International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 1 || January. 2014 || PP-41-48

On The Value Distribution of Some Differential Polynomials

[1,] Subhas.S.Bhoosnurmath, [2,] K.S.L.N.Prasad
[1,] *Department of Mathematics, Karnatak University, Dharwad-580003-INDIA*
[2,] *Lecturer, Department of Mathematics, Karnatak Arts College, Dharwad-580001-INDIA*

ABSTRACT: *We prove a value distribution theorem for meromorphic functions having few poles, which improves several results of C.C.Yang and others.Also we obtain a generalized form of Clunie's result.*

C. C. Yang [8] has stated the following.

Theorem A Let f(z) be a transcendental meromorphic function with N(r, f) = S(r, f).

If $P[f] = f^n + a_1\pi_{n-1}(f) + a_2\pi_{n-2}(f) + ... + a_n$ (1)

Where each $\pi_i(f)$ is a homogeneous differential polynomial in f of degree i, then $T(r, P[f]) = nT(r,f) + S(r,f)$.

This result is required at a later stage and we prove the above theorem on the lines of G. P. Barker and A. P. Singh [1].

Proof we have, $P[f] = f^n + a_1\pi_{n-1}(f) + a_2\pi_{n-2}(f) + ... + a_n$

$$= f^n\left\{1 + \frac{a_1\pi_{n-1}(f)}{f^n} + \frac{a_2\pi_{n-2}(f)}{f^n} + ... + \frac{a_n}{f^n}\right\}$$

$$= f^n\left\{1 + \frac{A_1}{f} + \frac{A_2}{f^2} + ... + \frac{A_n}{f^n}\right\} \quad (2)$$

Where $A_i = \frac{a_i\pi_{n-i}(f)}{f^{n-i}}, i = 1,2,......n$

Now, since each $\pi_n(f)$ is a homogeneous differential polynomial in f of degree n,

we have , $$m\left(r, \frac{\pi_n(f)}{f^n}\right) = m\left(r, \frac{\sum f^{l_0}\left(f^{(1)}\right)^{l_1}......\left(f^{(k)}\right)^{l_k}}{f^n}\right)$$

$$= m\left(r, \sum\left(\frac{f^{(1)}}{f}\right)^{l_1}\left(\frac{f^{(2)}}{f}\right)^{l_2}....\left(\frac{f^{(k)}}{f}\right)^{l_k}\right)$$

$$\le \sum m\left(r, \left(\frac{f^{(1)}}{f}\right)^{l_1}\left(\frac{f^{(2)}}{f}\right)^{l_2}....\left(\frac{f^{(k)}}{f}\right)^{l_k}\right) + O(1)$$

= S(r, f), using Milloux's theorem [10].

Hence $m(r, A_i) = S(r, f)$, for i = 1,2, . . . ,n.

Now on the circle |z | = r, let

$$A(re^{i\theta}) = \text{Max}\left\{ |A_1(re^{i\theta})|, |A_2(re^{i\theta})|^{1/2}, \ldots\ldots\ldots, |A_n(re^{i\theta})|^{1/n} \right\}$$

Then, m (r, A(z)) = S (r, f) (3)

Let $E_1 = \left\{ \theta \in [0,2\pi] \,\middle|\, |f(re^{i\theta})| > |2A(re^{i\theta})| \right\}$ and E_2 be the complement of E_1.

Then, on E_1, we have

$$|P[f]| = |f^n| \left| 1 + \frac{A_1}{f} + \frac{A_2}{f^2} + \ldots\ldots + \frac{A_n}{f^n} \right|$$

$$\geq |f|^n \left\{ 1 - \left|\frac{A_1}{f}\right| - \left|\frac{A_2}{f^2}\right| - \ldots\ldots\ldots - \left|\frac{A_n}{f^n}\right| \right\}$$

$$\geq |f|^n \left\{ 1 - \frac{1}{2} - \frac{1}{2^2} - \ldots\ldots\ldots - \frac{1}{2^n} \right\}$$

$$= \frac{1}{2^n} |f|^n$$

Hence, on E_1, $\log^+ 2^n |P[f]| \geq \log^+ \left[|f|^n \right]$

Therefore, $n \log^+ |f| \leq n \log 2 + \log^+ |P[f]|$

Therefore, $n\, m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} n \log^+ |f|\, d\theta$

$$= \frac{1}{2\pi} \int_{E_1} n \log^+ |f|\, d\theta + \frac{1}{2\pi} \int_{E_1} n \log^+ |f|\, d\theta$$

$$\leq \frac{1}{2\pi} \int_{E_1} \left(n \log 2 + \log^+ |P[f]| \right) d\theta + \frac{n}{2\pi} \int_{E_2} \log^+ |2A|\, d\theta$$

$$\leq \frac{1}{2\pi} \int_{E_1} \log^+ |P[f]|\, d\theta + \frac{n}{2\pi} \int_{E_2} \log^+ |A|\, d\theta + O(1)$$

$$= m(r, P[f]) + n\, m(r, A) + O(1)$$

$$= m(r, P[f]) + S(r, f).$$

Thus, n m (r, f) ≤ m (r, P([f]) + S(r, f). (4)

Adding n N(r, f) both sides and noting N(r, f) = S(r, f), we get,

nT(r, f) ≤ m (r, P[f]) + S(r, f).

Or n T (r, f) ≤ T(r, P(f)) + S(r, f) (5)

Now, m(r, P[f]) = $m\left(r, f^n + A_1 f^{n-1} + A_2 f^{n-2} + \ldots\ldots + A_{n-1} f + A_n\right)$, by (2)

$$\leq m\left(r, f\{f^{n-1} + A_1 f^{n-2} + \ldots\ldots + A_{n-1}\}\right) + m(r, A_n) + O(1)$$

$$\leq m(r, f) + m\left(r, f^{n-1} + A_1 f^{n-2} + \ldots\ldots + A_{n-1}\right) + S(r, f)$$

Proceeding on induction, we have

$$m(r, P[f]) \le n\ m(r, f) + S(r, f)$$

Therefore, T(r, P[f]) ≤ n m(r, f) + N(r, P[f] + S(r, f)

But, N(r, P[f]) = S(r, f), since N (r, f) = S(r, f).

Hence, T(r, P[f]) ≤ n T(r, f) + S(r, f) (6)

From (5) and (6) we have the required result.

We wish to prove the following result

Theorem 1 Let f(z) be a transcendental meromorphic function in the plane and $Q_1[f],\ Q_2[f]$ be differential polynomials in f satisfying $Q_1[f] \not\equiv 0, Q_2[f] \not\equiv 0.$ Let P(f) be as defined in (1).

If F = P[f] $Q_1[f] + Q_2[f]$, (7)

Then $\left(n - \gamma_{Q_2}\right)T(r, f) \le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r, f) + S(r, f)$

To prove the above theorem, we require the following Lemmas.

Lemma 1 [5]: If Q[f] is a differential polynomial in f with arbitrary meromorphic co-efficients $q_j, 1 \le j \le n,$ then

$$m(r, Q[f]) \le \gamma_Q\ m(r, f) + \sum_{j=1}^{n} m(r, q_j) + S(r, f).$$

Lemma 2 [5]: Let $Q^*[f]$ and $Q[f]$ denote differential polynomials in f with arbitrary meromorphic co-efficients $q_1^*, q_2^*, \ldots, q_n^*$ and $q_{1,} q_2, \ldots, q_s$ respectively.

If P[f] is a homogeneous differential polynomials in f of degree n and $P[f]Q^*[f] = Q[f]$ where $\gamma_Q \le n$, then

$$m(r, Q^*[f]) \le \sum_{j=1}^{n} m(r, q_j^*) + \sum_{j=1}^{s} m(r, q_j) + S(r, f).$$

This leads to a generalised form of Clunie's result.

Lemma 3 [5]: Suppose that M[f] is a monomial in f. If f has a pole at z = z_0 of order m, then z_0 is a pole of M[f] order (m-1) $\gamma_m + \Gamma_M$.

Lemma 4 [5]: Suppose that Q[f] is a differential polynomial in f. Let z_0 be a pole of f of order m and not a zero or a pole of the co-efficients of Q[f]. Then z_0 is a pole of Q[f] of order atmost m $\gamma_Q + \left(\Gamma_Q - \gamma_Q\right)$

Proof of Theorem 1

Let us suppose that $n > \gamma_{Q_2}$.

We have F = P[f] Q_1 [f] + Q_2 [f]

Therefore, $F' = \frac{F'}{F} P[f] Q_1[f] + \frac{F'}{F} Q_2[f]$

and $F' = P[f]' Q_1[f] + P[f](Q_1[f])' + (Q_2[f])'$

Hence, it follows that $P[f]Q^*[f] = Q[f]$ (8)

where $Q^*[f] = \frac{F'}{F} Q_1[f] - \frac{(P[f])'}{P[f]} Q_1[f] - (Q_1[f])'$ (9)

and $Q[f] = (Q_2[f])' - \frac{F'}{F} Q_2[f].$ (10)

We have by Lemma 2 and (8),

$$m(r, Q^*[f]) = S(r, f) \tag{11}$$

Again from (8), $P[f] = \frac{Q[f]}{Q^*[f]}$

Therefore, $m(r, P[f]) \leq m(r, Q[f]) + m\left(r, \frac{1}{Q^*[f]}\right)$ (12)

From (10) and Lemma 1, we have

$$m(r, Q[f]) \leq \gamma_{Q_2} m(r, f) + S(r, f). \tag{13}$$

By the First Fundamental Theorem and (11), we get

$$m\left(r, \frac{1}{Q^*(f)}\right) = N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) + S(r, f). \tag{14}$$

Clearly, the poles of $Q^*[f]$ occur only from the zeros and poles of F and P[f], the poles of f and the zeros and poles of the co efficients. Suppose that z_0 is a pole of f of order m, but not a zero or a pole of the co-efficients of $P[f]$, $Q_1[f]$ and $Q_2[f]$.

Hence, from Lemma 4, z_0 is a pole of Q[f] of order atmost $m\,\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$

If z_0 is a pole of $Q^*[f]$, we have from (8), $Q^*[f] = \frac{Q[f]}{P[f]}$

and hence z_0 is a pole of $Q^*[f]$ of order atmost $m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) - mn$

$= m(\gamma_{Q_2} - n) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$

Also, from (8), $\frac{1}{Q^*[f]} = \frac{P[f]}{Q[f]}$

Hence z_0 is a zero of Q*[f] of order atleast $mn - \{m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\}$

Thus, we have,

$$N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(\frac{1}{P[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r, f)$$
$$+ (\gamma_{Q_2} - n)N(r, f) + S(r, f) \tag{15}$$

we have from (8), $P[f] = \frac{Q[f]}{Q^*[f]}$

Therefore, $m(r, P[f]) = m\left(r, \frac{Q[f]}{Q*[f]}\right)$

Therefore, $m(r, P[f]) \le m(r, Q[f]) + m\left(r, \frac{1}{Q*[f]}\right)$

Using (14) we have,

$$m(r, P[f]) \le m(r, Q[f]) + N(r, Q*[f]) - N\left(r, \frac{1}{Q*[f]}\right) + S(r, f)$$

Using (4), (13) and (15) we have

$$n\, m(r, f) \le \gamma_{Q_2} m(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r, f)$$
$$+ \left(\gamma_{Q_2} - n\right) N(r, f) + S(r, f).$$

Hence, we get , $\left(n - \gamma_{Q_2}\right) T(r, f) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r, f) + S(r, f),$

which is the required result.

Remark (1) Putting P[f] = f^n in the above result, we get

$F = f^n Q_1[f] + Q_2[f]$ and

$$\left(n - \gamma_{Q_2}\right) T(r, f) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r, f) + S(r, f),$$

which is the result of Zhan Xiaoping [9]

(2) Putting $P[f] = a_n f^n + a_{n-1} f^{n-1} + ... + a_0$ in the above result,

We get, $F = P[f]\ Q_1[f] + Q_2[f]$ and

$$\left(n - \gamma_{Q_2}\right) T(r, f) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r, f) + S(r, f)$$

which is the result of Hong Xun Yi[4].

Theorem 2 Let $F = f^n Q[f]$, where Q[f] is a differential polynomial in f and Q[f] $\not\equiv 0$.

$$\text{If } \varlimsup_{r \to \infty} \frac{\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f)}{T,(r, f)} < n, \text{ then } \Theta(a, F) < 1 \quad \text{where } a \ne 0.$$

Proof Let $F = f^n\ Q[f]$. Putting $Q_2[f] \equiv 0$ in the result of Zhan Xiaoping (remark 1) we have,

$$n\, T(r, f) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f)$$

Hence $n - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r,f)} < n$ using hypothesis.

Therefore, $\overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r,f)} > 0$

Hence, $\overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F-a}\right)}{T(r,F)} > 0$, for $a \not\equiv 0$.

Therefore, $1 - \Theta(a,F) > 0$

Or $\Theta(a,F) < 1$ for $a \not\equiv 0$.

Hence the result.

Theorem 3 Let $F = f^n Q_1[f] + Q_2[f]$ where $Q_1[f]$ and $Q_2[f]$ are as defined in Theorem 1.

If $\overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_!} + 1\right)\overline{N}(r,f)}{T(r,f)} < \left(n - \gamma_{Q_2}\right)$ and

$Q_2[f] \not\equiv 0$, then $\Theta(a,F) < 1$ where $a \not\equiv 0$.

Proof Let $F = f^n Q_1[f] + Q_2[f]$ where $Q_1[f]$ and $Q_2[f]$ are as defined in Theorem .

By the result of Zhan Xiaoping [Remark 1], we have

$$(n - \gamma_{Q_2}) + T(r,f) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{f}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\,\overline{N}(r,f) + S(r,f).$$

Or $(n - \gamma_{Q_2}) - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r,f)} \le (n - \gamma_{Q_2})$ using hypothesis

Hence, $\overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F-a}\right)}{T(r,f)} > 0$

which implies $1 - \Theta(a,F) > 0$.

Or $\Theta(a,F) < 1$ for $a \not\equiv 0$.

Hence the result.

Theorem 4 Let $F = P[f]Q[f]$ where P[f] is as defined in (1) and Q[f] is a differential polynomial in f such that Q[f] $\not\equiv$ 0. If n > 1, then $\rho_F = \rho_f$ and $\lambda_F = \lambda_f$

Proof Let G = F – a, where a(≠0) is a finite complex number. Then by Theorem 1,

$$n\,T(r,f) \le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f)$$

Obviously, the zeros and poles of f are that of F respectively

Therefore, $\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,f)$

Thus, n T(r, f) $\le \overline{N}\left(r,\frac{1}{F-a}\right) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,f)$

$$\le 3T(r,F) + S(r,f)$$

Therefore, $T(r,f) = O\{T(r,F)\}$ as $r \to \infty$

Also, $T(r,F) = O\{T(r,f)\}$ as $r \to \infty$

Hence the theorem follows.

As an application of Theorem 1, we observe the following .

Theorem 5 No transcendental meromorphic function f can satisfy an equation of the form $a_1 P[f]Q[f] + a_2 Q[f] + a_3 = 0$,where $a_1 \not\equiv 0,\ a_3 \not\equiv 0,\ P[f]$ is as in (1) and Q[f] is a differential polynomial in f.

Putting $P[f] = f^n$ in the above theorem, we have the following.

Theorem 6 No transcendental meromorphic function f can satisfy an equation of the form $a_1 f^n Q[f] + a_2 Q[f] + a_3 = 0$,where $a_1 \ne 0,\ a_3 \ne 0,$ n is a positive integer and Q[f] is a differential polynomial in f.

This improves our earlier result namely ,

Theorem 7: No transcendental meromorphic function f with N(r, f) = S(r, f) can satisfy an equation of the form

$$a_1(z)[f(z)]^n \pi(f) + a_2(z)\pi(f) + a_3(z) = 0 \qquad (1)$$

where $n \ge 1, a_1(z) \not\equiv 0$ and $\pi(f) = M_i(f) + \sum_{j=1}^{i-1} a_j(z) M_j(f)$ is a differential polynomial in f of degree n and each $M_i(f)$ is a monomial in f.

(Communicated to Indian journal of pure and applied mathematics)

Acknowledgement: The second author is extremely thankful to University Grants Commission for the financial assistance given in the tenure of which this paper was prepared

REFERENCES

[1.] Barker G. P. And Singh A. P. (1980) : Commentarii Mathematici Universitatis : Sancti Pauli 29, 183.
[2.] Gunter Frank And Simon Hellerstein (1986) : 'On The Meromorphic Solutions Of Non Homogeneous Linear Differential Equation With Polynomial Co-Efficients', Proc. London Math. Soc. (3), 53, 407-428.
[3.] Hayman W. K. (1964) : Meromorphic Functions, Oxford Univ. Press, London.
[4.] Hong-Xun Yi (1990) : 'On A Result Of Singh', Bull. Austral. Math. Soc. Vol. 41 (1990) 417-420.
[5.] Hong-Xun Yi (1991) : "On The Value Distribution Of Differential Polynomials", Jl Of Math. Analysis And Applications 154, 318-328.
[6.] Singh A. P. And Dukane S. V. (1989) : Some Notes On Differential Polynomial Proc. Nat. Acad. Sci. India 59 (A), Ii.
[7.] Singh A. P. And Rajshree Dhar (1993) : Bull. Cal. Math. Soc. 85, 171-176.
[8.] Yang C. C. (1972) : 'A Note On Malmquist's Theorem On First Order Differential Equations, Ordering Differential Equations', Academic Press, New York.
[9.] Zhan-Xiaoping (1994) : "Picard Sets And Value Distributions For Differential Polynomials", Jl. Of Math. Ana. And Appl. 182, 722-730.