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The Values of k-th Order of Entire Functions Share 1 CM with its Linear Differential Polynomial

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Abstract

In this paper, we discuss the values of k-th order of the entire functions which with its linear differential polynomial share 1 CM. We prove that if $k(\geq 3)$ is an integer, for the given non-negative real number λ (may be infinity), then there exists an entire function $f(z)$ which with its linear differential polynomial share 1 CM, such that the k-th order of $f(z)$ is λ.

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1 Introduction and main results

In 1925, R. Nevanlinna developed a systematic study of the value distribution theory by means of his First and Second Fundamental Theorems. In this paper, we assume that the reader is familiar with fundamental results and standard notations of value distribution theory [8, 22]. The order and hyper-order of an entire function $f(z)$ are respectively defined by

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$
$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where, and in the sequel, $M(r,f)$ denotes the maximum modulus of $f(z)$ on the circle $\|z\| = r$.

Suppose that $f(z)$ and $g(z)$ are two non-constant meromorphic functions in the complex plane $\mathbb{C}$, and a is a finite complex number. We say that $f(z)$ and $g(z)$ share a CM, provided that $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. Similarly, we say that $f(z)$ and $g(z)$ share a IM, provided that $f(z) - a$ and $g(z) - a$ have the same zeros, where the multiplicities are not taken into account. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to +\infty$, possibly outside a set of r with finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ provided that $T(r,a) = S(r,f)$.

In 1977, L. A. Rubel and C. C. Yang proved the following result.

Theorem A [17]**.** Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.

This result has been generalized to sharing values IM by E. Mues and N. Steinmetz.

Theorem B [14]**.** Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two distinct values $a, b \in \mathbb{C}\backslash\{0\}$ IM, then $f(z) \equiv f'(z)$.

For examples (see [1]) are given by the solutions of the following differential equations

$$\frac{f'(z) - 1}{f(z) - 1} = e^{z^n}$$

and

$$\frac{g'(z) - 1}{g(z) - 1} = e^{e^z}.$$

In this case we note respectively that $\rho_2(f) = n$ $(n \in \mathbb{N})$ and $\rho_2(g) = +\infty$.

What can be said when a non-constant entire function $f(z)$ share one finite value CM with its derivative $f'(z)$? In 1996, R. Brück [1] proposed the following conjecture.

Brück Conjecture [1] Let $f(z)$ be a non-constant entire function of finite hyper-order $\rho_2(f)$ that is not a positive integer. If $f(z)$ and $f'(z)$ share a finite value a CM, then

$$\frac{f'(z) - a}{f(z) - a} = c, \tag{1.1}$$

for some constant $c \in \mathbb{C}\backslash\{0\}$.

Let $F(z) = f(z) - a$, then we can easily deduce

$$F'(z) = (f(z) - a)' = f'(z). \tag{1.2}$$

Substituting (1.2) into (1.1), we have

$$\frac{F'(z) - a}{F(z)} = c.$$

For simplicity of notation, we still denote $f(z) - a$ as $f(z)$, and let $a = 1$. Then Brück conjecture can be briefly described in the following form.

Brück Conjecture Let $f(z)$ be a non-constant entire function of finite hyper-order $\rho_2(f)$ that is not a positive integer. If

$$\frac{f'(z) - 1}{f(z)} = e^{\alpha(z)},$$

where $\alpha(z)$ is an entire function, then $\alpha(z) = c$ with c is some constant.

In full generality, this conjecture remains open. However, Brück himself proved the claim provided that either $a = 0$ or $N(r, \frac{1}{f'}) = S(r, f)$, see [1]. In the same paper, he also gave counterexamples to show that the conjecture fails when $f(z)$ and $f'(z)$ share a IM and that the restriction on the growth of $f(z)$ is necessary. In 1998, G. G. Gundersen and L. Z. Yang proved that the conjecture is true if $f(z)$ is of finite order, see [7]. In 2004, Z. X. Chen and K. H. Shon proved that the conjecture is also true for $f(z)$ of hyper-order $\rho_2(f) < \frac{1}{2}$, see [4]. In 2016, T. B. Cao affirmed the conjecture for the case where $f(z)$ is of hyper-order $\rho_2(f) = \frac{1}{2}$, see [2]. Recently, M. F. Chen, Z. S. Gao and J. L. Zhang affirmed the conjecture when $f(z)$ is of hyper-order $\rho_2(f) < 1$, see [3].

There are many results closely related to Brück conjecture, mainly in three directions. One replaces the shared value by a non-constant function, such as polynomial, small function with respect to $f(z)$, or entire functions with lower order than $f(z)$, see [6, 13, 18]. The second direction is to consider arbitrary k-th derivatives $f^{(k)}(z)$ instead of $f'(z)$, see [5, 11, 21]. The other direction is to extend $f'(z)$ to a linear differential polynomial, see [3, 15, 19].

By studying the growth of entire functions satisfying a certain functional equation, we have a new understanding of Brück conjecture. For further studying the growth of entire functions, we introduce the following concept. Let $f(z)$ be an entire function and

$$\tau_1 = \log^+ r \quad (r \geq 0), \tau_{k+1} = \log^+ \tau_k \quad (r \geq 0, \quad k = 1, 2, \cdots).$$

We denote by $\rho_k(f)$ the k-th order of $f(z)$ (see [9, 10]), i.e.,

$$\rho_k(f) = \overline{\lim_{r\to+\infty}} \frac{\tau_k(T(r,f))}{\tau_1(r)} = \overline{\lim_{r\to+\infty}} \frac{\tau_{k+1}(M(r,f))}{\tau_1(r)}$$

$$= \overline{\lim_{r\to+\infty}} \frac{\tau_k(\nu(r,f))}{\tau_1(r)} \quad (k=1,2,\cdots),$$

where, and in the sequel, $\nu(r,f)$ denotes the central index of $f(z)$. In particular, $\rho_1(f)$ denotes the order of $f(z)$, and $\rho_2(f)$ denotes the hyper-order of $f(z)$.

For convenience, we denote by $\mathcal{E}$ the ring of entire functions and by $\mathcal{E}_1$ the subring of $\mathcal{E}$ consisting of $f(z)$ and $f'(z)-1$ share 0 CM, that is,

$$\mathcal{E}_1{:=}\{f(z) : f(z) \text{ and } f'(z)-1 \text{ share 0 CM, and } f(z)\in\mathcal{E}\}.$$

Let $L_j(f(z)) = f^{(j)}(z) + a_{j-1}f^{(j-1)}(z) + \cdots + a_1f'(z) + a_0f(z)$, where j is a positive integer and $a_0, a_1, \cdots, a_{k-1}$ are complex numbers. And we denote by $\mathcal{E}_j$ the subring of $\mathcal{E}$ consisting of $f(z)$ and $L_j(f(z))-1$ share 0 CM, that is,

$$\mathcal{E}_j{:=}\{f(z) : f(z)\in\mathcal{E}, f(z) \text{ and } L_j(f(z))-1 \text{ share 0 CM}, j=1,2,3,\cdots\}.$$

Obviously, for the positive integer k, we have

$$\rho_k(\mathcal{E}) = \{\rho_k(f) : f(z)\in\mathcal{E}\} = [0,+\infty)\cup\{+\infty\}.$$

If Brück conjecture is valid if and only if

$$\rho_2(\mathcal{E}_1) = \{\rho_2(f) : f(z)\in\mathcal{E}_1\} = \{0\}\cup\mathbb{N}\cup\{+\infty\}.$$

It is easy to get that $\rho_2(\mathcal{E}_1)$ is a discrete point set. For the value of the order of $f(z)\in\mathcal{E}_j$ $(j=1,2,3,\cdots)$, we can get the following result.

Theorem 1 For any entire function $f(z)\in\mathcal{E}_j$ $(j=1,2,3,\cdots)$, we can get that the value of $\rho_1(f)$ is 0, 1 or $+\infty$.

Corollary 1 For any entire function $f(z)\in\mathcal{E}_1$, we can get that the value of $\rho_1(f)$ is 0, 1 or $+\infty$.

It is natural to ask the following question: If $k(\geq 3)$ is an integer, for the given non-negative real number λ (may be $+\infty$), is there an entire function $f(z)\in\mathcal{E}_j$ $(j=1,2,3,\cdots)$ such that $\rho_k(f)=\lambda$? In this paper, we give an affirmative answer to this question. In fact, we shall prove the following theorem.

Theorem 2 If the positive integer $k\geq 3$, we have

$$\rho_k(\mathcal{E}_j) = [0,+\infty)\cup\{+\infty\},$$

where j is a positive integer.

Corollary 2 If the positive integer $k\geq 3$, we have

$$\rho_k(\mathcal{E}_1) = [0,+\infty)\cup\{+\infty\}.$$

Let k be a positive integer, we define the following functional

$$\rho_k : \begin{array}{ccc} \mathcal{E}_1 & \longrightarrow & [0,+\infty)\cup\{+\infty\} \\ f & \longmapsto & \rho_k(f) \end{array}.$$

From our Theorem 1, 2 and Corrolary 1, 2, we can get to know the following facts. If Brück conjecture is valid, then we can get that only the values of functional ρ_1 and ρ_2 are discontinuous. However, we can get that the value of functional $\rho_k (k \geq 3)$ is continuous. In fact, we can obtain that the value of functional ρ_1 is $\{0, 1, +\infty\}$ and the value of functional $\rho_k (k \geq 3)$ is $[0, +\infty) \cup \{+\infty\}$. Under the premise that Brück conjecture is true, it is easy to get that the value of functional is $\{0, 1, 2, 3, \cdots, +\infty\}$. If we can get the value of functional ρ_2, then we can further investigate Brück conjecture. For example, if we can find an entire function $f(z) \in \mathcal{E}_1$, such that $\rho_2 \notin \{0\}\cup\mathbb{N}\cup\{+\infty\}$, then we can get Brück conjecture is not true. For any $f(z) \in \mathcal{E}_j$ $(j = 1, 2, 3, \cdots)$, if we can get $\rho_2(f) \in \{0\} \cup \mathbb{N} \cup \{+\infty\}$, then we can get Brück conjecture is true in the case of $f(z)$ is replaced by a linear differential polynomial.

2 Some lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1 [14, 22] Suppose $f(z)$ is a non-constant meromorphic function in the complex plane $\mathbb{C}$ and k a positive integer, and let

$$\psi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z),$$

where $a_1(z), a_2(z), \cdots, a_k(z)$ are small functions of $f(z)$. Then

$$m(r, \frac{\psi}{f}) = S(r, f)$$

and

$$T(r, \psi) \leq T(r, f) + k\overline{N}(r, f) + S(r, f) \leq (k+1)T(r, f) + S(r, f).$$

Lemma 2 [20] Suppose $f(z)$ is a non-constant entire function. Then for $0 \leq r < R < +\infty$, we have

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

Lemma 3 [10] Let $f(z)$ be a transcendental entire function and a set $E \subset (1, +\infty)$ have finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $\|f(z_k)\| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, $r_k \to \infty$,

(i) if $0 < \rho(f) < \infty$, then for any given $\varepsilon > 0$ and sufficiently large r_k, $r_k^{\rho(f)-\varepsilon} < v(r_k, f) < r_k^{\rho(f)+\varepsilon}$.

(ii) if $\rho(f) = \infty$, then for any given $M > 0$, and sufficiently large r_k, $v(r_k, f) > r_k^M$.

Lemma 4 Let $Q(z)$ ba a non-constant polynomial, $A_k(z)(\not\equiv 0), A_{k-1}(z), \cdots, A_0(z)$ be polynomials, and k be a positive integer. If $f(z)$ is a solution of the equation

$$A_k(z)f^{(k)}(z) + \cdots + A_1(z)f'(z) - (e^{Q(z)} - A_0(z))f(z) = B(z) \tag{2.1}$$

such that $\rho(f) > 1 + \max\limits_{0\leq j\leq k-1}\left\{\frac{degA_j - degA_k}{k-j}, 0\right\}$, where $B(z)$ is an entire function that is small with respect to $f(z)$, then $\rho(f) = \infty$ and $\rho_2(f) = \deg Q(z)$.

Proof Suppose that $f(z)$ is an entire solution of (2.1) with $\rho(f) > 1 + \max\limits_{0\leq j\leq k-1}\{\frac{\deg A_j - \deg A_k}{k-j}, 0\}$, by using the same reasoning as in the proof lemma 2.4 (see [15]), we obtain $\rho(f) = \infty$. Next, we only need to prove $\rho_2(f) = \deg Q(z)$, where $Q(z)$ is a non-constant polynomial.

It follows from that

$$A_k(z)\frac{f^{(k)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)} + A_0(z) - \frac{B(z)}{f(z)} = e^{Q(z)}. \tag{2.2}$$

By the Wiman-Valiron theory, there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure (that is $\int_{E_1} t^{-1}dt < +\infty$) such that for all z satisfying $\|z\| = r \notin [0,1] \cup E_1$ and $\|f(z)\| = M(r, f)$, we get

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v(r,f)}{z}\right)^j (1 + o(1)),\ j = 1, 2, \cdots, k,\ k \in \mathbb{N} \tag{2.3}$$

as $r \to \infty$.

By lemma 3, there exists $\{z_m = r_m e^{i\theta_m}\}$ with

$$\|f(z_m)\| = M(r_m, f),\ \theta_m \in [0, 2\pi],\ \lim_{m\to\infty} \theta_m = \theta_0 \in [0, 2\pi],\ r_m \notin [0,1] \cup E_1,$$

such that for any given large $M > 0$ and sufficiently large r_m, we have

$$v(r_m, f) > r_m^M \tag{2.4}$$

Since $B(z)$ is an entire function that is small with respect to $f(z)$ and $\|f(z_m)\| = M(r_m, f)$, we get

$$\left\|\frac{B(z_m)}{f(z_m)}\right\| = o(1) \tag{2.5}$$

for sufficiently large $r_m \notin E_1$. (We remark that $B(z)$ is identically to zero, the proof will still be valid.)

Substituting (2.3)-(2.5) into (2.2) yield

$$A_k(z_m)\left(\frac{v(r_m, f)}{z_m}\right)^k (1 + o(1)) = e^{Q(z_m)}. \tag{2.6}$$

By (2.6), we get

$$\begin{aligned}\log\|e^{Q(z_m)}\| &= \log\left\|A_k(z_m)(\frac{v(r_m,f)}{z_m})^k(1+o(1))\right\| \\ &= \log\|A_k(z_m)\| + k\log v(r_m, f) - k\log r_m + O(1),\end{aligned}$$

that is,

$$\log\|A_k(z_m)\| + k\log v(r_m, f) = \log\|e^{Q(z_m)}\| + k\log r_m + O(1). \tag{2.7}$$

Set $Q(z)=b_nz^n+b_{n-1}z^{n-1}+\cdots+b_1z+b_0$ is a polynomial with degree $\deg Q(z)=n$ and $b_n\neq 0$. Then, there exists $R(>0)$ such that for all $\|z_m\|=r_m>R$, we get

$$(1-o(1))\|b_n\|r_m^n\leq\|Q(z_m)\|\leq(1+o(1))\|b_n\|r_m^n \tag{2.8}$$

By combining $\log\|A(z_m)\|>0$ for sufficiently large $\|z_m\|=r_m\notin[0,1]\cup E_1$, with (2.7) and (2.8) we obtain

$$\begin{aligned}k\log v(r_m,f)&\leq\log\|e^{Q(z_m)}\|+k\log r_m+O(1)\\&\leq\|Q(z_m)\|+k\log r_m+O(1)\\&\leq(1+o(1))\|b_n\|r_m^n+k\log r_m+O(1)\end{aligned}$$

and thus

$$\log\log v(r_m,f)\leq n\log r_m+\log\log r_m+O(1)$$

for sufficiently large $\|z_m\|=r_m\notin[0,1]\cup E_1$. Then we have $\rho_2(f)\leq n=\deg Q(z)$.

Taking the principal branch of logarithm, (2.6) gives

$$\begin{aligned}Q(z_m)=&\log\left(A_k(z_m)\left(\frac{v(r_m,f)}{z_m}\right)^k(1+o(1))\right)\\&=\log\left\|A(z_m)(\frac{v(r_m,f)}{z_m})^k(1+o(1))\right\|+i\arg\left(A_k(z_m)\left(\frac{v(r_m,f)}{z_m}\right)^k(1+o(1))\right).\end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}(1-o(1))\|b_n\|r_m^n\leq\|Q(z_m)\|&\leq\left\|\log\left\|A_k(z_m)\left(\frac{v(r_m,f)}{z_m}\right)^k(1+o(1))\right\|\right\|+O(1)\\&\leq\deg A_k\cdot\log r_m+k\log v(r_m,f)+O(1)\\&\leq(k+1)\log v(r_m,f)\end{aligned}$$

for sufficiently large $\|z_m\|=r_m\notin[0,1]\cup E_1$. Thus we have $\deg Q(z)=n\leq\rho_2(f)$.

In light of the above discussion, we can get $\rho_2(f)=\deg Q(z)$. □

Lemma 5 [12] Let $f(z)$ be a non-constant entire function such that $\rho(f)<+\infty$ and $a(z)(\not\equiv 0)$ be an entire function such that $\rho(a)<\rho(f)$. Further suppose that $L(f(z))=f^{(k)}(z)+a_{k-1}f^{(k-1)}(z)+\cdots+a_1f'(z)+a_0f(z)$, where k is a positive integer and $a_0,a_1,\cdots,a_{k-1}$ are complex numbers. If $f(z)$ and $L(f(z))$ share the function $a(z)$, then $\rho(f)=1$ and one of the following two cases occurs:

(i)$L(f(z))-a(z)=c(f(z)-a(z))$ for some $c\in\mathbb{C}\backslash\{0\}$,

(ii)$f(z)$ is a solution of the equation $L(f(z))-a(z)=(f(z)-a(z))e^{\alpha z+\alpha}$ such that $\rho(f)=\mu(f)=1$, where $a_0,a_1,\cdots,a_{k-1}$ are not all zero and $\alpha(\neq 0),\alpha$ are complex numbers.

3 Proofs of theorems

Proof of Theorem 1 For any positive integer j, let $f(z)\in\mathcal{E}_j$. Since $f(z)$ and $L_j(f(z))-1$ share 0 CM, we can get

$$\frac{L_j(f(z))-1}{f(z)}=e^{\alpha(z)}, \tag{3.1}$$

where $\alpha(z)$ is an entire function.

From (3.1), we have

$$L_j(f(z)) - e^{\alpha(z)} f(z) = 1, \tag{3.2}$$

where $\alpha(z)$ is an entire function.

If $f(z)$ is a constant function, then $\rho_1(f) = 0$.

If $f(z)$ is a non-constant entire function, we distinguish the following three cases for discussion.

Case 1. Suppose that $\alpha(z)$ is a complex constant, that is, $e^{\alpha(z)} \equiv c$, where c is a non-zero complex constant.

By (3.2), we get

$$f^{(j)}(z) + a_{j-1} f^{(j-1)}(z) + \cdots + a_1 f'(z) + (a_0 - c) f(z) = 1.$$

It is easy to get that $f(z)$ is a non-constant exponential polynomials with constant coefficients.

Hence, we get $\rho_1(f) = 1$.

Case 2. Suppose that $\alpha(z)$ is a non-constant polynomial.

If $\rho(f) > 1$, then by Lemma 4, we can get $\rho_1(f) = +\infty$.

If $\rho(f) \leq 1$, then by Lemma 5, we can get $\rho_1(f) = 1$.

Thus, we have $\rho_1(f) = +\infty$.

Case 3. Suppose that $\alpha(z)$ is a transcendental entire function.

By Lemma 1, we can deduce

$$\begin{aligned} T(r, e^{\alpha}) &= T(r, \frac{L_j(f) - 1}{f}) \\ &\leq T(r, L_j(f)) + T(r, f) + S(r, f) \\ &\leq (j+1) T(r, f) + T(r, f) + S(r, f) \\ &= (j+2) T(r, f) + S(r, f). \end{aligned}$$

Therefore, $\rho_1(f) = +\infty$.

In light of the above discussion, we can get

$$\rho_1(\mathcal{E}_j) = \{0, 1, +\infty\} \ (j = 1, 2, 3, \cdots).$$

□

Proof of Theorem 2 For any positive integer j, let $f(z) \in \mathcal{E}_j$. By $f(z)$ and $f'(z) - 1$ share 0 CM, we can also get the differential equation (3.1), that is,

$$\frac{f^{(j)}(z)}{f(z)} + a_{j-1} \frac{f^{(j-1)}(z)}{f(z)} + \cdots + a_1 \frac{f'(z)}{f(z)} + a_0 - \frac{1}{f(z)} = e^{\alpha(z)}. \tag{3.3}$$

where $\alpha(z)$ is an entire function.

By Lemma 1, we can deduce

$$T(r, e^{\alpha}) \leq (j+2) T(r, f) + S(r, f).$$

Hence, we get

$$\rho_{k-1}(\alpha) = \rho_k(e^{\alpha}) \leq \rho_k(f) \quad (k = 2, 3, \cdots).$$

Note that $\rho_{k-2}(\alpha) \leq \rho_{k-1}(\alpha)$, then we have

$$\rho_{k-2}(\alpha) \leq \rho_k(f) \quad (k = 3, 4, \cdots). \tag{3.4}$$

From the Wiman-Valiron theory, there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure, such that for all $z_r = re^{i\theta}$ $(\theta \in [0, 2\pi))$ satisfying $\|z\| = r \notin E$ and $M(r, f) = \|f(z)\|$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^j (1 + o(1)) \quad (r \to +\infty). \tag{3.5}$$

From Lemma 3, there exists $\{z_m = r_m e^{i\theta_m}\}$ with $\|f(z_m)\| = M(r_m, f), \theta \in [0, 2\pi), \lim_{m\to\infty} \theta_m = \theta_0 \in [0, 2\pi), r_m \notin [0, 1] \cup E_1$, such that for any given large M and sufficiently lager r_m, we have

$$v(r_m, f) > r_m^M. \tag{3.6}$$

Since $f(z)$ is transcendental entire function and $\|f(z_m)\| = M(r_m, f)$, we get

$$\left\|\frac{1}{f(z_m)}\right\| = o(1) \quad (r_m \to +\infty, r_m \notin [0, 1] \cup E). \tag{3.7}$$

Substituting (3.5)-(3.7) into (3.3), we get

$$\left(\frac{v(r_m, f)}{z_m}\right)^j (1 + o(1)) = e^{\alpha(z_m)} \quad (r_m \to +\infty, r_m \notin [0, 1] \cup E). \tag{3.8}$$

From (3.8), we have

$$\left(\frac{\nu(r_m, f)}{r_m}\right)^j (1 + o(1)) \leq M(r_m, e^\alpha) + O(1) \quad (r \to +\infty, r_m \notin [0, 1] \cup E).$$

Thus, we have

$$j \log \nu(r_m, f) \leq \log M(r_m, e^\alpha) + j \log r + O(1)$$

for sufficiently large $r_m \notin [0, 1] \cup E$.

Therefore, we get

$$\tau_k(\nu(r, f)) \leq \tau_k(M(r, e^\alpha)) + \tau_k(r) + O(1)$$

for sufficiently large $r_m \notin [0, 1] \cup E$.

It is obvious that

$$\rho_k(f) \leq \rho_{k-1}(e^\alpha) = \rho_{k-2}(\alpha) \quad (k = 3, 4, \cdots). \tag{3.9}$$

Combing (3.4) and (3.9), we have $\rho_k(f) = \rho_{k-2}(\alpha) \quad (k = 3, 4, \cdots)$, That is,

$$[0, \infty) \cup \{+\infty\} \subset \rho_k(\mathcal{E}_j) \quad (j = 1, 2, \cdots, \ k = 3, 4, \cdots).$$

Noting that

$$\rho_k(\mathcal{E}_j) \subset [0, \infty) \cup \{+\infty\} \quad (j = 1, 2, \cdots, \ k = 3, 4, \cdots).$$

In light of the above discussion, we can get

$$\rho_k(\mathcal{E}_j) = [0, \infty) \cup \{+\infty\} \quad (j = 1, 2, \cdots, \ k = 3, 4, \cdots).$$

□

4 Conclusions

In this paper, for any positive integer j, we can obtain that the range of $\rho_1(\mathcal{E}_j)$ has discontinuity, that is $\rho_1(\mathcal{E}_j) = \{0, 1, +\infty\}$. And if Brück conjecture is objectively true, then we can get that the range of $\rho_2(\mathcal{E}_1)$ has discontinuity, that is $\rho_2(\mathcal{E}_1) = \{0\} \cup \mathbb{N} \cup \{+\infty\}$. However, for the positive integer j and $k \geq 3$, we can get that there is no discontinuity in the range of $\rho_k(\mathcal{E}_1)$, that is $\rho_k(\mathcal{E}_j) = [0, +\infty) \cup \{+\infty\}$ $(k \geq 3)$. If we can get the values of the hyper-order of an entire function $f(z)$ satisfying $\frac{f'(z)-1}{f(z)} = c$, where c is non-zero constant, then we can know that Brück conjecture is correct or not.

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