Nonlinearity **17** (2004) 447–457 PII: S0951-7715(04)56292-8

The variational principle for products of non-negative matrices

De-Jun Feng[1,2]

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

E-mail: dfeng@math.tsinghua.edu.cn and djfeng@math.gatech.edu

Received 11 November 2002, in final form 4 September 2003
Published 1 December 2003
Online at stacks.iop.org/Non/17/447 (DOI: 10.1088/0951-7715/17/2/004)

Recommended by V Baladi

Abstract
Let (Σ_A, σ) be a subshift of finite type and let $M(x)$ be a continuous function on Σ_A taking values in the set of non-negative matrices. We set up the variational principle between the pressure function, entropy and Lyapunov exponent for M on Σ_A. We also present some properties of equilibrium states.

Mathematics Subject Classification: 37D35, 34D20

1. Introduction

Let σ be the shift map on $\Sigma = \{1, 2, \ldots, m\}^{\mathbb{N}}$, $m \geqslant 2$. As usual, Σ is endowed with the metric $d(x, y) = m^{-n}$, where $x = (x_k)$, $y = (y_k)$ and n is the smallest of the k such that $x_k \neq y_k$. Given an $m \times m$ matrix A with entries 0 or 1, we consider the subshift of finite type (Σ_A, σ) (see [1]). We shall always assume that A is primitive.

Suppose M is a continuous function on Σ_A taking values in the set of all non-negative $d \times d$ matrices. Here, a matrix $A = (A_{i,j})_{1 \leqslant i,j \leqslant d}$ is said to be non-negative if $A_{i,j} \geqslant 0$ for all $1 \leqslant i, j \leqslant d$. Similarly, we say A is strictly positive if $A_{i,j} > 0$ for all $1 \leqslant i, j \leqslant d$. For $q \in \mathbb{R}$, the pressure function, $P(q)$, of M is defined by

$$P(q) := P(M, q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J \in \Sigma_{A,n}} \sup_{x\in[J]} \|M(x)M(\sigma x) \cdots M(\sigma^{n-1}x)\|^q, \tag{1.1}$$

where $\Sigma_{A,n}$ denotes the set of all admissible indices of length n over $\{1, \ldots, m\}$; for $J = j_1 \cdots j_n \in \Sigma_{A,n}$, $[J]$ denotes the cylinder set $\{x = (x_i) \in \Sigma_A : x_i = j_i,\ 1 \leqslant i \leqslant n\}$ and $\|\cdot\|$ denotes the matrix norm defined by $\|B\| := \mathbf{1}^t B\mathbf{1}$, $\mathbf{1}^t = (1, 1, \ldots, 1)$. By using a subadditive argument, it is easy to show that for $q > 0$, the limit in the above definition exists. With some additional conditions on the matrices (e.g. M is strictly positive), the limit exists for $q \in \mathbb{R}$.

[1] Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.
[2] The author was supported by the Special Funds for Major State Basic Research Projects in China.

The pressure function of a matrix-valued function is a natural generalization of that of the scalar case (i.e. $M(x) = \mathrm{e}^{\phi(x)}$, where $\phi(x)$ is a real-valued function called the potential of the subshift). The reader is referred to [1, 8, 13, 14] for the pressure and variational principle in the classical scalar case. In [5], Feng and Lau considered the pressure functions and Gibbs measures for the products of matrices, where the matrix function was assumed to be either strictly positive and Hölder continuous, or local non-negative constant, satisfying an irreducibility assumption. For instance, in the former setting, they proved the following theorem.

Theorem A. *Suppose that M is a Hölder continuous function on Σ_A taking values in the set of strictly positive $d \times d$ matrices. Then for any $q \in \mathbb{R}$, there is a unique σ-invariant, ergodic probability measure μ_q on Σ_A for which one can find constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \leqslant \frac{\mu_q([J])}{\exp(-nP(q)) \cdot \|M(x)M(\sigma x)\cdots M(\sigma^{n-1}x)\|^q} \leqslant C_2 \tag{1.2}$$

for any $n > 0$, $J \in \Sigma_{A,n}$ and $x \in [J]$.

In [3], the author used the pressure function to analyse the multifractal structure of the Lyapunov exponents for the products of matrices and proved the following theorem.

Theorem B. *Suppose M is a continuous function on Σ_A taking values in the set of strictly positive $d \times d$ matrices. For any $\alpha \in \mathbb{R}$, if the set $\{x \in \Sigma_A : \lambda_M(x) = \alpha\}$ is not empty, then*

$$\begin{aligned}\dim_{\mathrm{H}}\{x \in \Sigma_A : \lambda_M(x) = \alpha\} &= \frac{1}{\log m} \inf_{q\in\mathbb{R}}\{-\alpha q + P(q)\} \\ &= \frac{1}{\log m} \sup\{h(\mu) : \mu \in \mathcal{M}(\Sigma_A, \sigma), M_*(\mu) = \alpha\},\end{aligned}$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension, $\lambda_M(x)$ is the upper Lyapunov exponent of M at x defined by*

$$\lambda_M(x) = \lim_{n\to\infty} \frac{1}{n} \log \|M(x)M(\sigma x)\cdots M(\sigma^{n-1}x)\| \tag{1.3}$$

when the limit exists, $\mathcal{M}(\Sigma_A, \sigma)$ denotes the collection of all σ-invariant Borel probability measures on Σ_A and

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \|M(y)M(\sigma y)\cdots M(\sigma^{n-1}y)\| \,\mathrm{d}\mu(y). \tag{1.4}$$

Theorem B was also proved in [5] under an additional condition that M is Hölder continuous. For $\mu \in \mathcal{M}(\Sigma_A, \sigma)$, $M_*(\mu)$ is often called the upper Lyapunov exponent of μ associated with M. It was first proved by Furstenberg and Kesten [7] that $\lambda_M(x)$ exists for almost all x with respect to μ and $\int \lambda_M(x)\,\mathrm{d}\mu(x) = M_*(\mu)$.

The main purpose of this paper is to set up the variational principle for the non-negative matrix-valued functions. We prove the following general theorem, which does not need any additional smoothness condition or the strict positivity of M.

Theorem 1.1. *Suppose that M is a continuous function on Σ_A taking values in the set of non-negative $d \times d$ matrices. Then for any $q > 0$, we have*

$$P(q) = \sup\{h_\mu(\sigma) + qM_*(\mu) : \mu \in \mathcal{M}(\Sigma_A, \sigma)\} \tag{1.5}$$

and this supremum is attained.

If furthermore M is strictly positive, then (1.5) holds for any $q \in \mathbb{R}$, and the corresponding supremum is attained.

When $d = 1$, $M(x) = e^{\phi(x)}$ becomes a scalar function; and in this case (1.5) is just the classical variational principal formula for the potential $q\phi(x)$ (see, e.g. [8, theorem 4.4.11], where ϕ may take the value $-\infty$).

A member μ of $\mathcal{M}(\Sigma_A, \sigma)$ is called an equilibrium state for M with respect to q if

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Let $\mathcal{I}(M, q)$ denotes the collection of all equilibrium states of M with respect to q. It is interesting to consider under what condition $\mathcal{I}(M, q)$ contains only one element (in this case we say that M has a unique equilibrium state with respect to q). The following theorem establishes the derivative formula of the pressure function which is an extension of the classical Ruelle formula to matrix-valued functions (for the classical Ruelle formula, see [13, exercise 5, p 99], [11, lemma 4] and [8, theorem 4.3.5]).

Theorem 1.2. *Suppose that M is a continuous function on Σ_A taking values in the set of non-negative $d \times d$ matrices with $P(q) \neq -\infty$ for all $q > 0$. Then*

$$P'(q+) := \lim_{\epsilon\downarrow 0} \frac{P(q+\epsilon) - P(q)}{\epsilon} = \sup\{M_*(\mu) : \mu \in \mathcal{I}(M, q)\}, \tag{1.6}$$

$$P'(q-) := \lim_{\epsilon\downarrow 0} \frac{P(q-\epsilon) - P(q)}{-\epsilon} = \inf\{M_*(\mu) : \mu \in \mathcal{I}(M, q)\} \tag{1.7}$$

for any $q > 0$.

If furthermore M is strictly positive, then (1.6) and (1.7) hold for any $q \in \mathbb{R}$.

We remark that there are examples of $M \not\equiv \mathbf{0}$ satisfying $P(M, q) \equiv -\infty$. For instance, take $\Sigma = \{1, 2\}^{\mathbb{N}}$ and define $f \in C(\Sigma)$ by

$$f(x) = \begin{cases} 0 & \text{if } x_1x_2 = 00 \text{ or } 11, \\ 2^{-n} & \text{if } x_1\cdots x_{2n+1} = (01)^n 1 \text{ or } (10)^n 0, \end{cases}$$

where $x = (x_i)_{i=1}^\infty \in \Sigma$. Take $M(x) = f(x)I_d$, where I_d denotes the $d \times d$ identity matrix. Then $P(M, q) \equiv -\infty$. We point out that the condition $P(q) \neq -\infty$ for all $q > 0$ is equivalent to $P(q) \neq -\infty$ for some $q > 0$. A sufficient condition insuring $P(q) \neq -\infty$ is that there exists $x \in \Sigma_A$ such that

$$\bar{\lambda}_M(x) := \limsup_{n\to\infty} \frac{1}{n} \log \|M(x)M(\sigma x)\cdots M(\sigma^{n-1}x)\| \neq -\infty.$$

We also remark that for any fixed M and q, the pressure, $P(e^{\phi(x)}M, q)$, is a convex function of $\phi \in C(\Sigma_A)$. It can be derived directly by theorem 1.1 and the fact that $(e^{\phi(x)}M)^*(\mu) = \int \phi\, d\mu + M^*(\mu)$. In any case, we do not know whether there is any kind of convexity of $P(M, q)$ on M.

As a direct corollary of theorem 1.2, we have the following.

Corollary 1.3. *Let M be a continuous function on Σ_A taking values in the set of non-negative (strictly positive, resp.) $d \times d$ matrices. A necessary condition for M having a unique equilibrium state with respect to some $q > 0$ ($q \in \mathbb{R}$, resp.) is that $P(q)$ is differentiable at q.*

Under some additional assumptions, we can show the existence of a unique equilibrium state for M (see theorem 3.1, corollary 3.2).

As we have seen from theorems A and B, the pressure function, $P(q)$, is an important term in studying the Gibbs measures of $M(x)$ and the Hausdorff dimension of level sets of $\lambda_M(x)$. We should point out that $P(q)$ has also appeared naturally in the study of multifractal phenomena about measures. In [4] the author studied the multifractal structure of a class of self-similar measures with overlaps (namely, self-similar measures satisfying the finite type condition). He proved that these measures can locally be expressed as the product of a finite family of non-negative matrices, and their L^q-spectra, $\tau(q)$ (one of the basic ingredients in the study of multifractal phenomena, see [2, 12]), differ from $P(q)$ only by a factor (see [4, lemma 4.1 and theorem 5.2]). The readers are referred to [4, 6, 9, 10] and the references therein for the multifractal theory for self-similar measures with overlaps.

A first thought of proving theorem 1.1 is to re-express the pressure function, $P(q)$, of M as the classical pressure, P_f, for some scalar function f. However, this thought seems only possible for the case where $q = 1$ and M is strictly positive. In this case, we may enlarge the symbolic set $\{1, 2, \ldots, m\}$ to $S = \{(i, j) : i = 1, \ldots, d, j = 1, \ldots, m\}$ and define the 0–1 matrix $\hat{A} = \hat{A}_{S\times S}$ by

$$\hat{A}_{(i,j),(i',j')} = \begin{cases} 1, & \text{if } A_{j,j'} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

One may check that by definition $P(1)$ equals P_f for a scalar function f on the subshift space $S^{\mathbb{N}}_{\hat{A}}$ defined by

$$f(((i_1, j_1), (i_2, j_2), \ldots,)) = \log M_{i_1,i_2}(j_1 j_2 \cdots).$$

Even in this case we still have some difficulty in pulling back the variational result from $S^{\mathbb{N}}_{\hat{A}}$ to Σ_A.

Our proof of theorem 1.1 is essentially based on the existence of 'Gibbs' measures (see theorem A). In fact, by theorem A and a standard argument, we prove theorem 1.1 immediately in the special case where M is Hölder continuous and takes values in the set of strictly positive $d \times d$ matrices. The original part of our proof is the generalization of this result to functions that are continuous and with values in $d \times d$ non-negative matrices. We do this with two approximation steps: of continuous maps by Hölder continuous ones and of non-negative matrices by strictly positive ones.

We organize this paper as follows. In section 2, we prove theorem 1.1 (see propositions 2.6–2.8). In section 3, we consider the equilibrium states of M and give a proof of theorem 1.2.

2. The proof of theorem 1.1

For convenience, we use Γ_+ (Γ, resp.) to denote the collection of all continuous functions on Σ_A taking values in the set of all strictly positive (non-negative, resp.) $d \times d$ matrices. For $M \in \Gamma$, we write $\pi_n M(x)$ for the product $M(x)M(\sigma x) \cdots M(\sigma^{n-1}x)$.

Lemma 2.1. *Let $M \in \Gamma$. We have*

$$\|\pi_{n+\ell} M(x)\| \leqslant \|\pi_n M(x)\| \|\pi_\ell M(\sigma^n x)\|, \qquad \forall\, n, \ell \in \mathbb{N}, \quad x \in \Sigma_A.$$

Moreover, if $M \in \Gamma_+$, then there exists a constant $C > 0$ (depending on M) such that

$$\|\pi_{n+\ell} M(x)\| \geqslant C \|\pi_n M(x)\| \|\pi_\ell M(\sigma^n x)\|, \qquad \forall\, n, \ell \in \mathbb{N}, \quad x \in \Sigma_A.$$

Proof. The first inequality is trivial. The second one was proved in [5]. However for the reader's convenience, we include the detailed proof. Since $M \in \Gamma_+$, there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} \geqslant dC, \qquad \forall\, x \in \Sigma_A,$$

which implies that $M(x) \geqslant CEM(x)$ (here and afterwards we write $B^{(1)} \geqslant B^{(2)}$ for two matrices $B^{(1)}$, $B^{(1)}$ if $B^{(1)}_{i,j} \geqslant B^{(2)}_{i,j}$ for each index (i, j)); here, $E = (E_{i,j})_{1\leqslant i,j\leqslant d}$ is the matrix whose entries are all equal to 1. Let $\mathbf{1}$ be the d-dimensional column vector, each coordinate of which is 1. Then using $M(\sigma^n x) \geqslant CEM(\sigma^n x)$, we have

$$\begin{aligned}\|\pi_{n+\ell}M(x)\| &\geqslant \|(\pi_n M(x))CE(\pi_\ell M(\sigma^n x))\| \\ &= C\|(\pi_n M(x))\mathbf{1}^t\mathbf{1}(\pi_\ell M(\sigma^n x))\| \\ &= C\|\pi_n M(x)\|\|\pi_\ell M(\sigma^n x)\|. \end{aligned}$$

□

Lemma 2.2.

(i) If $M \in \Gamma$, then for $q > 0$ the limit

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q \tag{2.1}$$

exists and equals $\inf_n (1/n) \log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q$.

(ii) If $M \in \Gamma$, then for any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$,

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \|\pi_n M(x)\| \, \mathrm{d}\mu(x) = \inf_n \frac{1}{n} \int \log \|\pi_n M(x)\| \, \mathrm{d}\mu(x).$$

(iii) If $M \in \Gamma_+$, then for any $q \in \mathbb{R}$ the limit (2.1) exists. Moreover, let C be the constant as in lemma 2.1; then

$$P(q) = \begin{cases} \inf_n \dfrac{1}{n} \log \displaystyle\sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q, & \text{if } q \geqslant 0, \\ \inf_n \dfrac{1}{n} \log \displaystyle\sum_{J\in\Sigma_{A,n}} C^q \sup_{x\in[J]} \|\pi_n M(x)\|^q, & \text{if } q < 0. \end{cases}$$

Proof. Suppose $I \in \Sigma_{A,n}$, $J \in \Sigma_{A,\ell}$ with $IJ \in \Sigma_{A,n+\ell}$. By lemma 2.1, we have for $M \in \Gamma$ and $q \geqslant 0$,

$$\begin{aligned}\sup_{x\in[IJ]} \|\pi_{n+\ell}M(x)\|^q &\leqslant \sup_{x\in[IJ]} (\|\pi_n M(x)\|^q \|\pi_\ell M(\sigma^n x)\|^q) \\ &\leqslant \sup_{x\in[I]} \|\pi_n M(x)\|^q \cdot \sup_{y\in[J]} \|\pi_\ell M(y)\|^q, \end{aligned}$$

while for $M \in \Gamma_+$ and $q < 0$,

$$C^q \sup_{x\in[IJ]} \|\pi_{n+\ell}M(x)\|^q \leqslant \big(C^q \sup_{x\in[I]} \|\pi_n M(x)\|^q\big)\big(C^q \sup_{y\in[J]} \|\pi_\ell M(y)\|^q\big).$$

Using a subadditive argument, we obtain (i) and (iii). Statement (ii) is obtained similarly by using the fact $\|\pi_{n+\ell}M(x)\| \leqslant \|\pi_n M(x)\|\|\pi_\ell M(\sigma^n x)\|$. □

Lemma 2.3. *Let $M \in \Gamma_+$. For any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ and $n \in \mathbb{N}$, we have*

$$\int \frac{\log \|\pi_n M(x)\|}{n} \, \mathrm{d}\mu(x) + \frac{\log C}{n} \leqslant M_*(\mu) \leqslant \int \frac{\log \|\pi_n M(x)\|}{n} \, \mathrm{d}\mu(x),$$

where C is the constant in lemma 2.1.

Proof. Take any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. By lemma 2.1 and the invariance of μ, we have for any $n, \ell \in \mathbb{N}$,

$$\begin{aligned}\int \log \|\pi_{n+\ell}M(x)\| \, \mathrm{d}\mu(x) &\leqslant \int \log \|\pi_n M(x)\| \, \mathrm{d}\mu(x) + \int \log \|\pi_\ell M(\sigma^n x)\| \, \mathrm{d}\mu(x) \\ &= \int \log \|\pi_n M(x)\| \, \mathrm{d}\mu(x) + \int \log \|\pi_\ell M(x)\| \, \mathrm{d}\mu(x). \end{aligned}$$

A subadditive argument yields the second desired inequality. Similarly, we have

$$\int \log(C\|\pi_{n+\ell}M(x)\|)\,\mathrm{d}\mu(x) \geqslant \int \log(C\|\pi_n M(x)\|)\,\mathrm{d}\mu(x) + \int \log(C\|\pi_\ell M(x)\|)\,\mathrm{d}\mu(x),$$

which proves the first inequality by a super-additive argument. □

As a corollary, we have the following.

Corollary 2.4. *Suppose $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ converges to μ in the weak-star topology. Then for any $M \in \Gamma_+$,*

$$\lim_{k\to\infty} M_*(\mu_k) = M_*(\mu). \tag{2.2}$$

Proof. By lemma 2.3, for any $n \in \mathbb{N}$ we have

$$\left| M_*(\mu) - \frac{1}{n}\int \log\|\pi_n M(x)\|\,\mathrm{d}\mu(x)\right| \leqslant \frac{|\log C|}{n}$$

and

$$\left| M_*(\mu_k) - \frac{1}{n}\int \log\|\pi_n M(x)\|\,\mathrm{d}\mu_k(x)\right| \leqslant \frac{|\log C|}{n}.$$

Since $\lim_{k\to\infty}\int \log\|\pi_n M(x)\|\,\mathrm{d}\mu_k(x) = \int \log\|\pi_n M(x)\|\,\mathrm{d}\mu(x)$, we have

$$\limsup_{k\to\infty} |M_*(\mu) - M_*(\mu_k)| \leqslant \frac{2|\log C|}{n}.$$

Letting $n \to \infty$, we obtain the desired result. □

Lemma 2.5 (cf [14, lemma 9.9]). *Let $a_1, \ldots, a_k$ be given real numbers. If $p_i \geqslant 0$ and $\sum_{i=1}^k p_i = 1$, then*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \leqslant \log\left(\sum_{i=1}^k \mathrm{e}^{a_i}\right).$$

Proposition 2.6. *For any $M \in \Gamma_+$ and $q \in \mathbb{R}$ (resp., for any $M \in \Gamma$ and $q > 0$),*

$$P(q) \geqslant \sup\{h_\mu(\sigma) + qM_*(\mu) : \ \mu \in \mathcal{M}(\Sigma_A, \sigma)\}.$$

Proof. The following argument is classical. Let $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. By lemma 2.5, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\log \sum_{I\in\Sigma_{A,n}} \sup_{x\in[I]} \|\pi_n M(x)\|^q &\geqslant \sum_{I\in\Sigma_{A,n}} \Big[-\mu([I])\log\mu([I]) + \mu([I])\log\sup_{x\in[I]}\|\pi_n M(x)\|^q\Big] \\
&\geqslant \sum_{I\in\Sigma_{A,n}} (-\mu([I])\log\mu([I])) + \int \log\|\pi_n M(x)\|^q\,\mathrm{d}\mu(x) \\
&= \sum_{I\in\Sigma_{A,n}} (-\mu([I])\log\mu([I])) + q\int \log\|\pi_n M(x)\|\,\mathrm{d}\mu(x).
\end{aligned}$$

Dividing both sides by n and letting $n \to \infty$, we have

$$P(q) \geqslant h_\mu(\sigma) + qM_*(\mu).$$

□

Proposition 2.7. *For any $M \in \Gamma_+$ and $q \in \mathbb{R}$, there exists $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ such that*

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Proof. Fix $q \in \mathbb{R}$, we divide the proof into two steps.

Step 1. Assume M is Hölder continuous. In this case, let $\mu = \mu_q$ be the Gibbs measure in theorem A. Then for each $n \in \mathbb{N}$, $I \in \Sigma_{A,n}$ and $x \in [I]$,

$$\log C_1 \leqslant nP(q) + \log \mu([I]) - q \log \|\pi_n M(x)\| \leqslant \log C_2.$$

Integrating by μ and dividing both sides by n, we have

$$\frac{\log C_1}{n} \leqslant P(q) + \frac{1}{n} \sum_{I \in \Sigma_{A,n}} \mu([I]) \log \mu([I]) - q \int \frac{\log \|\pi_n M(x)\|}{n} \, \mathrm{d}\mu(x) \leqslant \frac{\log C_2}{n}.$$

Letting $n \to \infty$, we have $P(q) = h_\mu(\sigma) + M_*(\mu)$.

Step 2. Now let us consider M without the Hölder continuity assumption. For each $k \in \mathbb{N}$, define a matrix-valued function $M^{(k)}$ on Σ_A by

$$M^{(k)}_{i,j}(x) = \sup_{y \in I_k(x)} M_{i,j}(y), \qquad 1 \leqslant i, j \leqslant d,$$

where $I_k(x) = [x_1 x_2 \cdots x_k]$ for $x = (x_i)$.

By definition $M^{(k)}$ depends only on the first k coordinates of x, and thus it is Hölder continuous. As we proved in step 1, there exists $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ such that

$$P(M^{(k)}, q) = h_{\mu_k}(\sigma) + q(M^{(k)})_*(\mu_k).$$

Since M is strictly positive and continuous, there exists a sequence of positive numbers ϵ_k such that $\lim_k \epsilon_k = 0$ and

$$M(x) \leqslant M^{(k)}(x) \leqslant (1 + \epsilon_k) M(x), \qquad \forall x \in \Sigma_A,$$

from which we deduce that

$$|P(q) - P(M^{(k)}, q)| \leqslant |q| \log(1 + \epsilon_k)$$

and

$$\left| \frac{\log \|\pi_n M(x)\|}{n} - \frac{\log \|\pi_n M^{(k)}(x)\|}{n} \right| \leqslant \log(1 + \epsilon_k), \qquad \forall x \in \Sigma_A.$$

By the above two inequalities, we have

$$\begin{aligned} P(q) &= \lim_{k\to\infty} P(M^{(k)}, q) = \lim_{k\to\infty} [h_{\mu_k}(\sigma) + q(M^{(k)})_*(\mu_k)] \\ &= \lim_{k\to\infty} [h_{\mu_k}(\sigma) + qM_*(\mu_k)]. \end{aligned} \tag{2.3}$$

Since $\mathcal{M}(\Sigma_A, \sigma)$ is compact in the weak-star topology, there exists a subsequence $\{\mu_{k_i}\}$ of $\{\mu_k\}$ such that μ_{k_i} converges to some $\mu \in \mathcal{M}(\Sigma_A, \sigma)$.

By the upper semi-continuity of the measure-theoretic entropy on $M(\Sigma_A, \sigma)$ (cf [14, theorem 8.2]), we have

$$\limsup_{i\to\infty} h_{\mu_{k_i}}(\sigma) \leqslant h_\mu(\sigma). \tag{2.4}$$

On the other hand, by corollary 2.4, we have

$$\lim_{i\to\infty} M_*(\mu_{k_i}) = M_*(\mu). \tag{2.5}$$

Combining (2.3)–(2.5) yields

$$P(q) \leqslant h_\mu(\sigma) + qM_*(\mu)$$

and thus $P(q) = h_\mu(\sigma) + qM_*(\mu)$ by proposition 2.6. □

By propositions 2.6 and 2.7, to finish the proof of theorem 1.1, we only need to prove the following.

Proposition 2.8. *For any $M \in \Gamma$ and $q > 0$, there exists $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ such that*

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Now fix $M \in \Gamma$. For any $\epsilon > 0$, define a matrix-valued function M_ϵ on Σ_A by

$$M_\epsilon(x) = M(x) + \epsilon E,$$

where E is the $d \times d$ matrix, of which each entry equals 1. It is clear that M_ϵ is continuous and strictly positive. Note that since $\|\pi_n M_\epsilon(x)\|$ is a polynomial of ϵ with continuous coefficients, we have the following lemmas.

Lemma 2.9. *For a fixed $n \in \mathbb{N}$, there exist $a > 0$ and $\epsilon_0 > 0$ such that*

$$\|\pi_n M_\epsilon(x)\| \leqslant \|\pi_n M(x)\| + a\epsilon, \qquad \forall x \in \Sigma_A, \quad \epsilon < \epsilon_0. \tag{2.6}$$

To prove proposition 2.8, we still need the following simple lemma.

Lemma 2.10. *For any $q > 0$, $P(q) = \lim_{\epsilon \to 0} P(M_\epsilon, q)$.*

Proof. Fix $q > 0$. It is clear that $P(M_\epsilon, q) \geqslant P(q)$ for any $\epsilon > 0$. Let $\delta > 0$. By lemma 2.2, there exists $n_0 \in \mathbb{N}$ such that

$$P(q) \geqslant \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M(x)\|^q - \delta.$$

Since

$$\lim_{\epsilon \to 0} \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M_\epsilon(x)\|^q = \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M(x)\|^q,$$

it follows from lemma 2.2 that

$$\limsup_{\epsilon \to 0} P(M_\epsilon, q) \leqslant \lim_{\epsilon \to 0} \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M_\epsilon(x)\|^q \leqslant P(q) + \delta,$$

which implies the desired result. □

Proof of proposition 2.8. Fix $q > 0$. For any $k \in \mathbb{N}$, the matrix-valued function $M_{1/k}$ is continuous and strictly positive. Therefore, by proposition 2.7, there exists $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ such that

$$P(M_{1/k}, q) = h_{\mu_k}(\sigma) + q(M_{1/k})_*(\mu_k). \tag{2.7}$$

Let $\{\mu_{k_i}\}$ be a weak-star convergent subsequence of $\{\mu_k\}$ and μ be the limit point. We show below that $P(q) = h_\mu(\sigma) + qM_*(\mu)$. To see this, we first show that

$$\limsup_{i \to \infty} (M_{1/k_i})_*(\mu_{k_i}) \leqslant M_*(\mu). \tag{2.8}$$

Fix $n \in \mathbb{N}$. For any integer $N > 0$, define $g_N(x) = \max\{-N, (1/n) \log \|\pi_n M(x)\|\}$. By lemma 2.9, for any $\delta > 0$, there exists i_0 (depending on N) such that

$$\frac{1}{n} \log \|\pi_n M_{1/k_i}(x)\| \leqslant g_N(x) + \delta, \qquad \forall x \in \Sigma_A, \quad i \geqslant i_0.$$

Therefore,

$$(M_{1/k_i})_*(\mu_{k_i}) \leqslant \int \frac{1}{n} \log \|\pi_n M_{1/k_i}(x)\| \, \mathrm{d}\mu_{k_i}(x) \leqslant \int g_N(x) \, \mathrm{d}\mu_{k_i}(x) + \delta, \qquad \forall i \geqslant i_0.$$

Letting $i \to \infty$ and then $\delta \to 0$, we have

$$\limsup_{i\to\infty}(M_{1/k_i})_*(\mu_{k_i}) \leqslant \int g_N(x)\,\mathrm{d}\mu(x), \qquad \forall N \in \mathbb{N}. \tag{2.9}$$

Note that $\{g_N(x)\}_{N\geqslant 1}$ is a sequence of continuous functions on Σ_A having a uniform upper bound. By the Fatou theorem,

$$\limsup_{N\to\infty}\int g_N(x)\,\mathrm{d}\mu(x) \leqslant \int \limsup_{N\to\infty} g_N(x)\,\mathrm{d}\mu(x) = \int \frac{1}{n}\log\|\pi_n M(x)\|\,\mathrm{d}\mu(x).$$

Combining this with (2.9) yields

$$\limsup_{i\to\infty}(M_{1/k_i})_*(\mu_{k_i}) \leqslant \int \frac{1}{n}\log\|\pi_n M(x)\|\,\mathrm{d}\mu(x), \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain (2.8). By the upper semi-continuity of the measure-theoretic entropy on $M(\Sigma_A, \sigma)$, we have

$$\limsup_{i\to\infty} h_{\mu_{k_i}}(\sigma) \leqslant h_\mu(\sigma). \tag{2.10}$$

Combining (2.10), (2.8) and (2.7) yields

$$P(q) = \lim_{k\to\infty} P(M_{1/k}, q) \leqslant h_\mu(\sigma) + qM_*(\mu)$$

and thus $P(q) = h_\mu(\sigma) + qM_*(\mu)$ by proposition 2.6. □

3. The proof of theorem 1.2

In this section, we first give a proof of theorem 1.2, and then we give the existence result for the unique equilibrium state in some cases.

Proof of theorem 1.2. First assume that M is a continuous function on Σ_A satisfying $P(q) \neq -\infty$ for all $q > 0$. Then $P(q)$ is a convex continuous function. Therefore, $P'(q+)$ and $P'(q-)$ exist for any $q > 0$.

Fix $q > 0$. By theorem 1.1, $\mathcal{I}(M, q) \neq \emptyset$. For any $\mu \in \mathcal{I}(M, q)$ and $\epsilon > 0$, we have

$$P(q+\epsilon) \geqslant h_\mu(\sigma) + (q+\epsilon)M_*(\mu), \qquad P(q) = h_\mu(\sigma) + qM_*(\mu).$$

It follows that $P'(q+) \geqslant M_*(\mu)$ and thus

$$P'(q+) \geqslant \sup\{M_*(\mu) : \mu \in \mathcal{I}(M, q)\}. \tag{3.1}$$

Similarly, we have

$$P'(q-) \leqslant \inf\{M_*(\mu) : \mu \in \mathcal{I}(M, q)\}. \tag{3.2}$$

By (3.1) and (3.2), we know

$$M_*(\mu) = P'(q), \qquad \forall \mu \in \mathcal{I}(M, q) \tag{3.3}$$

if $P'(q)$ exists.

Since $P(\cdot)$ is convex, there exists a sequence of real numbers $q_k \downarrow q$ such that $P'(q_k)$ exist and $P'(q+) = \lim_{k\to\infty} P'(q_k)$. Take $\mu_k \in \mathcal{I}(M, q_k)$. Without loss of generality, we assume $\mu_k \to \mu$ in the weak-star topology. We claim that

$$\mu \in \mathcal{I}(M, q) \qquad \text{and} \qquad M_*(\mu) = \limsup_{k\to\infty} M_*(\mu_k). \tag{3.4}$$

To prove the claim, note that

$$\limsup_{k\to\infty} M_*(\mu_k) \leqslant \lim_{k\to\infty}\int \frac{1}{n}\log\|\pi_n M(x)\|\,\mathrm{d}\mu_k(x) = \int \frac{1}{n}\log\|\pi_n M(x)\|\,\mathrm{d}\mu(x)$$

for any $n \in \mathbb{N}$. Thus, we have

$$\limsup_{k\to\infty} M_*(\mu_k) \leqslant M_*(\mu). \tag{3.5}$$

This, combining $\limsup_{k\to\infty} h_{\mu_k}(\sigma) \leqslant h_\mu(\sigma)$, yields

$$P(q) = \lim_{k\to\infty} P(q_k) = \lim_{k\to\infty} (h_{\mu_k}(\sigma) + q_k M_*(\mu_k)) \leqslant h_\mu(\sigma) + qM_*(\mu),$$

which implies (3.4) (here we have used the positivity of q). By (3.3) and (3.4) we have

$$P'(q+) = \lim_{k\to\infty} P'(q_k) = \lim_{k\to\infty} M_*(\mu_k) = M_*(\mu),$$

which combined with (3.1) yields (1.6). An analogous argument proves (1.7).

Now assume M is strictly positive. The above argument (in which we use (2.2) to replace (3.5)) can prove (1.6) and (1.7) for $q \leqslant 0$. □

In the following we give two cases for which M has a unique equilibrium state with respect to q. Recall we have mentioned in theorem A the existence and uniqueness of Gibbs measures under the assumption that M is strictly positive and Hölder continuous. In [5], Feng and Lau also proved the existence and uniqueness of Gibbs measures when M is a function taking values in the set of non-negative $d \times d$ matrices satisfying the following assumptions:

(H1) $M(x) = M_i$ if $x \in [i]$, $i = 1, \ldots, m$;

(H2) M is irreducible in the following sense: there exists $r > 0$ such that for any $i, j \in \{1, 2, \ldots, m\}$,

$$\sum_{k=1}^{r} \sum_{K\in\Sigma_{A,k;i,j}} M_K > \mathbf{0}, \tag{3.6}$$

where $\Sigma_{A,k;i,j}$ denotes the set of all $K \in \Sigma_{A,k}$ such that $iKj \in \Sigma_{A,k+2}$ and $M_K = M_{u_1}M_{u_2}\cdots M_{u_k}$ for $K = u_1u_2\cdots u_k$.

More precisely, they proved the following theorem.

Theorem C. *Suppose M is a function on Σ_A taking values in the set of all $d \times d$ non-negative matrices and satisfies (H1) and (H2). Then, for any $q > 0$, there is a unique Gibbs measure, μ_q, on Σ_A as in theorem A.*

Now we can formulate our result about the existence of the unique equilibrium state.

Theorem 3.1.

(i) Suppose M satisfies the condition of theorem A, then $\mathcal{I}(M, q)$ contains only one element for any $q \in \mathbb{R}$;

(ii) Suppose M satisfies the condition of theorem C, then $\mathcal{I}(M, q)$ contains only one element for any $q > 0$.

We remark that theorem 3.1 follows from the uniqueness of Gibbs measures, by a proof very similar to that given in [1, theorem 1.22] for showing the uniqueness of the equilibrium states in the Hölder continuous real-valued functions case. The only, slight modification is to replace $S_n\phi(x)$ therein by $\log \|\pi_n M(x)\|$.

Combining corollary 1.3 and theorem 3.1, we have the following.

Corollary 3.2.

(i) Suppose M satisfies the condition of theorem A, then $P'(q)$ exists for any $q \in \mathbb{R}$;

(ii) Suppose M satisfies the condition of theorem C, then $P'(q)$ exists for any $q > 0$.

We remark that corollary 3.2 (not including the existence of $P'(0)$ in (i)) was also proved in [5] by a different method.

Acknowledgments

The author would like to thank the referees for their helpful comments and suggestions that led to the improvement of the manuscript.

References

[1] Bowen R 1975 Equilibrium states and the ergodic theory of Anosov diffeomorphisms (*Lecture Notes in Mathematics* vol 470) (Berlin: Springer)
[2] Falconer K J 1990 *Fractal Geometry. Mathematical Foundations and Applications* (Chichester: Wiley)
[3] Feng D-J 2003 Lyapunov exponent for products of matrices and multifractal analysis: Part I. Positive matrices *Israel J. Math.* at press
[4] Feng D-J 2003 Smoothness of the L^q-spectrum of self-similar measures with overlaps *J. Lond. Math. Soc.* **68** 102–18
[5] Feng D-J and Lau K-S 2002 The pressure function for products of non-negative matrices *Math. Res. Lett.* **9** 363–78
[6] Feng D-J and Olivier E 2003 Multifractal analysis of the weak Gibbs measures and phase transition—application to some Bernoulli convolutions *Ergod. Theory Dynam. Syst.* at press
[7] Furstenberg H and Kesten H 1960 Products of random matrices *Ann. Math. Stat.* **31** 457–69
[8] Keller G 1998 *Equilibrium States in Ergodic Theory* (Cambridge: Cambridge University Press)
[9] Lau K-S Multifractal structure and product of matrices *Preprint*
[10] Lau K-S and Ngai S-M 1999 Multifractal measures and a weak separation condition *Adv. Math.* **141** 45–96
[11] Olivier E 1999 Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures *Nonlinearity* **12** 1571–85
[12] Pesin Ya B 1997 *Dimension Theory in Dynamical Systems. Contemporary Views and Applications* (Chicago, IL: University of Chicago Press)
[13] Ruelle D 1978 Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics *Encyclopedia of Mathematics and its Applications* (Reading, MA: Addison-Wesley)
[14] Walters P 1982 *An Introduction to Ergodic Theory* (Berlin: Springer)