B_1^X(S_t^2, t)$ to be the boundary of the t-section $\mathcal{E}_1^X(t)$ and $B_2^X(S_t^1, t)$ to be the boundary of the t-section $\mathcal{E}_2^X(t)$. The optimal stopping time can now be characterized by $\tau = \inf\{t : S_t^1 \geq B_1^X(S_t^2, t) \text{ or } S_t^2 \geq B_2^X(S_t^1, t)\}$.

The characterization of $C_t^X(S_t^1, S_t^2)$ given in Proposition 2.6 enables us to derive a system of recursive integral equations for the optimal exercise boundaries and to infer the value of the max-option. Toward this end, define

$$c_t^X(S_t^1, S_t^2) = E_t^*\left[e^{-r(T-t)}(\max(S_T^1, S_T^2) - K)^+\right] \tag{2.5}$$

which represents the value of the European max-option and the functions

$$a_1^X(S_t^1, S_t^2) = \int_{v=t}^{T} e^{-r(v-t)} E_t^*\left[(\delta_1 S_v^1 - rK)1_{\{S_v^1 > B_1^X(S_v^2, v)\}}\right] dv \tag{2.6}$$

$$a_2^X(S_t^1, S_t^2) = \int_{v=t}^{T} e^{-r(v-t)} E_t^*\left[(\delta_2 S_v^2 - rK)1_{\{S_v^2 > B_2^X(S_v^1, v)\}}\right] dv \tag{2.7}$$

which are defined for a pair of continuous surfaces $\{(B_1^X(S_v^2, v), B_2^X(S_v^1, v) : v \in [t, T], S_v^1 \in \mathbb{R}^+, S_v^2 \in \mathbb{R}^+\}$. An explicit formula for the value of a European max-option in (2.5) is given in Johnson (1981) and Stulz (1982). Explicit expressions for (2.6) and (2.7) in terms of cumulative bivariate normal distributions can also be given.

PROPOSITION 2.7 (Early exercise premium representation for max-options). *The value of an American max-option is given by*

$$C_t^X(S_t^1, S_t^2) = c_t^X(S_t^1, S_t^2) + a_1^X(S_t^1, S_t^2, B_1^X(\cdot, \cdot)) + a_2^X(S_t^1, S_t^2, B_2^X(\cdot, \cdot)), \tag{2.8}$$

where $B_1^X(\cdot, \cdot)$ and $B_2^X(\cdot, \cdot)$ are solutions to the system of recursive integral equations

$$\begin{aligned} B_1^X(S_t^2, t) - K &= c_t^X(B_1^X(S_t^2, t), S_t^2) + a_1^X(B_1^X(S_t^2, t), S_t^2, B_1^X(\cdot, \cdot)) \\ &\quad + a_2^X(B_1^X(S_t^2, t), S_t^2, B_2^X(\cdot, \cdot)) \end{aligned} \tag{2.9}$$

$$\begin{aligned} B_2^X(S_t^1, t) - K &= c_t^X(S_t^1, B_2^X(S_t^1, t)) + a_1^X(S_t^1, B_2^X(S_t^1, t), B_1^X(\cdot, \cdot)) \\ &\quad + a_2^X(S_t^1, B_2^X(S_t^1, t), B_2^X(\cdot, \cdot)) \end{aligned} \tag{2.10}$$

subject to the boundary conditions

$$\lim_{t \uparrow T} B_1^X(S_t^2, t) = \max(B_T^1, S_T^2), \qquad \lim_{t \uparrow T} B_2^X(S_t^1, t) = \max(B_T^2, S_T^1) \tag{2.11}$$

$$B_1^X(0, t) = B_t^1, \qquad B_2^X(0, t) = B_t^2. \tag{2.12}$$

The sum $a_1^X(S_t^1, S_t^2, B_1^X(\cdot, \cdot)) + a_2^X(S_t^1, S_t^2, B_2^X(\cdot, \cdot))$ is the value of the early exercise premium.

The representation (2.8) shows that the value of the American max-option is the value of the European max-option plus the gains from early exercise. These gains have two components corresponding to the gains realized if exercise takes place in $\mathcal{E}_1^X$ or $\mathcal{E}_2^X$. Each component is the present value of the dividends net of the interest rate losses in the event of exercise.

Equations (2.8)–(2.12) have the potential to be used in a numerical valuation procedure, although the implementation may be a challenge. In the single asset case, Broadie and Detemple (1996) have shown that a numerical procedure based on the early exercise premium representation (the "integral method") is competitive with the standard binomial procedure. Boyle, Evnine, and Gibbs (1989) give a multinomial lattice procedure which is very useful for pricing American options on a small number of assets. For higher dimensional problems with a finite number of exercise opportunities Broadie and Glasserman (1994) have proposed a procedure based on Monte Carlo simulation. Alternatively, Dempster (1994) explores the numerical solution of the variational inequality formulation of some American option pricing problems. These methods may offer a practical numerical solution for the max-option using the formulation (2.4) in Proposition 2.6.

For ease of exposition we have focused on max-options on two underlying assets. However, as we show in Section 7, the results above extend to options on the maximum of n assets. Next we show that similar results hold for dual strike options.

American Dual Strike Options

Dual strike options have the payoff function $(\max(S_t^1 - K_1, S_t^2 - K_2))^+$, i.e., they pay the maximum of $S_t^1 - K_1$, $S_t^2 - K_2$, and zero upon exercise at time t. Dual strike options have optimal exercise policies that are similar to options on the maximum of two assets. In particular, there exist two exercise subregions that possess the properties of the subregions for the max-option. In this case, however, immediate exercise prior to maturity is always suboptimal along the translated diagonal $S_t^2 = S_t^1 + K_2 - K_1$.

PROPOSITION 2.8. *Let $\mathcal{E}^D$ represent the immediate exercise region for a dual strike option. Define the subregions $\mathcal{E}_i^D = \mathcal{E}^D \cap \{(S_t^1, S_t^2, t) : S_t^i - K_i = \max(S_t^1 - K_1, S_t^2 - K_2)\}$ for $i = 1, 2$. Then the following properties hold.*

(i) $(S_t^1, S_t^2, t) \in \mathcal{E}^D$ *implies* $(S_t^1, S_t^2, s) \in \mathcal{E}^D$ *for all* $t \leq s \leq T$.
(ii) $(S_t^1, S_t^2, t) \in \mathcal{E}_1^D$ *implies* $(\lambda S_t^1, S_t^2, t) \in \mathcal{E}_1^D$ *for all* $\lambda \geq 1$.
(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}_1^D$ *implies* $(S_t^1, \lambda S_t^2, t) \in \mathcal{E}_1^D$ *for all* $0 \leq \lambda \leq 1$.
(iv) $(S_t^1, 0, t) \in \mathcal{E}_1^D$ *implies* $S_t^1 \geq B_t^1$.
(v) *If* $S_t^2 = S_t^1 + K_2 - K_1$ *and* $\min(S_t^1, S_t^2) > 0$ *and* $t < T$ *then* $(S_t^1, S_t^2, t) \notin \mathcal{E}^D$.
(vi) $(S_t^1, S_t^2, t) \in \mathcal{E}_1^D$ *and* $(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}_1^D$ *implies* $\lambda(S_t^1, S_t^2, t) + (1-\lambda)(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}_1^D$ *for all* $0 \leq \lambda \leq 1$ *(subregion convexity).*

In (ii), (iii), (iv), *and* (vi) *analogous results hold for the subregion* $\mathcal{E}_2^D$.

The price function of the dual strike option can be characterized in terms of variational inequalities as in Proposition 2.6; an early exercise premium representation can also be derived as in Proposition 2.7.

3. AMERICAN SPREAD OPTIONS

A *spread option* is a contingent claim on two underlying assets that has a payoff upon exercise at time t of $(\max(S_t^2 - S_t^1, 0) - K)^+$. The payoff can be written more compactly as $(S_t^2 - S_t^1 - K)^+$. In the special case $K = 0$, the spread option reduces to the option to exchange asset 1 for asset 2. Exchange options were first studied by Margrabe (1978).

Let $C_t^S(S_t^1, S_t^2)$ denote the value of the spread option at time t with asset prices (S_t^1, S_t^2). As before, let B_t^i denote the immediate exercise boundary for a standard option with underlying asset i. Define the immediate exercise region for a spread option by $\mathcal{E}^S \equiv \{(S_t^1, S_t^2, t) : C_t^S(S_t^1, S_t^2) = (S_t^2 - S_t^1 - K)^+\}$.

PROPOSITION 3.1. *Let $\mathcal{E}^S$ represent the immediate exercise region for a spread option. Then $\mathcal{E}^S$ satisfies the following properties.*

(i) $(S_t^1, S_t^2, t) \in \mathcal{E}^S$ *implies* $S_t^2 > S_t^1 + K$.
(ii) $(S_t^1, S_t^2, t) \in \mathcal{E}^S$ *implies* $(S_t^1, S_t^2, s) \in \mathcal{E}^S$ *for all* $t \leq s \leq T$.
(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}^S$ *implies* $(S_t^1, \lambda S_t^2, t) \in \mathcal{E}^S$ *for all* $\lambda \geq 1$.
(iv) $(\lambda S_t^1, S_t^2, t) \in \mathcal{E}^S$ *for all* $0 \leq \lambda \leq 1$.

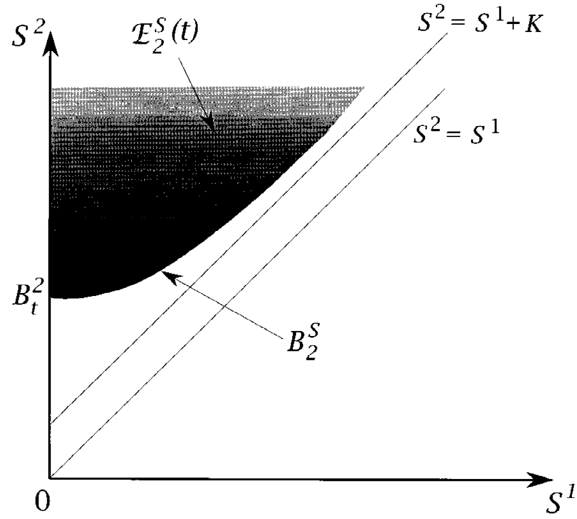

FIGURE 3.1. Illustration of $\mathcal{E}^S(t)$ for a spread option at time t with $t < T$.

(v) $(0, S_t^2, t) \in \mathcal{E}^S$ *implies* $S_t^2 \geq B_t^2$; $S_t^2 \geq B_t^2$ *and* $S_t^1 = 0$ *implies* $(0, S_t^2, t) \in \mathcal{E}^S$.

(vi) $(S_t^1, S_t^2, t) \in \mathcal{E}^S$ *and* $(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}^S$ *implies* $(S_t^1(\lambda), S_t^2(\lambda), t) \in \mathcal{E}^S$ *for all* $0 \leq \lambda \leq 1$, *where* $S_t^i(\lambda) = \lambda S_t^i + (1-\lambda)\tilde{S}_t^i$ *for* $i = 1, 2$.

Property (i) in Proposition 3.1 follows since immediate exercise at $S^2 \leq S^1 + K$ is dominated by any waiting policy which has a positive probability of giving a strictly positive payoff at some fixed future date. This property implies that the exercise region for the spread option can be thought of as a one-sided version of the exercise region for the max-option. The intuition behind properties (ii)–(vi) parallels the corresponding properties for the max-option. An illustration of the exercise region is given in Figure 3.1.

The price of the spread option can also be characterized in terms of variational inequalities as in Proposition 2.6. This characterization leads to the following early exercise premium representation of the value of the spread option. Define

$$c_t^S(S_t^1, S_t^2) = E_t^*\left[e^{-r(T-t)}(S_T^2 - S_T^1 - K)^+\right] \tag{3.1}$$

which represents the value of the European spread option and the function

$$a_2^S(S_t^1, S_t^2) = \int_{v=t}^{T} e^{-r(v-t)} E_t^*\left[(\delta_2 S_v^2 - \delta_1 S_v^1 - rK) 1_{\{S_v^2 > B_2^S(S_v^1, v)\}}\right] dv \tag{3.2}$$

which is defined for a continuous surface $\{B_2^S(S_v^1, v) : v \in [t, T], S_v^1 \in \mathbb{R}^+\}$.

PROPOSITION 3.2 (Early exercise premium representation for spread options). *The value of an American spread option is given by*

$$C_t^S(S_t^1, S_t^2) = c_t^S(S_t^1, S_t^2) + a_2^S(S_t^1, S_t^2, B_2^S(\cdot, \cdot)), \tag{3.3}$$

where $B_2^S(\cdot, \cdot)$ *is a solution to the integral equation*

$$B_2^S(S_t^1, t) - K = c_t^S(S_t^1, B_2^S(S_t^1, t)) + a_2^S(S_t^1, B_2^S(S_t^1, t), B_2^S(\cdot, \cdot)) \tag{3.4}$$

subject to the boundary conditions

$$\lim_{t \uparrow T} B_2^S(S_t^1, t) = \max\left(\frac{\delta_1}{\delta_2} S_T^1 + \frac{r}{\delta_2} K, S_T^1 + K\right) \tag{3.5}$$

$$B_2^S(0, t) = B_t^2. \tag{3.6}$$

Here $a_2^S(S_t^1, S_t^2, B_2^S(\cdot, \cdot))$ *is the value of the early exercise premium.*

American Options to Exchange One Asset for Another

When $K = 0$ the spread option becomes an American option to exchange one asset for another with payoff $(S_t^2 - S_t^1)^+$ upon exercise. This payoff can also be written as

$$(S_t^2 - S_t^1)^+ = S_t^1(R_t - 1)^+$$

where $R_t \equiv S_t^2/S_t^1$. Hence the exchange option can be thought of as S_t^1 options on an asset with price R and exercise price one. Of course, prior to the exercise date the random number of options S_t^1 is unknown. The next proposition summarizes important properties of the optimal exercise region for exchange options. Some of these properties are specific to exchange options and do not follow from Proposition 3.1. See Figure 3.2 for an illustration.

PROPOSITION 3.3. *Let* $\mathcal{E}^E$ *denote the optimal exercise region for an exchange option. Then* $\mathcal{E}^E$ *satisfies*

(i) $(S_t^1, S_t^2, t) \in \mathcal{E}^E$ *implies* $R_t > 1$
(ii) $(S_t^1, S_t^2, t) \in \mathcal{E}^E$ *implies* $(S_t^1, \lambda S_t^2, t) \in \mathcal{E}^E$ *for* $\lambda \geq 1$ *(up connectedness)*
(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}^E$ *implies* $(\lambda S_t^1, \lambda S_t^2, t) \in \mathcal{E}^E$ *for* $\lambda > 0$ *(ray connectedness)*
(iv) $S^1 = 0$ *implies immediate exercise is optimal for all* $S^2 > 0$.

Properties (i) and (ii) are particular cases of (i) and (iii) of Proposition 3.1. Property (iii) is new and states that if immediate exercise is optimal at a point (S^1, S^2) then it is optimal at every point of the ray connecting the origin to (S^1, S^2). This feature of the optimal exercise region is a consequence of the homogeneity of degree one of the payoff function with respect to (S^1, S^2). Properties (i)–(iii) imply that there exists $B^E(t) > 1$ such that

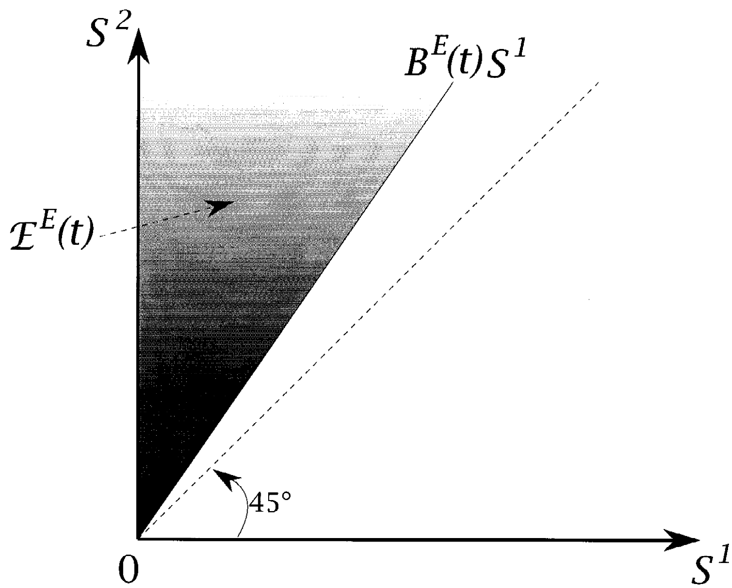

FIGURE 3.2. Illustration of $\mathcal{E}^E(t)$ for an American exchange option.

immediate exercise is optimal for all $S_t^1 > 0$ when $R_t \geq B^E(t)$. Hence, immediate exercise is optimal when $S_t^2 \geq B^E(t) S_t^1$ for all $S_t^1 \in \mathbb{R}^+$ and all $t \in [0, T]$. Property (iv) follows from (v) in Proposition 3.1 by noting that $B^2(t) = 0$ when $K = 0$.

Recall now that the price processes satisfy (2.1) and (2.2) and that the quadratic covariation process between z^1 and z^2 is $d[z^1, z^2]_t = \rho dt$. By Itô's lemma $R_t \equiv S_t^2 / S_t^1$ has the dynamics

$$dR_t = R_t[(r - \delta_R)dt + \sigma_R dz_t^R]$$

where $\delta_R \equiv \delta_2 + r - \delta_1 - \sigma_1^2 + \rho\sigma_1\sigma_2$, $\sigma_R^2 \equiv \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, and $dz_t^R = [\sigma_2 dz_t^2 - \sigma_1 dz_t^1]/\sigma_R$. The next proposition provides a valuation formula for the American exchange option. Rubinstein (1991) originally showed how the valuation of American exchange options could be simplified to the case of a single underlying asset in a binomial tree setting.

PROPOSITION 3.4 (Early exercise premium representation for exchange options). *The value of the American option to exchange one asset for another, with payoff $(S_t^2 - S_t^1)^+$ at the exercise date, is given by*

$$(3.7)\quad C^E(S^1, S^2, t) = c^E(S^1, S^2, t) + \int_t^T \delta_2 S_t^2 e^{-\delta_2(v-t)} N(-b(R_t, B_v^E, v-t, \delta_1 - \delta_2, \sigma_R))dv - \int_t^T \delta_1 S_t^1 e^{-\delta_1(v-t)} N(-b(R_t, B_v^E, v-t, \delta_1 - \delta_2, \sigma_R) - \sigma_R\sqrt{v-t})dv$$

where $c^E(S^1, S^2, t) \equiv E_t^*[e^{-r(T-t)}(S_T^2 - S_T^1)^+]$ *is the value of the European exchange option and*

$$(3.8) \quad b(R_t, B_v^E, v-t, \delta_1 - \delta_2, \sigma_R) \equiv \left[\log\left(\frac{B_v^E}{R_t}\right) - (\delta_1 - \delta_2 + \tfrac{1}{2}\sigma_R^2)(v-t)\right] \times \frac{1}{\sigma_R\sqrt{v-t}}.$$

The optimal exercise boundary $B^E(\cdot)$ *solves the recursive integral equation*

$$(3.9)\ B_t^E - 1 = c^E(1, B_t^E, t) + \int_t^T \delta_2 B_t^E e^{-\delta_2(v-t)} N(-b(B_t^E, B_v^E, v-t, \delta_1-\delta_2, \sigma_R))dv - \int_t^T \delta_1 e^{-\delta_1(v-t)} N(-b(B_t^E, B_v^E, v-t, \delta_1-\delta_2, \sigma_R) - \sigma_R\sqrt{v-t})dv$$

with boundary condition $B_T^E = \frac{\delta_1}{\delta_2} \vee 1$.

Formulas (3.7)–(3.9) reveal that the American exchange option with payoff $(S_t^2 - S_t^1)^+$ has the same value at time t as S_t^1 American options with exercise prices 1 on a *single* asset with value R_t, dividend rate δ_2, and volatility σ_R, in a financial market with interest rate δ_1.

Options on the Product with Random Exercise Price

This type of contract, which has a payoff of $(S_t^1 S_t^2 - K S_t^1)^+$, is an option to exchange one asset for another where the value of the asset to be received is a product of two prices. An example is an option on the Nikkei index with an exercise price (K) quoted in Japanese yen (see Dravid, Richardson, and Sun 1993). Then S_t^2 is the yen-value of the Nikkei, S_t^1 represents the \$/Y exchange rate, and K is the yen-exercise price. The payoff can also be written as

$$S_t^1(S_t^2 - K)^+.$$

Upon exercise, the contract produces a random number times the payoff on an option written on the asset S^2 only. When $\delta_P \equiv \delta_1 + \delta_2 - r - \rho\sigma_1\sigma_2$ equals zero, early exercise is suboptimal. When $\delta_P > 0$, the properties of the immediate exercise region can be inferred from Proposition 3.3 by replacing (S^1, S^2, R) by $(KS^1, S^1S^2, S^2/K)$. Replacing $(\delta_1, \delta_2, \delta_R, \sigma_1, \sigma_2, \sigma_R)$ in (3.7)–(3.9) by $(\delta_1, \delta_P, \delta_2, \sigma_1, \sigma_P, \sigma_2)$, together with the previous substitutions, produces a valuation formula and a recursive integral equation for the optimal exercise boundary.

4. AMERICAN EXCHANGE OPTIONS WITH PROPORTIONAL CAPS

This contract has a payoff equal to $(S^2 - S^1)^+ \wedge LS^1$ where $L > 0$. An example is a capped call option on an index or an asset which is traded on a foreign exchange or issued in a foreign currency. In the currency of reference the contract payoff is $(S - K)^+ \wedge L'$ where

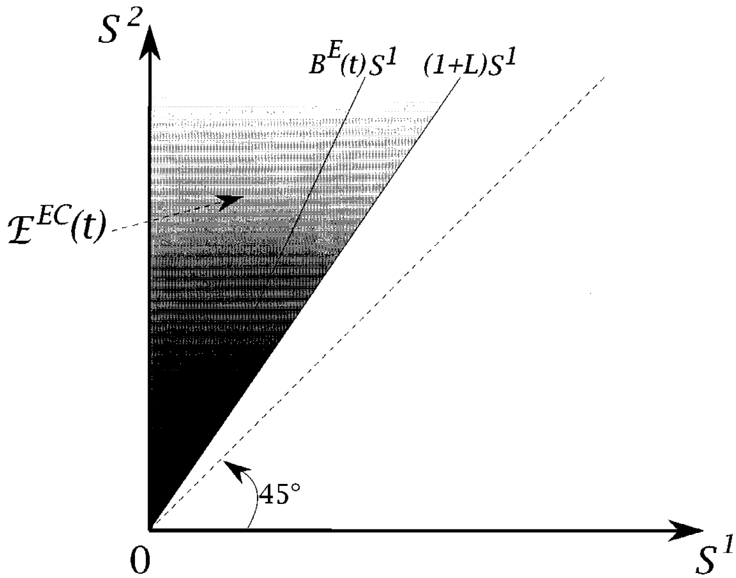

FIGURE 4.1. Exercise region for an American exchange option with a proportional cap.

S is the price of the asset in the foreign currency, K is the exercise price, and L' is the cap. From the perspective of a U.S. investor the payoff equals $e(S-K)^+ \wedge L'e$ or equivalently $(eS-Ke)^+ \wedge L'e$. With the identification $S^2 = eS$, $S^1 = Ke$, and $L = L'/K$ we obtain the payoff structure of an exchange option with a proportional cap.

Since the payoff of an exchange option with a proportional cap is nonconvex (and since the derivative of the payoff is discontinuous at the cap), the approach that derives the exercise boundary from the standard integral representation of the early exercise premium does not apply. However, it is still possible to identify the exercise boundary explicitly and to derive a valuation formula by using dominance arguments. Proposition 4.1 gives a characterization of the exercise boundary. See Figure 4.1 for an illustration.

PROPOSITION 4.1. *The immediate exercise boundary for an American exchange option with a proportional cap LS^1 is given by*

$$S_t^2 \geq B^{EC}(t)S_t^1 \equiv B^E(t)S_t^1 \wedge (1+L)S_t^1,$$

i.e., the immediate exercise boundary is the minimum of the exercise boundary for a standard uncapped exchange option ($B^E(t)S_t^1$) and the cap plus S^1.

Since the option payoff is bounded above by $(S^2 - S^1)^+ \wedge LS^1$ it is easy to verify that the option price is bounded above by the minimum of the price of an uncapped American exchange option $C^E(S^1, S^2, t)$ and LS_t^1. The optimality of immediate exercise when $S_t^2 \geq B^E(t)S_t^1 \wedge (1+L)S_t^1$ follows. If $S_t^2 < B^E(t)S_t^1 \wedge (1+L)S_t^1$ and $1 + L > (\delta_1/\delta_2) \vee 1$ it is always possible to find an uncapped exchange option with shorter maturity, T_0, whose optimal exercise boundary $B^E(t; T_0)$ lies below $(1+L)$ today and at all times s, $t \leq s \leq T_0$

and is greater than the ratio S_t^2/S_t^1 at date t. Hence the optimal exercise strategy of this short maturity exchange option is implementable for the holder of the capped exchange option. It follows that

$$C^{EC}(S^1, S^2, t) \geq C^E(S^1, S^2, t; T_0).$$

Since immediate exercise is suboptimal for the T_0-maturity option when $S_t^2 < B^E(t)S_t^1$, it is also suboptimal for the capped exchange option. If $S_t^2 < B^E(t)S_t^1 \wedge (1+L)S_t^1$ and $1 + L \leq (\delta_1/\delta_2) \vee 1$ immediate exercise is dominated by the strategy of exercising at the cap. This follows since the difference between these two strategies is the negative cash flows $\delta_2 S_v^2 - \delta_1 S_v^1$ on the event $\{t \leq v \leq \tau_L\}$, where τ_L is the hitting time of the cap (see equation (4.1) below). This proves Proposition 4.1.

PROPOSITION 4.2. *The value of an American exchange option with proportional cap is given by*

$$C^{EC}(S^1, S^2, t) = L\, E_t^*\left[e^{-r(\tau_L - t)} S_{\tau_L}^1\, 1_{\{\tau_L < t^*\}}\right] + E_t^*\left[e^{-r(t^* - t)} C^E(S_{t^*}^1, S_{t^*}^2, t^*)\, 1_{\{\tau_L \geq t^*\}}\right]$$

for $t \leq \tau_L \wedge t^*$ *where*

$$\tau_L \equiv \inf\{v \in [0, T] : S_v^2 = (1+L)S_v^1\}, \tag{4.1}$$

or $\tau_L = T$ *if no such time exists in* $[0, T]$*, and where* t^* *is the solution to the equation*

$$B^E(t) = 1 + L$$

if a solution exists. If $B^E(t) > 1 + L$ *for all* $t \in [0, T]$ *set* $t^* = T$*; if* $B^E(t) < 1 + L$ *for all* $t \in [0, T]$ *set* $t^* = 0$.

The proposition above provides a representation of the option value in terms of the value of an uncapped American exchange option and the payoff at the cap. We now seek to establish another decomposition of the option price which emphasizes the early exercise premium relative to an exchange option with automatic exercise at the cap.

Proposition 4.1 shows that immediate exercise is optimal when $S^2 \geq (1 + L)S^1$. Hence for $t < \tau_L$, the value of the American capped exchange option can also be written as

$$C^{EC}(S^1, S^2, t) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t^*\left[e^{-r(\tau_L \wedge \tau - t)}(S_{\tau_L \wedge \tau}^2 - S_{\tau_L \wedge \tau}^1)^+\right],$$

where $\mathcal{S}_{t,T}$ is the set of stopping times taking values in $[t, T]$. Thus, the American capped exchange option has the same value as an exchange option with automatic exercise at the cap that can be exercised prior to reaching the cap at the option of the holder of the contract.

The value function for this stopping time problem solves the variational inequality

$$\begin{cases} C^{EC}(S^1,S^2,t) \geq (S^2-S^1)^+, \ \dfrac{\partial C^{EC}}{\partial t} + \mathcal{L}C^{EC} \leq 0 & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^+ \cap \{(S^1,S^2)\colon \\ & \quad S^2 < (1+L)S^1\} \\ \left(\dfrac{\partial C^{EC}}{\partial t} + \mathcal{L}C^{EC}\right)((S^2-S^1)^+ - C^{EC}) = 0 & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^+ \cap \{(S^1,S^2)\colon \\ & \quad S^2 < (1+L)S^1\} \\ C^{EC}(S^1,S^2,T) = (S^2-S^1)^+ & \text{at} \quad t = T \\ C^{EC}(S^1,S^2,t) = LS^1 & \text{on} \quad S^2 = (1+L)S^1 \end{cases}$$

defined on the domain $\mathbb{R}^+ \times \mathbb{R}^+ \cap \{(S^1,S^2)\colon \ S^2 < (1+L)S^1\}$.

Consider now a capped exchange option with automatic exercise at the cap. The value of this contract is

$$C^{EL} = E_t^*[e^{-r(\tau_L - t)}(S^2_{\tau_L} - S^1_{\tau_L})^+] \tag{4.2}$$

for $t < \tau_L$, where τ_L is the stopping time defined in (4.1). Define the function

$$u(S^1,S^2,t) \equiv C^{EC}(S^1,S^2,t) - C^{EL}(S^1,S^2,t) \tag{4.3}$$

which represents the early exercise premium of the American capped exchange option over the capped option with automatic exercise at the cap. It is easy to show that (4.3) satisfies

$$\begin{cases} u \geq 0, \quad \dfrac{\partial u}{\partial t} + \mathcal{L}u \leq 0 & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^+ \cap \{(S^1,S^2)\colon \ S^2 < (1+L)S^1\} \\ \left(\dfrac{\partial u}{\partial t} + \mathcal{L}u\right)[(S^2-S^1)^+ - C^{EL} - u] = 0 & \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^+ \cap \{(S^1,S^2)\colon \ S^2 < (1+L)S^1\} \\ u(S^1,S^2,T) = 0 & \text{at} \quad t = T \\ u(S^1,S^2,t) = 0 & \text{on} \quad S^2 = (1+L)S^1. \end{cases}$$

An application of Itô's lemma enables us to prove the following representation formula.

PROPOSITION 4.3. *The value of an American capped exchange option has the representation*

$$C^{EC}(S^1,S^2,t) = C^{EL}(S^1,S^2,t) + E_t^*\left[\int_t^{\tau_L} e^{-r(v-t)}(\delta_2 S_v^2 - \delta_1 S_v^1) 1_{\{S_v^2 \geq B_v^{EC} S_v^1\}} dv\right] \tag{4.4}$$

for $t \leq \tau_L$, where $C^{EL}(S^1,S^2,t)$ represents the value of a capped exchange option with automatic exercise at the cap defined in (4.2). In (4.4) $\tau_L \equiv \inf\{v \in [0,T] : S_v^2 = (1+L)S_v^1\}$ or $\tau_L = T$ if no such v exists in $[0,T]$. The exercise boundary $B^{EC} \equiv \{B^{EC}(t), t \in [0,T]\}$

satisfies the recursive integral equation

$$(4.5)\quad S_t^1(B^{EC}(t)-1) = C^{EL}(S_t^1, S_t^1 B^{EC}(t), t) + E_t^*\left[\int_t^{\tau_L(t)} e^{-r(v-t)}(\delta_2 S_v^2 - \delta_1 S_v^1)1_{\{S_v^2 \geq B_v^{EC} S_v^1\}} dv\right]\Bigg|_{S_t^2 = S_t^1 B^{EC}(t)}$$

$$(4.6)\quad B^{EC}(T) = \left(1 \vee \frac{\delta_1}{\delta_2}\right) \wedge (1+L)$$

where $\tau_L(t) \equiv \inf\{v \in [t,T] : S_v^2 \geq (1+L)S_v^1\}$ *or* $\tau_L(t) = T$ *if no such time exists in* $[t,T]$.

It is easy to verify that the solution to the recursive integral equation (4.5) subject to (4.6) is the optimal exercise strategy $B^{EC} = B^E \wedge (1+L)$ of Proposition 4.1. Indeed, by the optional sampling theorem, the value of the uncapped exchange option can also be written as

$$C^E(S_t^1, S_t^2, t) = E_t^*\left[e^{-r(\tau^*-t)}(S_{\tau^*}^2 - S_{\tau^*}^1)\right] + E_t^*\left[\int_t^{\tau^*} e^{-r(v-t)}(\delta_2 S_v^2 - \delta_1 S_v^1)1_{\{S_v^2 \geq B^E(v) S_v^1\}} dv\right]$$

for any stopping time $\tau^* \in \mathcal{S}_{t,T}$ such that $\tau^* \geq \tau_{B^E} \equiv \inf\{v \in [0,T] : S_v^2 = B^E(v)S_v^1\}$ (or T if no such v exists in $[0,T]$). In particular if $t < \tau_L \wedge \tau_{B^E}$ and $\tau_{B^E} \leq \tau_L$ we can select $\tau^* = \tau_L$ to obtain a representation of the American exchange option which is similar to equation (4.4). Hence, as long as $B_t^E \leq 1+L$ and $t < \tau_L(t)$, newly issued capped and uncapped exchange options have the same representation. It follows that $(B_s^{EC}, s \in [t,T])$ and $(B_s^E, s \in [t,T])$ solve the same recursive equation subject to the same boundary condition. If $t \leq t^*$ we know that $B_t^E \geq 1+L$. Substitute $B^{EC}(t) \equiv 1+L$ in equation (4.5). At the point $S_t^2 = S_t^1(1+L)$, we have $C^{EL}(S^1, S^1(1+L), t) = LS_t^1$ and $\tau_L(t) = t$. It follows that the right-hand side of (4.5) equals LS_t^1. Hence $B^{EC}(t) = 1+L$ solves (4.5) when $t \leq t^*$.

The representation formula (4.4) differs from the standard early exercise premium representation since it relates the value of the option to a contract that expires when the asset price reaches the cap.

By setting $S^1 = K$ (i.e., $S_0^1 = K, \delta_1 = r, \sigma_1 = 0$) the American capped exchange option reduces to a capped option on a single underlying asset with exercise price K (see Broadie and Detemple 1995).[2] Proposition 4.3 then provides a new representation for an American capped call option (on a single underlying asset) in terms of the value of a capped call option with automatic exercise at the cap and of an early exercise premium. It also provides a recursive integral equation for the optimal exercise boundary of American capped options.

[2]In Broadie and Detemple (1995) the payoff on a capped option is written as $(S \wedge L' - K)^+$. This is equivalent to $(S-K)^+ \wedge (L'/K - 1)K$. Hence a cap of L in the analysis above corresponds to $L' = (1+L)K$ in our previous notation.

5. AMERICAN OPTIONS ON THE PRODUCT AND POWERS OF THE PRODUCT OF TWO ASSETS

In this section we consider options which are "essentially" written on the product of two assets. For instance, if S^1 and S^2 are the underlying asset prices the payoffs under consideration are

(i) product option: $(S_t^1 S_t^2 - K)^+ \equiv (P_t - K)^+$ where $P_t \equiv S_t^1 S_t^2$.
(ii) power-product option: $(P_t^\gamma - K)^+$ for some $\gamma > 0$.

Note that power-product options include as a special case product options ($\gamma = 1$) and options on a geometric average of assets ($\gamma = \frac{1}{2}$).

Define $Y_t \equiv P_t^\gamma \equiv (S_t^1 S_t^2)^\gamma$. An application of Itô's lemma yields

$$dY_t = Y_t[(r - \delta_Y)dt + \sigma_Y dz_t^P] \tag{5.1}$$

where $\delta_Y = \delta_P + (1-\gamma)(r - \delta_P + \frac{1}{2}\sigma_P^2)$, $\sigma_Y = \gamma\sigma_P = \gamma(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)^{\frac{1}{2}}$, $\delta_P = \delta_1 + \delta_2 - r - \rho\sigma_1\sigma_2$, and $dz_t^P = \frac{1}{\sigma_P}[\sigma_1 dz_t^1 + \sigma_2 dz_t^2]$. In the remainder of this section, we assume $\delta_Y \geq 0$. Now consider an American option on the single asset Y. Let $B_t(\delta_Y, \sigma_Y^2)$ denote its optimal exercise boundary and $C_t(Y_t)$ its value.

PROPOSITION 5.1. *The optimal exercise boundary for an American power-product option is*

$$B^{PP}(S_t^1, t) = \frac{(B_t)^{1/\gamma}}{S_t^1} \tag{5.2}$$

where $B_t = B_t(\delta_Y, \sigma_Y^2)$ is the exercise boundary on a standard American call option written on an asset whose price is Y satisfies (5.1). The power-product option value is

$$C^{PP}(S_t^1, S_t^2, t) = C_t(Y_t). \tag{5.3}$$

where $C_t(Y_t)$ is the American call option value on the single asset Y.

The shaded region in Figure 5.1 illustrates the exercise region for an American product option with $\gamma = 1$.

REMARK 5.1.

(i) If $\gamma = 1$ we get $\delta_Y = \delta_P$ and $\sigma_Y = \sigma_P$. In this case we recover the American option on a product of two assets.
(ii) If $\gamma = \frac{1}{2}$ we get $\delta_Y = \frac{1}{2}(\delta_P + r) + \frac{1}{8}\sigma_P^2$ and $\sigma_Y = \frac{1}{2}\sigma_P$. In this case we recover the American option on a geometric average of two asset prices.

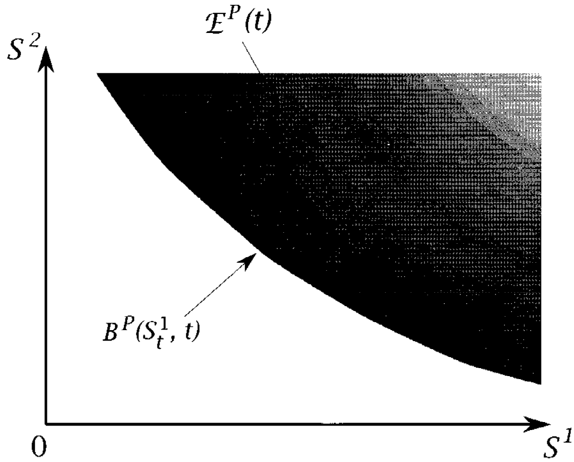

FIGURE 5.1. Illustration of the exercise region for a product option ($\gamma = 1$) at time t with $t < T$.

6. OPTIONS ON THE ARITHMETIC AVERAGE OF TWO ASSETS

We now consider American options which are written on an arithmetic average of assets.[3] For simplicity we focus on the case of two underlying assets. Consider an option with payoff $(\frac{1}{2}(S_t^1 + S_t^2) - K)^+$ upon exercise. The next proposition gives properties of the optimal exercise region.

PROPOSITION 6.1. *Let $\mathcal{E}^\Sigma$ denote the optimal exercise region. Then*

(i) $(0, S_t^2, t) \in \mathcal{E}^\Sigma$ *implies* $S_t^2 \geq 2B_t^2$ *where* B_t^2 *is the exercise boundary on* S^2*-option.*
(ii) $(S_t^1, 0, t) \in \mathcal{E}^\Sigma$ *implies* $S_t^1 \geq 2B_t^1$ *where* B_t^1 *is the exercise boundary on* S^1*-option.*
(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}^\Sigma$ *implies* $(\lambda_1 S_t^1, \lambda_2 S_t^2, t) \in \mathcal{E}^\Sigma$ *with* $\lambda_1 \geq 1$, $\lambda_2 \geq 1$ *(NE connectedness).*
(iv) $(S_t^1, S_t^2, t) \in \mathcal{E}^\Sigma$ *and* $(\tilde{S}_t^1, \tilde{S}_t^2, t) \in \mathcal{E}^\Sigma$ *implies* $(\lambda S_t^1 + (1-\lambda)\tilde{S}_t^1, \lambda S_t^2 + (1-\lambda)\tilde{S}_t^2) \in \mathcal{E}^\Sigma$ *(convexity).*
(v) $(S_t^1, S_t^2, t) \in \mathcal{E}^\Sigma$ *implies* $(S_t^1, S_t^2, s) \in \mathcal{E}^\Sigma$ *for* $T \geq s \geq t$.

Properties (i), (ii), (iv), and (v) are intuitive. Property (iii) states that the exercise region is connected in the northeast direction. Indeed, for $\lambda_1 > 1$ and $\lambda_2 > 1$ the payoff $(\frac{1}{2}(\lambda_1 S_t^1 + \lambda_2 S_t^2) - K)^+$ is bounded above by

$$(\tfrac{1}{2}(S_t^1 + S_t^2) - K)^+ + \tfrac{1}{2}((\lambda_1 - 1)S_t^1 + (\lambda_2 - 1)S_t^2).$$

It follows that the option value at $(\lambda_1 S_t^1, \lambda_2 S_t^2, t)$ is bounded above by the option value at (S_t^1, S_t^2, t) plus $\frac{1}{2}((\lambda_1 - 1)S_t^1 + (\lambda_2 - 1)S_t^2)$. For an illustration of the exercise region see Figure 6.1.

[3] An example of a related contract is the American option on the value-weighted S&P 100 index which has traded on the CBOE since 1983. The underlying stocks, however, pay dividends at discrete points in time.

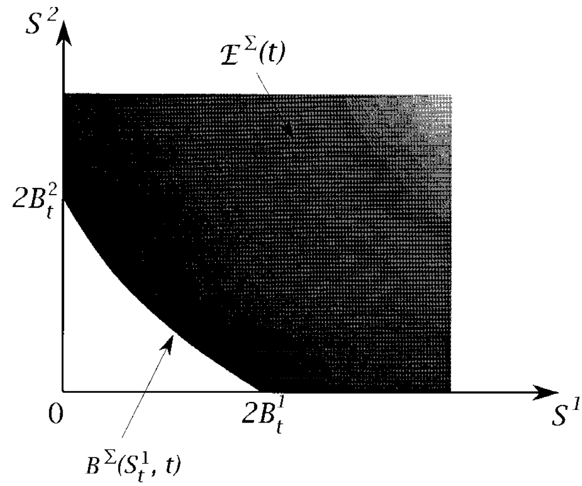

FIGURE 6.1. Illustration of the exercise region for an arithmetic average option at time t with $t < T$.

The next proposition provides a valuation formula for an American arithmetic average option.

PROPOSITION 6.2 (Early exercise premium representation for arithmetic average options). *The value of the American option on the arithmetic average of 2 assets is*

$$\begin{aligned} C^{\Sigma}(S^1, S^2, t) &= c^{\Sigma}(S^1, S^2, t) \\ &\quad + \int_t^T \tfrac{1}{2}\delta_1 S_t^1 e^{-\delta_1(v-t)} \tilde{\Phi}(S_t^2, B^{\Sigma}(\cdot, v), v-t, 0, \sigma_1\sqrt{v-t})dv \\ &\quad + \int_t^T \tfrac{1}{2}\delta_2 S_t^2 e^{-\delta_2(v-t)} \tilde{\Phi}(S_t^2, B^{\Sigma}(\cdot, v), v-t, \\ &\qquad\qquad \sigma_2\sqrt{1-\rho_{21}^2}\sqrt{v-t}, \sigma_2\rho_{21}\sqrt{v-t})dv \\ &\quad - \int_t^T rKe^{-r(v-t)}\ \tilde{\Phi}(S_t^2, B^{\Sigma}(\cdot, v), v-t, 0, 0)dv \end{aligned}$$

where $c^{\Sigma}(S^1, S^2, t)$ denotes the value of the European option on the arithmetic average of two assets and $\tilde{\Phi}(S_t^2, B^{\Sigma}(\cdot, v), v-t, x, y) \equiv \int_{-\infty}^{+\infty} n(w-y)N(-d(S_t^2, B^{\Sigma}(S_v^1(w), v), v-t, \rho, w)-x)dw$ and where $S_v^1(w) = S_t^1 \exp[(r-\delta_1-\frac{1}{2}\sigma_1^2)(v-t)+\sigma_1 w\sqrt{v-t}]$.

The optimal exercise boundary $B^{\Sigma}(S_t^1, t)$ solves

$$\begin{aligned} \tfrac{1}{2}(S_t^1 + B^{\Sigma}(S_t^1, t)) - K &= c^{\Sigma}(S_t^1, B^{\Sigma}(S_t^1, t), t) + \pi_t(S_t^1, B^{\Sigma}(S_t^1, t), t), \quad t \in [0, T] \\ \tfrac{1}{2}(\delta_1 S_T^1 + \delta_2 B^{\Sigma}(S_T^1, T)) &= rK \vee (\delta_2 K + \tfrac{1}{2}(\delta_1 - \delta_2)S_T^1 \\ B^{\Sigma}(2B_t^1, t) &= 0 \\ B^{\Sigma}(0, t) &= 2B_t^2, \quad t \in [0, T) \end{aligned}$$

where $\pi_t(S^1, S^2, t)$ denotes the early exercise premium.

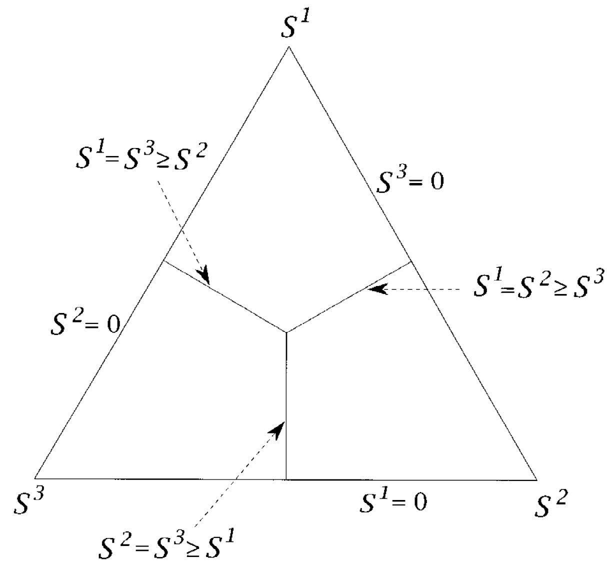

FIGURE 7.1.

7. AMERICAN OPTIONS WITH $n > 2$ UNDERLYING ASSETS

In this section we treat the case of American options with $n > 2$ underlying assets. We focus on the option on the maximum of n assets; optimal exercise policies and valuation formulas for other contracts, such as dual strike options and spread options, written on n assets can be deduced using similar arguments.

We use the following notation: $\mathcal{E}^{X,n}$ denotes the optimal exercise region for the max-option on n assets, $C^{X,n}$ is the corresponding price, $S \equiv (S^1, \ldots, S^n)$ denotes the vector of underlying asset prices, and $\mathcal{G}_i^{X,n} \equiv \{(S, t) : S_t^i = \max(S^1, \ldots, S^n)\}$ for $i = 1, \ldots, n$. Our first result parallels Proposition 2.1 of Section 2.

PROPOSITION 7.1. *If* $\max(S^1, \ldots, S^n) = S^i = S^j$ *for* $i \neq j$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$ *and if* $t < T$ *then* $(S, t) \notin \mathcal{E}^{X,n}$. *That is, prior to maturity immediate exercise is suboptimal if the maximum is achieved by two or more asset prices.*

Proposition 7.1 states that immediate exercise is suboptimal on all regions where the maximum asset price is achieved by two or more asset prices. The intuition for the result is straightforward. It is clear that $C^{X,n}(S, t) \geq C^{X,2}(S^i, S^j, t)$ where $C^{X,2}(S^i, S^j, t)$ is the value of an American option on the maximum of S^i and S^j. The result follows since immediate exercise of this option is suboptimal when $S^i = S^j$ (see Proposition 2.1). When $n = 3$ these regions are the two-dimensional semiplanes connecting the diagonal $(S^1 = S^2 = S^3)$ to the diagonals in the subspaces spanned by two prices $((S^1 = S^2, S^3 = 0)$, $(S^1 = S^3, S^2 = 0)$, $(S^2 = S^3, S^1 = 0))$. There are three such semiplanes. Figure 7.1 graphs the trace of these semiplanes on a simplex whose vertices lie on the three axes S^1, S^2, and S^3.

Figure 7.2 graphs the trace of the optimal exercise sets on this simplex. In the upper portion of the triangle, above the segments of line $S^1 = S^2 \geq S^3$ and $S^1 = S^3 \geq S^2$ the

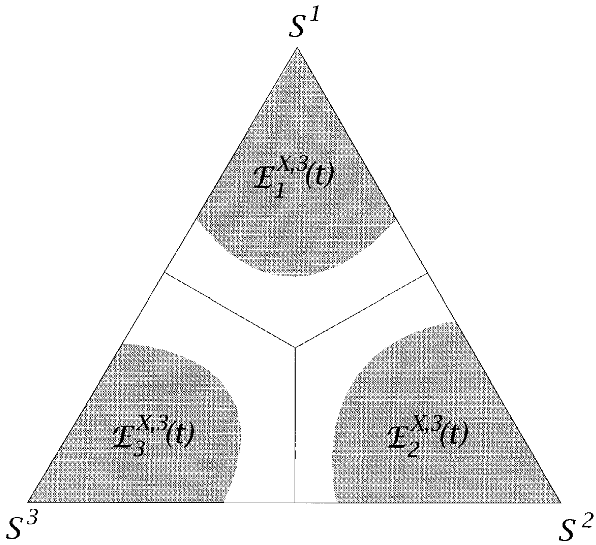

FIGURE 7.2.

maximum is achieved by S^1. Hence, $\mathcal{E}_1^{X,3}$ lies in this region. Similarly $\mathcal{E}_2^{X,3}$ lies in the lower right corner and $\mathcal{E}_3^{X,3}$ in the lower left corner with vertex S^3. The structure of these sets and in particular their convexity follows from our next propositions.

PROPOSITION 7.2 (Subregion Convexity). *Consider two vectors* $S \in \mathbb{R}_+^n$ *and* $\tilde{S} \in \mathbb{R}_+^n$. *Suppose that* $(S, t) \in \mathcal{E}_i^{X,n}$ *and* $(\tilde{S}, t) \in \mathcal{E}_i^{X,n}$ *for the same* $i \in \{1, \ldots, n\}$. *Given* λ *with* $0 \leq \lambda \leq 1$ *denote* $S(\lambda) \equiv \lambda S + (1 - \lambda)\tilde{S}$. *Then* $(S(\lambda), t) \in \mathcal{E}_i^{X,n}$. *That is, if immediate exercise is optimal at* S *and* $\tilde{S}$ *and if* $(S, t) \in \mathcal{G}_i^{X,n}$ *and* $(\tilde{S}, t) \in \mathcal{G}_i^{X,n}$ *then immediate exercise is optimal at* $S(\lambda)$.

PROPOSITION 7.3. $\mathcal{E}^{X,n}$ *satisfies the following properties.*

(i) $(S, t) \in \mathcal{E}^{X,n}$ *implies* $(S, s) \in \mathcal{E}^{X,n}$ *for all* $t \leq s \leq T$;
(ii) $(S, t) \in \mathcal{E}_i^{X,n}$ *implies* $(S^1, \ldots, \lambda S^i, \ldots, S^n, t) \in \mathcal{E}_i^{X,n}$ *for all* $\lambda \geq 1$;
(iii) $(S, t) \in \mathcal{E}_i^{X,n}$ *implies* $(\lambda^1 S^1, \lambda^2 S^2, \ldots, S^i, \lambda^{i+1} S^{i+1}, \ldots, \lambda^n S^n) \in \mathcal{E}_i^{X,n}$ *for all* $0 \leq \lambda^j \leq 1, \quad j = 1, \ldots, i-1, i+1, \ldots, n$;
(iv) $S_t^i = 0$ *and* $(S, t) \in \mathcal{E}_i^{X,n}$ *implies* $(S^1, \ldots, S^{i-1}, S^{i+1}, \ldots, S^n, t) \in \mathcal{E}_i^{X,n-1}$.

The proof of these results parallels the proofs of Propositions 2.2 and 2.3 for the case of two underlying assets. Combining Propositions 7.1, 7.2, and 7.3 we see that the properties of the max-option with two underlying assets extend naturally to the case of n underlying assets. Similarly, the characterizations of the price function in Propositions 2.5, 2.6, and 2.7 can be extended in a straightforward manner to the max-option written on n underlying assets.

8. CONCLUSIONS

In this paper we have identified the optimal exercise strategies and provided valuation formulas for various American options on multiple assets. Several of our valuation formulas express the value of the contracts in terms of an early exercise premium relative to a contract of reference. For the contracts with convex payoff functions that we have analyzed, the benchmarks are the corresponding European options with exercise at the maturity date only. For a nonconvex payoff with discontinuous derivatives, a relevant benchmark may be the corresponding contract with automatic exercise prior to maturity. For the case of an American exchange option with a proportional cap, the benchmark is a capped exchange option with automatic exercise at the cap. The early exercise premium in this case captures the benefits of exercising prior to reaching the cap. These representation formulas are also of interest since they can be used to derive hedge ratios and may be of importance in numerical applications. In addition our analysis of the optimal exercise strategies has produced new results of interest for the theory of investment under uncertainty. In particular we have shown that firms choosing among exclusive alternatives may optimally delay investments even when individual projects are well worth undertaking when considered in isolation.

One related contract that is not analyzed in the paper is a call option on the minimum of two assets. When one of the two asset prices, say S^1, follows a deterministic process this contract is equivalent to a capped option with growing cap written on a single underlying asset. The underlying asset is the risky asset with price S^2; the cap is the price of the riskless asset S^1. When the cap has a constant growth rate and the risky asset price follows a geometric Brownian motion process the optimal exercise policy is identified in Broadie and Detemple (1995). The extension of these results to the case in which both prices are stochastic is nontrivial. The determination of the optimal exercise boundary and the valuation of the min-option in this instance are problems left for future research.

APPENDIX A

Standard American Options

PROPOSITION A.1. *For a standard American option (i.e., on a single underlying asset), whose price follows a geometric Brownian motion process,*

$$C_t(\lambda S_t) - C_t(S_t) \leq (\lambda - 1)S_t$$

for all $\lambda \geq 1$.

Proof of Proposition A.1. Let $\lambda \geq 1$ and suppose that the price of the underlying asset is λS_t. Let τ denote the optimal exercise strategy. Using the multiplicative structure of geometric Brownian motion processes, we can write

$$\begin{aligned} C_t(\lambda S) &= E_t^*[e^{-r(\tau-t)}(\lambda S_\tau - K)^+] \\ &= E_t^*[e^{-r(\tau-t)}((\lambda - 1)S_\tau + (S_\tau - K))^+] \\ &\leq E_t^*[e^{-r(\tau-t)}((\lambda - 1)S_\tau + (S_\tau - K)^+)] \\ &\leq (\lambda - 1)S_t + C_t(S_t). \end{aligned}$$

The first inequality follows from $(a+b)^+ \leq a^+ + b^+$ for any real numbers a and b. The second inequality follows by the supermartingale property of S_t and by the suboptimality of the exercise policy τ for the standard American option. $\square$

REMARK A.1. For a standard American option, $(S,t) \in \mathcal{E}$ implies $(\lambda S, t) \in \mathcal{E}$ for all $\lambda \geq 1$. This follows immediately from Proposition A.1 by noting $(S,t) \in \mathcal{E}$ implies $C_t(S) = S - K > 0$ and so $C_t(\lambda S) \leq (\lambda - 1)S + C_t(S) = \lambda S - K$. Hence $(\lambda S, t) \in \mathcal{E}$.

American Options on Multiple Assets

Next we consider derivative securities written on n underlying assets. Throughout this appendix, we suppose that the price of asset i at time t satisfies

$$dS_t^i = S_t^i[(r - \delta_i)dt + \sigma_i dz_t^i] \tag{A.1}$$

where z^i, $i = 1, \ldots, n$ are standard Brownian motion processes and the correlation between z^i and z^j is ρ_{ij}. As before, r is the constant rate of interest, $\delta_i \geq 0$ is the dividend rate of asset i, and the price processes indicated in (A.1) are represented in their risk neutral form. We use this setting for ease of exposition. However, many of the results in this section hold in more general settings.

Consider an American contingent claim written on the n assets that matures at time T. Suppose that its payoff if exercised at time t is $f(S_t^1, S_t^2, \ldots, S_t^n) \geq 0$. For convenience, let S_t represent the vector $(S_t^1, S_t^2, \ldots, S_t^n)$. Denote the value of this "f-claim" at time t by $C_t^f(S_t)$. It follows from Bensoussan (1984) and Karatzas (1988) that

$$C_t^f(S_t) = \sup_{\tau \in \mathcal{S}_{t,T}} E_t^*[e^{-r(\tau - t)} f(S_\tau)]$$

where $\mathcal{S}_{t,T}$ is the set of stopping times of the filtration with values in $[t, T]$. The immediate exercise region for the f-claim is $\mathcal{E}^f \equiv \{(S_t, t) \in \mathbb{R}^n \times [0, T] : C_t^f(S_t) = f(S_t)\}$.

For any stopping time $\tau \in \mathcal{S}_{0,T}$ and for $i = 1, \ldots, n$ we can write

$$S_\tau^i = S^i \exp[(r - \delta_i - \tfrac{1}{2}\sigma_i^2)\tau + \sigma_i z^i \sqrt{\tau}] = S^i \exp[(r - \delta_i - \tfrac{1}{2}\sigma_i^2)\theta T + \sigma_i z^i \sqrt{\theta}\sqrt{T}]$$

where $\theta \in \mathcal{S}_{0,1}$. Now define $N_{\theta T}^i \equiv \exp[(r - \delta_i - \frac{1}{2}\sigma_i^2)\theta T + \sigma_i z^i \sqrt{\theta}\sqrt{T}]$, $i = 1, \ldots, n$, and let $N_{\theta T} \equiv (N_{\theta T}^1, \ldots, N_{\theta T}^n)$. In what follows, we write SN to indicate the product of two vectors. Using arguments similar to those in Jaillet, Lamberton, and Lapeyre (1990), it can be verified that

$$C_t^f(S) = \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f(SN_{\theta(T-t)})],$$

where the expectation is taken relative to the random variables z^i, $i = 1, \ldots, n$.

PROPOSITION A.2. *Suppose immediate exercise is optimal at time t with asset prices S, i.e., $(S, t) \in \mathcal{E}^f$. Then immediate exercise is optimal at all later times at the same asset prices. That is, $(S, s) \in \mathcal{E}^f$ for all s such that $t \le s \le T$.*

Proof of Proposition A.2. Consider the new stopping time $\theta' \equiv \theta \frac{T-t}{T-s}$. Since $\theta \in \mathcal{S}_{0,1}$ we have $\theta' \in \mathcal{S}_{0,k}$ where $k = \frac{T-t}{T-s} > 1$ for $t < s$. It follows that

$$\begin{aligned} C_t^f(S) &= \sup_{\theta' \in \mathcal{S}_{0,k}} E^*[e^{-r\theta'(T-s)} f(SN_{\theta'(T-s)})] \\ &\ge \sup_{\theta' \in \mathcal{S}_{0,1}} E^*[e^{-r\theta'(T-s)} f(SN_{\theta'(T-s)})] \\ &= C_s^f(S) \end{aligned}$$

where the inequality above holds since $\mathcal{S}_{0,1} \subset \mathcal{S}_{0,k}$ for $k > 1$. Suppose now that $(S, s) \notin E^f$. Then $C_s^f(S) > f(S)$ and the inequality above implies $C_t^f(S) > f(S)$. This contradicts $(S, t) \in \mathcal{E}^f$. □

Define $\lambda \circ_i S$ by

$$\lambda \circ_i S = (S^1, S^2, \ldots, S^{i-1}, \lambda S^i, S^{i+1}, \ldots, S^n).$$

Proposition A.3 gives a sufficient condition for immediate exercise to be optimal at time t with asset prices $\lambda \circ_i S_t$ and $\lambda \ge 1$ if immediate exercise is optimal at time t with asset prices S_t.

PROPOSITION A.3 (Right/up connectedness). *Consider an American f-claim with maturity T that has a payoff on exercise at time t of $f(S_t)$. Suppose immediate exercise is optimal at time t with asset prices S_t, i.e., $(S_t, t) \in \mathcal{E}^f$, or equivalently, $C_t^f(S_t) = f(S_t)$. Fix an index i and $\lambda \ge 1$. Suppose that the payoff function f satisfies*

$$f(\lambda \circ_i S_t) = f(S_t) + cS_t^i \tag{A.2}$$

where $c \ge 0$ is a constant that is independent of S_t^i, but may depend on λ and S_t^j for $j \ne i$. Also suppose that

$$f(\lambda \circ_i S) \le f(S) + cS^i \tag{A.3}$$

for all $S \in \mathbb{R}_+^n$ (with the same c as in (A.2)). Then $(\lambda \circ_i S_t, t) \in \mathcal{E}^f$.

Proof of Proposition A.3. Suppose not; i.e., suppose $C_t^f(\lambda \circ_i S_t) > f(\lambda \circ_i S_t)$ for some fixed i and $\lambda \geq 1$. We have

$$\begin{aligned}
C_t^f(\lambda \circ_i S) &= \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f((\lambda \circ_i S) N_{\theta(T-t)})] \\
&\leq \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} (f(SN_{\theta(T-t)}) + cS^i N^i_{\theta(T-t)})] \text{ (by (A.3))} \\
&\leq C_t^f(S) + cS^i \\
&= f(S) + cS^i \qquad \text{(since } (S, t) \in \mathcal{E}^f) \\
&= f(\lambda \circ_i S) \qquad \text{(by assumption A.2).}
\end{aligned}$$

This contradicts our assumption $C_t^f(\lambda \circ_i S_t) > f(\lambda \circ_i S_t)$. □

Conditions (A.2) and (A.3) are satisfied by the following option payoff functions (for the indicated values of i):

	Option payoff function	Valid i
(a)	$f(S_t) = (\max(S_t^1, \ldots, S_t^n) - K)^+$	$\{i : S_t^i = \max(S_t^1, \ldots, S_t^n)\}$
(b)	$f(S_t^1, S_t^2) = (S_t^2 - S_t^1 - K)^+$	$i = 2$

First consider payoff function (a). We prove that conditions (A.2) and (A.3) hold for all i such that $S_t^i = \max(S_t^1, \ldots, S_t^n)$. Note that (S_t, t) belonging to $\mathcal{E}^f$ implies $f(S_t) = (S_t^i - K)^+ = S_t^i - K > 0$. For $\lambda > 1$ we have

$$\begin{aligned}
f(\lambda \circ_i S_t) &= \lambda S_t^i - K \\
&= S_t^i - K + (\lambda - 1) S_t^i \\
&= f(S_t) + cS_t^i.
\end{aligned}$$

So (A.2) holds for $c = \lambda - 1$. To prove (A.3), define $l = \text{argmax}_{j=1,\ldots,n} \lambda \circ_i S_\tau^j$ and note that if $l \neq i$,

$$\begin{aligned}
f(\lambda \circ_i S_\tau) &= (S_\tau^l - K)^+ \\
&\leq (S_\tau^l - K)^+ + (\lambda - 1) S_\tau^i \\
&= f(S_\tau) + cS_\tau^i.
\end{aligned}$$

If $l = i$, then

$$\begin{aligned}
f(\lambda \circ_i S_\tau) &= (\lambda S_\tau^i - K)^+ \\
&= [(S_\tau^i - K) + (\lambda - 1) S_\tau^i]^+ \\
&\leq (S_\tau^i - K)^+ + (\lambda - 1) S_\tau^i \\
&\leq f(S_\tau) + cS_\tau^i.
\end{aligned}$$

The first inequality follows since $(a + b)^+ \leq a^+ + b^+$ for any real numbers a and b.

For payoff function (b), conditions (A.2) and (A.3) hold for $i = 2$. To prove this, note that $(S_t, t) \in \mathcal{E}^f$ implies $f(S_t) = S_t^2 - S_t^1 - K > 0$. Thus, for $\lambda > 1$ we have

$$\begin{aligned} f(\lambda \circ_i S_t) &= \lambda S_t^2 - S_t^1 - K \\ &= S_t^2 - S_t^1 - K + (\lambda - 1)S_t^2 \\ &= f(S_t) + cS_t^2, \end{aligned}$$

so (A.2) holds for $c = \lambda - 1$. To prove (A.3), note that

$$\begin{aligned} f(\lambda \circ_i S_\tau) &= (\lambda S_\tau^2 - S_\tau^1 - K)^+ \\ &= [(S_\tau^2 - S_\tau^1 - K) + (\lambda - 1)S_\tau^2]^+ \\ &\leq (S_\tau^2 - S_\tau^1 - K)^+ + (\lambda - 1)S_\tau^2 \\ &= f(S_\tau) + cS_\tau^2. \end{aligned}$$

Proposition A.4 gives a sufficient condition for the optimality of immediate exercise at time t with asset prices $\lambda \circ_i S_t$ and $0 \leq \lambda \leq 1$ if immediate exercise is optimal at time t with asset prices S_t.

PROPOSITION A.4. *Consider an American f-claim with maturity T that has a payoff on exercise at time t of $f(S_t)$. Suppose immediate exercise is optimal at time t with asset prices S_t, i.e., $(S_t, t) \in \mathcal{E}^f$, or equivalently, $C_t^f(S_t) = f(S_t)$. Fix an index i and fix λ with $0 \leq \lambda \leq 1$. Suppose that the payoff function f satisfies*

$$f(\lambda \circ_i S_t) = f(S_t). \tag{A.4}$$

Also suppose that

$$f(\lambda \circ_i S) \leq f(S) \tag{A.5}$$

for all $S \in \mathbb{R}_+^n$. Then $(\lambda \circ_i S_t, t) \in \mathcal{E}^f$.

Proof of Proposition A.4. The proof is similar to the proof of Proposition A.3. Suppose not; i.e., suppose $C_t^f(\lambda \circ_i S_t) > f(\lambda \circ_i S_t)$. We have

$$\begin{aligned} C_t^f(\lambda \circ_i S) &= \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f((\lambda \circ_i S)N_{\theta(T-t)})] \\ &\leq \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f(SN_{\theta(T-t)})] \quad \text{(by assumption (A.5))} \\ &= C_t^f(S) \\ &= f(S) \qquad\qquad \text{(since } (S,t) \in \mathcal{E}^f) \end{aligned}$$

Hence $C_t^f(\lambda \circ_i S) \leq f(S) = f(\lambda \circ_i S)$ by (A.4). This contradicts $C_t^f(\lambda \circ_i S) > f(\lambda \circ_i S)$. □

Conditions (A.4) and (A.5) are satisfied by the following option payoff functions (for the indicated values of i):

	Option payoff function	Valid i
(a)	$f(S_t) = (\max(S_t^1, \ldots, S_t^n) - K)^+$	$\{i : S_t^i < \max(S_t^1, \ldots, S_t^n)\}$
(b)	$f(S_t^1, S_t^2) = (S_t^2 - S_t^1 - K)^+$	$i = 1$

It is trivial to verify that conditions (A.4) and (A.5) hold for payoff functions (a) and (b) for the indices indicated.

Define αS by the usual scalar multiplication

$$\alpha S = (\alpha S^1, \alpha S^2, \ldots, \alpha S^n).$$

Proposition A.5 gives a sufficient condition for immediate exercise to be optimal at time t with asset prices αS_t ($\alpha \geq 1$) if immediate exercise is optimal at time t with asset prices S_t.

PROPOSITION A.5 (Ray connectedness). *Consider an American f-claim with maturity T that has a payoff on exercise at time t of $f(S_t)$. Suppose immediate exercise is optimal at time t with asset prices S_t, i.e., $(S_t, t) \in \mathcal{E}^f$, or equivalently, $C_t^f(S_t) = f(S_t)$. Also suppose that for all $\alpha \geq 1$ the payoff function f satisfies*

$$f(\alpha S_t) = \alpha f(S_t) + c \tag{A.6}$$

where $c \geq 0$ is a constant that is independent of S_t, but may depend on α. Also suppose that

$$f(\alpha S) \leq \alpha f(S) + c \tag{A.7}$$

for all $S \in \mathbb{R}_+^n$. Then for all $\alpha \geq 1$ we have $(\alpha S_t, t) \in \mathcal{E}^f$.

Proof of Proposition A.5. Suppose not; i.e., suppose $C_t^f(\alpha S_t) > f(\alpha S_t)$ for some $\alpha > 1$. A contradiction follows from the string of inequalities

$$\begin{aligned}
C_t^f(\alpha S_t) &= \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f(\alpha S_t N_{\theta(T-t)})] \\
&\leq \sup_{\theta \in \mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} (\alpha f(S_t N_{\theta(T-t)}) + c)] && \text{(by assumption (A.7))} \\
&\leq \alpha C_t^f(S_t) + c \\
&= \alpha f(S_t) + c && \text{(since } (S_t, t) \in \mathcal{E}^f\text{)} \\
&= f(\alpha S_t) && \text{(by (A.6))} \ \square
\end{aligned}$$

Conditions (A.6) and (A.7) are satisfied by the option payoff functions

$$\begin{aligned} &\text{(a)} \quad f(S_t) = (\max(S_t^1, \ldots, S_t^n) - K)^+ \\ &\text{(b)} \quad f(S_t^1, S_t^2) = (S_t^2 - S_t^1 - K)^+ \end{aligned}$$

For payoff function (a), conditions (A.6) and (A.7) hold. To prove this, note that $(S_t, t) \in \mathcal{E}^f$ implies $f(S_t) > 0$. We then have

$$\begin{aligned} f(\alpha S_t) &= \max_{j=1,\ldots,n} \alpha S_t^j - K \\ &= \alpha(\max_{j=1,\ldots,n} S_t^j - K) + (\alpha - 1)K \\ &= \alpha f(S_t) + c, \end{aligned}$$

so (A.6) holds for $c = (\alpha - 1)K$. To prove (A.7), define $l = \operatorname{argmax}_{j=1,\ldots,n} S^j$ and note that

$$\begin{aligned} f(\alpha S) &= (\alpha S^l - K)^+ \\ &= [\alpha(S^l - K) + (\alpha - 1)K]^+ \\ &\leq \alpha(S^l - K)^+ + (\alpha - 1)K \\ &= \alpha f(S) + c. \end{aligned}$$

For payoff function (b), conditions (A.6) and (A.7) hold. To prove this, note that $(S_t, t) \in \mathcal{E}^f$ implies $f(S_t) = S_t^2 - S_t^1 - K > 0$. Then

$$\begin{aligned} f(\alpha S_t) &= \alpha S_t^2 - \alpha S_t^1 - K \\ &= \alpha(S_t^2 - S_t^1 - K) + (\alpha - 1)K \\ &= \alpha f(S_t) + c, \end{aligned}$$

so (A.6) holds for $c = (\alpha - 1)K$. To prove (A.7),

$$\begin{aligned} f(\alpha S) &= (\alpha S^2 - \alpha S^1 - K)^+ \\ &= [\alpha(S^2 - S^1 - K) + (\alpha - 1)K]^+ \\ &\leq \alpha(S^2 - S^1 - K)^+ + (\alpha - 1)K \\ &= \alpha f(S) + c. \end{aligned}$$

PROPOSITION A.6 (Convexity). *Consider an American f-claim with maturity T that has a payoff on exercise at time t of $f(S_t)$. Suppose that f is a (strictly) convex function. Then $C_t^f(S)$ is (strictly) convex with respect to S.*

Proof of Proposition A.6. Using the convexity of the payoff function, we can write

$$\begin{aligned} C_t^f(S(\lambda)) &= \sup_{\theta\in\mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} f(\lambda S N_{\theta(T-t)} + (1-\lambda)\tilde{S} N_{\theta(T-t)})] \\ &\le \sup_{\theta\in\mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} (\lambda f(S N_{\theta(T-t)}) + (1-\lambda) f(\tilde{S} N_{\theta(T-t)}))] \\ &\le \sup_{\theta\in\mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} \lambda f(S N_{\theta(T-t)})] + \sup_{\theta\in\mathcal{S}_{0,1}} E^*[e^{-r\theta(T-t)} (1-\lambda) f(\tilde{S} N_{\theta(T-t)})] \\ &= \lambda C_t^f(S) + (1-\lambda) C_t^f(\tilde{S}). \end{aligned}$$

□

APPENDIX B

Proof of Proposition 2.1. Suppose not; i.e., suppose $C_t^X(S_t^1, S_t^1) = (S_t^1 - K)^+$ for some $t < T$. Consider a portfolio consisting of (1) a long position in one max-option, (2) a short position of one unit of asset 1, and (3) $\$K$ invested in the riskless asset. The value of this portfolio at time t, denoted V_t, is zero since S_t^1 must be greater than K for the assumption to hold.[4]

Let u be a fixed time greater than t. Since exercise of the max-option at time u may not be optimal, the value of the portfolio at time t, V_t, satisfies

$$V_t \ge E_t^*[e^{-r(u-t)}(\max(S_u^1, S_u^2) - K)^+] - S_t^1 + K.$$

Next we show that the right-hand side of the previous inequality is strictly positive for some $u > t$. That is, $V_t > 0$ which contradicts $V_t = 0$ asserted earlier.

To show $V_t > 0$, first let $A(u)$ denote $E_t^*[e^{-r(u-t)}(\max(S_u^1, S_u^2) - K)^+]$. Then

$$\begin{aligned} A(u) &\ge E_t^*[e^{-r(u-t)}(\max(S_u^1, S_u^2) - K)] \\ &= E_t^*\big[e^{-r(u-t)}[S_u^1 - K + 1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)]\big] \\ &= e^{-r(u-t)}\big(E_t^*(S_u^1) - K + E_t^*[1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)]\big) \\ &= S_t^1 e^{-\delta_1(u-t)} - K e^{-r(u-t)} + e^{-r(u-t)} E_t^*[1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)]. \end{aligned}$$

Clearly (a) $S_t^1 e^{-\delta_1(u-t)} - K e^{-r(u-t)} - (S_t^1 - K) \to 0$ as $u \to t$. Also, (b) $e^{-r(u-t)} E_t^*[1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)] \downarrow 0$ as $u \to t$. However, Lemma B.1 below shows that convergence is faster in case (a). That is, there exists a $u > t$ such that $A(u) > S_t^1 - K$. This implies $V_t \ge A(u) - S_t^1 + K > 0$ which contradicts $V_t = 0$. Hence $C_t^X(S_t^1, S_t^1) > (S_t^1 - K)^+$ for all $t < T$. □

LEMMA B.1. *Suppose $S_t^1 = S_t^2 > 0$ and $t < T$. Then there exists a time u, $t < u < T$,*

[4] If $S_t^1 = S_t^2 < K$ we can always find an exercise strategy whose value is strictly positive. It follows that $C_t^X(S_t^1, S_t^1) > 0$.

such that

$$S_t^1(e^{-\delta_1(u-t)} - 1) - K(e^{-r(u-t)} - 1) + e^{-r(u-t)} E_t^*[1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)] > 0.$$

Proof of Lemma B.1. Let $u = t + \Delta t$ and $B(\Delta t) = e^{-r\Delta t} E_t^*[1_{\{S_u^2 > S_u^1\}}(S_u^2 - S_u^1)]$, where the expectation is evaluated at $S_t^1 = S_t^2$. A straightforward computation gives

$$\begin{aligned} B(\Delta t) \;=\; & \int_{-\infty}^{\infty} S_t^2 \exp[-(\delta_1 + \tfrac{1}{2}\sigma_2^2(1-\rho^2))\Delta t] N\Big(-d(w) + \frac{\sigma_2 - \rho\sigma_1}{\sqrt{1-\rho^2}}\sqrt{\Delta t}\Big) n(w)dw \\ & - \int_{-\infty}^{\infty} S_t^2 \exp[-\delta_1 \Delta t] N\Big(-d(w) - \frac{\sigma_1(\sigma_1 - \rho\sigma_2)}{\sigma_2\sqrt{1-\rho^2}}\sqrt{\Delta t}\Big) n(w)dw \end{aligned}$$

where $d(w) = [(\delta_2 - \delta_1 + \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_1^2)\sqrt{\Delta t} + w(\sigma_1 - \rho\sigma_2)]/(\sigma_2\sqrt{1-\rho^2})$. It can be shown that $B(0) = 0$ and $B'(0) = +\infty$. Let $\Phi(\Delta t) \equiv S_t^1(e^{-\delta_1\Delta t} - 1) - K(e^{-r\Delta t} - 1)$. Then $\Phi(0) = 0$ and Φ has a finite derivative at zero given by $\Phi'(0) = rK - \delta_1 S_t^1$. Hence, there exists a $\Delta t > 0$ (or equivalently, $u > t$) such that the assertion of the lemma holds. □

Proof of Proposition 2.2. Since $(S, t) \in \mathcal{E}_i^X$ and $(\tilde{S}, t) \in \mathcal{E}_i^X$ we have $C_t^X(S) = S^i - K$ and $C_t^X(\tilde{S}) = \tilde{S}^i - K$. Since $(S^1 \vee S^2 - K)^+$ is convex in S^1 and S^2 we can apply Proposition A.6 and write

$$C_t^X(S(\lambda)) \le \lambda C_t^X(S) + (1-\lambda)C_t^X(\tilde{S}) = \lambda(S^i - K) + (1-\lambda)(\tilde{S}^i - K) = S^i(\lambda) - K.$$

On the other hand, since immediate exercise is a feasible strategy $C_t^X(S(\lambda)) \ge (S^1(\lambda) \vee S^2(\lambda) - K)^+ = S^i(\lambda) - K$ when $(S, t) \in \mathcal{E}_i^X$ and $(\tilde{S}, t) \in \mathcal{E}_i^X$. Combining these two inequalities implies $(S(\lambda), t) \in \mathcal{E}_i^X$. □

Proof of Proposition 2.3.

(i) This assertion follows immediately from Proposition A.2 in Appendix A.
(ii) This is immediate from Proposition A.3 and the remarks for payoff function (a) which follow that proposition.
(iii) This assertion follows from Proposition A.4 and the remarks for payoff function (a) which follow that proposition.
(iv) If $S_t^2 = 0$ then $S_v^2 = 0$ for all $v \ge t$. Hence the max-option is equivalent to a standard option on the single asset S^1. By definition, the optimal exercise boundary for this standard option is B_t^1. □

Proof of Proposition 2.4. The proof uses the following lemmas.

LEMMA B.2. *Let $K_1 > K_2$ denote two exercise prices and let $\mathcal{E}^X(t, K_1)$ and $\mathcal{E}^X(t, K_2)$ represent the corresponding exercise regions at time t. Then $\mathcal{E}^X(t, K_1) \subset \mathcal{E}^X(t, K_2)$. In particular, for $K > 0$ we have $\mathcal{E}^X(t, K) \subset \mathcal{E}^X(t, 0)$.*

Proof of Lemma B.2. Suppose immediate exercise is optimal at time t for the K_1 option but not for the K_2 option. Then

$$\begin{aligned}
(S^1 \vee S^2 - K_2)^+ &< C^X(S^1, S^2, K_2) \\
&= E^*[e^{-r(\tau-t)}(S^1_\tau \vee S^2_\tau - K_2)^+] \\
&= E^*[e^{-r(\tau-t)}(S^1_\tau \vee S^2_\tau - K_1 + K_1 - K_2)^+] \\
&\leq E^*[e^{-r(\tau-t)}(S^1_\tau \vee S^2_\tau - K_1)^+] + E^*[e^{-r(\tau-t)}(K_1 - K_2)^+] \\
&\leq C^X(S^1, S^2, K_1) + K_1 - K_2 \\
&= (S^1 \vee S^2 - K_1)^+ + K_1 - K_2
\end{aligned}$$

where the last line follows from the optimality of immediate exercise for the K_1-option. The contradiction obtained shows that immediate exercise is optimal for the K_2-option. □

LEMMA B.3 (Ray connectedness). *If* $(S^1, S^2, t) \in \mathcal{E}^X(t, 0)$ *then* $(\lambda S^1, \lambda S^2, t) \in \mathcal{E}^X(t, 0)$ *for all* $\lambda > 0$.

Proof of Lemma B.3. Suppose $(\lambda S^1, \lambda S^2, t) \notin \mathcal{E}^X(t, 0)$ for some $\lambda > 0$. Then there exists $\tau_\lambda \in \mathcal{S}_{t,T}$ such that

$$\lambda S^1 \vee \lambda S^2 < C(\lambda S^1, \lambda S^2, 0) = E^*_t[e^{-r(\tau_\lambda - t)}(\lambda S^1_\tau \vee \lambda S^2_\tau)] = \lambda E^*_t[e^{-r(\tau_\lambda - t)}(S^1_\tau \vee S^2_\tau)].$$

It follows that $S^1 \vee S^2 < E^*_t[e^{-r(\tau_\lambda - t)}(S^1_\tau \vee S^2_\tau)]$; i.e., the stopping time strategy τ_λ dominates immediate exercise at (S^1, S^2, t). This contradicts the hypothesis. □

LEMMA B.4. $(S, S, t) \notin \mathcal{E}^X(t, 0)$ *for* $t < T$.

Proof of Lemma B.4. This follows from the proof of Proposition 2.1 with $K = 0$. □

Now to prove the proposition, Lemma B.4 states that $(S, S, t) \notin \mathcal{E}^X(t, 0)$. Since $\mathcal{E}^X(t, 0)$ is a closed set, there exists an open neighborhood of (S, S) such that $(S^1, S^2, t) \notin \mathcal{E}^X(t, 0)$ for all (S^1, S^2) in the neighborhood. The ray connectedness of Lemma B.3 implies the existence of an open cone $R(\lambda_1, \lambda_2)$ such that $R(\lambda_1, \lambda_2) \cap \mathcal{E}^X(t, 0) = \emptyset$. Finally, Lemma B.2 implies $R(\lambda_1, \lambda_2) \cap \mathcal{E}^X(t, K) = \emptyset$. □

Proof of Proposition 2.6.

(i) Uniform boundedness of the spatial derivatives: We focus on the derivative relative to S^1. The argument for S^2 follows by symmetry. Consider two asset values

(S_t^1, S_t^2, t) and $(\tilde{S}_t^1, S_t^2, t)$. For any stopping time $\tau \in \mathcal{S}_{t,T}$ we have

$$\text{(B.1)}\quad \begin{aligned} |(S_\tau^1 \vee S_\tau^2 - K)^+ - (\tilde{S}_\tau^1 \vee S_\tau^2 - K)^+| &\le |(S_\tau^1 \vee S_\tau^2) - (\tilde{S}_\tau^1 \vee S_\tau^2)| \\ &\le |S_\tau^1 - \tilde{S}_\tau^1| \\ &= |S_t^1 - \tilde{S}_t^1| \exp\big[(r-\delta_1)(\tau - t) \\ &\quad - \tfrac{1}{2}\sigma_1^2(\tau - t) + \sigma_1(z_\tau^1 - z_t^1)\big] \\ &\le |S_t^1 - \tilde{S}_t^1|\ \exp\big[r(\tau - t) \\ &\quad - \tfrac{1}{2}\sigma_1^2(\tau - t) + \sigma_1(z_\tau^1 - z_t^1)\big]. \end{aligned}$$

Without loss of generality, suppose $S_t^1 > \tilde{S}_t^1$. Let $\tau_1 \in \mathcal{S}_{t,T}$ represent the optimal stopping time for (S_t^1, S_t^2, t). We have

$$\begin{aligned} |C^X(S_t^1, S_t^2, t) - C^X(\tilde{S}_t^1, S_t^2, t)| &\le E_t^*[e^{-r(\tau_1 - t)}|(S_{\tau_1}^1 \vee S_{\tau_1}^2 - K)^+ \\ &\quad -(\tilde{S}_{\tau_1}^1 \vee S_{\tau_1}^2 - K)^+|] \\ &\le |S_t^1 - \tilde{S}_t^1| E_t^*[\exp(-\tfrac{1}{2}\sigma_1^2(\tau_1 - t) \\ &\quad +\sigma_1(z_{\tau_1}^1 - z_t^1))] \quad \text{(by (A.8))} \\ &= |S_t^1 - \tilde{S}_t^1|. \end{aligned}$$

Hence, $[|C^X(S_t^1, S_t^2, t) - C^X(\tilde{S}_t^1, S_t^2, t)|]/(|S_t^1 - \tilde{S}_t^1|) \le 1$; i.e., one is a uniform upper bound.

(ii) Local boundedness of the time derivative: Define $u(t) \equiv C^X(S^1, S^2, t)$ and let $\theta(t) \in \mathcal{S}_{0,1}$ denote the optimal stopping time for this problem. We have

$$\begin{aligned} \text{(B.2)}\, |u(t) - u(s)| &\le \Big|E^*\big[e^{-r\theta(t)(T-t)}(\max_i S^i N^i_{\theta(t)(T-t)} - K)^+ \\ &\quad -e^{-r\theta(t)(T-s)}(\max_i S^i N^i_{\theta(t)(T-s)} - K)^+\big]\Big| \\ &\qquad\qquad \text{(since } \theta(t) \text{ is suboptimal for } u(s)\text{)} \\ &\le E^*\big[|e^{-r\theta(t)(T-t)} - e^{-r\theta(t)(T-s)}|(\max_i S^i N^i_{\theta(t)(T-t)} - K)^+ \\ &\quad + e^{-r\theta(t)(T-s)}|(\max_i S^i N^i_{\theta(t)(T-t)} - K)^+ \\ &\quad -(\max_i S^i N^i_{\theta(t)(T-s)} - K)^+|\big]. \end{aligned}$$

Since $G(t) \equiv e^{-r\theta(t)(T-t)}$ is convex in t, we can write

$$\text{(B.3)}\ |e^{-r\theta(t)(T-t)} - e^{-r\theta(t)(T-s)}| \le [\sup_{\substack{\theta\in[0,1]\\ v\in[0,T]}} (r\theta e^{-r\theta(T-v)})]|r\theta(t)(t-s)| \le k|t-s|$$

for some constant k. Also

$$\text{(B.4)}\quad E^*(\max_i S^i N^i_{\theta(t)(T-t)} - K)^+ \le \sum_{i=1}^{2} E^*(S^i N^i_{\theta(t)(T-t)}) \le \sum_{i=1}^{2} S^i \exp[|r-\delta_i|(T-t)] \equiv \sum_{i=1}^{2} k_i S^i$$

for some constants k_i.

Finally, let $\alpha_i \equiv r - \delta_i - \frac{1}{2}\sigma_i^2$, $i = 1, 2$, and define $a_i(s) \equiv \alpha_i \theta(t)(T-s) + \sigma_i z^i \sqrt{\theta(t)}\sqrt{T-s}$. We can write

$$\begin{aligned}
\text{(B.5)}\Psi &\equiv |(S^1 e^{a_1(t)} \vee S^2 e^{a_2(t)} - K)^+ - (S^1 e^{a_1(s)} \vee S^2 e^{a_2(s)} - K)^+| \\
&\le |S^1 e^{a_1(t)} \vee S^2 e^{a_2(t)} - S^1 e^{a_1(s)} \vee S^2 e^{a_2(s)}| \\
&\le |S^1 e^{a_1(t)} \vee S^2 e^{a_2(t)} - S^1 e^{a_1(s)} \vee S^2 e^{a_2(t)}| \\
&\quad + |S^1 e^{a_1(s)} \vee S^2 e^{a_2(t)} - S^1 e^{a_1(s)} \vee S^2 e^{a_2(s)}| \\
&\le S^1 (e^{a_1(t)} \vee e^{a_1(s)})|a_1(t) - a_1(s)| + S^2 (e^{a_2(t)} \vee e^{a_2(s)})|a_2(t) - a_2(s)| \\
&\le (S^1 + S^2) e^{|a_1(t)|+|a_1(s)|+|a_2(t)|+|a_2(s)|}(|a_1(t) - a_1(s)| + |a_2(t) - a_2(s)|),
\end{aligned}$$

where the third inequality follows from the convexity of the exponential function. But $|a_i(s)| \le |\alpha_i|\theta(t)(T-s) + \sigma_i|z^i|\sqrt{\theta(t)}\sqrt{T-s} \le |\alpha_i|T + \sigma_i|z^i|\sqrt{T}$, and $\sum_i |a_i(t) - a_i(s)| \le \sum_i (|\alpha_i|\theta(t)(t-s) + \sigma_i|z^i|\sqrt{\theta(t)}(\sqrt{T-t} - \sqrt{T-s})) \le A(|t-s| + \sum_i |z_i|(\sqrt{T-t} - \sqrt{T-s})) \equiv h$. Substituting these inequalities in (A.12), taking expectations, and using the Cauchy–Schwartz inequality yields

$$\begin{aligned}
\text{(B.6)}\qquad E^*[\Psi] &\le (S^1 + S^2) E^*[e^{(\sum_i |\alpha_i|)T + (\sum_i \sigma_i |z_i|)\sqrt{T}} h] \\
&\le (S^1 + S^2)(E^*[e^{2(\sum_i |\alpha_i|)T + 2(\sum_i \sigma_i|z_i|)\sqrt{T}}] E^*|h|^2)^{\frac{1}{2}} \\
&\le B(S^1 + S^2)(E^*|h|^2)^{\frac{1}{2}},
\end{aligned}$$

for some constant B. Furthermore

$$E^*|h|^2 \le D\left(|s-t|^2 + E^*\left(|z^1| + |z^2|\right)^2 \left(\sqrt{T-t} - \sqrt{T-s}\right)^2\right),$$

for some constant D. Since $\phi(t) \equiv \sqrt{T-t}$ has $\phi'(t) = -\frac{1}{2}(T-t)^{-1/2} < 0$ and $\phi''(t) = -\frac{1}{4}(T-t)^{-3/2} < 0$, we have $0 \le \phi(t) - \phi(s) \le \frac{1}{2}(T-s)^{-1/2}|s-t|$ for $t \le s$. It follows that

$$\begin{aligned}
\text{(B.7)}\quad E^*|h|^2 &\le D\left(|s-t|^2 + 2(E^*(z^1)^2 + E^*(z^2)^2)\frac{1/4}{T-s}|s-t|^2\right) \\
&\equiv \bar{D}|s-t|^2.
\end{aligned}$$

Substituting (B.3), (B.4), (B.6), and (B.7) in (B.2) yields

$$|u(t) - u(s)| \leq (S^1 + S^2) N_s |t - s|$$

where N_s depends on s. Local boundedness of $\partial u/\partial t$ follows.

Theorem 3.2 in Jaillet, Lamberton, and Lapeyre (1990) shows that C^X satisfies the variational inequalities (2.4). These variational inequalities can be combined with the convexity of the price function (Proposition 2.5, (iv)), the local boundedness of $\partial C^X/\partial t$, and the uniform boundedness of $\partial C^X/\partial S^i$, $i = 1, 2$, to prove that the second partial derivatives are locally bounded (see equation (B.9) below). □

Proof of Corollary 2.1. Using the transformation $S^1 = e^{y_1}$ and $S^2 = e^{y_2}$ we can rewrite equation (2.4) as

$$\text{(B.8)} \quad \frac{1}{2}\left[\frac{\partial^2 C^X}{\partial y_1^2}\sigma_1^2 + 2\frac{\partial^2 C^X}{\partial y_1 \partial y_2}\sigma_1\rho\sigma_2 + \frac{\partial^2 C^X}{\partial y_2^2}\sigma_2^2\right] \leq rC^X - \alpha_1\frac{\partial C^X}{\partial y_1} - \alpha_2\frac{\partial C^X}{\partial y_2} - \frac{\partial C^X}{\partial t},$$

where $\alpha_i = r - \delta_i - \frac{1}{2}\sigma_i^2$, $i = 1, 2$. Convexity also implies $z'Hz \geq 0$ for all $z \in \mathbb{R}^2$ where H represents the Hessian of C^X. Let $C_{ij}^X \equiv \frac{\partial^2 C^X}{\partial y_i \partial y_j}$ for $i, j = 1, 2$. For $z' \equiv (\rho\sigma_1, \sigma_2)$ we get

$$(\rho\sigma_1, \sigma_2)\begin{pmatrix} C_{11}^X & C_{12}^X \\ C_{21}^X & C_{22}^X \end{pmatrix}\begin{pmatrix} \rho\sigma_1 \\ \sigma_2 \end{pmatrix} = \rho^2\sigma_1^2 C_{11}^X + 2\rho\sigma_1\sigma_2 C_{12}^X + \sigma_2^2 C_{22}^X \geq 0,$$

which implies $\sigma_1^2 C_{11}^X + 2\rho\sigma_1\sigma_2 C_{12}^X + \sigma_2^2 C_{22}^X \geq (1-\rho^2)\sigma_1^2 C_{11}^X \geq 0$. Combining this inequality with (B.8) yields

$$\text{(B.9)} \qquad 0 \leq \tfrac{1}{2}(1-\rho^2)\sigma_1^2 C_{11}^X \leq rC^X - \alpha_1\frac{\partial C^X}{\partial y_1} - \alpha_2\frac{\partial C^X}{\partial y_2} - \frac{\partial C^X}{\partial t}.$$

Now consider the domain

$$\Sigma_t \equiv \{(y_1, y_2) : y_2^- \leq y_2 \leq y_2^+, y_1^-(y_2) \equiv B_1^X(y_2, t) - \epsilon \leq y_1 \leq B_1^X(y_2, t) + \epsilon \equiv y_1^+(y_2)\}$$

for given constants $y_2^- \leq y_2^+$ and $\epsilon > 0$. Integrating (B.9) over $\Sigma_t \times [t_1, t_2]$ yields

$$\begin{aligned} 0 \;\leq\; & \tfrac{1}{2}(1-\rho^2)\sigma_1^2 \int_{t_1}^{t_2}\int_{y_2^-}^{y_2^+}\int_{y_1^-(y_2)}^{y_1^+(y_2)} C_{11}^X dy_1\, dy_2\, dt \leq r\int_{t_1}^{t_2}\int_{\Sigma_t} C^X dy_1\, dy_2\, dt \\ & - \alpha_1\int_{t_1}^{t_2}\int_{\Sigma_t}\frac{\partial C^X}{\partial y_1}dy_1\, dy_2\, dt - \alpha_2\int_{t_1}^{t_2}\int_{\Sigma_t}\frac{\partial C^X}{\partial y_2}dy_1\, dy_2\, dt \\ & - \int_{t_1}^{t_2}\int_{\Sigma_t}\frac{\partial C^X}{\partial t}dy_1\, dy_2\, dt, \end{aligned}$$

for all $\epsilon > 0$. Equivalently

$$
\begin{aligned}
0 \;\le\;& \tfrac{1}{2}(1-\rho^2)\sigma_1^2 \int_{t_1}^{t_2}\int_{y_2^-}^{y_2^+} \left(C_1^X(y_1^+(y_2), y_2) - C_1^X(y_1^-(y_2), y_2)\right) dy_2\, dt \\
\le\;& r\int_{t_1}^{t_2} \left(\sup_{\Sigma_t} C^X\right) \lambda(\Sigma_t) dt \\
& - \alpha_1 \int_{t_1}^{t_2}\int_{y_2^-}^{y_2^+} \left(C^X(y_1^+(y_2), y_2) - C^X(y_1^-(y_2), y_2)\right) dy_2\, dt \\
& - \alpha_2 \int_{t_1}^{t_2}\int_{y_1^-}^{y_1^+} \left(C^X(y_1, y_2^+(y_1)) - C^X(y_1, y_2^-(y_1))\right) dy_1\, dt \\
& + \int_{t_1}^{t_2} \sup_{\Sigma_t} \left(-\frac{\partial C^X}{\partial t}\right) \lambda(\Sigma_t) dt
\end{aligned}
$$

where $\lambda(\Sigma_t)$ is the Lebesgue measure of the set Σ_t. To obtain the integral relative to y_1, we reversed the order of integration: y_1^-, y_1^+, $y_2^-(y_1)$, and $y_2^+(y_1)$ denote the edges of the domain Σ_t under this transformation.

As $\epsilon \downarrow 0$ all four terms on the righthand side converge to zero since $\lambda(\Sigma_t) \downarrow 0$, C^X is locally bounded, and $\partial C^X/\partial t$ is locally bounded (see Proposition 2.6). We conclude that $C_1^X(y_1^+(y_2), y_2) - C_1^X(y_1^-(y_2), y_2) \downarrow 0$ as $\epsilon \downarrow 0$ for all $t \in [t_1, t_2]$ and all $y_2 \in [y_2^-, y_2^+]$. Since $C_1^X(y_1^+(y_2), y_2) = 1$ it follows that $C_1^X(y_1^-(y_2), y_2) = 1$ for all $t \in [t_1, t_2]$, $y_2 \in [y_2^-, y_2^+]$. Proceeding along the same lines we can show $C_2^X(y_1, y_2^+(y_1)) = 0$ across the boundary $B_1^X(y_2, t)$. □

Proof of Proposition 2.7. Since the partial derivatives exist and since the spatial derivatives are continuous on $[0, T) \times \mathbb{R}^+ \times \mathbb{R}^+$ (by Proposition 2.6 and Corollary 2.1) we can apply Itô's lemma and write

$$
\text{(B.10)} \quad e^{-r(T-t)} C^X(S_T^1, S_T^2, T) = C^X(S_t^1, S_t^2, t) + \int_{s=t}^{T} e^{-r(s-t)} \sum_{i=1}^{2} \frac{\partial C^X}{\partial S^i} \sigma_i S_s^i dz_s^i + \int_{s=t}^{T} \left(\mathcal{L}[e^{-r(s-t)} C_s^X] + e^{-r(s-t)} \frac{\partial C^X}{\partial s}\right) ds.
$$

On the continuation region $\mathcal{C}$ we have $\partial C^X/\partial t + \mathcal{L}C^X = 0$. On the immediate exercise region $\mathcal{E}^X$ we have $C^X(S_t^1, S_t^2, t) = \max(S_t^1, S_t^2) - K$. Thus

$$
\frac{\partial C^X}{\partial t} + \mathcal{L}C^X = \begin{cases} -(\delta_1 - r)S_t^1 - r(S_t^1 - K) = -\delta_1 S_t^1 + rK & \text{on } \mathcal{E}_1^X \\ -(\delta_2 - r)S_t^2 - r(S_t^2 - K) = -\delta_2 S_t^2 + rK & \text{on } \mathcal{E}_2^X. \end{cases}
$$

Also $C^X(S_T^1, S_T^2, T) = (\max(S_T^1, S_T^2) - K)^+$. Substituting and taking expectations on both sides of (B.10) gives

(B.11)

$$
\begin{aligned}
E_t^*[e^{-r(T-t)}(\max(S_T^1, S_T^2) - K)^+] &= C^X(S_t^1, S_t^2, t) \\
&\quad + \int_{s=t}^{T} E_t^*[e^{-r(s-t)}(rK - \delta_1 S_s^1) 1_{\{S_s^1 \geq B_1^X(S_s^2, s)\}} \\
&\quad + e^{-r(s-t)}(rK - \delta_2 S_s^2) 1_{\{S_s^2 \geq B_2^X(S_s^1, s)\}}] ds.
\end{aligned}
$$

Rearranging (B.12) produces the representation (2.8). The recursive equations (2.9) and (2.10) for the optimal exercise boundaries are obtained by imposing the boundary conditions $C_t^X(B_1^X(S_t^2, t), S_t^2) = B_1^X(S_t^2, t) - K$ and $C_t^X(S_t^1, B_2^X(S_t^1, t)) = B_2^X(S_t^1, t) - K$. The boundary conditions (2.11) hold since the max-option converges to an option on one asset as $t \uparrow T$. Similarly, (2.12) holds since the max-option is a standard option on a single asset when one price is zero. □

Proof of Proposition 3.1.

(i) Clearly immediate exercise is suboptimal if $S_t^2 \leq S_t^1 + K$.
(ii) This assertion follows immediately from Proposition A.2 in Appendix A.
(iii) This is immediate from Proposition A.3 and the remarks for payoff function (b) which follow that proposition.
(iv) This assertion follows from Proposition A.4 and the remarks for payoff function (b) which follow that proposition.
(v) If $S_t^1 = 0$ then $S_v^1 = 0$ for all $v \geq t$. Hence the spread option is equivalent to a standard option on the single asset S^2. By definition, the optimal exercise boundary for this standard option is B_t^2.
(vi) The proof is similar to the proof of Proposition 2.2. □

Proof of Proposition 3.3.

(i) If $R_t \leq 1$ there exists a waiting policy which has positive value.
(ii) Let $\lambda > 1$ and suppose that $(S_t^1, \lambda S_t^2, t) \notin \mathcal{E}^E$. Then there exists a stopping time τ such that $\tau \in \mathcal{S}_{t,T}$ and

$$
\begin{aligned}
C(S_t^1, \lambda S_t^2, t) &= E_t^*[e^{-r(\tau-t)} S_\tau^1 (\lambda R_\tau - 1)^+] \\
&= E_t^*[e^{-r(\tau-t)} S_\tau^1 (R_\tau - 1 + (\lambda - 1) R_\tau)^+] \\
&\leq E_t^*[e^{-r(\tau-t)} S_\tau^1 (R_\tau - 1)^+] + (\lambda - 1) E_t^*[e^{-r(\tau-t)} S_\tau^1 R_\tau] \\
&\leq C(S_t^1, S_t^2, t) + (\lambda - 1) S_t^2 \\
&= S_t^2 - S_t^1 + (\lambda - 1) S_t^2 = \lambda S_t^2 - S_t^1.
\end{aligned}
$$

(iii) Consider $\lambda > 0$ and suppose that $(\lambda S_t^1, \lambda S_t^2, t) \notin \mathcal{E}^E$. Then there exists $\tau \in \mathcal{S}_{t,T}$

with $\tau > t$ such that

$$\begin{aligned} C(\lambda S_t^1, \lambda S_t^2, t) &> \lambda S_t^1 \left(\frac{\lambda S_t^2}{\lambda S_t^1} - 1 \right) \\ \iff \quad & E_t^*[e^{-r(\tau-t)} \lambda S_\tau^1 (R_\tau - 1)^+] > \lambda S_t^1 (R_t - 1)^+ \\ \iff \quad & E_t^*[e^{-r(\tau-t)} S_\tau^1 (R_\tau - 1)^+] > S_t^1 (R_t - 1)^+. \end{aligned}$$

Since $C(S_t^1, S_t^2, t) \geq E_t^*[e^{-r(\tau-t)} S_\tau^1 (R_\tau - 1)^+]$ we get $C(S_t^1, S_t^2, t) > S_t^1 (R_t - 1)^+$. This contradicts the assumption $(S_t^1, S_t^2, t) \in \mathcal{E}^E$.

(iv) If $S_t^1 = 0$ we have $S_v^1 = 0$ for all $v \geq t$. Hence, $S_\tau^2 - S_\tau^1 = S_\tau^2$ for all stopping times τ. But $S_t^2 \geq E_t^*[e^{-r(\tau-t)} S_\tau^2]$ for all stopping times τ. The result follows. □

Proof of Proposition 3.4. The value of the option in the exercise region is $S_t^2 - S_t^1$ which has dynamics

$$d(S_t^2 - S_t^1) = S_t^2[(r - \delta_2)dt + \sigma_2 dz_t^2] - S_t^1[(r - \delta_1)dt + \sigma_1 dz_t^1] \quad \text{on } \{R_t \geq B_t^E\}.$$

The value of the option can then be written as

$$C^E(S_t^1, S_t^2, t) = c^E(S_t^1, S_t^2, t) + E_t^* \left[\int_t^T e^{-r(v-t)} (\delta_2 S_v^2 - \delta_1 S_v^1) 1_{\{R_v \geq B_v^E\}} dv \right]$$

where $c^E(S_t^1, S_t^2, t) \equiv E_t^*[e^{-r(T-t)} (S_T^2 - S_T^1)^+]$ is the value of the European exchange option.

But $R_v \geq B_v^E$ if and only if $z^R \geq d(R_t, B_v^E, v - t)$, where

$$d(R_t, B_v^E, v - t) \equiv \left[\log \left(\frac{B_v^E}{R_t} \right) - (r - \delta_R - \tfrac{1}{2}\sigma_R^2)(v - t) \right] \frac{1}{\sigma_R \sqrt{v - t}}.$$

For $i = 1, 2$, define $z^i = \rho_{iR} z^R + \sqrt{1 - \rho_{iR}^2}\, u^{iR}$ where

$$u^{iR} \equiv \frac{z^i - \rho_{iR} z^R}{\sqrt{1 - \rho_{iR}^2}} \quad \text{and} \quad \rho_{iR} dt \equiv \frac{1}{\sigma_i \sigma_R} \mathrm{Cov} \left(\frac{dS_t^i}{S_t^i}, \frac{dR}{R} \right) = \frac{1}{\sigma_i \sigma_R} [\sigma_i^2 - \rho \sigma_1 \sigma_2] dt.$$

Let $d(R_t, B_v^E, v - t) \equiv d$. Taking account of the fact that u^{2R} and u^{1R} have standard normal distributions and are each independent of z^R, we can write the early exercise premium as

$$
\begin{aligned}
&\int_t^T \int_{\left\{ \substack{z^R \geq d \\ u^{2R} \in (-\infty, +\infty)} \right\}} \delta_2 S_t^2 e^{-\delta_2(v-t)} \exp[-\tfrac{1}{2}\sigma_2^2(v-t) + \sigma_2(\rho_{2R} z^R \\
&\quad + \sqrt{1-\rho_{2R}^2}\, u^{2R})\sqrt{v-t}] n(z^R) n(u^{2R}) dz^R du^{2R} dv \\
&\quad - \int_t^T \int_{\left\{ \substack{z^R \geq d \\ u^{1R} \in (-\infty, +\infty)} \right\}} \delta_1 S_t^1 e^{-\delta_1(v-t)} \exp[-\tfrac{1}{2}\sigma_1^2(v-t) + \sigma_1(\rho_{1R} z^R \\
&\quad + \sqrt{1-\rho_{1R}^2}\, u^{1R})\sqrt{v-t}] n(z^R) n(u^{1R}) dz^R du^{1R} dv \\
&= \int_t^T \int_d^\infty \int_{-\infty}^\infty \delta_2 S_t^2 e^{-\delta_2(v-t)} n(z^R - \sigma_2 \rho_{2R}\sqrt{v-t}) \\
&\quad \times\ n(u^{2R} - \sigma_2\sqrt{1-\rho_{2R}^2}\sqrt{v-t}) dz^R du^{2R} dv \\
&\quad - \int_t^T \int_d^\infty \int_{-\infty}^\infty \delta_1 S_t^1 e^{-\delta_1(v-t)} n(z^R - \sigma_1 \rho_{1R}\sqrt{v-t}) \\
&\quad \times\ n(u^{1R} - \sigma_1\sqrt{1-\rho_{1R}^2}\sqrt{v-t}) dz^R du^{1R} dv \\
&= \int_t^T \int_{d-\sigma_2\rho_{2R}\sqrt{v-t}}^\infty \int_{-\infty}^\infty \delta_2 S_t^2 e^{-\delta_2(v-t)} n(w^R) n(w) dw^R dw dv \\
&\quad - \int_t^T \int_{d-\sigma_1\rho_{1R}\sqrt{v-t}}^\infty \int_{-\infty}^\infty \delta_1 S_t^1 e^{-\delta_1(v-t)} n(w^R) n(w) dw^R dw dv \\
&= \int_t^T \delta_2 S_t^2 e^{-\delta_2(v-t)} N(-d(R_t, B_v^E, v-t) + \sigma_2\rho_{2R}\sqrt{v-t}) dv \\
&\quad - \int_t^T \delta_1 S_t^1 e^{-\delta_1(v-t)} N(-d(R_t, B_v^E, v-t) + \sigma_1\rho_{1R}\sqrt{v-t}) dv
\end{aligned}
$$

where

$$
\begin{aligned}
d(R_t, B_v^E, v-t) - \sigma_2\rho_{2R}\sqrt{v-t} &= \left[\log\left(\frac{B_v^E}{R_t}\right) - (\delta_1 - \delta_2 + \tfrac{1}{2}\sigma_R^2)(v-t)\right] \frac{1}{\sigma_R\sqrt{v-t}} \\
&\equiv b(R_t, B_v^E, v-t, \delta_1 - \delta_2, \sigma_R)
\end{aligned}
$$

and

$$
d(R_t, B_v^E, v-t) - \sigma_1\rho_{1R}\sqrt{v-t} = \left[\log\left(\frac{B_v^E}{R_t}\right) - (\delta_1 - \delta_2 + \tfrac{1}{2}\sigma_R^2)(v-t)\right]
$$

$$\times \frac{1}{\sigma_R\sqrt{v-t}} + \sigma_R\sqrt{v-t}$$
$$= b(R_t, B_v^E, v-t, \delta_1-\delta_2, \sigma_R) + \sigma_R\sqrt{v-t}.$$

The recursive integral equation for the optimal boundary is obtained by dividing by S_t^1 throughout and setting $C^E(S_t^1, S_t^2, t) = S_t^1(B_t^E - 1)$ at the point $S_t^2/S_t^1 \equiv R_t = B_t^E$. □

The proof of Proposition 4.1 follows from the next lemma.

LEMMA B.5. *The price of the exchange option with proportional cap satisfies the following inequalities,*

$$0 \leq (S^2 - S^1)^+ \wedge LS^1 \;\leq\; C^{EC}(S^1, S^2, t) \;\leq\; C^E(S^1, S^2, t) \wedge V(LS^1, t)$$

where $V(LS^1, t)$ is the date t value of an American contingent claim which pays LS^1 upon exercise. When $\delta_1 > 0$ we have $V(LS^1, t) = LS_t^1$.

Proof of Lemma B.5. The lower bound on the price follows since immediate exercise is always a feasible strategy. To obtain the upper bound note that $(S^2-S^1)^+ \wedge LS^1 \leq (S^2-S)^+$. Hence $C^{EC}(S^1, S^2, t) \leq C^E(S^1, S^2, t)$. On the other hand $(S^2 - S^1)^+ \wedge LS^1 \leq LS^1$. This yields $C^{EC}(S^1, S^2, t) \leq V(LS^1, t)$. Combining these two bounds yields the upper bound in the lemma. Finally note that when $\delta_1 > 0$ it does not pay to delay buying the asset S^1 since this amounts to a loss of dividend payments. □

Proof of Proposition 4.1. From the lemma it is straightforward to see that immediate exercise is optimal if $S_t^2 \geq B^E(t)S_t^1 \wedge (1+L)S_t^1$. When $S_t^2 < B^E(t)S_t^1 \wedge (1+L)S_t^1$, the suboptimality of immediate exercise is proved in the text. □

Proof of Proposition 4.3. We first establish the continuity of the derivatives of $C^{EC}(S^1, S^2, t)$ across the exercise boundary B^{EC}.

LEMMA B.6. *The spatial derivatives $(\partial C^{EC}/\partial S^i)(S^1, S^2, t)$, $i = 1, 2$ are continuous on $\{S^2 = B^{EC}S^1\} \cap \{S^2 < (1+L)S^1\}$.*

Proof of Lemma B.6. On $\{S^2 = B^{EC}S^1\} \cap \{S^2 < (1+L)S^1\}$ we know that $B^{EC} = B^E$. Thus if $S^2 > B^E S^1$ we can write $(S^2 - S^1)^+ = C^{EC}(S^1, S^2, t) = C^E(S^1, S^2, t)$. On the other hand, if $S^2 < B^E S^1$ we have $(S^2 - S^1)^+ \leq C^{EC}(S^1, S^2, t) \leq C^E(S^1, S^2, t)$.

Consider now $S^2 = B^E S^1$ and let $S_+^2 = S^2 + \epsilon$, $S_-^2 = S^2 - \epsilon$ for $\epsilon > 0$. The following bounds hold

$$\frac{(S_+^2 - S^1)^+ - (S_-^2 - S^1)^+}{2\epsilon} \geq \frac{C^{EC}(S^1, S_+^2, t) - C^{EC}(S^1, S_-^2, t)}{2\epsilon}$$
$$\geq \frac{C^E(S^1, S_+^2, t) - C^E(S^1, S_-^2, t)}{2\epsilon}$$

for all $\epsilon > 0$. Taking the limit as $\epsilon \downarrow 0$ yields

$$1 \geq \frac{1}{2}\left[\frac{\partial C_{+}^{EC}}{\partial S^2} + \frac{\partial C_{-}^{EC}}{\partial S^2}\right] \geq \frac{1}{2}\left[\frac{\partial C_{+}^{E}}{\partial S^2} + \frac{\partial C_{-}^{E}}{\partial S^2}\right]$$

where the subscripts $+$ and $-$ denote the right and left derivatives, respectively. By continuity of $\partial C^E/\partial S^2$ across the boundary and since $\partial C^E/\partial S^2 = 1$ at that point the result follows. A similar argument holds for the derivative relative to S^1. □

To prove the proposition it now suffices to apply Itô's lemma noting that $\partial u/\partial t + \mathcal{L}u = 0$ in the continuation region and $\partial u/\partial t + \mathcal{L}u = -\delta_2 S^2 + \delta_1 S^1$ in the exercise region. This establishes (4.4). The recursive equation (4.5) follows by imposing the boundary condition $C^{EC}(S^1, S^2, t) = S_t^1(B^{EC}(t) - 1)$ when $S^2 = B^{EC} S^1$. □

Proof of Proposition 6.1. (i) and (ii) are obvious. To prove (iii), suppose that there exists $\tau \in \mathcal{S}_{t,T}$ with $\tau > t$ such that $C^{\Sigma}(\lambda_1 S_t^1, \lambda_2 S_t^2, t) = E_t^*[e^{-r(\tau-t)}(\frac{1}{2}(\lambda_1 S_\tau^1 + \lambda_2 S_\tau^2) - K)^+]$. Then

$$\begin{aligned}
C^{\Sigma}(\lambda_1 S_t^1, \lambda_2 S_t^2, t) &= E_t^*[e^{-r(\tau-t)}(\tfrac{1}{2}S_\tau^1 + \tfrac{1}{2}S_\tau^2 - K + \tfrac{1}{2}(\lambda_1 - 1)S_\tau^1 + \tfrac{1}{2}(\lambda_2 - 1)S_\tau^2)^+] \\
&\leq E_t^*[e^{-r(\tau-t)}(\tfrac{1}{2}(S_\tau^1 + S_\tau^2) - K)^+] + \tfrac{1}{2}(\lambda_1 - 1)E_t^*[e^{-r(\tau-t)}S_\tau^1] \\
&\quad + \tfrac{1}{2}(\lambda_2 - 1)E_t^*[e^{-r(\tau-t)}S_\tau^2] \\
&\leq C^{\Sigma}(S_t^1, S_t^2, t) + \tfrac{1}{2}(\lambda_1 - 1)S_t^1 + \tfrac{1}{2}(\lambda_2 - 1)S_t^2 \\
&= \tfrac{1}{2}(S_t^1 + S_t^2) - K + \tfrac{1}{2}(\lambda_1 - 1)S_t^1 + \tfrac{1}{2}(\lambda_2 - 1)S_t^2 \\
&= \tfrac{1}{2}\lambda_1 S_t^1 + \tfrac{1}{2}\lambda_2 S_t^2 - K.
\end{aligned}$$

Assertion (iv) follows from the convexity of the payoff function and Proposition A.6.

To prove (v), note that if $(S_s^1, S_s^2, s) \notin \mathcal{E}^{\Sigma}$ then there exists $\tau \in \mathcal{S}_{s,T}$ with $\tau > s$ such that waiting until τ dominates immediate exercise. But since $t \leq s \leq T$, the strategy τ is feasible at t, and dominates immediate exercise. This contradicts $(S_t^1, S_t^2, t) \in \mathcal{E}^{\Sigma}$. □

Proof of Proposition 6.2. We have

$$\begin{aligned}
C^{\Sigma}(S_t^1, S_t^2, t) &= E_t^*[e^{-r(T-t)}(\tfrac{1}{2}(S_T^1 + S_T^2) - K)^+] \\
&\quad + \int_t^T e^{-r(T-t)} E_t^*[(\tfrac{1}{2}(\delta_1 S_v^1 + \delta_2 S_v^2) - rK)1_{\{S_v^2 \geq B^{\Sigma}(S_v^1, v)\}}]dv.
\end{aligned}$$

Let π_t denote the early exercise premium. We have

$$\begin{aligned}
S_v^2 \geq B^{\Sigma}(S_v^1, v) &\iff z^2 \geq \left[\log\left(\frac{B^{\Sigma}(S_v^1, v)}{S_t^2}\right) - (r - \delta_2 - \tfrac{1}{2}\sigma_2^2)(v - t)\right]\frac{1}{\sigma_2\sqrt{v - t}} \\
&\iff z^2 \geq d(S_t^2, B^{\Sigma}(S_v^1, v), v - t)
\end{aligned}$$

$$\begin{aligned}
&\iff \rho z^1 + \sqrt{1-\rho^2}u^{21} \geq d(S_t^2, B^\Sigma(S_v^1, v), v-t) \\
&\iff u^{21} \geq d(S_t^2, B^\Sigma(S_v^1, v), v-t)\frac{1}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}z^1 \\
&\iff u^{21} \geq d(S_t^2, B^\Sigma(S_v^1, v), v-t, \rho, z^1).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
\pi_t &= \int_t^T e^{-r(v-t)}\Bigg[\tfrac{1}{2}\delta_1 S_t^1 e^{(r-\delta_1)(v-t)}\int_{-\infty}^{\infty}\int_d^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z^1-\sigma_1\sqrt{v-t})^2}n(u^{21})du^{21}dz^1dv \\
&\qquad + \tfrac{1}{2}\delta_2 S_t^2 e^{(r-\delta_2)(v-t)} \\
&\qquad \int_{-\infty}^{\infty}\int_d^{\infty} e^{-\frac{1}{2}\sigma_2^2(v-t)+\sigma_2(\rho_{21}z^1+\sqrt{1-\rho_{21}^2}u^{21})\sqrt{v-t}}n(z^1)n(u^{21})du^{21}dz^1dv \\
&\qquad - rK\int_{-\infty}^{\infty}\int_d^{\infty} n(z^1)n(u^{21})du^{21}dz^1dv\Bigg] \\
&= \int_t^T \tfrac{1}{2}\delta_1 S_t^1 e^{-\delta_1(v-t)} \\
&\qquad \int_{-\infty}^{+\infty} n(w-\sigma_1\sqrt{v-t})N(-d(S_t^2, B^\Sigma(S_v^1(w), v), v-t, \rho, w))dwdv \\
&\quad + \int_t^T \tfrac{1}{2}\delta_2 S_t^2 e^{-\delta_2(v-t)}\int_{-\infty}^{\infty}\int_d^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z^1-\sigma_2\rho_{21}\sqrt{v-t})^2-\frac{1}{2}(u^{21}-\sigma_2\sqrt{1-\rho_{21}^2}\sqrt{v-t})^2}\frac{1}{\sqrt{2\pi}} \\
&\qquad e^{-\frac{1}{2}\sigma_2^2(v-t)+\frac{1}{2}\sigma_2^2\rho_{21}^2(v-t)+\frac{1}{2}\sigma_2^2(1-\rho_{21}^2)(v-t)}du^{21}dz^1dv \\
&\quad - \int_t^T rKe^{-r(v-t)}\int_{-\infty}^{\infty} n(w)N(-d(S_t^2, B^\Sigma(S_v^1(w), v), v-t, \rho, w))dw\,dv.
\end{aligned}$$

It is easy to verify that the double integral in the second term equals

$$\begin{aligned}
&\int_{-\infty}^{+\infty} n(w-\sigma_2\rho_{21}\sqrt{v-t})N(-d(S_t^2, B^\Sigma(S_v^1(w), v), v-t, \rho, w) \\
&\qquad + \sigma_2\sqrt{1-\rho_{21}^2}\sqrt{v-t})dw\,dv.
\end{aligned}$$

Defining $\tilde{\Phi}(S_t^2, B^\Sigma(\cdot, v), v-t, \rho, x, y) \equiv \int_{-\infty}^{\infty} n(w-y)N(-d(S_t^2, B^\Sigma(S_v^1(w), v), v-t, \rho, w)+x)dw$ and substituting in the expression above yields the formula in the proposition. □

Proof of Proposition 7.1. Let $S^{(m)}$ denote an m-dimensional subset of $\{S^1, \ldots, S^n\}$. Then $\forall m < n$ we have,

$$C^{X,n}(S, t) \geq C^{X,m}(S^{(m)}, t)$$

In particular for $m = 2$ the lower bound is $C^{X,2}(S^{(2)}, t)$. Now suppose that there exists i and j, $i \neq j$, $(i, j) \in \{1, \ldots, n\}$ such that $\max(S^1, \ldots, S^n) = S^i = S^j$. Then selecting $S^{(2)} = (S^i, S^j)$ yields $C^{X,n}(S, t) \geq C^{X,2}(S^i, S^j, t)$. An application of Proposition 2.1 now shows that $C^{X,2}(S^i, S^j, t) > (S^i - K)^+ = (S^j - K)^+$. The result follows. $\square$

REFERENCES

BENSOUSSAN, A. (1984): "On the Theory of Option Pricing," *Acta Appl. Math.*, 2, 139–158.

BENSOUSSAN, A., and J. L. LIONS (1978): *Applications des Inéquations Variationnelles en Contrôle Stochastique*. Paris: Bordas (Dunod).

BOYLE, P. P., J. EVNINE, and S. GIBBS (1989): "Numerical Evaluation of Multivariate Contingent Claims," *Rev. Financial Stud.*, 2, 241–250.

BROADIE, M., and J. DETEMPLE (1995a): "American Capped Call Options on Dividend-Paying Assets," *Rev. Financial Stud.*, 8, 161–191.

BROADIE, M., and J. DETEMPLE (1996): "American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods," *Review of Financial Studies*, 9, 1211–1250.

BROADIE, M., and P. GLASSERMAN (1994): "Pricing American-Style Securities Using Simulation," unpublished manuscript, Columbia Univerity.

CARR, P., R. JARROW, and R. MYNENI (1992): "Alternative Characterizations of American Put Options," *Math. Finance*, 2, 87–106.

DEMPSTER, M. A. H. (1994): "Fast Numerical Valuation of American, Exotic, and Complex Options," unpublished manuscript, University of Essex.

DIXIT, A., and R. PINDYCK (1994): *Investment Under Uncertainty*. Princeton, NJ: Princeton University Press.

DRAVID, A., M. RICHARDSON, and T. S. SUN (1993): "Pricing Foreign Index Contingent Claims: An Application to Nikkei Index Warrants," *J. Derivatives*, 1, 1, 33–51.

EL KAROUI, N., and I. KARATZAS (1991): "A New Approach to the Skorohod Problem and its Applications," *Stoch. Stoch. Rep.*, 34, 57–82.

GELTNER, D., T. RIDDIOUGH, and S. STOJANOVIC (1994): "Insights on the Effect of Land Use Choice: The Perpetual Option on the Best of Two Underlying Assets," unpublished manuscript, MIT.

JACKA, S. D. (1991): "Optimal Stopping and the American Put," *Math. Finance*, 1, 1–14.

JAILLET, P., D. LAMBERTON, and B. LAPEYRE (1990): "Variational Inequalities and the Pricing of American Options," *Acta Appl. Math.*, 21, 263–289.

JOHNSON, H. (1981): "The Pricing of Complex Options," unpublished manuscript, Louisiana State University.

JOHNSON, H. (1987): "Options on the Maximum or the Minimum of Several Assets," *J. Financial Quant. Anal.*, 22, 227–283.

KARATZAS, I. (1988): "On the Pricing of American Options," *Appl. Math. Optim.*, 17, 37–60.

KIM, I. J. (1990): "The Analytic Valuation of American Options," *Rev. Financial Stud.*, 3, 547–572.

MARGRABE, W. (1978): "The Value of an Option to Exchange One Asset for Another," *J. Finance*, 33, 177–186.

MERTON, R. C. (1973): "Theory of Rational Option Pricing," *Bell J. Econ. Mgt. Sci.*, 4, 141–183.

MYNENI, R. (1992): "The Pricing of the American Option," *Ann. Applied Prob.*, 2, 1–23.

RUBINSTEIN, M. (1991): "One for Another," *Risk*, July–August.

RUTKOWSKI, M. (1994): "The Early Exercise Premium Representation of Foreign Market American Options," *Math. Finance*, 4, 313–325.

STULZ, R. M. (1982): "Options on the Minimum or the Maximum of Two Risky Assets," *J. Financial Econ.*, 10, 161–185.

TAN, K., and K. VETZAL, (1994): "Early Exercise Regions for Exotic Options," unpublished manuscript, University of Waterloo.

VAN MOERBEKE, P. (1976): "On Optimal Stopping and Free Boundary Problems," *Arch. Rational Mech. Anal.*, 60, 1976, 101–148.