THE VEC-PERMUTATION MATRIX, THE VEC OPERATOR AND KRONECKER PRODUCTS: A REVIEW

by

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ABSTRACT

The vec-permutation matrix $I_{m,n}$ is defined by the equation $vecA_{m \times n} = I_{m,n} vecA'$, where vec is the vec operator such that $vecA$ is the vector of columns of A stacked one under the other. The variety of definitions, names and notations for $I_{m,n}$ are discussed, and its properties are developed by simple proofs in contrast to certain lengthy proofs in the literature that are based on descriptive definitions. For example, the role of $I_{m,n}$ in reversing the order of Kronecker products is succinctly derived using the vec operator. The matrix $M_{m,n}$ is introduced as $M_{m,n} = I_{m,n} M$; it is the matrix having for rows, every n'th row of M, of order mn X c, starting with the first, then every n'th row starting with the second, and so on. Special cases of $M_{m,n}$ are discussed.

1. NOTATION

Let A and B be matrices of order $m \times n$ and $p \times q$, respectively. Rows of a matrix will be denoted by Greek letters transposed and columns by Roman letters:

$$A = \{a_{ij}\} = [a_1 \cdots a_n] = \begin{bmatrix} \alpha_1' \\ \vdots \\ \alpha_m' \end{bmatrix} \text{ and } B = \{b_{rs}\} = [b_1 \cdots b_q] = \begin{bmatrix} \beta_1' \\ \vdots \\ \beta_p' \end{bmatrix} . \quad (1)$$

I_n is the identity matrix of order n, with e_i denoting its i'th column and then

$$I_n = \sum_{i=1}^{n} e_i e_i' .$$

2. THE VEC OPERATOR AND KRONECKER PRODUCTS

2.1. The vec operator and its many names and notations

The vec operator stacks the columns of a matrix one underneath the other to form a single vector. Thus for A of (1)

$$\text{vec}A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} , \quad (2)$$

with vecA (for "vector of columns of A") being the notation currently in vogue. The equivalent notations vecA and vec(A) are used interchangeably, the parentheses being employed only when deemed necessary for clarity.

An early reference to this idea of stacking elements of a matrix in a vector is Sylvester [1884a,b,c] who used it in connection with linear equations. Roth [1934], using the notation A^c, develops results for using the operation on a product matrix, Aitken [1949] mentions the idea in connection with Jacobians, and Koopmans et al. [1950] introduce the notation vec. More recently, the concept has been exploited in a variety of ways, for example, in solving linear matrix equations and in matrix differentiation, from which it is seen to be useful in deriving Jacobians of matrix transformations. The paper by Henderson and Searle [1979] outlines these applications and highlights several uses in statistics; e.g., in rewriting multivariate linear models in a univariate form, in developing the dispersion matrix of elements of a matrix such as a Wishart matrix and, from this, in deriving fourth moments in a general linear model.

As a result of these applications, the concept has, in recent years, been used by numerous writers. There is vecA or vec(A) used by Neudecker [1968, 1969a,b], Browne [1974] (who also uses a), Swain [1975] (who also uses A), Conlisk [1976], Balestra [1976], Anderson et al. [1977], Anderson [1978], Searle [1978, 1979], Brewer [1978], Magnus and Neudecker [1979] and Henderson and Searle [1979]. Equivalent notations are S(A), for stacking columns of A, used by Nissen [1968] who also uses a; and L(A) used by Conlisk [1969]; also, A for the column-rolled-out form of A, as it is described by Cole [1969]; and csA as the column string of A used by Vetter [1970, 1973, 1975], Kucera [1974] and Mitra [1977]. There is also

A_c of Tracy and Dwyer [1969], Singh [1972] and Tracy and Singh [1972a,b], col(A) of Hartwig [1972, 1975] and Hartwig and Morris [1975], $\mathscr{P}$(A) for the pack of A, in MacRae [1974] and in Rogers and Young [1978]; and vc$A$ as the vector of columns of A by Legault-Giguère [1974] and Giguère and Styan [1974] and a_c by Nel [1978].

Variations on what we have defined as vec$A$ are also available: for example, having it be the column vector derived from writing each transposed row of A one under the other. This stacks the elements of A in lexicon order and is, in our notation, equivalent to vecA', where A' is the transpose of A. Notations for this include a used by Lancaster [1969, 1970], A_r by Tracy and Dwyer [1969], Singh [1972] and Tracy and Singh [1972a,b], v(A) by Barnett [1973], vr$A$ as the vector of transposed rows of A by Legault-Giguère [1974] and a_r by Nel [1978]; and even vec$A$ by McDonald and Swaminathan [1973], Bentler and Lee [1975], McDonald [1976] and Pukelsheim [1977]. Other variations include the row vector of rows of A, (vecA')' in our notation, which Roth [1934] denotes by A^R and Vetter [1970, 1973, 1975] by rs$A$, as the row string of A. A final form, the row vector of transposed columns of A, (vec$A$)' has not yet been seen in the literature!

2.2. Origins of the Kronecker product

The Kronecker product of two matrices is defined for A and B of (1) as the mp X nq matrix

$$A \otimes B = \{a_{ij}B\} \ , \tag{3}$$

which is a particular case of the tensor product for transformations as discussed, for example, by Halmos [1958, p. 97]. In its original setting it seems to have been first studied by Zehfuss [1858], and although he dealt only with its determinant, Rutherford [1933] appropriately calls $A \otimes B$ the Zehfuss matrix of A and B. Loewy [1910, pp. 149-150] refers to Zehfuss's determinantal result as Kronecker's theorem which, according to Hensel [1891, p. 319], Kronecker (1823-91) had for sometime given in his algebra lectures in Berlin, which Bell [1937, p. 478] notes, he presented regularly from 1861 "principally on his personal researches, after the necessary introductions". Thus, although the name of Kronecker is now generally associated with the $A \otimes B$ operation, with an early use as pointed out by Dr. George Styan being Murnaghan [1938, p. 68], the exact origin of this association is hard to find.

Basic properties of Kronecker products are collected, with references to sources after 1889, by Loewy [1910, pp. 148-151] and by MacDuffee [1933, p. 82] using the names "product transformation" and "direct product", respectively. More recent summaries are Searle [1966], Neudecker [1968, 1969a,b] and Graybill [1969] in the context of statistical applications, and Barnett [1979] with engineering applications.

2.3. The vec and Kronecker product operators are connected

The definitions of vec and of Kronecker product show that $\text{vec}(ab') = b \otimes a$, so that $\text{vec}[(Aa)(b'C)] = (C' \otimes A)\text{vec}(ab')$. This, together with $B = \{b_{ij}\} = \Sigma\Sigma b_{ij} e_i e_j'$, leads to

$$\text{vec}(ABC) = (C' \otimes A)\text{vec}B , \qquad (4)$$

a result derived by Roth [1934], and hence called Roth's column lemma by Hartwig [1975], and rediscovered by Aitken [1949], Koopmans et al. [1950], Nissen [1968] and Neudecker [1969b].

3. VEC-PERMUTATION MATRICES

The vectors vecA and vecA' contain the same elements, in different sequences. We define $I_{m,n}$ as the permutation matrix such that, for A of order m by n,

$$\text{vec}A = I_{m,n}\text{vec}A' , \qquad (5)$$

and give it the name vec-permutation matrix.

There has been increasing interest over the last decade in vec-permutation matrices in matrix algebra, mathematics, statistics, econometrics and psychometrics. This wide interest in what we call

$I_{\sim m,n}$ of (5) is partly responsible for the diversity of its liter-
ature, for duplication of published results, for varying defin-
itions and for a wide range of notation and nomenclature. For
example, (5) defines what $I_{\sim m,n}$ does, in contrast to definitions
discussed later that describe what $I_{\sim m,n}$ looks like. This is an
important distinction because operational definitions of $I_{\sim m,n}$, like
(5), lead to succinct derivations of its properties, in contrast to
some recent and rather tedious proofs based on descriptive defi-
nitions. An example of this is the important role of $I_{\sim m,n}$ in re-
versing the order of Kronecker products; e.g., for $A_{\sim m\times n}$ and $B_{\sim p\times q}$

$$B \otimes A = I_{\sim m,p}(A \otimes B)I_{\sim q,n} . \tag{6}$$

In these circumstances, and because of the topical nature of
the subject, a cohesive account and succinct development of proper-
ties of vec-permutation matrices is needed. This is attempted in
Section 4. But first we discuss the variety of definitions, names
and notations to be found in the literature and summarized in Table
1, for a vec-permutation matrix, using our notation $I_{\sim m,n}$.

(Show Table 1 here)

3.1. <u>Definitions from reversing the order of Kronecker products of</u>
<u>matrices and vectors</u>

Ledermann [1936], in material from his Ph.D. thesis supervised
by A. C. Aitken, gives an elegant proof of (6), although without
explicit development of the vec-permutation matrices. Using column
vectors r, x, s and y of order m, n, p and q, respectively, with
$r = Ax$ and $s = By$, he proceeds as follows.

"The two products $A \otimes B$ and $B \otimes A$ are related to each other
by an identity

$$Q(A \otimes B)P^{-1} = (B \otimes A) \, ,$$

where P and Q are permutation matrices which depend only
on the types of A and B and not on their elements.
Proof: Apart from the order, the vectors $x \otimes y$ and $y \otimes x$
contain the same elements $\cdots$. We can therefore find a
permutation matrix P of degree nq such that

$$(y \otimes x) = P(x \otimes y)$$

and similarly

$$(s \otimes r) = Q(r \otimes s) \, ,$$

where Q is a permutation matrix of degree mp. Evidently
P and Q do not depend on the elements of $x, \; y, \; r, \; s$ but
only on the numbers m, n, p. q. By $[(A \otimes B)(x \otimes y) =$
$Ax \otimes By]$ we have

$$(r \otimes s) = (A \otimes B)(x \otimes y),$$
$$(s \otimes r) = (B \otimes A)(y \otimes x).$$

On premultiplying the first equation by Q and substi-
tuting $\cdots$ we get

$$(s \otimes r) = Q(A \otimes B)(x \otimes y) = (B \otimes A)P(x \otimes y) \, .$$

Since there is obviously no linear relation between the
elements of $x \otimes y$, we obtain

$$Q(A \otimes B) = (B \otimes A)P$$

or

$$Q(A \otimes B)P^{-1} = (B \otimes A) \, ,$$

as was to be proved."

[For consistent notation we have replaced X in the original

form of this quotation by ⊗, and changed subscripts to conform with

(1); and we do this in all quotations, without further comment.]

Ledermann [1936] then, in our notation, defines $\underset{\sim m, n}{I}$ as the

permutation matrix such that

$$\underset{\sim m, n}{I}(\underset{\sim}{a} \otimes \underset{\sim}{b}) = \underset{\sim}{b} \otimes \underset{\sim}{a} , \tag{7}$$

for any column vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ of order m and n, respectively.

Conlisk [1976] has a similar idea in mind for $\underset{\sim n, n}{I}$, using the nota-

tion $\underset{\sim}{M}$. [Actually, he writes (p. 760) $(\underset{\sim}{a} \otimes \underset{\sim}{b})M = \underset{\sim}{b} \otimes \underset{\sim}{a}$, which is

clearly incorrect, from dimension considerations alone. However,

he correctly claims (p. 763) that $\underset{\sim}{M}(\underset{\sim}{a} \otimes \underset{\sim n \times n}{B}) = \underset{\sim}{B} \otimes \underset{\sim}{a}$.] Defining

$\underset{\sim m, n}{I}$, as in (7), in terms of reversing the order of Kronecker pro-

ducts of vectors, was also suggested by a referee of Henderson and

Searle [1979]. The suggestion has merit, but it emphasizes a

feature of $\underset{\sim m, n}{I}$ that is second to the more fundamental and more

general definition given in (5) in terms of the vec operator. In

fact, (7) is the special case of (5) with $\underset{\sim}{A}$ being $\underset{\sim\sim}{ba}'$, of rank 1.

Murnaghan [1938] was also interested in reversing the order of

Kronecker products and, in this connection, gives an explicit formu-

lation of what we call a vec-permutation matrix. He denotes the

position of $a_{ij}b_{rs}$ in $\underset{\sim}{A} \otimes \underset{\sim}{B}$ by the row index-pair (i, r) and column

index-pair (j, s), and notes (p. 68-69) that for $\underset{\sim}{A}$ and $\underset{\sim}{B}$ square of

order m and n, respectively,

"$B \otimes A$ is obtainable from $A \otimes B$ by applying the <u>same</u>
permutation to the rows and columns of the latter. Let
us denote by $(i,r)^*$ the position of (r,i) when the
ordering is dictionary-like, the label with the range n
coming first; e.g., if $m = 2$, $n = 3$ the dictionary order
where the label with the range 2 comes first is $(1,1)$,
$(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(2,3)$ whilst the dictionary
order when the label with the range 3 comes first is
$(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,1)$, $(3,2)$. Hence $(1,1)^*$
$= (1,1)$, $(1,2)^* = (1,3)$, $(1,3)^* = (2,2)$, $(2,1)^* = (1,2)$,
$(2,2)^* = (2,1)$, $(2,3)^* = (2,3)$. Then if the (j,s) column
of $A \otimes B$ is transferred to the $(j,s)^*$ position and the
(i,r) row to the $(i,r)^*$ position we obtain $B \otimes A$. In
other words

$$(B \otimes A) = P(A \otimes B)P^{-1} \tag{8}$$

where P is the permutation matrix associated with $\left(\dfrac{(i,r)^*}{(i,r)} \right)$.
Notice that P is quite independent of the elements of
A and B being completely determined by their dimensions."

A clearer statement of Murnaghan's [1938] development of P,
which is our $I_{m,n}$, is that

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row (i,j) of $\underset{\sim}{I}_{m,n}$ is row (j,i) of $\underset{\sim}{I}_{mn}$, (9)

for $i = 1, \cdots, n$ and $j = 1, \cdots, m.$ Note that $\underset{\sim}{I}_{mn}$ is the identity matrix of order mn, to be distinguished from $\underset{\sim}{I}_{m,n}$, a vec-permutation matrix of the same order. Searle [1966, p. 216] makes a more explicit statement of this situation:

"It is apparent from (3) that for $\underset{\sim}{A}_{m \times n} = \{a_{ij}\}$ and $\underset{\sim}{B}_{p \times q} = \{b_{rs}\}$ the elements of both $\underset{\sim}{A} \otimes \underset{\sim}{B}$ and $\underset{\sim}{B} \otimes \underset{\sim}{A}$ consist of all possible products $a_{ij}b_{rs}$. In fact, $\underset{\sim}{B} \otimes \underset{\sim}{A}$ is simply $\underset{\sim}{A} \otimes \underset{\sim}{B}$ with the rows and columns each in a different order. Thus [in general]

$$\underset{\sim}{B} \otimes \underset{\sim}{A} = \underset{\sim}{P}(\underset{\sim}{A} \otimes \underset{\sim}{B})\underset{\sim}{Q} \qquad (10)$$

where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are each a product of E-type elementary operators [permutation matrices].

For any values of i, j, r and s the element $a_{ij}b_{rs}$ is located in $\underset{\sim}{A} \otimes \underset{\sim}{B}$ in the r'th row and s'th column of the i,j'th sub-matrix $[a_{ij}\underset{\sim}{B}] \cdots$. It is therefore in row $[p(i - 1) + r]$ and column $[q(j - 1) + s]$ of $\underset{\sim}{A} \otimes \underset{\sim}{B}$. In $\underset{\sim}{B} \otimes \underset{\sim}{A}$, however, $b_{rs}a_{ij}$ is in row i and column j of the r,s'th sub-matrix $[b_{rs}\underset{\sim}{A}] \cdots$. It is therefore in row $[m(r - 1) + i]$ and column $[n(s - 1) + j]$ of $\underset{\sim}{B} \otimes \underset{\sim}{A}$. Consequently the interchanging of rows and columns implied in (10), to obtain $\underset{\sim}{B} \otimes \underset{\sim}{A}$ from $\underset{\sim}{A} \otimes \underset{\sim}{B}$, can be specified as follows. The $[i + (j - 1)m]$'th row of $\underset{\sim}{P}$ is the $[(i - 1)p + j]$'th row of $\underset{\sim}{I}_{mp}$, for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, p$, and the $[i + (j - 1)n]$'th column of $\underset{\sim}{Q}$

is the $[(i - 1)q + j]$'th column of $\underset{\sim}{I}_{nq}$, for i = 1, 2,
$\cdots$, n and j = 1, 2, $\cdots$, q. "

Comparison of (6) with (10) shows that $\underset{\sim}{P} = \underset{\sim}{I}_{m,p}$ and $\underset{\sim}{Q} = \underset{\sim}{I}_{q,n}$; so
that Searle's [1966] expression for $\underset{\sim}{P}$ may be restated, in our no-
tation, as

$$\text{row } (i - 1)m + j \text{ of } \underset{\sim}{I}_{m,n} \text{ is row } (j - 1)n + i \text{ of } \underset{\sim}{I}_{mn} , \quad (11)$$

for i = 1, $\cdots$, n and j = 1, $\cdots$, m. This is, of course, equiva-
lent to (9) because row (i,j) of $\underset{\sim}{I}_{m,n}$ is its $[(i - 1)m + j]$'th row.
Expressions (9) and (11) show consecutive rows of $\underset{\sim}{I}_{m,n}$ to be every
n'th row of $\underset{\sim}{I}_{mn}$ starting with the first, and so on; a fact which
Tracy and Dwyer [1969] rediscovered and used as its definition, as
is now discussed.

3.2. Defining the row permutations

Tracy and Dwyer [1969, p. 1579] introduce

$$\underset{\sim}{I}_{(n)} \text{ as the matrix obtained by rearranging}$$

the rows of $\underset{\sim}{I}_r$ by taking

every n'th row starting with the first, then

every n'th row starting with the second, and so on.

$$(12)$$

Tracy and Singh [1972a,b] and Singh [1972], restricting $\underset{\sim}{I}_{(n)}$ to be of order mn, establish (5) and (6) and call $\underset{\sim}{I}_{(n)}$ a permuted identity matrix, a name which actually applies to any permutation matrix. We call $\underset{\sim}{I}_{(n)}$ of (12) a <u>generalized</u> <u>vec-permutation</u> <u>matrix</u> because it is defined for any order, not necessarily a multiple of n, whereas

$$\underset{\sim}{I}_{m,n} = \underset{\sim}{I}_{(n)} \text{ of order mn .} \tag{13}$$

For example, the rows of

$$\underset{\sim}{I}_{2,3} = \begin{bmatrix} 1 & . & . & | & . & . & . \\ . & . & . & | & 1 & . & . \\ \hline . & 1 & . & | & . & . & . \\ . & . & . & | & . & 1 & . \\ \hline . & . & 1 & | & . & . & . \\ . & . & . & | & . & . & 1 \end{bmatrix} \tag{14}$$

(where dots represent zeros), are every third row, namely rows 1, 4, 2, 5, 3 and 6, respectively, of $\underset{\sim}{I}_6$.

3.3. Descriptive definitions of $I_{\sim m, n}$

In contrast to operational definitions like (5) and (7) through (13), there is a number of what can be called descriptive definitions based on describing what $I_{\sim m, n}$ looks like. They amount to defining

$I_{\sim m, n}$ as a square matrix of order mn, partitioned
into an n by m array of submatrices each of order
m by n, such that the (i, j)'th submatrix $S_{\sim ij}$, say,
has unity in its (j, i)'th position and zeros elsewhere. (15)

For example, $I_{\sim 2, 3}$ of (14) is shown partitioned in accord with (15).

Variations of (15) include Vetter's [1970] $\langle U^{ji}_{\sim \ ij} \rangle$,
Hartwig's [1972, p. 540] $P_{\sim}$ (which should be $P_{\sim}^{-1}$) for his use in
$P_{\sim}^{-1}(A_{\sim m \times m} \otimes B_{\sim n \times n})P_{\sim} = B_{\sim} \otimes A_{\sim}$; Vetter's [1973, 1975] $E_{\sim n \times m}^{m \times n}$; the "permuted identity matrix" $I_{\sim (m, n)}$ of MacRae [1974], who notes (p. 338) that it "is identical to the matrix $I_{\sim (n)}$ [of order mn] defined by Tracy and Dwyer [1969]" and the universal flip matrix $P_{\sim m, n}$ of Hartwig and Morris [1975]. Balestra [1976, p. 21], using $P_{\sim m, n}$, clarifies the partitioning in MacRae's [1974] definition. He also gives a symbolic version of (15):

$$I_{\sim m, n} = P_{\sim m, n} = \{S_{\sim ij}\} \quad i = 1, \cdots, n; \ j = 1, \cdots, m,$$

where (16)

$$S_{\sim ij} = e_{\sim j}e_{\sim i}' = e_{\sim j} \otimes e_{\sim i}' = e_{\sim i}' \otimes e_{\sim j},$$

which is a generalization of $I_{\sim (n, n)}$ in Anderson et al. [1977], with $S_{\sim ij} = e_{\sim j}e_{\sim i}'$ presented in Anderson [1978, p. 53].

A further formulation of (15) and of (16), where $H'_{\sim ij}$ is $S_{\sim ij}$, is

$$\underset{\sim m,n}{I} = \underset{\sim nm}{K} = \sum_{i=1}^{n} \sum_{j=1}^{m} (\underset{\sim ij}{H} \otimes \underset{\sim ij}{H'})$$

where $H_{\sim ij}$ is the $n \times m$ matrix with a 1 in position (i,j) and zeros elsewhere. $\qquad$ (17)

Vetter [1973, 1975] presents (17) as does Brewer [1977, 1978], who uses $U_{\sim n \times m}$ for $\underset{\sim m,n}{I}$. Magnus and Neudecker [1979] also use (17), which they derive from (5). They call it a "commutation matrix" because of its role in reversing ("commuting") the order of Kronecker products, and denote it by $K_{\sim nm}$.

Definition (15) and its variations impose an artificial partitioning on $\underset{\sim m,n}{I}$, which disguises its inherent permutation features. Only the partitioning into n blocks of m rows or m blocks of n columns, but not both, is natural. This natural partitioning is implicit in the operational definitions and is explicit in a further formulation given by Balestra [1976]:

$$\underset{\sim m,n}{I} = \{ \underset{\sim m}{I} \otimes \underset{\sim i}{e'} \}_{i=1 \cdots n} = \{ \underset{\sim n}{I} \otimes \underset{\sim j}{e} \}_{j=1 \cdots m} , \qquad (18)$$

the second of which is also given by Swain [1975] who uses $L_{\sim mn}$ for $\underset{\sim m,n}{I}$. Denoting the direct sum $\underset{\sim 1}{A} \oplus \cdots \oplus \underset{\sim n}{A}$ by $\bigoplus_{i=1}^{n} \underset{\sim i}{A}$, we see that (18) may be written as

$$\underset{\sim m,n}{I} = \left\{ \bigoplus_{j=1}^{m} \underset{\sim i}{e'} \right\}_{i=1 \cdots n} = \left\{ \bigoplus_{i=1}^{n} \underset{\sim j}{e} \right\}_{j=1 \cdots m} , \qquad (19)$$

where in the first partitioning in (19) the $e'_{\sim i}$ have order n, and in the second the $e_{\sim j}$ have order m.

Results (18) and (19) are nicely illustrated by (14). Two further

formulations of $I_{m,n}$ developed by Magnus and Neudecker [1979] are

$$I_{m,n} = \sum_{i=1}^{m} (e_i' \otimes I_n \otimes e_i) \quad \text{and} \quad I_{m,n} = \sum_{j=1}^{n} (e_j \otimes I_m \otimes e_j') . \quad (20)$$

Although Hartwig and Morris [1975] give a block formulation

definition of $I_{m,n}$, they give an ingenious card shuffling interpre-

tation which is in fact based on (5), our definition. They identify

the permutation implicit in $I_{m,n}$ with the "generalized out faro-

shuffle" for a one dimensional deck of mn cards. Their description,

on p. 451, with further details in Morris and Hartwig [1976], is as follows.

> "In this shuffle a deck of mn cards, labeled from
> top to bottom, is cut into m portions of n cards, and
> each portion is given in clockwise fashion to one of m
> players seated at a circular table, starting with the
> dealer. If, starting with the dealer, in clockwise
> fashion, each player plays his top card when it is his
> turn, until all cards have been played, we obtain the
> permutation Π , labeled from the bottom cards up.
>
> When m = 2 this reduces to the classical out faro-
> shuffle (in which an even deck of cards is cut in halves
> and then ruffled such that the first and the last cards
> remain in fixed positions) which is the basis to several
> remarkable card tricks."

A further application of these permutations, kindly brought to

our attention by Dr. Stephen Barnett, is that of Whelchel and Guinn

[1970] who are concerned with shuffling data in computer storage.

In this context, they refer to $I_{n,n}$ as the "shuffle matrix", and

denote it by S_{n^2}.

3.4. Defining $\underset{\sim m,n}{I}$ as a matrix derivative operator

McDonald and Swaminathan [1973, p. 39] denote $\underset{\sim m,n}{I}$ by $\underset{\sim}{E}'$, and subsequently in McDonald [1976] by $\underset{\sim n,m}{E}$ to eliminate ambiguity, defined as the derivative operator

$$\underset{\sim m,n}{I} = \underset{\sim n,m}{E} = \partial \underset{\sim}{X}'/\partial \underset{\sim}{X}, \text{ where } \underset{\sim m \times n}{X} \text{ is m.i.v.} \tag{21}$$

[A matrix is said to be mathematically independent and variable (m.i.v.) if no elements are functionally dependent or constant.] They recognize $\underset{\sim m,n}{E}$ as a permutation matrix of $\underset{\sim mn}{I}$, and give (1973, p. 39) its form:

"the general element of $\underset{\sim}{E}[\underset{\sim n,m}{I}]$, e_{gh}, is
equal to unity if $j = k'$ and $k = j'$ with

$g = n(j - 1) + k$, $0 < j \leq m$, $0 < k \leq n$,

$h = m(j' - 1) + k'$, $0 < j' \leq n$, $0 < k' \leq m$,

and is zero otherwise."

This is a less concise statement than each of (9), (11) and (13).

Rather than define $\underset{\sim m,n}{I}$ as the outcome of differentiation, as in (21) and which demands the m.i.v. property, and which fails to highlight the permutation properties of $\underset{\sim m,n}{I}$, we prefer to derive (21) from the more fundamental definition (5) using the standard result $\partial \underset{\sim\sim}{Ax}/\partial \underset{\sim}{x} = \underset{\sim}{A}'$ for $\underset{\sim}{x}$ being m.i.v.; and then for $\underset{\sim}{X}$ being m.i.v.

$$\frac{\partial \underset{\sim}{X}'}{\partial \underset{\sim}{X}} \equiv \frac{\partial \text{vec}\underset{\sim}{X}'}{\partial \text{vec}\underset{\sim}{X}} = \frac{\partial \underset{\sim n,m}{I} \text{vec}\underset{\sim}{X}}{\partial \text{vec}\underset{\sim}{X}} = (\underset{\sim n,m}{I})' = \underset{\sim m,n}{I} \cdot$$

4. PROPERTIES OF VEC-PERMUTATION MATRICES

Except for (6) which we consider separately, well-known

properties of $I_{\sim m,n}$ are given without proof by Tracy and Singh

[1972a,b], Singh [1972], Vetter [1973, p. 354] who suggests veri-

fication "by construction", MacRae [1974, p. 339] who comments

they "can be verified by direct examination", Balestra [1976] and

Swain [1975], and very recently by Brewer [1978] and Magnus and

Neudecker [1979] whose proofs seem somewhat lengthy. Using $I_{\sim m,n}$ as

defined in (5), we develop considerably shorter and simpler proofs.

Applying (5) to vecA' in (5) itself gives $\text{vec}A_{\sim} = I_{\sim m,n}I_{\sim n,m}\text{vec}A_{\sim}$

so that

$$I_{\sim m,n}I_{\sim n,m} = I_{\sim mn} . \tag{22}$$

Then, because $I_{\sim m,n}$ is a permutation matrix and so is orthogonal, we

have

$$(I_{\sim m,n})^{-1} = (I_{\sim m,n})' = I_{\sim n,m} . \tag{23}$$

Also because $\text{vec}a_{\sim} = \text{vec}a_{\sim}'$

$$I_{\sim m,1} = I_{\sim 1,m} = I_{\sim m} . \tag{24}$$

[We use the notation $(I_{\sim m,n})'$ and $I'_{\sim m,n}$ interchangeably.]

4.1. Reversing the order of Kronecker products of matrices

Vec-permutation matrices are related very directly to Kronecker

products through the identity (6):

$$B_{\sim p\times q} \otimes A_{\sim m\times n} = I_{\sim m,p}(A_{\sim} \otimes B_{\sim})I_{\sim q,n} . \tag{25}$$

Derivation of (25) comes from using the two fundamental properties of the vec operator, (4) and (5), as follows:

$$(B \otimes A)\text{vec}X = \text{vec}AXB' = I_{m,p}\text{vec}BX'A'$$

$$= I_{m,p}(A \otimes B)\text{vec}X'$$

$$= I_{m,p}(A \otimes B)I_{q,n}\text{vec}X \;,$$

for any X of order n by q. Letting vecX be in turn the columns of an identity matrix of order nq yields (25).

This concise proof was motivated by Barnett's [1973] use of the vec operator to establish (25) but without explicit development of the permutation matrices. Similar developments are also given by Hartwig and Morris [1975] and more recently by Magnus and Neudecker [1979], even though the latter do not use vec to prove results like (22) and (23).

Result (25) shows the exact form of the relationship between $A \otimes B$ and $B \otimes A$, which simply entails a resequencing of rows and of columns of $A \otimes B$ to obtain $B \otimes A$. Postmultiplying (25) by $I_{n,q}$ and using (22) gives

$$I_{m,p}(A \otimes B) = (B \otimes A)I_{n,q} \;. \qquad (26)$$

An early reference to reversing the order of Kronecker products of rectangular matrices is Ledermann [1936] although without explicit development of permutation matrices. But Murnaghan [1938, pp. 68-69] deals with square matrices, noting for square A and B, the validity of $B \otimes A = P(A \otimes B)P^{-1}$ as in (8). Vartak [1955] generalizes this to (9), namely $B \otimes A = P(A \otimes B)Q$, for rectangular A and B, indicating that a generalization of Murnaghan's proof can be con-

structed.   Searle [1966, p. 216] shows that $a_{ij}b_{rs}$ is in row

i + (r - 1)m of $\underset{\sim}{B} \otimes \underset{\sim}{A}$ and row (i - 1)p + r of $\underset{\sim}{A} \otimes \underset{\sim}{B}$, with a similar

analysis for columns, and is thus able to give the explicit form of

$\underset{\sim}{P}$ and $\underset{\sim}{Q}$, as detailed after equation (10). Hartwig [1972, p. 540]

gives Murnaghan's result but misprints the form for $\underset{\sim}{P}^{-1}$ as being

that for $\underset{\sim}{P}$. Singh [1972, p. 22] identifies columns in (26) and

thence establishes (25). Tracy and Singh [1972b], MacRae [1974]

and Swain [1975] give (25) without proof; so does Vetter [1973,

p. 354] who comments that it is "well-known $\cdots$ though the explicit

transposition relationship is not usually given". In similar vein

is the remark that "the proof [not given] $\cdots$ is detailed and rather

tedious" by Bentler and Lee [1975, p. 148], to whom McDonald [1976,

p. 90] attributes (25). Balestra [1976, p. 23] uses (18), and

Brewer [1978] suggests substituting (17). These developments are

tedious in comparison to our proof using the vec operator.

4.2.   Trace and determinant of $\underset{\sim m, n}{I}$
_____

The case of m = n is easy. Because $\underset{\sim n, n}{I}$ is real, orthogonal

and symmetric, it has eigenvalues $\pm$ 1 with multiplicities $\frac{1}{2}n(n \pm 1)$,

respectively. The determinant and trace, being the product and sum

of the eigenvalues, respectively, are

$$\left| \underset{\sim n, n}{I} \right| = (-1)^{\frac{1}{2}n(n-1)} \quad \text{and} \quad \text{tr}(\underset{\sim n, n}{I}) = n \ .$$

For the more general situation, when m and n are not necessarily

equal, the recurrence relation $\left| \underset{\sim m, n}{I} \right| = (-1)^{\frac{1}{2}m(m-1)(n-1)} \left| \underset{\sim m, n-1}{I} \right|$

yields

$$\left| \underset{\sim}{I}_{m,n} \right| = (-1)^{\frac{1}{4}m(m-1)n(n-1)} \ .$$

This derivation is given by Hartwig and Morris [1975, p. 450] and Magnus and Neudecker [1979, p. 383] with an inductive proof presented by Swain [1975, Appendix A].

An expression for the trace is more difficult. Magnus and Neudecker [1979] prove that

$$\text{tr}(\underset{\sim}{I}_{m,n}) = 1 + \gcd(m - 1, \ n - 1) \ ,$$

where $\gcd(m,n)$ is the greatest common divisor of m and n. A less compact form of this result is available in Hartwig and Morris [1975] who additionally derive expressions for the characteristic and minimal polynomials.

4.3. A generalization: $\underset{\sim}{M}_{(n)}$ and $\underset{\sim}{M}_{m,n}$

Tracy and Dwyer [1969] introduce

$\underset{\sim}{M}_{(n)}$ as the r X c matrix formed by rearranging the rows of $\underset{\sim}{M}$, of order r X c, by taking every n'th row starting with the first, then every n'th row starting with the second, and so on,   (27)

so that $\underset{\sim}{I}_{(n)}$ of (12) performs the row permutations on $\underset{\sim}{M}$ to obtain $\underset{\sim}{M}_{(n)}$ as

$$\underset{\sim}{M}_{(n)} = \underset{\sim}{I}_{(n)} \underset{\sim}{M} \ . \tag{28}$$

For example, with dots denoting zeros, for

$$\underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{a}' \\ \underset{\sim}{b}' \\ \underset{\sim}{c}' \end{bmatrix} , \qquad \underset{\sim}{M}(2) = \begin{bmatrix} \underset{\sim}{a}' \\ \underset{\sim}{c}' \\ \underset{\sim}{b}' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix} \begin{bmatrix} \underset{\sim}{a}' \\ \underset{\sim}{b}' \\ \underset{\sim}{c}' \end{bmatrix} = \underset{\sim}{I}(2)\underset{\sim}{M} .$$

Papers subsequent to Tracy and Dwyer [1969] have focused on $\underset{\sim}{I}(n)$, rather than on the more general notion of $\underset{\sim}{M}(n)$ that has wider applicability. In this context, we now define $\underset{\sim}{M}_{m,n}$ for $\underset{\sim}{M}$ with mn rows as

$$\underset{\sim}{M}_{m,n} = \underset{\sim}{I}_{m,n}\underset{\sim}{M} \tag{29}$$

$$= \underset{\sim}{M}(n) \quad \text{when } \underset{\sim}{M} \text{ has } mn \text{ rows} . \tag{30}$$

Our symbol $\underset{\sim}{I}_{m,n}$ for the vec-permutation matrix conforms with $\underset{\sim}{M}_{m,n}$ of (29) because it is $\underset{\sim}{M}_{m,n}$ with $\underset{\sim}{M} \equiv \underset{\sim}{I}$; i.e.,

$$\underset{\sim}{I}_{m,n} = \underset{\sim}{I}_{m,n}\underset{\sim}{I} , \tag{31}$$

so motivating use of the letter $\underset{\sim}{I}$ for the matrix $\underset{\sim}{I}_{m,n}$ in preference to any other letter. In the special case of $m = n$ there might sometimes be convenience in alternative symbols; e.g., $\underset{\sim}{P}_n \equiv \underset{\sim}{I}_{n,n}$ or $\underset{\sim}{K}_n$ as introduced by Magnus and Neudecker [1979].

Within the framework of the definition (29) of $\underset{\sim}{M}_{m,n}$ it can be noted that the defining property, (5), and results (25) and (26) can be rewritten:

$$\text{vec}\underset{\sim}{A} = (\text{vec}\underset{\sim}{A}')_{m,n} , \tag{32}$$

$$\underset{\sim}{B} \otimes \underset{\sim}{A} = [(\underset{\sim}{A} \otimes \underset{\sim}{B})'_{m,p}]'_{n,q} = [(\underset{\sim}{A}' \otimes \underset{\sim}{B}')'_{n,q}]_{m,p} , \tag{33}$$

and

$$(\underset{\sim}{A} \otimes \underset{\sim}{B})_{m,p} = (\underset{\sim}{B}' \otimes \underset{\sim}{A}')'_{q,n} . \tag{34}$$

4.4. Equalities from reversing the order of Kronecker products

Alternative forms of (26) are now developed. In (34), $(\underset{\sim}{A} \otimes \underset{\sim}{B})_{m,p}$ is by definition a reordering of the rows of $\underset{\sim}{A} \otimes \underset{\sim}{B}$ by taking every p'th row starting with the first, and so on. Since $\underset{\sim}{B}$ has p rows, the i'th block of rows of $(\underset{\sim}{A} \otimes \underset{\sim}{B})_{m,p}$ is $\underset{\sim}{A} \otimes \underset{\sim}{\beta}'_i$, for $\underset{\sim}{\beta}'_i$ of (1) so that

$$\underset{\sim}{I}_{m,p}(\underset{\sim}{A} \otimes \underset{\sim}{B}) = (\underset{\sim}{A} \otimes \underset{\sim}{B})_{m,p} = \{\underset{\sim}{A} \otimes \underset{\sim}{\beta}'_i\}_{i=1\cdots p} \tag{35}$$

$$= \begin{bmatrix} \underset{\sim}{A} \otimes \underset{\sim}{\beta}'_1 \\ \vdots \\ \underset{\sim}{A} \otimes \underset{\sim}{\beta}'_p \end{bmatrix} = \{\underset{\sim}{a}_j \otimes \underset{\sim}{\beta}'_i\}_{\substack{i=1\cdots p \\ j=1\cdots n}} = \{\underset{\sim}{a}_j \underset{\sim}{\beta}'_i\}_{\substack{i=1\cdots p \\ j=1\cdots n}} \tag{36}$$

Notice that $(\underset{\sim}{A} \otimes \underset{\sim}{B})_{m,p}$ affords a compact notation for $\begin{bmatrix} \underset{\sim}{A} \otimes \underset{\sim}{\beta}'_1 \\ \vdots \\ \underset{\sim}{A} \otimes \underset{\sim}{\beta}'_p \end{bmatrix}$.

Applying (35) and (36) to the right-hand side of (34) yields the further equalities:

$$\underset{\sim}{I}_{m,p}(\underset{\sim}{A} \otimes \underset{\sim}{B}) = (\underset{\sim}{B} \otimes \underset{\sim}{A})\underset{\sim}{I}_{n,q}$$

$$= (\underset{\sim}{B}' \otimes \underset{\sim}{A}')'_{q,n} \qquad = \{\underset{\sim}{B}' \otimes \underset{\sim}{a}'_j\}'_{j=1\cdots n} = \{\underset{\sim}{B} \otimes \underset{\sim}{a}_j\}_{j=1\cdots n} \tag{37}$$

$$= [\underset{\sim}{B} \otimes \underset{\sim}{a}_1 \cdots \underset{\sim}{B} \otimes \underset{\sim}{a}_n] = \{\underset{\sim}{\beta}'_i \otimes \underset{\sim}{a}_j\}_{\substack{i=1\cdots p \\ j=1\cdots n}} = \{\underset{\sim}{a}_j \underset{\sim}{\beta}'_i\}_{\substack{i=1\cdots p \\ j=1\cdots n}} \tag{38}$$

These equalities lead to the equivalent descriptive formulations of $\underset{\sim}{I}_{m,n}$, (15), (16), (17) and (18), on substituting $\underset{\sim}{A} = \underset{\sim}{I}_m$ and $\underset{\sim}{B} = \underset{\sim}{I}_n$ in (35) - (38) to give $\underset{\sim}{I}_{m,n} = (\underset{\sim}{I}_m \otimes \underset{\sim}{I}_n)_{m,n}$.

4.5. Reversing the order of Kronecker products of vectors

The special cases of $\underset{\sim}{A}$ and/or $\underset{\sim}{B}$ being vectors are worthy of note. For column vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ of order m and p, respectively, putting $n = 1$, then $q = 1$ and then $n = q = 1$, in (26) gives

$$(\underset{\sim}{a} \otimes \underset{\sim}{B})_{m,p} = \underset{\sim m,p}{I}(\underset{\sim}{a} \otimes \underset{\sim}{B}) = \underset{\sim}{B} \otimes \underset{\sim}{a} , \tag{39}$$

$$(\underset{\sim}{A} \otimes \underset{\sim}{b})_{m,p} = \underset{\sim m,p}{I}(\underset{\sim}{A} \otimes \underset{\sim}{b}) = \underset{\sim}{b} \otimes \underset{\sim}{A} , \tag{40}$$

and

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})_{m,p} = \underset{\sim m,p}{I}(\underset{\sim}{a} \otimes \underset{\sim}{b}) = \underset{\sim}{b} \otimes \underset{\sim}{a} . \tag{41}$$

Transposing (39) - (41), or putting $p = 1$ and $m = 1$ in (26), gives the corresponding results for row vectors. Using $n = 1$ and $p = 1$ in (26), (36) and (38) reveals the familiar commutativity property of Kronecker products of vectors:

$$\underset{\sim}{a} \otimes \underset{\sim}{b}' = \underset{\sim\sim}{ab}' = \underset{\sim}{b}' \otimes \underset{\sim}{a} .$$

4.6. The vec-permutation matrix and Kronecker products of 3 or

more matrices

Extending the vec-permutation matrix to cyclically permute order in Kronecker products of three or more matrices is straight-forward, as indicated by Magnus and Neudecker [1979]. Balestra [1976, p. 24] introduced $\underset{\sim mp,s}{I}$, for use in this connection but perhaps surprisingly did not explicitly give its use to cyclically permute a Kronecker product of three matrices. We present this result by immediate application of (25) to $(\underset{\sim}{A} \otimes \underset{\sim}{B}) \otimes \underset{\sim}{C} = \underset{\sim}{A} \otimes (\underset{\sim}{B} \otimes \underset{\sim}{C})$ for $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ of order $m \times n$, $p \times q$ and $s \times t$, respectively:

$$\underset{\sim}{C} \otimes \underset{\sim}{A} \otimes \underset{\sim}{B} = \underset{\sim mp, s}{I} [(\underset{\sim}{A} \otimes \underset{\sim}{B}) \otimes \underset{\sim}{C}] \underset{\sim t, nq}{I} \tag{42}$$

$$= \underset{\sim p, ms}{I} [\underset{\sim}{B} \otimes (\underset{\sim}{C} \otimes \underset{\sim}{A})] \underset{\sim nt, q}{I}$$

$$= \underset{\sim p, ms}{I} \underset{\sim m, ps}{I} [\underset{\sim}{A} \otimes (\underset{\sim}{B} \otimes \underset{\sim}{C})] \underset{\sim qt, n}{I} \underset{\sim nt, q}{I} \tag{43}$$

$$= (\underset{\sim m, s}{I} \otimes \underset{\sim p}{I})(\underset{\sim m}{I} \otimes \underset{\sim p, s}{I})[\underset{\sim}{A} \otimes (\underset{\sim}{B} \otimes \underset{\sim}{C})](\underset{\sim n}{I} \otimes \underset{\sim t, q}{I})(\underset{\sim t, n}{I} \otimes \underset{\sim q}{I}). \tag{44}$$

Properties of vec-permutation matrices with the same set of three indices, developed by Balestra [1976, pp. 24-25] and Magnus and Neudecker [1979] with lengthy algebra, are now shown to be easy consequences of (42) - (44): put $n = q = t = 1$ in (42) - (44), so that $\underset{\sim}{A} \otimes \underset{\sim}{B} \otimes \underset{\sim}{C}$ becomes $\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}$ and the final vec-permutation matrices on the right-hand sides of (42) - (44) reduce to $\underset{\sim}{I}$, giving

$$\underset{\sim mp, s}{I}(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}) = \underset{\sim p, ms}{I} \underset{\sim m, ps}{I}(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c})$$

$$= (\underset{\sim m, s}{I} \otimes \underset{\sim p}{I})(\underset{\sim m}{I} \otimes \underset{\sim p, s}{I})(\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}). \tag{45}$$

Let $\underset{\sim}{a} \otimes \underset{\sim}{b} \otimes \underset{\sim}{c}$ take in turn the columns of $\underset{\sim mps}{I}$, and hence

$$\underset{\sim mp, s}{I} = \underset{\sim p, ms}{I} \underset{\sim m, ps}{I} = (\underset{\sim m, s}{I} \otimes \underset{\sim p}{I})(\underset{\sim m}{I} \otimes \underset{\sim p, s}{I}). \tag{46}$$

Since $\underset{\sim mp, s}{I} = \underset{\sim pm, s}{I}$, interchanging m and p in (46) yields

$$\underset{\sim mp, s}{I} = \underset{\sim pm, s}{I} = \underset{\sim m, ps}{I} \underset{\sim p, ms}{I} = (\underset{\sim p, s}{I} \otimes \underset{\sim m}{I})(\underset{\sim p}{I} \otimes \underset{\sim m, s}{I}). \tag{47}$$

Postmultiplying the middle two equalities in (47) by $\underset{\sim s, mp}{I}$ gives, using (22),

$$\underset{\sim m, ps}{I} \underset{\sim p, sm}{I} \underset{\sim s, mp}{I} = \underset{\sim mps}{I}. \tag{48}$$

This development, based on (45), is in contrast to the lengthy manipulations of Balestra [1976] who uses (18) to establish (46) - (48), and Magnus and Neudecker [1979] who give (42) and use (20) to develop (48) and all of (47) except its final equality.

A special case of (42) is

$$(\underset{\sim}{a} \otimes \underset{\sim}{b}' \otimes \underset{\sim}{C})_{m,s} = \underset{\sim}{b}' \otimes \underset{\sim}{C} \otimes \underset{\sim}{a} \ . \tag{49}$$

Using $\underset{\sim}{a} \otimes \underset{\sim}{b}' = \underset{\sim\sim}{ab}'$, (49) becomes

$$(\underset{\sim\sim}{ab}' \otimes \underset{\sim}{C})_{m,s} = \underset{\sim}{b}' \otimes \underset{\sim}{C} \otimes \underset{\sim}{a} \tag{50}$$

which, applied on

$$\underset{\sim}{I}_{m,n} = (\underset{\sim}{I}_m \otimes \underset{\sim}{I}_n)_{m,n} = \left(\sum_{i=1}^{m} \underset{\sim}{e}_i \underset{\sim}{e}_i' \otimes \underset{\sim}{I}_n \right)_{m,n} , \tag{51}$$

with $\underset{\sim}{C} = \underset{\sim}{I}_n$ and $\underset{\sim}{a} = \underset{\sim}{b} = \underset{\sim}{e}_i$, yields the Magnus and Neudecker [1979] formulations for $\underset{\sim}{I}_{m,n}$ in (20).

Generalization to Kronecker products of four or more matrices and vec-permutation matrices with four or more indices is straightforward.

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Table 1: Definitions, notations and names for what, in this paper, is called the vec-permutation matrix, $I_{m,n}$.

Definition	Writer	Notation	Name
Reversing Kronecker products (a) vectors	Ledermann [1936]	P and Q	Permutation matrix
	Conlisk [1976] (incorrect)	M for $I_{n,n}$	Permutation matrix
(b) matrices	Murnaghan [1938]	P	Permutation matrix
	Vartak [1955]	P and Q	Permutation matrix
	Searle [1966]	P and Q	Elementary operators
	Whelchel and Guinn [1970]	S_{n^2} for $I_{n,n}$	Shuffle matrix
Defining row permutations	Tracy and Dwyer [1969]		
	Tracy and Singh [1972a,b]	$I_{(n)}$	Permuted identity matrix
	Singh [1972]		
The derivative $\partial X'/\partial X$	McDonald and Swaminathan [1973]	E	Permutation matrix
	Bentler and Lee [1975]	E	
	McDonald [1976]	$E_{n,m}$	
Description of elements	Vetter [1970]	$\langle U^{ji}{}_{ij}\rangle$	
	Hartwig [1972] (incorrect)	P	Permutation matrix
	Vetter [1973, 1975]	$E^{m \times n}_{n \times m}$	Permutation matrix
	MacRae [1974]	$I_{(m,n)}$	Permuted identity matrix
	Hartwig and Morris [1975]	$P_{m,n}$	Universal flip matrix
	Swain [1975]	L_{mn}	Permutation matrix
	Balestra [1976]	$P_{m,n}$	Permuted identity matrix
	Brewer [1977, 1978]	$U_{n \times m}$	Permutation matrix
	Magnus and Neudecker [1979]	K_{nm}	Commutation matrix
vecA related to vecA'	Barnett [1973]	P and Q	Permutation matrix
	Henderson and Searle [1979]	$I_{(m,n)}$	vec-permutation matrix
	Henderson and Searle (this paper)	$I_{m,n}$	vec-permutation matrix