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The Vertex Coloring Problem and its Generalizations

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Chapter 1

Introduction

The main topic of this thesis is the *Vertex Coloring Problem* and its generalizations, for which models, algorithms and bounds are proposed in the First Part.

The Second Part is dedicated to a different problem on graphs, namely a *Routing Problem* in telecommunication networks where not only the efficiency, but also the *fairness* of the solution are considered.

1.1 The Vertex Coloring Problem and its Generalizations

Consider the following problems:

1. Color the map of England, in such a way that no two counties touching with a common stretch of boundary are given the same color, by using the smallest number of colors [1].

2. Organize the timetable of examinations of a university. Each examination needs a time slot, and the university wants to organize as many examinations in parallel as possible, without exceeding the availability of classrooms, in order to reduce the number of time slots. Since students can take more than one course, and they must be able to take part in the exams of all the courses they have followed, two examinations cannot be scheduled at the same time, if there is at least one student taking both the corresponding courses.

3. Radio spectrum has to be assigned to broadcast emitting stations, in such a way that adjacent stations, which could interfere, use different frequencies (each station may need one or more frequencies). In general it is required that two interfering stations use frequencies that are far each other, with a distance depending on propagation phenomena. Since radio spectrum is a very scarce and expensive resource, the allocation of frequency must be the most efficient, i.e. the total number of frequencies has to be minimized.

4. In the metal industry, metal coils are heated in furnaces. Each coil has to be heated for at least a given amount of time (different for each coil), and coils heated together must

[1] Four are enough for any map, see Appel, Haken and Koch [10], the Four Color Conjecture was proposed by Francis Guthrie in 1852

be compatible, i.e. they must have similar heights. The problem is to decide which coils will be heated together, in order to minimize the total heating time.

5. An nXn traffic matrix has to be transmitted through an analogical satellite. Each entry of the matrix represents the amount of traffic (i.e. the connection duration) to be sent from a transmitting antenna to a receiving antenna. In order to be transmitted, the matrix must be decomposed into mode matrices, i.e. matrices with at most one non zero element (corresponding to the traffic sent for each pair of transmitting-receiving antennas) per row and per column, in such a way that the sum of the mode matrices corresponds to the original traffic matrix. The transmitting time of each mode matrix corresponds to its largest element, and the problem is to minimize the total transmitting time.

6. Some vehicles are used to deliver items. Some items cannot travel on the same vehicle, because they are dangerous or require special equipment. The problem is to minimize the number of vehicles, by considering that each item has a weight and the capacity of vehicles is bounded.

7. Aircrafts are approaching an airport. The traffic control system assigns them an altitude, where they wait their landing time. If the arrival intervals of two planes overlap, they cannot use the same altitude. The available altitudes are limited, and they have to be assigned efficiently.

At first sight, the problems listed above have nothing in common. They consider coloring a map, telecommunications, heating in a furnace, timetabling, delivery, assignment of altitudes to aircrafts, etc. However, they all are optimization problems with a common structure.

A resource is shared among users. Some users can access the resource simultaneously, while others are pairwise incompatible, and the resource must be duplicated. The problem asks how to group the users that will access the resource simultaneously, in such a way that the number of copies of the resource is minimized.

Problems with this structure have been represented as Vertex Coloring Problems. Formally, consider an undirected graph $G = (V, E)$, where V is the set of vertices and E the set of edges, having cardinality n and m, respectively. The Vertex Coloring Problem (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized. Vertex Coloring is a well known NP-hard problem (see Garey and Johnson [53]), and has received a large attention in the literature, not only for its real world applications in many engineering fields, a subset of those is reported above as an example, but also for its theoretical aspects and for its difficulty from the computational point of view. Actually, exact algorithms proposed for VCP are able to solve consistently only small instances, with up to 100 vertices for random graphs. On the other hand, real world applications commonly deal with graphs of hundreds or thousands of vertices, for which the use of heuristic and metaheuristic techniques is necessary.

Since n colors will always suffice for coloring any graph, a straightforward Integer Linear Programming (ILP) model for VCP can be obtained by defining the following two sets of binary variables: variables x_{ih} ($i \in V,\ h = 1, \ldots, n$), with $x_{ih} = 1$ iff vertex i is assigned to color h, and variables y_h ($h = 1, \ldots, n$) denoting if color h is used in the solution. A possible model for VCP reads:

$$\min \sum_{h=1}^{n} y_h \tag{1.1}$$

$$\sum_{h=1}^{n} x_{ih} = 1 \qquad \forall i \in V \tag{1.2}$$

$$x_{ih} + x_{jh} \leq y_h \quad \forall (i,j) \in E,\ h = 1, \ldots, n \tag{1.3}$$

$$x_{ih} \in \{0,1\} \quad \forall i \in V,\ h = 1, \ldots, n \tag{1.4}$$

$$y_h \in \{0,1\} \qquad h = 1, \ldots, n \tag{1.5}$$

Objective function (1.1) minimizes the number of colors used. Constraints (1.2) require that each vertex is colored, while (1.3) impose that at most one of a pair of adjacent vertices receive a color, when the color is used. Finally, (1.5) and (1.4) impose the integrality of the variables. Albeit more sophisticated models can lead to better computational results when solved by means of exact or heuristic techniques, and are discussed in the following of this thesis, the model using binary variables x_{ih} and y_h has the advantage of the clarity, and can be easily extended to VCP generalizations, as discussed in the following.

From the list of proposed problems, Problem 1 is a classical VCP, by defining a vertex for each county of England, and an edge connecting two vertices if the corresponding counties are touching with a common stretch of boundary. Problem 7 is a VCP too, if we associate a vertex to each aircraft and an edge connecting two vertices if the arrival intervals of the corresponding aircrafts overlap.

In many practical situation, the number of users that can access to a resource is bounded, or it may happen that each user consumes a (possibly different) fraction of the resource, and the total capacity of the resource is limited. We can model this situation by assigning a positive weight w_i to each vertex, and imposing a capacity constraint on the total weight of the vertices that receive the same color. The corresponding problem is known as Bounded Vertex Coloring Problem (BVCP) or Bin Packing Problem with Conflicts (where the Bin Packing Problem requires to assign a set of items, each one with a positive weight, to the smallest number of bins, each bin with the same capacity C, see Martello and Toth [88]). If C is the capacity of each color, we can impose a capacity constraint as follows:

$$\sum_{i=1}^{n} w_i x_{ih} \leq C \quad \forall h = 1, \ldots, n \tag{1.6}$$

Model (1.1)–(1.4), with constraint (1.6) is a BVCP. When all the weights of the vertices are equal to 1, constraint (1.6) determines the maximum number of vertices which can receive the same color. This models for example Problem 2, if a vertex i of weight 1 is associated to each examination, and a color h corresponds to a time slot: vertex i is assigned color h iff examination i is scheduled in time slot h; two vertices are adjacent if the corresponding examinations cannot be scheduled in the same time slot, because there is at least one student who may want to take part in both the examinations. Each time slot has a maximum capacity C, corresponding to the number of classrooms available for the examinations. Problem 6 can be modelled as a BVCP as well, if a vertex is associated to each item, a color to each vehicle, and C represents the capacity of a vehicle with respect to a given dimension.

In the *Bandwidth Coloring Problem* (BCP) distance constraints are imposed between adjacent vertices, replacing the difference constraints (1.3) of model (1.1)–(1.4), and the largest color used is minimized. A distance $d(i,j)$ is defined for each edge $(i,j) \in E$, and the absolute value of the difference between the colors assigned to i and j must be at least equal to this distance: $|c(i) - c(j)| \geq d(i,j)$ (in this problem more than n colors may be necessary, so let H be the set of available colors).

A possible model for BCP, using binary variables x_{ih} and y_h defined above, and a continuous variable k, reads:

$$\min\ k \tag{1.7}$$

$$k \geq y_h h \qquad h \in H \tag{1.8}$$

$$\sum_{h \in H} x_{ih} = 1 \qquad i \in V \tag{1.9}$$

$$x_{ih} + x_{jl} \leq 1 \qquad (i,j) \in E,\ h \in H,\ l \in \{h - d(i,j) + 1, ..., h + d(i,j) - 1\} \tag{1.10}$$

$$x_{ih} \leq y_h \qquad i \in V,\ h \in H \tag{1.11}$$

$$x_{i,h} \in \{0,1\} \qquad i \in V,\ h \in H \tag{1.12}$$

$$y_h \in \{0,1\} \qquad h \in H \tag{1.13}$$

The objective function (1.7) (in conjunction with constraints (1.8)) asks for minimizing the maximum color used. Note that in BCP the number of colors assigned to the vertices can be smaller than maximum color used. Constraints (1.10) state that the absolute value of the difference between the colors assigned to vertices i and j must be at least equal to $d(i,j)$. Constraints (1.11) ensure that if a vertex i uses a color h, then color h results as used.

In the *Multicoloring Problem* (MCP) a positive request r_i is defined for each vertex $i \in V$, representing the number of colors that must be assigned to vertex i, so that for each $(i,j) \in E$ the intersection of the color sets assigned to vertices i and j is empty. The *Bandwidth Multicoloring Problem* (BMCP) is the combination of the two problems above. Each vertex i must be assigned r_i colors, and each of these colors must respect the distance $d(i,j)$ with all the colors assigned to any adjacent vertex j. In this case, loop $d(i,i)$ represents the minimum distance between different colors assigned to the same vertex i. The three problems defined above, and in particular the BMCP (which generalizes the BCP and MCP) received a wide interest in telecommunications [4], where they model frequency assignment problems, like the one proposed in Problem 3

In all the problems and corresponding models considered up to now, the cost of each color has been set equal to one. In this thesis we consider also a Weighted version of the Vertex Coloring Problem (WVCP) in which each vertex i of a graph G has associated a positive weight w_i, and the objective is to minimize the sum of the costs of the colors used, where the cost of each color is given by the maximum weight of the vertices assigned to that color. The most natural model for this problem requires, in addition to the binary x_{ih} variables, saying if vertex i receives color h, continuous variables z_h $(h = 1, \ldots, n)$, denoting the cost of color h in the solution. The corresponding model for WVCP is:

$$\min \sum_{h=1}^{n} z_h \tag{1.14}$$

$$z_h \geq w_i \, x_{ih} \qquad i \in V, \; h = 1, \ldots, n \tag{1.15}$$

$$\sum_{h=1}^{n} x_{ih} = 1 \qquad i \in V \tag{1.16}$$

$$x_{ih} + x_{jh} \leq 1 \qquad (i,j) \in E, h = 1, \ldots, n \tag{1.17}$$

$$x_{ih} \in \{0,1\} \qquad i \in V, h = 1, \ldots, n \tag{1.18}$$

where objective function (1.14) minimizes the sum of the costs of the colors, which are defined by constraints (1.15). This model can be used to represent Problem 4, where each metal coil is associated to a vertex, its heating time to the vertex weight, and the coils which are heated together receive the same color, while coils which cannot enter the furnace together are connected by an arc. The heating time of a subset of the coils which enter the furnace together corresponds to the largest heating time, and to the cost of the color as well. The same model represents also Problem 5, when a vertex is associated to each non zero element of the traffic matrix, and an edge connects each vertex to all vertices appearing on the same row and on the same column. In this case a color corresponds to a so called mode matrix.

The first part of this thesis is devoted to the Vertex Coloring Problem and its generalizations, namely those introduced in this Chapter. The interest is mainly on models and efficient heuristic and metaheuristic algorithms for the approximate solution of large instances, which could not be tackled by means of exact techniques. All proposed algorithms have been implemented, and extensive computational experiments have been performed on benchmark instances from the literature, in order to evaluate the performance of the proposed approaches.

In detail, in Chapter 2 we consider the classical VCP, for which a two-phases metaheuristic approach is proposed: the first phase is based on an Evolutionary Algorithm, while the second one is a post-optimization phase based on the Set Covering formulation of the problem. Computational results on a set of benchmark instances conclude that the approach represents the state of the art heuristic algorithm for the problem.

Chapter 3 considers the BMCP, for which a metaheuristic algorithm, inspired to the one proposed in Chapter 2, is presented. The algorithm outperforms, on a set of benchmark instances, other metaheuristic approaches from the literature.

Chapter 4 is devoted to the WVCP. We propose a straightforward formulation for WVCP, and two alternative ILP models: the first one is used to derive, dropping integrality requirement for the variables, a tight lower bound on the solution value, while the second one is used to derive a 2-phase heuristic algorithm, also embedding fast refinement procedures aimed at improving the quality of the solutions found. Computational results on a large set of instances from the literature are reported.

Finally, Chapter 5 considers the BVCP, for which we present new lower and upper bounds, and investigate their behavior by means of computational experiments on benchmark instances.

1.2 Fair Routing

The term routing, in its broadest meaning, refers to selecting paths in a network along which to send data, vehicles, flow, etc., depending on the specific problem. In the second part of this thesis we consider the routing of packets in a telecommunication network, i.e. the selection of paths to send packets from their source in the network, toward their ultimate destination through intermediary nodes.

In particular we will consider a wireless ad hoc network, i.e. a network formed by a set of nodes which communicate through wireless connections, and do not make use of any preexisting infrastructure. Wireless ad hoc networks are characterized by two main aspects:

- the lack of preexisting infrastructure, and the possibility that the network topology changes over time, preventing the use of centralized solutions for the control of these networks. Decisions are usually taken at node level, where only local information, about the node condition and its close neighbors, is normally available;

- the use of wireless connections, which, in all situations where the network nodes are feed by an internally owned limited energy supply (e.g. a battery), raises problems about the energetic efficiency of the network.

Examples of ad hoc networks are sensor networks, used for geographical surveys, or temporary networks present during meetings or happenings. Wireless ad hoc networks will be highly pervasive in the next future, and it is not unlikely that the Internet network will be often extended through wireless ad hoc networks, instead of using wired connections.

A wide literature is available on ad-hoc networks (see Tonguz and Ferrari [105] for an introduction to ad-hoc networks), mainly devoted to the study of the efficiency of networks, and to the design of mechanisms to obtain a desired behavior from the network nodes. The efficiency of an ad hoc network is highly related to the routing protocols that the network uses. Since transmitting packets through the network has an energetic cost for the nodes, the routing of packets should be the most efficient one, in order to minimize the energy cost of the network, and to ensure its survival. Concerning the behavior of nodes, it must be considered that a node participates to the network by sending its own traffic, and, in addition, by forwarding the traffic of other nodes, thus ensuring the connectivity and improving the efficiency of the network. Forwarding traffic has no tangible benefit for the node; of course, the node has a benefit if other nodes forward too. The absence of immediate benefit for nodes contributing to the network raises the problem of nodes that, acting selfishly, do not forward network traffic. This led to the design of mechanisms to obtain a desired behavior from nodes.

The study of network efficiency and the design of forcing mechanisms do not consider that routing decisions in the network may be very unfair. The definition of the fairness of ad hoc networks is far from trivial and is part of this work, however, intuitively, the fairness of a network should measure the contribution that each node gives to the network with respect to the benefit it obtains from being in the network. It may happen that a very efficient routing, for the network as a whole, leads a node to spend all its energy to forward packets from other nodes, thus draining its energy source without benefit, and this is unfair.

The second part of this thesis is devoted to the study of possible measures for the fairness of ad hoc networks, to the relation existing between an efficient routing algorithm and a fair one, and to the design of fair and efficient routing algorithms.

To this aim, we first propose a model for the routing of packets in ad hoc networks. The network is represented as a weighted digraph $G = (V, A)$, where each node corresponds to a vertex $i \in V$, and the network links correspond to arcs $a \in A$. The weight of each arc a represents the energy needed to send a unit of information through the arc, and depends on the adopted propagation model. Each node has a capacity, depending on its remaining energy, which limits the amount of traffic it can send and forward. We want the analysis to be independent of the specific transmitting protocol, and then we use a fluid model representation, i.e. we describe the traffic in the network as a flow (of bits). Bits will be grouped into packets, but how the bits are grouped depends on the chosen protocol. Thus, from the feasibility viewpoint, the routing problem is tackled as a *Splittable MultiCommodity Flow with node Capacity*, i.e., given the quantity of information to be routed for a set of origin-destination pairs, the information can be split into multiple paths, and the routing is constrained by the capacities given by the residual battery life associated with the nodes.

If fairness is disregarder, the problem asks for finding the routing of minimum cost on graph G, i.e. the most energetically efficient, such that all the demand is transmitted. In this thesis we give two alternative measures for the fairness of a routing in such networks, and discuss how an efficient routing can be computed by satisfying a minimum fairness constraint, through the solution of a Linear Programming Model. The computation of this routing, however, requires a set of information which is normally not available at single node level, where the routing decisions are taken. Thus, the routing obtained through the solution of the proposed model can be considered as a benchmark on the best possible routing, and could be implemented only by a centralized control of the system, which is impossible for the intrinsic decentralized nature of ad hoc network.

So, we propose also a distributed routing algorithm, which uses only local information, available at node level. The algorithm is aimed at computing an efficient routing, while taking into consideration the fairness experienced by the nodes in the network.

The cost of fairness and the efficiency of the proposed distributed routing algorithm are evaluated through extensive computational experiments on randomly generated networks, which represent various network and traffic configurations.

Part I

Vertex Coloring Problems

Chapter 2

A Metaheuristic Approach for the Vertex Coloring Problem

[1]

Given an undirected graph $G = (V, E)$, the *Vertex Coloring Problem* (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized. In this paper we propose a metaheuristic approach for VCP which performs two phases: the first phase is based on an Evolutionary Algorithm, while the second one is a post-optimization phase based on the Set Covering formulation of the problem. Computational results on the DIMACS set of instances show that the overall algorithm is able to produce high quality solutions in a reasonable amount of time. For 4 instances, the proposed algorithm is able to improve the best known solution, while for almost all the remaining instances it finds the best known solution in the literature.

2.1 Introduction

Given an undirected graph $G = (V, E)$, the Vertex Coloring Problem (VCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized.

Vertex Coloring is a well known NP-hard problem (see Garey and Johnson [53]) with real world applications in many engineering fields, including scheduling [80], timetabling [39], register allocation [30], frequency assignment [52] and communication networks [112]. This suggests that effective algorithms would be of great importance. Despite its relevance, few exact algorithms for VCP have been proposed, and are able to solve consistently only small instances, with up to 100 vertices for random graphs [72, 101, 103, 41]. On the other hand, several heuristic and metaheuristic algorithms have been proposed which are able to deal with graphs of hundreds or thousands of vertices. We review below, after some useful definitions, the most important classes of known heuristics and metaheuristics proposed for VCP.

Let n and m be the cardinalities of vertex set V and edge set E, respectively; let $\delta(v)$ be the degree of a given vertex v. A subset of V is called an *independent set* if no two adjacent vertices belong to it. A *clique* of a graph G is a complete subgraph of G. A k *coloring* of G is a partition of V into k independent sets. An optimal coloring of G is a k *coloring* with the

[1] The results of this chapter appear in [84].

smallest possible value of k (the *chromatic number* $\chi(G)$ of G). The *chromatic degree* of a vertex is the number of different colors of its adjacent vertices.

The first approaches to VCP were based on greedy constructive algorithms. These algorithms sequentially color the vertices of the graph following some rule for choosing the next vertex to color and the color to use. They are generally very fast but produce poor results, which can be very sensitive to some input parameter, like the ordering of the vertices. Beyond the simple greedy sequential algorithm SEQ, the best known techniques are the *maximum saturation degree* DSATUR and the *Recursive Largest First* RLF procedures proposed by Brèlaz [20] and by Leighton [80], respectively (see Section 2.1.2 for a short description of these algorithms). Culberson and Luo [37] proposed the *iterated greedy algorithm* IG which can be combined with various techniques. In [18] Bollobàs and Thomason proposed algorithm MAXIS that recursively selects the maximun independent set from the set of uncolored vertices.

Many effective metaheuristic algorithms have been proposed for VCP. They are mainly based on simulated annealing (Johnson, Aragon, McGeoch and Schevon [72] compared different neighborhoods and presented extensive computational results on random graphs; Morgenstern [92] proposed a very effective neighborhood search) or Tabu Search (Hertz and De Werra [62]; Dorne and Hao [43]; Caramia and Dell'Olmo [25] proposed a local search with priorities rules, inspired from Tabu Search techniques). Funabiki and Higashino [50] proposed one of the most effective algorithms for the problem, which combines a Tabu Search technique with different heuristic procedures, color fixing and solution recombination in the attempt to expand a feasible partial coloring to a complete coloring. Hybrid algorithms integrating local search and diversification via crossover operators were proposed (Fleurent and Ferland [49]; Galinier and Hao [51] proposed to combine an effective crossover operator with Tabu Search), showing that diversification is able to improve the performance of local search.

As a general observation, two main strategies can be identified in the literature, which correspond to different formulations of the problem. The first strategy tackles the problem in the most natural way, trying to assign a color to each vertex. This leads to fast greedy algorithms but seems to produce poor results. The second strategy tackles the problem of feasibly coloring the graph by partitioning the vertex set into independent sets. Algorithms based on this strategy build different color classes by identifying different independent sets in the graph, and try to cover all the vertices by using the minimum number of independent sets. All the algorithms able to find good solutions on large graphs are based on the latter strategy.

The paper is organized as follows: in the remaining part of this section a new two-phase heuristic approach for VCP is presented. Sections 2.2 and 2.3 describe the first phase, based on an Evolutionary algorithm, and the second phase, which is a post optimization procedure based on the Set Covering formulation of the problem, respectively. Extensive computational experiments on literature instances are presented in Section 2.4. Concluding remarks are discussed in Section 2.5.

2.1.1 The Heuristic Algorithm MMT

The approach we propose is based on the second strategy and performs, in sequence, an initialization step and two optimization phases. In the initialization step, some fast lower bounding procedures and greedy heuristics from the literature are executed to derive a lower and an upper bound (LB and UB, respectively) on the optimal solution value. In the first

optimization phase (*Evolutionary Generation*), an effective Evolutionary Algorithm, based on the concept of partitioning the vertex set into independent sets, is executed. This algorithm works in decision version (i.e., given as input the number k of colors to use, it looks for a k *coloring* in the graph G), trying to improve on the best valued solution found by the greedy procedures executed in the initialization step. Sometimes the Evolutionary Algorithm is able to find a provably optimal solution; in any case, this algorithm generally improves the best incumbent solution, and during the search generates a very large number of independent sets (*columns*). When optimality of the incumbent solution is not proved, such columns are stored in a family $\mathcal{S}'$. The second optimization phase (*Column Optimization*) considers the Set Covering Problem (SCP) associated with the columns in $\mathcal{S}'$ and heuristically solves it through the Lagrangian heuristic algorithm CFT proposed by Caprara, Fischetti and Toth [23], improving many times the best incumbent solution.

Both optimization phases can be stopped as soon as a solution which is proven to be optimal is found, i.e., if the value of the best solution found so far is equal to a lower bound for the original problem.

The overall algorithm MMT is structured as follows:

begin
Initialization Step
1. compute lower bound LB;
2. compute upper bound UB;
3. $\mathcal{S}' := \emptyset$;
Phase 1: Evolutionary Algorithm (Evolutionary Generation)
4. **while** (not time limit)
   apply the Evolutionary Algorithm;
   update UB and $\mathcal{S}'$;
   if $LB = UB$ **stop**
5. **endwhile**;
Phase 2: Column Optimization
6. apply heuristic algorithm CFT to the Set Covering instance corresponding to subfamily $\mathcal{S}'$ with a given time limit (possibly updating UB)

end.

2.1.2 Initialization Step

Lower Bounding

As lower bound LB we use the cardinality of a maximal clique K of G. Although this is the simplest lower bound for the problem, better lower bounds would require a big computational effort (see for instance Caramia and Dell'Olmo [26], [27]). We compute LB as the maximum cardinality of the maximal cliques of G obtained by executing several times (say 10), with different random orderings of the vertices, the following greedy algorithm, which defines a maximal clique K. Let v_i be the i-th vertex of the considered ordering, LB the incumbent value of the lower bound and $\eta(v_i)$ the *K-degree* of v_i, i.e. the number of vertices in K that are adjacent to v_i. While the incumbent clique K can be expanded, we insert in K the first vertex of the considered ordering having maximum *K-degree*, if this insertion can improve on the best incumbent LB:

begin
1. $K := \emptyset$;
2. **while** $(|K| = max_{i:v_i \in V \setminus K}\ \eta(v_i))$
3. $j := min(arg\ max_{i:v_i \in V \setminus K \textbf{ and } \delta(v_i) > LB}\ \eta(v_i))$;
6. **if** no such j exists **then break**;
4. $K := K \cup \{v_j\}$
7. **end while**

end.

Upper Bounding

To derive an initial upper bound UB for the problem we perform one iteration of the greedy procedures SEQ, DSATUR, RLF [72].

SEQ is the simplest greedy algorithm for VCP. Assume that the vertices are labelled $v_1, ..., v_n$. Vertex v_1 is assigned to the first color class, and thereafter, vertex v_i $(i = 2, ..., n)$ is assigned to the lowest indexed color class that contains no vertices adjacent to v_i.

DSATUR [20, 72] is similar to SEQ, but dynamically chooses the vertex to color next, picking the first vertex that is adjacent to the largest number of distinctly colored vertices (i.e. the vertex with maximum chromatic degree).

The Recursive Largest First (RLF) algorithm [80, 72] colors the vertices, one class at a time, in the following greedy way. Let C be the next color class to be constructed, V' the set of uncolored vertices that can legally be placed in C, and U the set (initially empty) of uncolored vertices that cannot legally be placed in C.

- Choose the first vertex $v_0 \in V'$ that has the *maximum* number of adjacent vertices in V'. Place v_0 in C and move all the vertices $u \in V'$ that are adjacent to v_0 from V' to U.
- While V' remains nonempty, do the following: choose the first vertex $v \in V'$ that has the maximum number of adjacent vertices in U; add v to C and move all the vertices $u \in V'$ that are adjacent to v from V' to U.

We use these algorithms also during the initialization of the Evolutionary Algorithm (see Section 2.2.2).

Complexity

The time complexity of the initialization procedure is analyzed in the following.

- The maximal clique algorithm of Section 2.1.2 asks for choosing at most n times the vertex v_j with maximum value of $\eta(v_j)$, with a total complexity of $O(n^2)$. Every time a new vertex is inserted into the clique K the value of η of its adjacent vertices must be updated. The total time of this update is $O(m)$. The overall time complexity of the algorithm is $O(n^2)$.
- In the implementation of the SEQ algorithm a data structure of size $O(n^2)$ is used to store, for every vertex v_i and for every color h, if at least one vertex adjacent to v_i has color h. When trying to assign vertex v_i to a color class, the algorithm checks in this data structure if the color is available for v_i. So, the algorithm has to check $O(n)$ colors

for n vertices, with a total time complexity of $O(n^2)$. Every time a new vertex is colored the information stored in the data structure is updated for all its adjacent vertices, with a total complexity of $O(m)$. The overall time complexity of the SEQ algorithm is $O(n^2)$.

- The DSATUR algorithm uses the same data structure, the only difference being that, for n times, the vertex with the maximum chromatic degree is picked (with total complexity of $O(n^2)$). The update of the chromatic degree for every vertex is performed together with the update of the data structure, and the corresponding total complexity is $O(m)$. The overall time complexity of the DSATUR algorithm is $O(n^2)$.

- The RLF algorithm chooses every time the next vertex to be colored as the vertex v_i with the maximum number of adjacent vertices (in V' or in U), the total complexity of this search is $O(n^2)$. For every chosen vertex v_i, all its adjacent vertices w_j are moved in U with a total complexity of $O(m)$. Every time one adjacent vertex w_j is moved, its adjacent vertices z_l are adjacent to one more vertex in U. The update of this information has a total complexity of $O(m^2/n)$, since for every chosen vertex the algorithm has to retrieve all its adjacent vertices and all the vertices adjacent to them. The overall complexity of the RLF algorithm is $O(m^2/n)$.

2.2 PHASE 1: Evolutionary Algorithm

To find high quality solutions our first idea was to use a Tabu Search procedure, a metaheuristic technique that showed a very good experimental behavior on hard combinatorial optimization problems. The first results were quite encouraging but showed some drawbacks of our approach. In particular, for several instances, the Tabu Search procedure was unable to explore different regions of the whole solution space. So we decided to use it as a component of a more complex Evolutionary Algorithm.

2.2.1 Tabu Search Algorithm

A local search procedure can be seen as the result of three main components:

- the definition of a solution S;
- the solution evaluating function $f(S)$;
- the solution neighborhood $N(S)$.

In the simple local search procedures, given a solution S the algorithm explores its neighborhood $N(S)$ and moves to the best (according to the evaluating function $f(S)$) improving solution $S' \in N(S)$. If a solution S is the best of its neighborhood, i.e. it is a local optimum, the local search algorithm is not able to move and the search is stopped. In Tabu Search procedures, to avoid local optimum traps, the algorithm moves to the best solution S' in the neighborhood, even if it is not improving the current solution. To avoid cycling, some attributes of solution S' are stored in a *Tabu List*; for a specified number of iterations (the so called *Tabu Tenure*) a solution which presents tabu attributes is declared tabu and is not considered, except in the case it would improve the best incumbent solution (*aspiration criterion*). Most of the Tabu Search algorithms proposed so far for VCP move between

infeasible solutions, i.e. they partition the set V in subsets which are not necessary independent sets, trying to reduce the number of infeasibilities in every subset. Following an idea by Morgenstern [92], we propose a Tabu Search procedure which moves between *partial feasible colorings*, i.e. solutions in which each vertex subset is an independent set but not all vertices are assigned to subsets. In [92] Morgenstern defines the *Impasse Class Neighborhood*, a structure used to improve a partial k *coloring* to a complete coloring of the same value. The *Impasse Class* requires a target value k for the number of colors to be used. A solution S is a partition of V in $k+1$ color classes $\{V_1, ..., V_k, V_{k+1}\}$ in which all classes, but possibly the last one, are independent sets. This means that the first k classes constitute a partial feasible k *coloring*, while all vertices that do not fit in the first k classes are in the last one. Making this last class empty gives a complete feasible k *coloring*. To move from a solution S to a new solution $S' \in N(S)$ one can randomly choose an uncolored vertex $v \in V_{k+1}$, assign v to a different color class, say h, and move to class $k+1$ all vertices v' in class h that are adjacent to v. This assures that color class h remains feasible. Class h is chosen by comparing different target classes by mean of the evaluating function $f(S)$. Rather than simply minimizing $\mid V_{k+1} \mid$ it seems a better idea to minimize the value:

$$f(S) = \sum_{w \in V_{k+1}} \delta(w) \tag{2.1}$$

This forces vertices having small degree, which are easier to color, to enter class $k+1$. Morgenstern uses this idea, together with a procedure for the recombination of the solutions, to build a simulated annealing algorithm. We use the same idea within a Tabu Search approach. At every iteration we move from a solution S to the best solution $S' \in N(S)$ (even if $f(S) < f(S')$). To avoid cycling, we use the following tabu rule: a vertex v cannot take the same color h it took at least one of the last T iterations; for this purpose we store in a tabu list the pair (v, h). While pair (v, h) remains in the tabu list, vertex v cannot be assigned to color class h. We also use an *Aspiration Criterion*: a tabu move can be performed if it improves on the best solution encountered so far. A Tabu Search algorithm based on the same neighborhood structure was experimented by Blöchliger and Zufferey [15]: in this work the next vertex to color is not chosen randomly, but selected so that it, entering the best color class, produces the best solution in the neighborhood. This approach increases the size of the neighborhood reducing at the same time the randomness introduced in the search. Thus, to avoid premature convergence, the authors use an evaluating function that simply minimizes $\mid V_{k+1} \mid$.

Our Tabu Search algorithm takes in input:

- graph $G(V, E)$;
- the target value k for the coloring;
- a feasible partial k *coloring*;
- the maximum number L of iterations to be performed ;
- the tabu tenure T.

If the algorithm solves the problem within L iterations it gives on output a feasible coloring of value k, otherwise it gives on output the best scored partial coloring found during the search.

Let S be the current solution and S^* the best incumbent solution. The Tabu Search algorithm works as follows:

begin
1. initialize a solution $S := \{V_1, ..., V_k, V_{k+1}\}$;
2. $S^* := S$;
3. $tabulist := \emptyset$;
4. **for** ($iterations = 1$ **to** L)
5. randomly select an uncolored vertex $v \in V_{k+1}$;
6. **for each** $j \in \{1, ..., k\}$ (explore the neighborhood of S)
7. $V_j^{'} := V_j \setminus \{w \in V_j : (v, w) \in E\} \cup \{v\}$;
8. $V_{k+1}^{'} := V_{k+1} \setminus \{v\} \cup \{w \in V_j : (v, w) \in E\}$;
9. $S_j := S \setminus \{V_j, V_{k+1}\} \cup \{V_j^{'}, V_{k+1}^{'}\}$
10. **end for**;
11. $h := \arg\min_{j \in \{1,...,k\}:(v,j) \notin tabulist}$ **or** $_{f(S_j) < f(S^*)}\, f(S_j)$;
12. **if** no such h exists **then** $h := \arg\min_{j \in \{1,...,k\}} f(S_j)$;
13. $S := S_h$;
14. insert (v, h) in $tabulist$, (v, h) is tabu for T iterations;
15. **if** $f(S) < f(S^*)$ **then** $S^* := S$;
16. **if** $V_{k+1} = \emptyset$ **then return** S^*
17. **end for**;
18. **return** S^*

end.

At line 10 we try to select the best color class which improves on the best solution so far or does not represent a tabu move. If all moves are tabu, at line 11 we simply select the best color class.

Our Tabu Search algorithm is very simple and requires as parameter to be experimentally tuned only the tabu tenure T. At the same time it has a good experimental behavior, since it is often able to find good solutions in very short computing times (see Section 2.4.1). Computational experiments showed that the algorithm generally needs a small number of iterations to solve the problem, and when this does not occur, seldom the algorithm is able to solve the problem even if a bigger number of iterations is allowed. This behavior can be explained by the aggressive strategy adopted: we start with a partial feasible coloring and iteratively try to insert uncolored vertices in color classes. If a colored vertex is not conflicting (adjacent) with an uncolored one, its color is not changed, and possibly it will never be changed during the execution of the algorithm. The main drawback of this strategy is that in some cases it is not able to explore different regions of the whole solution space. This can be explained with an example: suppose that a pair of vertices belonging to different color classes are not conflicting with any of the uncolored vertices nor conflicting each other: their assignment to a color class will never be changed, and the algorithm will not explore the feasible solutions where the two vertices are in the same class. This consideration suggests that this Tabu Search scheme could be much more effective if combined with a suitable diversification strategy.

Complexity

The Tabu Search procedure represents the most time consuming part of the proposed approach, actually millions of Tabu Search iterations are performed to solve hard instances. An iteration is composed by four main operations: the random choice of the vertex v to color,

performed in constant time; the computation of how much would cost (according to the evaluating function (2.1)) to insert v in each color class, requiring the retrieve of all the vertices adjacent to v, with an average complexity of $O(m/n)$ ($O(n)$ if the graph is complete); the choice of the best color class V_h which is not tabu, with a complexity of $O(k)$; the update of the current coloring (i.e. the movement of the vertices adjacent to v which are in color class V_h to color class V_{k+1}), requiring the retrieve of all the vertices adjacent to v, is performed on average in $O(m/n)$ (in $O(n)$ if the graph is complete). Thus the total time complexity of one Tabu Search iteration is $O(n)$ in the worst case.

2.2.2 Evolutionary Diversification

Our Tabu Search procedure is simple, very quick in exploring a portion of the search space and often able to find good solutions in short times. To improve its performance we use it together with a diversification operator, trying to extend the search to the whole solution space. Diversification is usually used in genetic algorithms, in which a pool of solutions (population) is stored during the computation. Solutions in the pool evolve through interactions with other solutions during the *diversification phase*, when they mix together (parent solutions) to generate new solutions (offspring). In addition they evolve by themselves during the *mutation phase*, when, to avoid premature convergence and to preserve diversity, they are randomly perturbed. In general, solutions in the pool are improved by using some local search technique. Every solution is evaluated according to a fitness function so that, when new good solutions are generated, the worst solutions can be removed from the population. As shown by Davis [38], the classical genetic algorithms give poor results for VCP.

A recent development of these algorithms is represented by Evolutionary Algorithms. In this case the evolution of the population is obtained by means of two elements: an efficient local search procedure and a specialized *crossover operator*. The crossover operator should be able to create new and potentially good solutions to be improved through the local search procedure. For this a reason it cannot be a general operator but it must be designed specifically for the considered problem. In addition it must be able to transmit interesting properties from parents to new offspring. The main idea behind the use of a specialized crossover operator is that good solutions share part of their structure with optimal ones, and a specialized crossover should be able to identify properties that are meaningful for the problem.

In our algorithm we start with an initial pool of partial feasible solutions of value k (in the following simply *solutions*) obtained by using different methods (greedy and Tabu Search procedures initialized with different parameters). Then we apply the Tabu Search algorithm to improve these solutions during the local search phase. We implemented a variation of the specialized crossover operator *Greedy Partition Crossover* proposed by Galinier and Hao [51] to generate new solutions and diversify the search. Our purpose is to extend a feasible partial k *coloring* to a complete coloring. The general procedure is summarized as follows: given a pool of solutions, we randomly choose two parents from the pool and generate an offspring, which is improved by means of the Tabu Search algorithm and finally inserted in the pool, deleting the worst parent. After the initialization, the generation-improvement-insertion procedure is iterated until the problem is solved (i.e. a partial feasible solution is extended to a complete solution) or the number of iterations equals a given threshold.

Initialization

We initialize the pool by generating *poolsize* initial solutions (partial feasible k *colorings*). In this phase it is crucial to start with solutions which are far each other, thus exploring the whole search space and avoiding premature convergence of the search. For this purpose we generate the initial pool by using three different algorithms:

- The sequential algorithm SEQ applied with different random orderings of the vertices to generate the first third of the solutions in the pool. Each application of the algorithm is stopped as soon as k color classes are built (uncolored vertices being in class $k+1$).
- The maximum saturation degree algorithm DSATUR applied with different random orderings of the vertices to generate the second third of the solutions in the pool. Each application of the algorithm is stopped as soon as k color classes are built (uncolored vertices being in class $k+1$).
- The Tabu Search algorithm applied starting from a dummy solution (all vertices in class $k+1$) to generate the last third of the solutions in the pool (the random choice of the next vertex to color in the Tabu Search procedure leads the algorithm to obtain different initial partial colorings).

Every initial solution is improved with Tabu Search before being inserted in the pool. The use of an off-line procedure to compute diversity in the pool confirms that this choice is able to generate a well diversified pool.

Crossover Operator

Given two parent solutions randomly chosen from the pool, the crossover operator outputs an offspring sharing "interesting properties" with the parents. A solution is a partition of the vertices in $k+1$ sets where the first k are independent sets. It seems reasonable that interesting structures of the parents could be identified in these independent sets (in the following we will refer to independent sets or color classes). In [51] Galinier and Hao proposed a crossover operator which, given two parents (partition of the vertices in k sets, not necessarily independent), alternatively considers each parent to generate the next color class of the offspring in this way: the color class of maximum cardinality of the considered parent becomes the next color class of the offspring; all the vertices in this color class are deleted from the parents. When k steps are performed, some vertices may remain unassigned. These vertices are then assigned to a class randomly chosen. We modified this operator according to our purpose. Indeed in our case the offspring must be a (possibly partial) k *coloring* . Given two parents $S^1 = \{V_1^1, ..., V_k^1, V_{k+1}^1\}$ and $S^2 = \{V_1^2, ..., V_k^2, V_{k+1}^2\}$ the crossover operator outputs the offspring $S^3 = \{V_1^3, ..., V_k^3, V_{k+1}^3\}$ as follows:

begin
1. $CurrentColor := 1$;
2. **while** ($CurrentColor \leq k$ **and** $V_1^1 \cup ... \cup V_k^1 \cup V_1^2 \cup ... \cup V_k^2 \neq \emptyset$
3. $A := SelectParent()$;
6. $h := \arg\max_{i=1,...,k} |V_i^A|$;
7. $V_{CurrentColor}^3 := V_h^A$;
8. remove the vertices of V_h^A from S^1 and S^2;

9. $CurrentColor := Currentcolor + 1$
9. **end while**;
10. **for each** vertex $v \in \mathbf{V} \setminus (V_1^3 \cup ... \cup V_k^3)$ try to color v in a greedy way (i.e. try to insert v in one of the k color classes $V_1^3, ..., V_k^3$);
11. $V_{k+1}^3 := \mathbf{V} \setminus (V_1^3 \cup ... \cup V_k^3)$

end.

Function $SelectParent()$, which returns the parent chosen to generate the next color class, works as follows:

begin
1. **if** $((V_1^1, ..., V_k^1) \neq \emptyset$ **and** $(V_1^2, ..., V_k^2) \neq \emptyset)$ **then**
2. **if** $CurrentColor$ is odd **then** $A := 1$ **else** $A := 2$
3. **else if** $(V_1^1, ..., V_k^1) \neq \emptyset$ **then** $A := 1$ **else** $A := 2$

end.

This function takes into account that one parent can terminate the available colored vertices, in this case it considers only color classes from the parent who still has colored vertices. When both parents terminate the available colored vertices or the offspring has used k colors, we try to insert each uncolored vertex v in one of the offspring color classes in the following sequential greedy way:

begin
1. **for each** color class $h = 1, ..., k$
2. **if** $\nexists w \in V_h^3 : (v, w) \in E$ **then** $V_h^3 := V_h^3 \cup \{v\}$ and **exit**
3. **end for**

end.

Solution evaluation

The quality (score) of every solution S in the pool is evaluated through the function $f(S)$ defined by (2.1) and used during the Tabu Search algorithm. This allows us to compare the solutions and to tune the quality of the pool during the computation.

Pool Update

Every offspring is first of all improved by means of the Tabu Search algorithm and then inserted in the pool, substituting the worst parent. It can occur that the offspring is similar to one of the parents or to a solution yet present in the pool (i.e. it has the same score and the same number of uncolored vertices). In this case, with a probability p_{greedy} proportional to the percentage number of colored vertices in the population (see Table 2.3), we do not insert the offspring in the population but we insert a completely new greedy solution, avoiding premature convergence. This new solution is built by using a sequential greedy algorithm which gives priority to the vertices that, during the computation, were more often left uncolored (i.e. inserted in class $k + 1$). We call this algorithm *Priority Greedy*. More in detail we order the vertices according to decreasing values of the number of times they were left uncolored in the pool. We locally perturb this ordering: with a probability $p = 0.5$ we swap every vertex with the next one in the ordering and then apply the SEQ algorithm. This perturbation prevents

the generation of the same coloring at different calls of the SEQ algorithm. In this way we build different solutions where vertices that were more often left uncolored are in color classes of low order, while in class $k+1$ we have vertices more often colored during the computation (see how SEQ works).

To summarize, the Evolutionary Algorithm takes in input:

- graph $G(V, E)$;
- the target value k for the coloring;
- the maximum number L of Tabu Search iterations between the application of two consecutive crossover operators;
- the cardinality of the pool *poolsize*;
- the tabu tenure T ;
- the *timelimit*.

and it works as follows:

begin
1. generate the initial *pool* of solutions;
2. **if** $\exists S^h \in pool : V^h_{k+1} = \emptyset$ **then stop**;
3. **while** (not *timelimit*)
4. randomly select 2 solutions S^1 and S^2 from the *pool*;
5. generate $S^3 := Crossover(S^1, S^2)$;
6. **if** $V^3_{k+1} = \emptyset$ **then stop**;
7. [Improve the offspring] $S^3 := TabuSearch(S^3)$;
8. **if** $V^3_{k+1} = \emptyset$ **then stop**;
9. [Update the *pool*:] **if** S^3 is similar to a solution S^j in the *pool* **then**
with probability p_{greedy} $S^3 := PriorityGreedy()$;
10 **if** $V^3_{k+1} = \emptyset$ **then stop**;
11. insert S^3 in the *pool*, delete the worst parent
12. **end while**

end.

2.2.3 Evolutionary Algorithm as part of the Overall Algorithm

As anticipated in the previous section we apply the Evolutionary Algorithm in the first phase of the overall algorithm MMT. It must be noted that the Evolutionary Algorithm works in decision version, while we are approaching the problem from the optimization point of view. In other words the Evolutionary Algorithm requires as input the value k (the number of colors to be used) while we are trying to minimize this value. To solve this problem we use the information obtained from the initial greedy heuristics: if the current upper bound UB for the problem is $k+1$, we apply the Evolutionary Algorithm with k as input parameter. If the Evolutionary Algorithm solves the problem for the target value k within the given *timelimit*, we apply it again with $k-1$ as input parameter, and we iterate until the Evolutionary Algorithm is unable to solve the problem.

The Evolutionary Algorithm is very effective but largely dependent on the input parameters, i.e. L (number of tabu search iterations between two crossover steps) and *poolsize*. Computational esperiments show that difficult instances require a longer Tabu Search phase and the use of a wider population. In general, high density graphs with many vertices tend to be difficult to solve, but it seems to exist no explicit correlation between effective input parameters and some intrinsic property of the graph.

Thus, we implemented a procedure that dynamically modifies these input parameters for every execution of the Evolutionary Algorithm, based on the results obtained in the previous execution. Suppose we are solving an instance using k colors: if the algorithm finds a solution within *UpdateLimit* applications of the crossover operator, we consider the instance to be easy and we do not update the input parameters when trying to solve the same instance using $k-1$ colors; otherwise we increase the values of L and $poolsize$ of *DeltaTabuIterations* and *DeltaPoolSize*, respectively. *UpdateLimit* is a parameter dependent on the graph properties (see computational analysis in Section 2.4).

2.3 PHASE 2: Set Covering Formulation

If the incumbent solution found in Phase 1 (i.e. during the execution of the Evolutionary Algorithm) is not proved to be optimal, a further optimization phase is executed in order to improve the value of the solution. This phase is based on an Integer Linear Programming (ILP) formulation of VCP and uses a subset of the independent sets found by the Evolutionary Algorithm during Phase 1.

A natural ILP model for VCP is the one having a binary variable for each independent set and a constraint for each vertex. This model is often referred to as the *Set Covering* (or *Set Partitioning*) formulation. The relevance of this model lays in the fact that it can describe all those problems in which one is required to partition a given set of items into subsets having special features and minimizing the sum of the costs associated with the subsets. This can be done not only for VCP (see Merhotra and Trick [90]) but, for instance, for Bin Packing Problems [91], Vehicle Routing Problems [75], Crew Scheduling Problems [87, 16, 111, 23] as well.

We present a post-optimization phase based on the Set Covering formulation for VCP.

Let $\mathcal{S}$ be the family of all the Independent Sets of G. Each independent set (column) $s \in \mathcal{S}$ has associated a binary variable x_s having value 1 iff all the vertices of s receive the same color. VCP can be formulated as follows:

$$\min \sum_{s \in S} x_s \tag{2.2}$$

$$\sum_{s: i \in s} x_s \geq 1 \qquad i \in V \tag{2.3}$$

$$x_s \in \{0,1\} \qquad s \in \mathcal{S} \tag{2.4}$$

The objective function (2.2) asks to minimize the total number of independent sets (and hence of colors) used. Constraints (2.3) state that every vertex i in the graph must belong to at least one independent set (i.e., must receive at least one color). Indeed, if a vertex i is assigned to more than one independent set in a feasible solution, it can be removed from all the independent sets but one (in other words if a vertex is assigned more than one color,

a feasible solution of the same value can be obtained using any one of these colors for the vertex). Finally, constraint (2.4) impose variables x_s to be binary.

The advantage of the Set Covering formulation, w.r.t. alternative descriptive formulations, is that it avoids symmetries in the solution and its continuous relaxation leads to tighter lower bounds. The main drawbacks are that the number of variables can grow exponentially with the cardinality of vertex set V (even if one is allowed to consider only the *maximal* independent sets in the definition of S) and that SCP is an NP-hard problem, whose exact solution could require very large computing times. Our approach to the problem is heuristic in the sense that during Phase 1 we store only a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ of all the independent sets of graph G, and that in Phase 2 we solve model (2.2)–(2.4) corresponding to subfamily $\mathcal{S}'$ through a heuristic algorithm from the literature.

In particular, subfamily $\mathcal{S}'$ is defined during the execution of the Evolutionary Algorithm of Section 2.2.2 in the following way. For each (partial or complete) feasible k *coloring* found in Phase 1, we can generate k independent sets (columns) for Phase 2. Every independent set (not necessary maximal) is first of all completed to a maximal independent set by using a greedy procedure that simply tries to insert in the set vertices that are currently not in it, following the input order of the vertices. This ordering is perturbed at every call of the procedure, thus if the procedure is called on the same set different times, it is generally able to complete the set by introducing different vertices and hence obtaining different columns.

Generally the global number of independent sets (columns) generated in Phase 1 is very large and could ask for excessive memory requirements. Hence we decided to insert in $\mathcal{S}'$ only the independent sets generated by the initial greedy algorithms and those corresponding to the feasible solutions found by the Evolutionary Algorithm, and to the solutions contained in the pool at the beginning and at the end of each Evolutionary Algorithm iteration. We also insert all the independent sets corresponding to solutions built to preserve diversity during the computation. An hashing technique is used to remove identical columns (see [91] for further details).

Computational experiments showed that this choice, that privileges independent sets corresponding to solutions which tend to have high diversity each other, did not affect the effectiveness of Phase 2 while reducing considerably the computation time and avoiding memory problems.

As to the solution of the corresponding Set Covering instance, we use the Lagrangian heuristic algorithm CFT proposed by Caprara, Fischetti and Toth [23]. This iterative algorithm can handle very large Set Covering instances, producing good (possibly optimal) solutions within a reasonable amount of computing time. Moreover, algorithm CFT computes an "internal" lower bound (not valid for VCP) on the value of the optimal solution of the corresponding Set Covering instance and its execution can be stopped as soon as this lower bound equals the value of the best incumbent solution for VCP. Of course, optimality for SCP does not imply optimality for the original problem, because we do not enumerate all the independent sets of G.

A similar approach has been used to derive effective heuristic algorithms for bin packing problems in [91], the main difference with respect to that approach being the aim of Phase 1. While in [91] the first phase (Column Generation Phase) was mainly aimed at generating "good" columns for the second phase and the effectiveness of the approach was essentially due to the second phase (Column Optimization Phase), in the current algorithm the first phase is crucial for its effectiveness. This aspect is stressed by the computational results (see Section 2.4), showing that the Evolutionary Algorithm of Phase 1 is very effective on the instances

of our test bed and that Phase 2 can be considered as a post-optimization tool which turned out to improve the upper bound on a subset of instances for which Phase 1 fails in proving optimality of the solution.

2.3.1 The General Structure of the Algorithm

begin
Initialization
1. compute lower bound LB ;
2. compute upper bound UB by means of greedy heuristics;
3. **if** $UB = LB$ **stop**;
4. insert the columns corresponding to the greedy solutions in $\mathcal{S}'$;
Phase 1: Evolutionary Algorithm
5. $k := UB - 1$:
6. **while** $k \geq LB$
7. call $EvolutionaryAlgorithm(k, poolsize, L, TimeLimit)$:
8. generate the initial *pool* of solutions;
9. insert the columns corresponding to the *pool* in $\mathcal{S}'$;
10. **if** $\exists\ S^h \in pool : V^h_{k+1} = \emptyset$ **then** $k := k - 1$ **goto** 24;
11. **while** (not *timelimit*)
12. randomly select 2 parent solutions S^1 and S^2 from the *pool*;
13. generate $S^3 := Crossover(S^1, S^2)$;
14. improve $S^3 := TabuSearch(S^3)$;
15. **if** $V^3_{k+1} = \emptyset$ **then goto** 22;
16. [update the *pool*:] **if** S^3 is similar to a solution S^j in the *pool* **then**
17. with probability p_{greedy} $S^3 := PriorityGreedy()$;
18. insert columns corresponding to S^3 in $\mathcal{S}'$;
19. **if** $V^3_{k+1} = \emptyset$ **then goto** 22
20. insert S^3 in the *pool*, delete the worst parent solution
21. **end while**;
22. insert the columns corresponding to the final *pool* in $\mathcal{S}'$;
23. **if** no feasible solution of value k has been found **then break**;
24. update UB;
25. if $LB = UB$ **stop**;
26. insert the columns corresponding to the feasible solution in $\mathcal{S}'$;
27. dynamically modify $L, poolsize$;
28. $k := k - 1$
29. **endwhile**;
Phase 2: Column Optimization
30. apply heuristic algorithm CFT (with a given time limit) to the Set Covering instance corresponding to subfamily $\mathcal{S}'$;
31. update UB
end.

2.4 Computational Analysis

The Evolutionary Algorithm described in Section 2.2.2 was coded in ANSI C and compiled with full optimization option; all other procedures, including algorithm CFT [23], were coded in ANSI FORTRAN77 and compiled with full optimization option. The programs were run on a PIV 2.4MHz with 512MB RAM under Windows XP and tested on the *DIMACS benchmark graph instances* [1],[73]. These instances correspond to different graph types used for evaluating the performance of VCP algorithms. In particular this set of instances contains random graphs (DSJC$n.x$), geometric random graphs (DSJR$n.x$ and R$n.x$[c]), "quasi-random" graphs (flatn_x_0), artificial graphs (len_x and latin_square_10), graphs from real life applications (school1 and school1_nsh). All the computing times reported in this section are expressed in seconds of a PIV 2.4GHz. To allow a meaningful - although approximate - comparison on results obtained with different machines a benchmark program (dfmax), together with a benchmark instance (r500.5), are available. Computing times obtained on different machines can be scaled w.r.t. the performance obtained on this program (our machine spent 7 seconds user time). To perform our computational experiments we selected the subset of DIMACS instances considered by the papers describing the most effective heuristic algorithms for VCP.

2.4.1 Performance of the Tabu Search Algorithm

In this section we report the experimental results obtained with the Tabu Search algorithm described in Section 2.2.1. Since our algorithm uses random numbers, for each instance we performed 4 runs with 4 different seeds for the random number generator. We try to solve every instance starting from a value of k equal to the chromatic number χ or trying to improve on the best known solution value in the literature when χ is unknown. We report experimental results starting from the value of k for which we have at least one successful run and ending with the value of k for which we have 4 successful runs. The Tabu Search algorithm always uses a fixed tabu tenure of 45 and is initialized with a partial feasible solution built by the SEQ algorithm (the ordering of the vertices given in input to SEQ , and hence the initial solution, depends on the seed). In Table 2.1 we report, for every considered instance, the best known solution value ever found in the literature (in bold when in is the proven optimal value), the number of successful runs within a limit of 100 millions iterations (no information is given if all the 4 runs were successful), the target value k, the average computing time (we report 0 when the time is lower than 1 second) and the average number of iterations for the successful runs. The main aspect turning out from these experiments is that the Tabu Search algorithm is quite fast in finding good solutions but, when this does not happen within a short number of iterations, rarely the algorithm is able to solve the problem even if a bigger number of iterations is allowed. This behavior is particulary true for geometric random instances (DSJRx.y and Rx.y[c]); in some cases the same instance is solved in few iterations with one seed and not solved within the iteration limit with a different seed. As discussed in Section 2.2.1, this is mainly due to the aggressive strategy adopted, which makes the final solution much dependent on the starting one.

We compare our algorithm with the Local Search algorithm HCD, inspired by Tabu Search techniques, proposed by Caramia and Dell'Olmo [25]. Both algorithms are fast and simple, thus they can be used as subroutines in real-time systems or as part of more complex procedures. HCD works in optimization version and stops when a given number of iterations is reached. Since we have access to the C source code of HCD we performed the computational

experiments on our machine. There are 5 slightly different versions of the code, in the last two columns of Table 2.1 we report the best value of k obtained performing one run with every one of these versions, with an iteration limit of 10 millions. We also report the computing time of the last improvement corresponding to the best solution found. Although HCD has a good performance w.r.t. other Tabu Search approaches proposed for the problem [25], our Tabu Search algorithm is able to find better solutions for all the instances but DSJR500.5, confirming the effectiveness of our approach.

2.4.2 Performance of the Evolutionary Algorithm

In this section we report the experimental results obtained with the Evolutionary Algorithm described in Section 2.2.2. We fix a common set of parameters working for most of the instances, but we have to adjust parameters *poolsize* and L (number of Tabu Search iterations between two crossover steps) to obtain good results on hard instances, while we use a fixed tabu tenure T of 45 for all instances. Since our algorithm uses random numbers, for each instance we performed 4 runs with 4 different seeds for the random number generator. The Evolutionary Algorithm works in decision version, and it requires as input the target value k of the coloring. We try to solve every instance starting from a value of k equal to the chromatic number χ or trying to improve on the best known solution value in the literature when χ is unknown. We report experimental results starting from the value of k for which we have at least one successful run and ending with the value of k for which we have 4 successful runs. In Table 2.2 we report, for every considered instance, the best known solution value ever found in the literature (in bold when it is the proven optimal value), the input parameters (*poolsize* and L), the number of successful runs with a time limit of 6000 seconds (with the exception of instance DSJC1000.5, where we tried a big population with a time limit of 40000 seconds, and instance flat1000_76_0, for which we experimented a long computation with a time limit of 40000 seconds), the target value k and the average computing time for the successful runs.

It is clear from the experimentation that the evolutionary diversification is effective, and able to bring the Tabu Search algorithm to the exploration of regions of the solution space containing high quality solutions. More in detail, we solved for the first time, to proven optimality, instances le450_25c and le450_25d and improved the best solution known in the literature for instance flat1000_76_0. In synthesis the Evolutionary Algorithm, on the complete set of the 42 considered instances, 3 times improves on the best known solution, 36 times finds the best known solution and for only 3 instances finds a worse solution.

2.4.3 Performance of the Overall Algorithm

As was observed by Morgenstern [92], reporting timings for algorithms that require the value of k and other parameters as input does not reflect the real effort required to find the corresponding values, and this hidden cost can increase substantially the actual computing time. When we built the overall algorithm MMT (Evolutionary Algorithm and Column Optimization) our objective was actually to solve the problem in optimization version with a unique setting of the parameters (with the exception of the time limits of the Evolutionary Algorithm and of CFT), able to solve any instance. The idea is that a robust algorithm should have a good performance not only on well studied instances from the literature (for which the best bounds, the structure, etc. are known) but also on "unknown" ones.

As described in Section (2.2.3) we implemented a procedure that, starting from a common

Table 2.1: Performance of the Tabu Search Algorithm.

Instance name	n	m	best (χ)	succ./runs	k	avg time	avg iter	HCD	time
DSJC125.1	125	736	5		5	0	21055	6	0
DSJC125.5	125	3891	17		17	1	1433498	19	0
DSJC125.9	125	6961	44		44	0	5270	44	0
DSJC250.1	250	3218	8		8	0	157652	9	2
DSJC250.5	250	15668	28		29	1	343024	31	1
DSJC250.9	250	27897	72		72	100	31826164	75	3
DSJC500.1	500	12458	12	1/4	12	22	21218513	14	3
					13	0	42751		
DSJC500.5	500	62624	48		51	14	4157673	53	141
DSJC500.9	500	112437	127		130	88	14672257	132	73
DSJC1000.1	1000	49629	20		21	4	2588732	22	140
DSJC1000.5	1000	249826	83	2/4	91	252	37474963	95	531
					92	165	24775243		
DSJC1000.9	1000	449449	224	2/4	235	404	33939239	241	857
				3/4	236	294	24738984		
					237	350	29558204		
DSJR500.1	500	3555	**12**		**12**	0	208	**12**	2
DSJR500.1C	500	121275	85		85	196	30826284	85	7
DSJR500.5	500	58862	**122**	3/4	130	21	6798990	129	269
				2/4	131	0	85		
				2/4	132	0	198		
				3/4	133	0	383		
					134	0	60		
le450_15a	450	8168	**15**		**15**	0	46035	16	42
le450_15b	450	8169	**15**		**15**	0	34561	16	19
le450_15c	450	16680	**15**		**15**	0	86865	17	213
le450_15d	450	16750	**15**		**15**	0	259805	18	68
le450_25c	450	17343	26		26	0	65662	27	59
le450_25d	450	17425	26		26	0	35547	27	74
le450_5a	450	5714	**5**		**5**	0	20597	8	6
le450_5b	450	5734	**5**		**5**	0	55677	8	0
le450_5d	450	9757	**5**		**5**	0	3196	7	0
r125.1	125	209	**5**		**5**	0	2	**5**	0
r125.1c	125	7501	**46**	3/4	**46**	0	908	**46**	0
				3/4	47	0	64		
					48	0	39		
r125.5	125	3838	**36**	2/4	**36**	0	649	**36**	17
				3/4	37	0	162		
				3/4	38	0	70		
					39	0	18		
r250.1	250	867	**8**		**8**	0	2	**8**	0
r250.1c	250	30227	**64**	3/4	**64**	0	15321	**64**	0
				3/4	65	0	6671		
				3/4	66	0	7872		
				3/4	67	0	1838		
				3/4	68	0	975		
				3/4	69	0	246		
					70	0	464		

Instance name	n	m	best (χ)	succ./runs	k	avg time	avg iter	HCD	time
r250.5	250	14849	**65**	2/4	68	0	19888	68	5
				2/4	69	0	639		
					70	0	6424		
r1000.1	1000	14378	**20**		**20**	0	13297	**20**	13
r1000.1c	1000	485090	98		98	96	8005753	99	173
r1000.5	1000	238267	237	1/4	250	0	268	256	727
				1/4	251	0	1078		
					252	0	438		
school1	385	19095	**14**		**14**	0	75247	**14**	0
school1_nsh	352	14612	**14**		**14**	0	1601	**14**	1
latin_square_10	900	307350	99	1/4	107	136	15763447	107	687
				2/4	108	310	35541496		
					109	108	12320495		
flat300_20_0	300	21375	**20**		**20**	0	6655.25	**20**	2
flat300_26_0	300	21633	**26**		**26**	1	364307	35	1
flat300_28_0	300	21695	**28**		31	6	2552809	33	38
flat1000_50_0	1000	245000	**50**		**50**	7	709627	92	
flat1000_60_0	1000	245830	**60**		**60**	26	2775283	93	
flat1000_76_0	1000	246708	83	1/4	90	198	59319600	94	293
					91	50	7429507		

set of parameters, dynamically modifies the parameters *poolsize* and L for every call of the Evolutionary Algorithm, based on the results obtained in the previous calls. In Table 2.3, where *dens* denotes the density of graph G, we report the values found during the experimental set-up of this procedure.

For each considered instance, 4 runs, with different seeds for the random number generator, were performed. The corresponding computational results are reported in Table 2.4. Since the overall algorithm MMT works in optimization version, it always gives on output a feasible solution and it does not require a target value k as input. For every considered instance, we report the best known solution value ever found in the literature (in bold when it corresponds to the optimal value), the lower bound LB computed during the initialization step, the average solution value after phase 1, the average solution value after phase 2 (this value is reported only if it is better than the corresponding value obtained at the end of phase 1), the best solution value found during the overall computation, the average total computing time, the time limit of the Evolutionary Algorithm.

Comparing these results with those of the previous section the reader should remember that in this case the only input parameters are the time limits and that the problem is approached from the optimization point of view, so, in some way, the problem is much more "difficult", because the algorithm does not take advantage of all the information available for the well known instances from the literature. When optimality is not proven in phase 1 (i.e. when the solution value after phase 1 is greater than LB) and phase 2 does not improve the incumbent solution, the Evolutionary Algorithm has performed one useless iteration up to the time limit, spending an important amount of computing time, after the last successful iteration; e.g. for instance DSJC125.1 the final solution is found on average after 1 second, but optimality is not proven and 20 seconds are spent trying to improve on this solution. On the contrary, when phase 2 improves the solution or optimality is proven in phase 1 (which happens for all the le450_x instances and for school1), the time of the last improvement

Table 2.2: Performance of the Evolutionary Algorithm.

Instance name	n	m	best (χ)	*poolsize*	L	succ./runs	k	avg time
DSJC125.1	125	736	5	10	10000		5	0
DSJC125.5	125	3891	17	10	10000		17	1
DSJC125.9	125	6961	44	10	10000		44	0
DSJC250.1	250	3218	8	10	10000		8	1
DSJC250.5	250	15668	28	10	10000		28	28
DSJC250.9	250	27897	72	10	10000		72	193
DSJC500.1	500	12458	12	10	20000		12	78
DSJC500.5	500	62624	48	10	20000	3/4	48	84
				10	20000		49	27
DSJC500.9	500	112437	127	30	40000		127	967
DSJC1000.1	1000	49629	20	10	20000		20	303
DSJC1000.5	1000	249826	83	100	300000	2/4	83	22573
				40	50000	3/4	84	1632
				40	50000		85	845
DSJC1000.9	1000	449449	224	50	50000	2/4	226	3340
				50	50000	1/4	227	1734
				50	50000		228	3235
DSJR500.1	500	3555	**12**	10	10000		**12**	0
DSJR500.1C	500	121275	85	20	20000		85	142
DSJR500.5	500	58862	**122**	20	10000		**122**	30
le450_15a	450	8168	**15**	10	10000		**15**	0
le450_15b	450	8169	**15**	10	10000		**15**	0
le450_15c	450	16680	**15**	10	10000		**15**	0
le450_15d	450	16750	**15**	10	10000		**15**	1
le450_25c	450	17343	26	10	10000		**25**	1321
le450_25d	450	17425	26	10	10000		**25**	424
le450_5a	450	5714	**5**	10	10000		**5**	0
le450_5b	450	5734	**5**	10	10000		**5**	0
le450_5d	450	9757	**5**	10	10000		**5**	0
r125.1	125	209	**5**	10	10000		**5**	0
r125.1c	125	7501	**46**	10	10000		**46**	0
r125.5	125	3838	**36**	10	10000		**36**	0
r250.1	250	867	**8**	10	10000		**8**	0
r250.1c	250	30227	**64**	10	10000		**64**	0
r250.5	250	14849	**65**	10	10000		**65**	8
r1000.1	1000	14378	**20**	10	10000		**20**	0
r1000.1c	1000	485090	98	20	20000		98	101
r1000.5	1000	238267	237	30	10000	3/4	237	168
				30	10000		238	4
school1	385	19095	**14**	10	10000		**14**	0
school1_nsh	352	14612	**14**	10	10000		**14**	0
latin_square_10	900	307350	99	50	50000	3/4	103	3263
				50	50000		104	1820
flat300_20_0	300	21375	**20**	10	10000		**20**	0
flat300_26_0	300	21633	**26**	10	10000		**26**	4
flat300_28_0	300	21695	**28**	10	10000		31	54
flat1000_50_0	1000	245000	**50**	10	100000		**50**	33
flat1000_60_0	1000	245830	**60**	10	200000		**60**	73
flat1000_76_0	1000	246708	83	10	250000	1/4	82	34056
				10	200000	3/4	83	2226
				10	200000		84	1387

Table 2.3: Parameters of Algorithm MMT.

tabu tenure T (fixed)	45
initial $poolsize$	10
initial L	10000
$UpdateLimit$	$40000/n$
$DeltaTabuIterations$	initial L x $dens$ x 0.55
$DeltaPoolSize$	initial $poolsize$ x $dens$
time limit of CFT (s)	150
p_{greedy}	$poolsize$/n. of uncolored vertices in the pool

corresponds to the total computing time.

Table 2.4 shows that, for large size instances like DSJC1000.9, latin_square_10 and flat1000_76_0, giving a larger time limit to phase 1 leads to slightly better solutions. Of course, working in the optimization version of the problem leads to longer computing times, although the solution quality is slightly better than that obtained by the Evolutionary Algorithm. More in detail, the dynamic set up of the parameters is not always able to bring the Evolutionary Algorithm, executed during phase 1, to high quality solutions as done by the ad hoc tuning reported in Table 2.2. This is true in particular for instances flat1000_50_0 and flat1000_60_0 where the performance is quite poor. But for these instances phase 2 is able to improve on the incumbent solution up to the solution reported in Table 2.2, and in three instances, namely DSJC1000.9, r1000.5 and latin_square_10, the solution is better than that found by the Evolutionary Algorithm stand alone. In particular, the complete MMT algorithm solved for the first time, to proven optimality, instance r1000.5. In synthesis, algorithm MMT, on the complete set of the 42 considered instances, 3 times improves on the best known solution in the literature, 35 times finds the best known solution in the literature and for only 4 instances (DSJC1000.5, DSJC1000.9, latin_square_10, flat300_28_0) finds a worse solution.

2.4.4 Comparison with the most effective heuristic algorithms

In Tables 2.5 and 2.6 we compare the approaches described in Sections 2.2.2 and 2.3, respectively, with the heuristic algorithms that, to the best of our knowledge, represent the state of the art for VCP. For every considered instance we report the value of the best known solution found in the literature (in bold when it is the proven optimal value). All these solutions were found by the algorithms considered in our comparison. For the VCP in decision version, we report in Table 2.5 the computational results of:

- The *Impasse* algorithm by Morgenstern [92], which is actually composed by three different algorithms based on the idea of *Impasse Class Neighborhood*. These algorithms work in decision version and require as input the target value k for the coloring and a couple of other parameters that are tuned for every instance. For each instance considered in [92] we report the smallest value of k for which no failure occurred over 5 runs and the average running time to solve the instance, scaled w.r.t the benchmark problem.

- The HCA (Hybrid Coloring Algorithm) by Galinier and Hao [51]. HCA requires as input the target value k for the coloring and a couple of others parameters that are tuned for every instance. For each instance considered in [51] we report the smallest value of k for which there was at least one successful run over 10 (or 5) runs (succ.), and an

Table 2.4: Performance of the Algorithm MMT.

Instance name	n	m	best (χ)	LB	avg value of k phase 1	avg value of k phase 2	best k	avg total time	time limit Evol.	time limit Alg.
DSJC125.1	125	736	5	3	5.00		5	21		20
DSJC125.5	125	3891	17	8	17.00		17	122		20
DSJC125.9	125	6961	44	28	44.00		44	121		20
DSJC250.1	250	3218	8	3	8.00		8	21		20
DSJC250.5	250	15668	28	9	28.00		28	117		80
DSJC250.9	250	27897	72	34	72.50	72.00	72	89		80
DSJC500.1	500	12458	12	4	12.25		12	210		200
DSJC500.5	500	62624	48	9	48.25		48	388		200
DSJC500.9	500	112437	127	42	128.75	127.75	127	433		200
DSJC1000.1	1000	49629	20	4	20.25		20	260		200
DSJC1000.5	1000	249826	83	11	84.25		84	8407		3000
DSJC1000.9	1000	449449	224	47	233.75	226.00	225	3234		800
					229.75	225.50	225	9476		3000
DSJR500.1	500	3555	**12**	11	**12.00**		**12**	25		20
DSJR500.1C	500	121275	85	67	86.00	85.00	85	88		60
DSJR500.5	500	58862	**122**	111	**122.00**		**122**	163		100
le450_15a	450	8168	**15**	15	**15.00**		**15**	0		20
le450_15b	450	8169	**15**	15	**15.00**		**15**	0		20
le450_15c	450	16680	**15**	15	**15.00**		**15**	3		20
le450_15d	450	16750	**15**	15	**15.00**		**15**	4		20
le450_25c	450	17343	26	25	**25.00**		**25**	1321		4000
le450_25d	450	17425	26	25	**25.00**		**25**	436		4000
le450_5a	450	5714	**5**	5	**5.00**		**5**	0		20
le450_5b	450	5734	**5**	5	**5.00**		**5**	0		20
le450_5d	450	9757	**5**	5	**5.00**		**5**	0		20
r125.1	125	209	**5**	5	**5.00**		**5**	0		20
r125.1c	125	7501	**46**	44	**46.00**		**46**	21		20
r125.5	125	3838	**36**	33	**36.00**		**36**	21		20
r250.1	250	867	**8**	8	**8.00**		**8**	26		20
r250.1c	250	30227	**64**	55	**64.00**		**64**	21		20
r250.5	250	14849	**65**	61	**65.00**		**65**	64		50
r1000.1	1000	14378	**20**	17	**20.00**		**20**	37		20
r1000.1c	1000	485090	98	68	98.25	98.00	98	518		200
r1000.5	1000	238267	237	225	238.75	**234.00**	**234**	753		400
school1	385	19095	**14**	14	**14.00**		**14**	0		20
school1_nsh	352	14612	**14**	13	**14.00**		**14**	21		20
latin_square_10	900	307350	99	90	103.50	102.00	101	5156		2000
					103.50	101.75	101	6346		3000
flat300_20_0	300	21375	**20**	9	**20.00**		**20**	21		20
flat300_26_0	300	21633	**26**	9	**26.00**		**26**	36		20
flat300_28_0	300	21695	**28**	9	31.00		31	212		150
flat1000_50_0	1000	245000	**50**	11	78.50	**50.00**	**50**	1417		150
flat1000_60_0	1000	245830	**60**	11	74.50	**60.00**	**60**	3645		300
flat1000_76_0	1000	246708	83	10	84.50		84	3709		1000
					83.50		83	7325		3000

approximate average running time for the successful runs (we divided the reported time, obtained on an UltraSPARC-IIi 333MHz with 128MB RAM, for which the authors do not report the performance on the benchmark problem, by 6, following the performance ratio reported by Dongarra [42] for machines similar to those used in [51] and in our experiments). We report also the best solution values found during the complete set of computational experiments performed on the algorithm (for which the computing times and the number of successful runs are not given).

- The PC (Partialcol) and RPC (React-Partialcol) algorithms by Blöchliger and Zufferey [15] are Tabu search algorithms, based on the idea of *Impasse Class Neighborhood* [92], which implement a dynamic and reactive tabu tenure, respectively, and require as input parameter only the target value k for the coloring. For each instance considered in [15] we report the smallest value of k for which there was at least one successful run over 10 runs (succ.) of each of the two algorithms, and an approximate average running time for the successful runs of the best algoritmh (considering as best the algorithm which finds the best solution value, breaking ties by considering first the number of successful runs and then the computing time). The computational experiments were carried out on different Linux systems mostly running on a PIV 2GHz with 512MB RAM, whose performance is similar (and directly comparable) to that of the machine used in our experiments.

- The Evolutionary Algorithm, whose performance is summarized reporting the smallest value of k for which there was at least one successful run over 4, the corresponding number of successful runs (succ.), and the average running time for the successful runs.

The Evolutionary Algorithm clearly outperforms *Impasse*, this should not surprise since it uses the same neighborhood structure in a more complex procedure. The comparison with HCA is harder due to the scarcity of instances reported in [51]; the Evolutionary algorithm finds 2 better solutions (le450_25c and flat1000_76_0) vs 1 better solution found by HCA (DSJC1000.9 for which the computing time is not reported) and shows a more robust behavior. The computing times, when reported, are comparable for equivalent values of the coloring; in particular the Evolutionary Algorithm spends less than 1 second to color instance le450_25c with 26 colors and, as reported in Table 2.2, it spends 2226 seconds to color instance flat1000_76_0 with 83 colors. The Evolutionary algorithm spends on average 22573 seconds to color instance DSJC1000.5 with 2 successful runs over 4. This time is 10 times longer than the one spent by HCA, but for this instance the number of attempts and successful runs are not reported in [51]. The Evolutionary Algorithm almost always finds solution values that are better or equal to those found by PC and RPC, in the latter case with computing times that are generally shorter. The only exception is instance flat300_28_0 which is solved to optimality ($k = 28$) for the first time by [15], improving the previous best known solution of value 31.

For the problem in optimization version we report in Table 2.6 the computational results of:

- The MIPS-CLR (MInimal-state Processing Search algorithm for the graph CoLoRing problem) by Funabiki and Higashino [50]. This algorithm works in optimization version but requires as input the target value k_{init} for the coloring, together with some other parameters whose values are tuned for hard instances. If the algorithm is not able to

solve the problem with k_{init} colors, it dynamically modifies this target value. In the first 4 columns we report the results obtained giving the target value k_{init} as external input, the best value (best$k$) found over the 5 runs, the average solution value (avg$k$) and the average running time approximately scaled w.r.t the benchmark problem (we run the benchmark problem on a machine similar to the one used in [50], which spent 17 seconds user time to solve the benchmark problem). In the following 4 columns we report the results obtained by setting the target value k_{init} equal to the cardinality of a maximal clique in the graph, computed during the initialization of the algorithm: the best value (best$k$) found over the 5 runs, the average value of k (avg$k$) and the average running time approximately scaled w.r.t the benchmark problem. The solution quality is slightly worse than that of the previous initialization of the target value (k_{init}), whereas the computing time is larger when k_{init} is far from the final solution value.

- The MMT algorithm, whose performance is summarized reporting the best value (best$k$) over 4 runs, the average value of k (avg$k$) and the average computing time up to the time limit or to proven optimality, to allow a comparison with the timings of MIPS-CLR. Since when we use the MMT algorithm we do not take advantage of any information from the literature (such as the expected value of the coloring), we think that a fair timing comparison with MIPS-CLR should be done with the values obtained by setting the target value k_{init} equal to the cardinality of a maximal clique, i.e. with the values reported in the last 4 columns of MIPS-CLR.

Algorithm MMT always finds solutions that are better or equal to those found by MIPS-CLR, considering both the best value (best$k$) or the average solution value (avg$k$), with the exception of instance latin_square_10 where MIPS-CLR finds the best known solution. In the last line of Table 2.6 we report the sum of the times on the common subset of instances; it can be concluded that these times are practically the same, by considering the approximation introduced by scaling the time w.r.t the performance obtained on the benchmark problem.

To compare with a single index the performance of the different algorithms considered in this paper, we compute the average ratio between the solution value and the best known solution value from the literature $k/best$. This ratio always refers to the best results reported, for the corresponding instance, in the associated paper (i.e. k for Impasse [92], HCA [51] and PC-RPC [15], best$k$ for MIPS-CLR [50]) and the best solution value found by the Tabu Search algorithm, the Evolutionary algorithm and algorithm MMT. Since Morgenstern [92], Galinier and Hao [51] and Blöchliger and Zufferey [15] did not consider the entire set of instances, in Table 2.7 we compare our results with those of the other algorithms on the common subset of instances.

Table 2.7 shows the improvements obtained on the simple Tabu Search algorithm by the evolutionary diversification and the post optimization procedures, and confirms that the proposed approaches outperform the others algorithms w.r.t. the solution quality, with comparable computing times.

2.5 Conclusions

In this paper we presented the two phase metaheuristic algorithm MMT for the Vertex Coloring Problem. The first phase of MMT is based on an Evolutionary Algorithm which combines an effective Tabu Search with a diversification procedure based on a specialized crossover op-

Table 2.5: Performance of the most effective heuristics in decision version.

		Impasse [92]		HCA [51]			PC-RPC [15]			Evolutionary Alg.		
Instance name	best	k	time	succ.	k	time	succ	k	time	succ	k	time
DSJC125.1	5										5	0
DSJC125.5	17	17	1								17	1
DSJC125.9	44										44	0
DSJC250.1	8										8	1
DSJC250.5	28	28	22	9/10	28	13					28	28
DSJC250.9	72										72	193
DSJC500.1	12						10/10	12	120		12	78
DSJC500.5	48	49	660	5/10	48	268	1/10	49	720	3/4	48	84
DSJC500.9	127						2/10	127	1560		127	967
DSJC1000.1	20			?/?	20	?	1/10	20	2640		20	303
DSJC1000.5	83	89	1148	?/?	83	2258	2/10	88	14400	2/4	83	22573
DSJC1000.9	224			?/?	224	?	4/10	226	18000	2/4	226	3340
DSJR500.1	**12**	**12**	0								**12**	0
DSJR500.1C	85	85	5								85	142
DSJR500.5	**122**	123	14								**122**	30
le450_15a	**15**	**15**	0								**15**	0
le450_15b	**15**	**15**	0								**15**	0
le450_15c	**15**	**15**	5	6/10	**15**	8	10/10	**15**	2		**15**	0
le450_15d	**15**	**15**	3				10/10	**15**	8		**15**	1
le450_25c	26			10/10	26	55	10/10	27	1		**25**	1321
le450_25d	26						10/10	27	1		**25**	424
le450_5a	**5**										**5**	0
le450_5b	**5**										**5**	0
le450_5d	**5**										**5**	0
r125.1	**5**	**5**	0								**5**	0
r125.1c	**46**	**46**	0								**46**	0
r125.5	**36**	**36**	0								**36**	0
r250.1	**8**	**8**	0								**8**	0
r250.1c	**64**	**64**	0								**64**	0
r250.5	**65**	**65**	7								**65**	8
r1000.1	**20**	**20**	1								**20**	0
r1000.1c	98	98	46								98	101
r1000.5	237	241	77							3/4	237	168
school1	**14**										**14**	0
school1_nsh	**14**										**14**	0
latin_square_10	99									3/4	103	3263
flat300_20_0	**20**	**20**	0				10/10	**20**	0		**20**	0
flat300_26_0	**26**	**26**	1				10/10	**26**	0		**26**	4
flat300_28_0	**28**	31	156	6/10	31	20	3/10	**28**	420		31	54
flat1000_50_0	**50**	**50**	0				10/10	**50**	18		**50**	33
flat1000_60_0	**60**	**60**	0				10/10	**60**	90		**60**	73
flat1000_76_0	83	89	897	4/5	83	1471	5/10	87	18000	1/4	82	34056

erator; the second phase is a post optimization phase based on the Set Covering formulation of the problem.

Extensive computational experiments performed on 42 hard instances from the well known DIMACS benchmark graph instances show that the Evolutionary Algorithm is very effective and has a robust behavior on all the considered instances, still requiring the experimental set-up of a couple of input parameters. This Evolutionary Algorithm is subsequently combined, in the overall MMT Algorithm, with a procedure which performs the dynamic set-up of the

Table 2.6: Performance of the most effective heuristics in optimization version.

		MIPS_CLR [50]				MMT						
Instance name	best	k_{init}	best$k$	avg$k$	time	k_{init}	best$k$	avg$k$	time	best$k$	avg$k$	time
DSJC125.1	5	5	5	5.0	0	4.0	5	5.0	1	5	5.00	21
DSJC125.5	17	17	17	17.0	1	9.6	17	17.2	15	17	17.00	122
DSJC125.9	44	44	44	44.0	0	32.8	44	44.0	22	44	44.00	121
DSJC250.1	8	7	8	8.0	5	4.0	8	8.0	30	8	8.00	21
DSJC250.5	28	28	28	28.4	14	11.0	28	28.6	80	28	28.00	117
DSJC250.9	72	72	72	72.4	31	41.6	72	72.4	148	72	72.00	89
DSJC500.1	12	10	12	12.4	84	5.0	12	12.8	137	12	12.25	210
DSJC500.5	48	47	49	49.4	349	12.2	49	50.0	454	48	48.25	388
DSJC500.9	127	125	127	127.8	480	53.2	128	128.8	999	127	127.75	433
DSJC1000.1	20	20	21	21.0	90	5.2	21	21.0	776	20	20.25	260
DSJC1000.5	83	82	88	89.0	4658	14.2	89	89.6	2634	84	84.25	8407
DSJC1000.9	224	228	228	229.6	1565	62.0	228	229.8	7087	225	226.00	3234
DSJR500.1	**12**	12	**12**	**12.0**	0	12.0	**12**	**12.0**	0	**12**	**12.00**	25
DSJR500.1C	85	85	85	85.0	6					85	85.00	88
DSJR500.5	**122**	122	**122**	123.4	276	122.0	**122**	123.4	276	**122**	**122.00**	163
le450_15a	**15**	15	**15**	**15.0**	1	15.0	**15**	**15.0**	1	**15**	**15.00**	0
le450_15b	**15**	15	**15**	**15.0**	1	15.0	**15**	**15.0**	1	**15**	**15.00**	0
le450_15c	**15**	15	**15**	15.2	11	15.0	**15**	15.2	11	**15**	**15.00**	3
le450_15d	**15**	15	**15**	**15.0**	5	15.0	**15**	**15.0**	5	**15**	**15.00**	4
le450_25c	26	26	26	26.0	7					**25**	**25.00**	1321
le450_25d	26	26	26	26.4	1					**25**	**25.00**	436
le450_5a	**5**	5	**5**	**5.0**	1	5.0	**5**	**5.0**	1	**5**	**5.00**	0
le450_5b	**5**	5	**5**	**5.0**	2	5.0	**5**	**5.0**	2	**5**	**5.00**	0
le450_5d	**5**	5	**5**	**5.0**	3	5.0	**5**	**5.0**	3	**5**	**5.00**	0
r125.1	**5**	5	**5**	**5.0**	0	5.0	**5**	**5.0**	0	**5**	**5.00**	0
r125.1c	**46**	46	**46**	**46.0**	0	46.0	**46**	**46.0**	0	**46**	**46.00**	20
r125.5	**36**	36	**36**	**36.0**	0	36.0	**36**	**36.0**	0	**36**	**36.00**	21
r250.1	**8**	8	**8**	**8.0**	0	8.0	**8**	**8.0**	0	**8**	**8.00**	26
r250.1c	**64**	64	**64**	**64.0**	2					**64**	**64.00**	21
r250.5	**65**	65	**65**	65.8	16	65.0	**65**	65.8	16	**65**	**65.00**	64
r1000.1	**20**	20	**20**	**20.0**	0	20.0	**20**	**20.0**	0	**20**	**20.00**	37
r1000.1c	98	98	98	98.8	557					98	98.00	518
r1000.5	237	234	237	238.6	1345					**234**	**234.00**	753
school1	**14**	14	**14**	**14.0**	0	14.0	**14**	**14.0**	0	**14**	**14.00**	0
school1_nsh	**14**	14	**14**	**14.0**	1	14.0	**14**	**14.0**	1	**14**	**14.00**	21
latin_square_10	99	90	99	100.2	938	90.0	99	100.2	938	101	102.00	5156
flat300_20_0	**20**	20	**20**	**20.0**	2	10.2	**20**	**20.0**	114	**20**	**20.00**	21
flat300_26_0	**26**	26	**26**	**26.0**	1	11.0	**26**	**26.0**	104	**26**	**26.00**	36
flat300_28_0	**28**	30	31	31.0	133	11.4	31	31.2	1355	31	31.00	212
flat1000_50_0	**50**	50	**50**	**50.0**	14	13.0	**50**	**50.0**	2351	**50**	**50.00**	1417
flat1000_60_0	**60**	60	**60**	**60.0**	59	13.6	**60**	**60.0**	2436	**60**	**60.00**	3645
flat1000_76_0	83	84	87	87.8	2499	13.6	87	88.2	7087	83	83.50	7325
								Σ =27085			Σ =31619	

parameters during the computation. A post optimization phase is finally performed. The overall MMT Algorithm approaches the problem in optimization version, requiring no input parameter, with the exception of the time limits of the 2 phases. Computational experiments show that the quality of the solutions found by Algorithm MMT (with the dynamic set-up of the parameters and the post optimization phase), is slightly better than that of the Evolutionary Algorithm (with the ad hoc set-up of the parameters), leading the proposed

Table 2.7: Average gap on the common subset of instances.

	Instance set from [92]	Instance set from [51]	Instance set from [15]	Full instance set [50]
Impasse	1.0114			
HCA		1.0163		
PC-RPC			1.0086	
MIPS_CLR				1.0072
Tabu Search Algorithm	1.0226	1.0579	1.0296	1.0189
Evolutionary Algorithm	1.0041	1.0086	1.0017	1.0018
MMT Algorithm	1.0046	1.0095	1.0029	1.0013

approach to be the state of the art heuristic algorithm for the problem. More in detail, 3 instances from the literature (namely le450_25c, le450_25d and r1000.5.) were solved, for the first time, to proven optimality, the best known solution for instance flat1000_76_0 was improved, and the best known solution values are still found on 35 of the remaining 38 instances.

The main deficiency of our approach is that, when optimality is not proven, the computation continues up to the time limit, even if the problem is yet solved or the diversity of the population of the Evolutionary Algorithm is low and unlikely it will find new improved solutions . Thus, future work should deal with the development of a procedure to stop the computation when the probability of improving the best incumbent solution becomes too low, and with the search for improved lower bounds for the problem.

Chapter 3

An Evolutionary Approach for Bandwidth Multicoloring Problems

[1]

In this paper we consider some generalizations of the Vertex Coloring Problem, where distance constraints are imposed between adjacent vertices (Bandwidth Coloring Problem) and each vertex has to be colored with more than one color (Bandwidth Multicoloring Problem). We propose an Evolutionary metaheuristic approach for the first problem, combining an effective Tabu Search Algorithm with Population Management procedures. The approach can be applied to the second problem as well, after a simple transformation. Computational results on instances from the literature show that the overall algorithm is able to produce high quality solutions in a reasonable amount of time, outperforming the most effective algorithms proposed for the Bandwidth Coloring Problem, and improving the best known solution of many instances of the Bandwidth Multicoloring Problem.

Keywords: Combinatorial Optimization, Bandwidth Coloring Problem, Bandwidth Multicoloring Problem, Evolutionary Algorithm, Tabu Search.

3.1 Introduction

Given an undirected graph $G = (V, E)$, where V is the vertex set and E is the edge set, the classical *Vertex Coloring Problem* (VCP) requires to assign a *color* to each vertex (i.e. to label each vertex $i \in V$ with an integer $c(i)$ corresponding to a color) in such a way that colors on adjacent vertices are different and the maximum color used is minimized.

In the *Bandwidth Coloring Problem* (BCP) distance constraints are imposed between adjacent vertices, replacing the difference constraints. A distance $d(i, j)$ is defined for each edge $(i, j) \in E$, and the absolute value of the difference between the colors assigned to i and j must be at least equal to this distance: $|c(i) - c(j)| \geq d(i, j)$.

In the *Multicoloring Problem* (MCP) a positive weight $w(i)$ is defined for each vertex $i \in V$, representing the number of colors that must be assigned to vertex i, so that for each $(i, j) \in E$ the intersection of the color sets assigned to vertices i and j is empty.

The *Bandwidth Multicoloring Problem* (BMCP) is the combination of the two problems

[1] The results of this chapter appear in [86].

above. Each vertex i must be assigned $w(i)$ colors, and each of these colors must respect the distance $d(i,j)$ with all the colors assigned to any adjacent vertex j. In this case, loop $d(i,i)$ represents the minimum distance between different colors assigned to the same vertex i.

Some definitions will be useful in the following: let n and m be the cardinalities of vertex set V and edge set E, respectively; let $\delta(i)$ be the degree of a given vertex i. A subset of V is called an *independent set* if no two adjacent vertices belong to it. A *clique* of a graph G is a complete subgraph of G. A k *coloring* of G is a coloring in which all the constraints concerning colored vertices are satisfied and the maximum color used is k. A k coloring of a graph G can be represented as a vector c of n components where, for each colored vertex $i \in V$, the corresponding color $c(i) \in \{1,\ldots,k\}$ is indicated, and where $c(i) = 0$ denotes that vertex i is uncolored. A *feasible* k *coloring* is a k coloring with all the vertices colored, while in a *partial* k *coloring* some vertices can be uncolored. An *optimal coloring* of G is a feasible k coloring with the smallest possible value of k (the *chromatic number* $\chi(G)$ of G). A *color class* is a set of vertices of the same color. The *chromatic degree* of a vertex is the number of different colors of its adjacent vertices.

BCP, MCP and BMCP are NP-hard because they generalize VCP, that is known to be NP-hard (see Garey and Johnson [53]). Clearly, a VCP instance is a BMCP instance where all the distances are equal to 1 and each vertex must receive only one color. The BCP where the distances between adjacent vertices are the same (i.e. $d(i,j) = T$ for any edge $(i,j) \in E$) is also known as $T-Coloring$, see e.g. Roberts [100]. BCP, MCP and BMCP allow complex situations to be modelled, like for example the assignment of frequencies to different cells in a mobile network.

More in detail, the *Frequency Assignment Problems* (FAP) concern the allocation of frequencies to transmitters, with the aim of avoiding or minimizing interference (see the survey by Aardal et al. [4] for an overview of the models and exact or heuristic methods proposed in the literature; see the web page [2] for an updated bibliography and a collection of FAP test instances). In FAP, either the radio spectrum used should be minimized with an interference not larger than a given threshold, or the interference should be minimized for a fixed allocation of frequency channels. The BMCP models the first situation, where vertices correspond to transmitters which have to receive a given number of frequencies, and distances correspond to the minimum distance between frequencies that can be re-used by adjacent (i.e. possibly interfering) transmitters. Different objective functions can be considered in FAP, the BMCP corresponding to the *Minimum Span Frequency Assignment Problem* (MS-FAP), the problem where the span, i.e. the range of frequencies, has to be minimized. Different heuristic approaches were proposed for MS-FAP, like greedy algorithms (see Zoellner and Beall [114], describing and comparing various greedy algorithms), Local Search (Wang and Rushforth [110], Tsang and Voudouris [107]), Tabu Search (Costa [36], Hao et al. [61]), Simulated Annealing [36], Genetic Algorithm (Velanzuela et al. [108]) and Constraint Programming approaches (Walser [109]). Most of these papers describe algorithms designed to solve instances with a special structure, representing real MS-FAP cases arising in telecommunications, like e.g. the well studied Philadelphia instances, firstly proposed by Anderson [9] in 1973. In all these instances n is equal to 21 and each transmitter (i.e. vertex) must receive a large number of frequencies (i.e. colors), so they correspond to the Multicoloring version of the problem we consider.

In 2002 a Computational Symposium on Graph Coloring and its Generalizations was organized in order to promote computational research on these problems [106]. Different computational approaches for VCP and its generalizations were presented during the sympo-

sium, and a set of instances of VCP, BCP, MCP and BMCP was proposed. The published results on this set of instances allow a direct comparison with the state of the art algorithms proposed for the problems. At the computational symposium, Phan and Skiena [93] proposed to solve VCP and BCP by means of a general heuristic, called *Discropt* (designed for "black box optimization"), adapted to the specific coloring problems. During the same symposium, Prestwich [95] proposed a combination of local search and constraint propagation in a method called FCNS to solve generalized graph coloring problems. In a successive work, the same author proposed a hybrid local search for VCP and BMCP [96]. In [82], Lim et al. proposed a method for solving VCP, BCP, MCP and BMCP combining hill-climbing techniques and Squeaky Wheel Optimization, a general heuristic approach for optimization, originally proposed by Joslin and Clements [74]. In a recent work Lim et al. [81] studied the performance of different heuristic methods, including Squeaky Wheel and Tabu Search and their hybridization, for the solution of BCP, MCP and BMCP.

Although the approaches we propose are explicitly designed for BCP, they can be applied to BMCP instances as well. Indeed, in BMCP each vertex $i \in V$ must be assigned a number of colors corresponding to its weight $w(i)$. To deal with these constraints, we transform each instance of BMCP to the corresponding instance of BCP, as suggested for example in [81]. In particular, we split each vertex i into a clique of cardinality $w(i)$, with each edge of the clique having distance $d(i,i)$, corresponding to the distance of the loop edge of vertex i in the original graph. The new graph will then have $\sum_{i=1,...,n} w(i)$ vertices. This transformation introduces extra symmetry, and hence our approaches are effective only when the number of colors to be assigned to any vertex is "small" (which is the case for the instances proposed in [106], but not for the MS-FAP instances).

The paper is organized as follows: Section 3.2 describes a possible Integer Linear Program BCP, Section 3.3 describes fast constructive heuristics for BCP and Section 3.4 proposes a Tabu Search approach for BCP. Section 3.5 describes how this approach can be integrated in a more complex Evolutionary algorithm, where a pool of solutions is managed. Extensive computational experiments on BCP and BMCP instances from the literature are presented in Section 3.6. Concluding remarks are discussed in Section 3.7.

3.2 An ILP Model for the Bandwidth Coloring Problem

A possible ILP for BCP is the following. Let $H = \{1, 2, ..., t\}$ be the set of available colors (where t represents an upper bound on the value of the maximum color used). Consider the binary variables $x_{i,h}$ having value 1 iff vertex i is colored with color $h \in H$, and the binary variables y_h having value 1 iff color h is used. Then the model reads:

$$\min \; k \tag{3.1}$$

$$k \geq y_h h \qquad h \in H \tag{3.2}$$

$$\sum_{h \in H} x_{ih} = 1 \qquad i \in V \tag{3.3}$$

$$x_{ih} + x_{jl} \leq 1 \qquad (i,j) \in E,\ h \in H,\ l \in \{h - d(i,j) + 1, ..., h + d(i,j) - 1\} \tag{3.4}$$

$$x_{ih} \leq y_h \qquad i \in V,\ h \in H \tag{3.5}$$

$$x_{i,h} \in \{0,1\} \qquad i \in V,\ h \in H \tag{3.6}$$

$$y_h \in \{0,1\} \qquad h \in H \tag{3.7}$$

The objective function (3.1) (in conjunction with constraints (3.2)) asks for minimizing the maximum color used. Note that in BCP the number of colors assigned to the vertices can be smaller than maximum color used; consider for example an edge (i, j) with $d(i, j) > 1$: two different colors are assigned to vertices i and j but the maximum color used is larger than two. Constraints (3.3) state that every vertex i in the graph must receive one color. Constraints (3.4) state that the absolute value of the difference between the colors assigned to vertices i and j must be at least equal to $d(i, j)$. Constraints (3.5) assure that if a vertex i uses a color h, the color h results as used.

It is known that models based on the Set Covering formulation for the classical VCP (see e.g. [90]) lead to stronger lower bounds when the integrality constraints are relaxed. However they cannot easily be extended to BCP, where constraints are not only imposed on vertices having the same color, but also on the relations between vertices having different colors (Constraints (3.4)).

3.3 Constructive Heuristics

In order to fast compute feasible solutions of BCP, different greedy algorithms proposed for VCP can be adapted to this generalization of the problem.

SEQ is a sequential greedy algorithm. Assume that the vertices are labelled $1, \ldots, n$. Vertex 1 is assigned to the first color class, and thereafter, vertex v ($v = 2, \ldots, n$) is assigned to the lowest indexed color class such that, for every adjacent vertex w: $|c(v)-c(w)| \geq d(v, w)$.

DSATUR [20, 72] is similar to SEQ, but dynamically chooses the vertex to color next, by picking the first vertex, in the given input ordering, that maximizes a given score. In the classical VCP the score of each vertex v is given by the number of distinctly colored adjacent vertices (i.e. the chromatic degree of v). This should force "difficult" vertices to be colored at the beginning. In BCP the "difficulty" of a vertex is given not only by the number of distinctly colored adjacent vertices but also by the distance to be respected between the color of the vertex and the colors of its adjacent vertices. Thus we consider as score $s(v)$ of vertex v the sum of the maximum distances between the vertex and each adjacent color:

$$s(v) = \sum_{h=1,\ldots,k} \max_{w:(v,w)\in E \text{ and } c(w)=h} d(v, w) \tag{3.8}$$

where k denotes the maximum color currently used.

3.4 Tabu Search Algorithm

Tabu Search is a local search technique that showed a very good experimental behavior on hard combinatorial optimization problems. The basic idea of the local search procedures is to start from a solution S and iteratively move to the best improving solution S' in a given neighborhood $N(S)$, until a local optimum is reached. The quality of different solutions is measured by means of a solution evaluating function $f(S)$. Tabu Search is a local search procedure that, in order to avoid local optimum traps, allows moves to the best solution S' in the neighborhood, even if it is not improving the current solution. To avoid cycling, some attributes of solution S' are stored in a *Tabu List*; for a specified number of iterations (the so called *tabu tenure*) a solution which presents tabu attributes is declared tabu and is not

considered, except in the case it would improve on the best incumbent solution (*aspiration criterion*).

The *Tabu Search Algorithm* proposed for BCP applies iteratively a Tabu Search Procedure (TS-P) for BCP working in decision version, i.e. requiring as input parameter the maximum color k available to color graph G, and returning, if successful, a feasible k coloring. The Tabu Search Algorithm computes an upper bound UB by means of a fast greedy procedure (described in the previous Section), and then calls the Tabu Search Procedure setting the maximum available color k equal to $UB - 1$, and starting the search from an initial solution. If the problem is solved within a given time limit, k is decreased to $k - 1$, the time counter is set to zero, and the process is iterated, starting from a new initial solution, until a value of k is reached for which the procedure is not able to solve the problem within the given time limit. If one prefers to perform multiple restarts of the Tabu Search Procedure, instead of using all the computing time in a single long run, an iteration limit can be used to control the execution of the procedure. When the iteration limit is reached and the problem is not yet solved for the given value of k, the Tabu Search Procedure is restarted from scratch, for the same value of k, starting from a different initial solution. The new initial solution is obtained by means of the constructive heuristics described in the previous Section, which can be executed with perturbed orderings of the vertices, so as to obtain different solutions.

The Tabu Search Procedure we propose moves between partial k colorings, i.e. solutions in which the maximum color used is k, all the distance constraints are satisfied, but not all the vertices are colored. For each value of k, the starting partial k coloring is obtained from the initial solution by "uncoloring" all the vertices having a color exceeding k. Note that also a dummy solution (i.e. a solution in which all the vertices are uncolored) is a partial k coloring which can be used as starting k coloring. When all the vertices of G are colored with the available k colors (i.e. a feasible k coloring is found), the problem is solved for the given value of k. The first idea of partial coloring neighborhood was proposed by Morgenstern [92] and was implemented in Tabu Search procedures for the classical VCP by Blöchliger and Zufferey [15] and by Malaguti et al. [84]. In particular, the Tabu Search procedure we propose in this paper is a generalization of the Tabu Search procedure proposed in [84] for the VCP.

To move from a solution $\overline{S}$ to a new solution $S' \in N(\overline{S})$ one can randomly choose an uncolored vertex v ($c(v) = 0$), assign v to a color class, say h, and uncolor (i.e. assign to class 0) all the vertices w that are adjacent to v and such that $|c(w) - h| < d(v, w)$. This ensures that the new solution S' is still a partial k coloring. Class h is chosen by comparing different target classes, and choosing the class leading to a solution S that minimizes the corresponding value of the *evaluating function* $f(S)$. We do not simply minimize $| \{v : c(v) = 0\} |$, because this does not consider that some vertices, having an higher degree and larger distances to be maintained with the adjacent vertices, are more difficult to color. To take into consideration both aspects we propose the following evaluating function for solution S:

$$f(S) = \sum_{v:c(v)=0} \sum_{w:(v,w)\in E} d(v, w) \tag{3.9}$$

To avoid cycling, we use the same tabu rule used in [84]: a vertex v cannot take the same color h it took in at least one of the last T iterations (where T represents the tabu tenure); for this purpose we store in a tabu list the pair (v, h) for T iterations. While the move remains tabu, vertex v cannot be assigned to color class h. We also use an *Aspiration Criterion*: a tabu move can be performed if it improves on the best solution encountered so far.

Tabu Search Procedure TS-P takes in input:

- a graph $G(V, E)$ with distances on edges;
- the target value k for the maximum color used;
- an initial partial k coloring $S := \{c(v_1), \ldots, c(v_n)\}$ (obtained by applying a greedy procedure, see Section 3.3);
- the maximum number L of iterations to be performed;
- the tabu tenure T.

If the procedure solves the problem within L iterations then it gives on output a feasible k coloring.

Let S be the current solution and S^* the best incumbent solution. The Tabu Search Procedure TS-P works as follows:

begin
1. $S^* := S$;
2. $tabulist := \emptyset$;
3. **for** ($iterations = 1$ **to** L)
4. randomly select an uncolored vertex v ($c(v) = 0$);
5. **for each** $j \in \{1, \ldots, k\}$ (explore the neighborhood of S)
6. $S_j = (c_j(v_1), \ldots, c_j(v_n)) := S$;
7. $c_j(v) := j$;
8. **for each** $w : (v, w) \in E$ **and** $|c_j(v) - c_j(w)| < d(v, w)$
9. $c_j(w) := 0$
10. **end for**
11. **end for**;
12. $h := \arg\min_{j \in \{1,\ldots,k\} : (v,j) \notin tabulist}$ **or** $_{f(S_j) < f(S^*)} f(S_j)$;
13. **if** no such h exists **then** $h := \arg\min_{j \in \{1,\ldots,k\}} f(S_j)$;
14. $S := (c(v_1), \ldots, c(v_n)) := S_h$;
15. insert (v, h) in $tabulist$ ((v, h) is tabu for T iterations);
16. **if** $f(S) < f(S^*)$ **then** $S^* := S$;
17. **if** $\{v : c(v) = 0\} = \emptyset$ **then return** S^*
18. **end for**
19. **return** S^*

end.

At line 12 we try to select the best color class which improves on the best solution so far or does not represent a tabu move. If all moves are tabu, at line 13 we simply select the best color class. In both cases (line 12 and 13), ties are broken by selecting the first best color class.

The computational experiments presented in Section 3.6 show that the proposed Tabu Search Algorithm is able to find high quality solutions. However we observed that when, for a given value of k, the Tabu Search Procedure TS-P is not able to solve the problem within the given number of iterations L, rarely it is able to find a feasible k coloring, even if more iterations are allowed. In this case a restart from a different initial solution is usually more effective than continuing the search from the current solution. This behavior shows that the search strategy is quite aggressive, and able to obtain fast descent of the objective function

in short computing time, but, when the starting solution is in a poor region of the search space, the Tabu Search Procedure is not able to escape from this region. More in detail, it can happen that a vertex v is assigned to a color at the very beginning of the computation, and this choice is never changed during the Tabu Search iterations. These considerations suggest that a procedure able to extend the search to the whole solution space would be of great efficacy. This procedure is described in the next Section.

3.5 The Evolutionary Algorithm

The Tabu Search Procedure TS-P described in the previous Section can be combined with a population management procedure, obtaining a more complex population based Evolutionary Procedure E-P for BCP. This procedure still works in decision version, and handles a pool of solutions composed by partial k colorings. Like the Tabu Search Procedure TS-P, the Evolutionary Procedure E-P as well can be called iteratively starting from an upper bound, in order to solve the optimization version of the problem, thus leading to an overall *Evolutionary Algorithm*.

The Evolutionary Procedure E-P works as follows: first an initial *pool* is defined, i.e. a population of solutions (partial k colorings) is initialized by means of fast greedy heuristic algorithms (see Section 3.3). Then two solutions (parents) are randomly selected from the population and combined through a *crossover operator* in order to obtain a new solution (*offspring*). Every offspring solution is improved by means of the Tabu Search Procedure TS-P before to be introduced into the population, replacing the worst parent, i.e. the parent having the worst fitness (the fitness of each solution being computed by means of the evaluating function (3.9)). This algorithm can solve both BCP and BMCP, after the transformation of the instances of the latter problem to BCP instances. The algorithm can of course solve also classical VCP instances, that are a special case of BMCP, arising when all the distances and weights are equal to 1. However, specialized metaheuristic algorithms designed for the classical VCP (see e.g. Galinier and Hao [51], Funabiki and Higashino [50], Malaguti et al. [84] for classical metaheuristic approaches; see Barbosa et al. [12] for two novel evolutionary formulations of the problem and for effective heuristics based on these formulations) have a better performance, since the crossover operator we propose is based on the tight satisfaction of the distance constraints, and hence it would perform poorly on VCP.

The Evolutionary Procedure E-P takes in input:

- a graph $G(V, E)$ with distances on edges;
- the target value k for the maximum color used;
- the number L of iterations to be performed by procedure TS-P between two calls of the crossover operator;
- the dimension $poolsize$ of the pool;
- the tabu tenure T;
- the maximum computing time ($timelimit$) allowed for the execution of the procedure.

The Evolutionary Procedure E-P is structured as follows:

begin
1. generate the initial *pool* containing *poolsize* solutions (partial k colorings);
2. **if** the problem is solved (i.e. a feasible k coloring is found) during the initialization **then stop**;
3. **while** (not *timelimit*)
4. randomly select 2 solutions S^1 and S^2 from the *pool*;
5. generate $S^3 := Crossover(S^1, S^2)$;
6. **if** the problem is solved **then stop**;
7. [Improve the offspring] $S^3 := TabuSearch(S^3)$;
8. **if** the problem is solved **then stop**;
9. insert S^3 in the *pool* and delete its worst parent
10. **end while**

end.

3.5.1 Pool Management

We use the Tabu Search Procedure described in Section 3.4 as a component of the more complex Evolutionary Procedure, where a pool of different solutions is managed, and solutions are combined, by means of a crossover operator, in order to generate new solutions. These new solutions can be used as starting points to extend the Tabu Search to different regions of the whole solution space. In this framework, the Tabu Search Procedure is used to improve on the quality of the different solutions of the pool.

The use of Pool Management procedures is justified by the experimental observation that, given a fixed number of Tabu Search iterations to be executed, it is better to perform multiple restarts from different initial solutions than to spend all the available iterations in a single run. Indeed, the Tabu Search Procedure seems unable to move away from a limited region of the solution space, and the final solution is usually close to the initial one. At the same time, a naive multiple restart strategy does not intensify the search in more promising regions of the solution space. Thus we use a specialized crossover operator to generate new starting solutions, sharing interesting properties with their parents, and we improve these solutions by means of the Tabu Search Procedure. It is known that generic crossover operators perform poorly on VCP (see, e.g., Davis [38]), while it was experimentally shown that specialized crossover operators can be effective in the case of the classical VCP (see [51, 84]), and the computational experiments presented in Section 3.6 show that this is true in the case of VCP generalizations as well. The main idea behind the use of a specialized crossover operator is that "good" solutions share part of their structure with the optimal ones, and that a specialized crossover is generally able to identify properties that are meaningful for the problem.

Initialization

During the initialization phase of the Evolutionary Procedure E-P, we generate *poolsize* initial solutions (partial k colorings) that will be handled by the Pool Management procedures and improved by means of the Tabu Search Procedure. For the success of the overall search it is crucial to start with solutions having different structures, thus allowing the Pool Management procedures to access the whole solution space, by recombining solutions through the crossover operator.

To diversify as much as possible the solutions contained in the initial pool, during the

initialization phase we use three different algorithms: two thirds of the solutions are generated by applying the two greedy algorithms SEQ and DSATUR described in Section 3.3; the last third is generated by the Tabu Search Procedure TS-P applied starting from a dummy solution (all vertices uncolored) and maximum number k of colors. The three algorithms are applied with different random orderings of the vertices, so as to generate different solutions. To obtain partial k colorings, all the vertices colored by the greedy algorithms with colors greater than k are uncolored before to insert the corresponding solution in the pool.

Every initial solution generated by SEQ or DSATUR is improved by means of the Tabu Search Procedure before to be inserted in the pool.

Solutions Selection

Once the pool of solutions is initialized, two solutions must be selected and combined to extend the search. The solutions to be combined through the crossover operator are randomly chosen among the solutions of the pool. The new offspring solution, improved by means of the Tabu Search Procedure, is then introduced into the population, replacing the parent which results to have the worst fitness with respect to the solution evaluating function (3.9) used in the Tabu Search Procedure.

Crossover Operators

An effective crossover operator must take in input two parent solutions $S1$ and $S2$ and transmit to the offspring solution $S3$ the parents's properties that are meaningful for the problem. In the case of the classical VCP, the structure of a solution is given by the partition of the vertices into independent sets (each corresponding to a different color). The color that is given to a set is completely arbitrary. This leads to very effective crossover operators where new solutions are built by combining the independent sets of the corresponding parents (see [51, 84]).

In the case of BCP, the color that is assigned to each set of vertices, with respect to the other sets, becomes important for the feasibility of the solution, as imposed by the distance constraints. Thus the structure of a solution is given not only by the partition of the vertices into independent sets, but by the mutual distances between colored vertices as well.

We computationally experimented different crossover operators that, given two parent solutions $S1$ and $S2$ represented by partial k colorings, output as offspring a partial k coloring $S3$. The characteristics of these crossovers are summarized in the following:

- Crossover #1: Given two parents S1 and S2, the color classes of the offspring S3 are built by iteratively picking the color class of maximum cardinality from one parent (in turn $S1$ or $S2$) and assigning the corresponding vertices to the offspring. These vertices are inserted in the first free color class of $S3$, starting from color 1, if they do not violate the distance constraints. Every vertex violating the distance constraints, as well as vertices uncolored in both parents, are left uncolored. The inserted vertices are then deleted from the other parent. The process is iterated from the other parent while there are available colors in the offspring. This crossover was designed for VCP (see [51, 84] for more details) and gives poor performance in the case of BCP, due to the fact that many vertices cannot be colored in the offspring without violating the distance constraints.

- Crossover #2: This crossover is similar to crossover #1, but, in this case, the color classes of the offspring are filled in the following order: 1, k, 2, $k-1$, etc. This way of filling the color classes generally reduces the number of vertices that cannot be colored in the offspring because of the distance constraints.

- Crossover #3: This crossover as well is similar to crossover #1, but, in this case, the maximum cardinality color class picked from the parent is forced to keep the same position (color) when copied to $S3$. This choice generally avoids the infeasibilities between color classes picked from the same parent, thus reducing the number of uncolored vertices in the offspring.

- Crossover #4: Fixed a value p, a given percentage p of the vertices of the offspring $S3$, randomly selected, are colored with the same color they have in $S1$, the remaining percentage $1-p$ is colored with the color they have in $S2$, if this does not violate the distance constraints.

- Crossover #5: Fixed a value h, the first h color classes of parent $S1$ are copied to the offspring $S3$, the last $k-h$ color classes of $S2$ (with the exception of already colored vertices) are copied to $S3$, leaving uncolored the vertices that would violate the distance constraints.

By considering the test instances described in Section 3.6, we observed that the last two crossovers perform better than the previous ones, but their performance is not comparable with that of the crossover we finally included in our algorithm and that was extensively tested as reported in Section 3.6.2. We call it *distance_crossover* because it is based on the idea that the important structure to be transferred from the parents to the offspring is the relative color distance between the vertices, in particular when the distance constraints are satisfied tightly. The *distance_crossover* works as follows (let $c_1(v)$, $c_2(v)$ and $c_3(v)$ denote the color assigned to vertex $v \in V$ in solution $S1$, $S2$ and $S3$, respectively; initially all the vertices in $S3$ are uncolored):

begin
1. $\mathcal{E}_1 := \{(v,w) \in E : \ v < w, \ |c_1(v) - c_1(w)| = d(v,w)\}$;
2. **for each** $(v,w) \in \mathcal{E}_1 : \ c_3(v) := c_1(v), \ c_3(w) := c_1(w)$;
3. $\mathcal{E}_2 := \{(v,w) \in E : \ v < w, \ |c_2(v) - c_2(w)| = d(v,w)\}$;
4. **for each** $(v,w) \in \mathcal{E}_2$:
5. **if** both v and w are currently uncolored in $S3$:
6. **if** feasible, set $c_3(v) = c_2(v); \ c_3(w) = c_2(w)$;
7. **else** try to color v and w with any color, starting from 1, still maintaining $|c_3(v) - c_3(w)| = d(v,w)$
8. **else if** one vertex of the pair (v,w)(say v) is already colored:
9. **if** feasible, set $c_3(w) := c_3(v) + d(v,w)$
10. **else if** feasible, set $c_3(w) := c_3(v) - d(v,w)$
11. **end for**

end.

We copy all the "tight distance" pairs from $S1$ to the offspring, keeping the same color assignment. Then we try to do the same with "tight distance" pairs from $S2$, keeping the

same color assignment, when this does not violate the distance constraints with respect to the already colored vertices of the offspring. As in the other crossovers that we experimented, when a vertex cannot feasibly be colored, or it was uncolored in both parents, we leave it uncolored.

After the crossover, that is mainly aimed at generating new solutions, sharing important structures with their parents (and not simply at generating an offspring with many colored vertices), we could use a greedy algorithm in order to color uncolored vertices. We prefer to leave these vertices uncolored and directly apply the Tabu Search Procedure to the offspring $S3$, which will insert the uncolored vertices in the first color class where they can be inserted without moving other vertices (if any).

3.6 Computational Analysis

The Tabu Search Algorithm and the Evolutionary Algorithm described, respectively, in Sections 3.4 and 3.5 were coded in ANSI C and compiled with full optimization option. The programs were run on a PIV 2.4MHz computer, with 512MB RAM, under Windows XP, and tested on the instances proposed during the Computational Symposium on Graph Coloring and its Generalizations, held in 2002. These instances (which can be found at [106]) correspond to different graph types used for evaluating the performance of VCP algorithms. To perform our computational experiments, we considered the instance sets GEOMn, GEOMna and GEOMnb (where n represents the number of vertices of the graph), proposed for BCP and BMCP. In these geometric graphs, the vertices are uniformly randomly generated in a 10,000 by 10,000 grid, and are connected by an edge if they are close enough. Edge distances are inversely proportional to the Euclidean distance between the corresponding vertices. Vertex weights are uniformly randomly generated, between 1 and 10 for sets GEOMn and GEOMna, and between 1 and 3 for set GEOMnb. GEOMn instances correspond to sparse graphs; GEOMna and GEOMnb instances correspond to denser graphs. We also considered the Philadelphia instances [9], that were extensively studied in the MS-FAP literature (see, e.g., [4]). In these BMCP instances, each of the 21 vertices of the graph must receive a large number of colors (32.2 on average).

All the computing times reported in this section are expressed in seconds of our machine. To allow a meaningful - although approximate - comparison on results obtained with different machines, a benchmark program (dfmax), together with a benchmark instance (r500.5), are available. Computing times obtained on different machines can be scaled with respect to the performance obtained on this program (our machine spent 7 seconds "user time").

3.6.1 Performance of the Tabu Search Algorithm

In this section we report the experimental results obtained by the Tabu Search Algorithm (described in Section 3.4) on the BCP and BMCP instances presented in [106]. In order to fast compute an initial upper bound UB, we perform 20 runs of the greedy algorithm SEQ (described in Section 3.3), with perturbed input orderings of the vertices, and set UB equal to the best solution value found. As explained in Section 3.4, we run the Algorithm with multiple restarts of the Tabu Search Procedure TS-P. The maximum number of iterations L of the Tabu Search Procedure TS-P is set to 100,000 in the case of BCP, and to 400,000 in the case of BMCP. After L iterations, if the problem is not solved for the current value of k, we restart procedure TS-P from a new solution (partial k coloring). Every partial k coloring

is obtained in turn by applying the SEQ or DSATUR algorithms, where we uncolor all the vertices whose current color exceeds the given k value, or starting from a dummy solution (all vertices uncolored). If we are able to solve the problem, for the current value of k, within the time limit (which is set to 500 seconds in the case of BCP and to 3000 seconds in the case of BMCP), we decrease the value of k and set to zero the time counter. We iterate until, for the current value of k, the problem cannot be solved within the time limit.

In Table 3.1 we report, for every considered BCP instance from [106], the name of the instance, the number of vertices (n) and edges (m) of the considered graph, the best known solution value ever published in the literature [93, 95, 82, 96, 81] ("best"), the initial upper bound found by the SEQ algorithm ("UB"), the solution value ("k"), the number of iterations needed to find the last improving solution starting from the initial UB ("iter"), the total computing time (including the initial UB computation) corresponding to the last improvement ("time", we report 0 when the time is lower than 1 second). The tabu tenure T is fixed to a value of 50 for the BCP instances, while better results can be obtained in the case of the BMCP instances with a tabu tenure T equal to the number of vertices of the graph, after the transformation described in Section 3.1. In Table 3.2 we report the results obtained on the BMCP instances. The last two rows of Tables 3.1 and 3.2 report the average ratio between the solution value and the best known solution value from the literature ($k/best$) and the average computing time.

For the set-up of the parameters T and L, we selected the values that, on different computational tests performed, lead on average to the best results in terms of solution value and time. Of course, better results can be obtained by using an ad hoc set up tuned on each instance, but our opinion is that the robustness of the algorithm, i.e. the capacity of solving different instances without requiring a special set-up on each instance, has to be preferred.

The Tabu Search Algorithm we propose has a very good experimental behavior in the case of BCP (Table 3.1). The quality of the solutions found is comparable with that of the solutions found by the state of the art algorithms. Over 33 considered instances, the Tabu Search Algorithm is able to improve on the best published solution in the literature 4 times, in 21 cases it finds a solution whose value equals the best published one and in the remaining 8 cases it is worse, but always using only one more color. Also in the case of BMCP (Table 3.2) the quality of the solutions found is comparable with that of the solutions found by the state of the art algorithms. Over 33 considered instances, the Tabu Search Algorithm is able to improve on the best published solution in the literature 8 times, in 11 cases it finds a solution whose value equals the best published one, and in the remaining 14 cases it is worse.

We tested the Tabu Search Algorithm also on the 9 Philadelphia instances [9], representing BMCP instances corresponding to the Minimum Span Frequency Assignment Problem (MS-FAP), where 21 vertices must receive a large number of colors (32.2 on average). As it could be expected, our algorithm performs very poorly on these instances, obtaining solutions values that are on average 6.5% larger than the optimal ones (which are known, and can be found by specialized metaheuristics approaches, see [4]). This is due to the extra symmetry introduced in the problem by transforming BCMP instances to BCP ones, i.e. by transforming each vertex i which must receive $w(i)$ colors to a clique of cardinality $w(i)$. This transformation leads to BCP instances where our approach is completely ineffective, because large cliques are built, and our Tabu Search approach risks to spend all the iterations by moving vertices of the same clique (representing originally a single vertex) to different color classes, without really changing the solution.

Table 3.1: Tabu Search Algorithm: Bandwidth Coloring Instances.

Instance name	n	m	best	UB	k	iter	time
GEOM20	20	40	20	21	21	0	0
GEOM20a	20	57	20	22	20	1613	0
GEOM20b	20	52	13	15	13	10	0
GEOM30	30	80	27	28	28	0	0
GEOM30a	30	111	27	31	27	13051	0
GEOM30b	30	111	26	26	26	0	0
GEOM40	40	118	27	29	28	15	0
GEOM40a	40	186	37	41	37	24753	0
GEOM40b	40	197	33	40	33	3745	0
GEOM50	50	177	28	29	28	697	0
GEOM50a	50	288	50	55	50	32125	0
GEOM50b	50	299	35	44	35	500360	0
GEOM60	60	245	33	35	33	3038	0
GEOM60a	60	339	50	57	50	24038	0
GEOM60b	60	426	43	54	41	96611523	147
GEOM70	70	337	38	42	38	2021	0
GEOM70a	70	529	62	70	61	15810658	29
GEOM70b	70	558	48	62	48	44923788	76
GEOM80	80	429	41	46	41	76460	0
GEOM80a	80	692	63	77	63	4474619	9
GEOM80b	80	743	61	78	61	14347412	30
GEOM90	90	531	46	50	46	10594	0
GEOM90a	90	879	64	77	63	32340177	73
GEOM90b	90	950	72	89	72	17708445	47
GEOM100	100	647	50	58	50	1184211	2
GEOM100a	100	1092	68	89	69	33992244	73
GEOM100b	100	1150	73	96	74	45945300	114
GEOM110	110	748	50	60	50	371993	1
GEOM110a	110	1317	73	94	74	570556	1
GEOM110b	110	1366	79	105	79	129462538	338
GEOM120	120	893	60	70	59	1975913	3
GEOM120a	120	1554	84	103	85	202323831	470
GEOM120b	120	1611	86	116	87	18180399	50
avg. gap (k/best)					1.0029		
avg. time							44

Table 3.2: Tabu Search Algorithm: Bandwidth Multicoloring Instances.

Instance name	n	m	best	UB	k	iter	time
GEOM20	20	40	149	155	149	939426	3
GEOM20a	20	57	169	176	169	123721114	405
GEOM20b	20	52	44	46	44	10552157	13
GEOM30	30	80	160	172	160	195505	1
GEOM30a	30	111	209	242	211	767838021	3684
GEOM30b	30	111	77	79	77	121319	0
GEOM40	40	118	167	180	167	4416693	18
GEOM40a	40	186	213	230	215	302864957	1639
GEOM40b	40	197	74	87	74	160164	1
GEOM50	50	177	224	235	225	24019956	130
GEOM50a	50	288	318	359	320	102725774	792
GEOM50b	50	299	86	103	83	118972111	304
GEOM60	60	245	258	266	258	40375817	202
GEOM60a	60	339	358	404	363	764735785	5632
GEOM60b	60	426	116	139	114	878357241	2969
GEOM70	70	337	273	312	270	251841925	1487
GEOM70a	70	529	469	509	473	106820588	1084
GEOM70b	70	558	119	142	119	62714747	208
GEOM80	80	429	383	422	388	269267418	2088
GEOM80a	80	692	379	406	370	154315259	1530
GEOM80b	80	743	141	164	141	95766220	330
GEOM90	90	531	332	354	334	197265420	1460
GEOM90a	90	879	377	416	384	352269766	3024
GEOM90b	90	950	147	178	146	36450728	153
GEOM100	100	647	404	424	412	439473614	3425
GEOM100a	100	1092	453	507	452	437615679	4471
GEOM100b	100	1150	159	201	160	163576744	804
GEOM110	110	748	383	421	382	748917011	7325
GEOM110a	110	1317	494	535	492	40969388	455
GEOM110b	110	1366	206	240	207	108643060	645
GEOM120	120	893	402	430	405	333840804	2718
GEOM120a	120	1554	556	611	559	257342109	3149
GEOM120b	120	1611	195	224	195	176508745	951
avg. gap (k/best)					1.0009		
avg. time							1548

3.6.2 Performance of the Evolutionary Algorithm

In this section we describe the experiments performed with the Evolutionary Algorithm (described in Section 3.5) and compare it with the most effective algorithms proposed in the literature on the BCP and BMCP instances presented in [106]. The corresponding computational results are reported, respectively, in Tables 3.3 and 3.4. The first two columns of the tables report the instance name and the corresponding best published solution value ("best").

The solution values ("k") and the corresponding computing times obtained by Lim et al. [82], with an algorithm combining hill-climbing techniques and Squeaky Wheel Optimization, are reported in the third and forth columns of Tables 3.3 and 3.4. The algorithm works in optimization version (i.e. it does not require as input the maximum color k to be used). We report the best solution value found over a single run and the computing time needed to get this result, scaled with respect to the time obtained on the benchmark problem (25.32 seconds "user time"), and thus directly comparable with the other reported times.

The fifth column of Table 3.3 reports the solution values ("k") obtained by Phan and Skiena by means of the *Discropt* general heuristic [93] in the case of BCP (they report no results for BMCP). The algorithm works in optimization version. The reported results represent the best value obtained over 3 runs, using 3 different versions of the algorithm. The total computing time was 300 seconds (100 seconds for each run), on a Athlon K7 AMD processor with 768 MB of RAM under RedHat 7.2.

The solution values ("k") and the corresponding computing times obtained by Lim et al. [81], with an algorithm combining Squeaky Wheel Optimization with Tabu Search, are reported in the fifth and sixth columns of Table 3.4 in the case of BMCP (results for BCP are not competitive and were not reported in detail in [81]). The algorithm works in optimization version. We report, for each instance, the best solution value found over a run with 10 restarts and the computing time needed to get this result, scaled with respect to the time obtained on the benchmark problem (74.12 seconds "user time").

The solution values ("k") and the corresponding computing times obtained by Prestwich with an algorithm which hybridizes Local Search and Constraint Programming [96] are reported in the sixth and seventh columns of Table 3.3 in the case of BCP, and in the seventh and eighth columns of Table 3.4 in the case of BMCP. The times are scaled with respect to the time obtained on the benchmark problem (27.43 seconds "user time"). The algorithm, working in optimization version, requires some input parameters, whose values are tuned, among a limited set of possibilities, for each instance.

Finally we report the results obtained by means of the Evolutionary Algorithm, used to solve the optimization version of the problem. Like we did with the Tabu Search Algorithm, we compute an initial upper bound UB by performing 20 runs of the greedy algorithm SEQ (described in Section 3.3), with perturbed input orderings of the vertices, and set UB equal to the best solution value found.

Then, like in the case of the Tabu Search Algorithm, we run the Evolutionary Algorithm, trying to improve on this UB. We use a time limit, for every value of k, of 500 seconds in the case of BCP and of 3000 seconds in the case of BMCP. If the problem is solved, for the current value of k, we decrease it to $k-1$ and set to zero the time counter. We iterate until, for the current value of k, the problem cannot be solved within the time limit.

The last columns of Table 3.3 and Table 3.4 report the results obtained by the Evolutionary Algorithm in the case of BCP and BMCP, respectively. We report the solution value ("k"), the number of iterations needed to find the last improving solution starting from the initial

UB ("iter"), the total computing time (including the initial UB computation) corresponding to the last improvement, and the difference between the solution value ("k") and the best solution value ("$best$") published in the literature on instances from [106]. The value of the initial upper bound UB and the setting of the tabu tenure T are equal to those used for the Tabu Search Algorithm. The $poolsize$ was fixed to a value of 40 for both BCP and BMCP. Instead of performing multiple restarts, the Evolutionary Algorithm calls the cross over operator. The number of iterations (L) of the Tabu Search Procedure, between two consecutive calls of the cross over operator, is equal to that used when testing multiple restarts for the Tabu Search Algorithm, i.e. 100,000 in the case of BCP and 400,000 in the case of BMCP. The use of the same set-up of the parameters T and L and of the same time limit should ensure a fair comparison between the Tabu Search Algorithm and the Evolutionary Algorithm.

The last two rows of Tables 3.3 and 3.4 report, for each considered algorithm, the average ratio between the solution value and the best known solution value from the literature ($k/best$), and the average computing time.

Over the 33 considered instances of BCP, the Evolutionary Algorithm is able to improve on the best published solution in the literature 9 times, in 21 cases it finds a solution whose value equals the best published one, and in the remaining 3 cases it is worse. It clearly outperforms the other proposed algorithms, singularly considered, from the point of view of the solution quality. The computing times are much longer, but, in our opinion, acceptable, since they improve on the best known solutions. The Evolutionary Algorithm improves 8 times on the best solution found by using the Tabu Search Algorithm, thus showing the efficacy of the overall diversification approach with respect to a simple multiple restart strategy.

Over the 33 considered instances of BMCP, the Evolutionary Algorithm is able to improve on the best published solution in the literature 13 times, in 13 cases it finds a solution whose value equals the best published one, and in the remaining 7 cases it is worse. Its performance, from the solution quality point of view, is better than that of the best considered algorithm, namely the one proposed by Lim et al. [81]. The latter finds 7 times a better solution than the one found by the Evolutionary Algorithm, while 13 times the proposed algorithm finds a better solution. Our computing times are in general larger on bigger instances, but still acceptable. The Evolutionary Algorithm improves 17 times on the best solution found by using the Tabu Search Algorithm, while the latter finds 3 times a better solution. This shows again the efficacy of the overall diversification approach with respect to a simple multiple restart strategy.

In our experiments, we used time limits which lead to long computational times with respect to other algorithms proposed in the literature: we think that this choice is justified, since we are able to improve some of the best know solution values published in the literature. It must be observed that the hard part of the computation very often corresponds to those improvements. In other words, when we spend a long time to improve the best known solution value of only few colors, a large part of the computation is devoted to solve the problem for the last values of k. This means that, when shorter computing times are imposed, our algorithm is still able to find feasible solutions of good quality. During our experiments, we observed that allowing even longer time limits would lead to better solutions on some instances, but with an average computational cost exceeding the acceptable threshold.

We tested the Evolutionary Algorithm also on the 9 Philadelphia (MS-FAP) instances, obtaining solution values that are on average 6.6% far from the optimal ones. This behavior, due to the transformation of BMCP instances to BCP instances, should not surprise, as

discussed in the previous Section.

3.7 Conclusions

In this paper we present an Evolutionary heuristic algorithm for two generalizations of the well known VCP, namely the Bandwidth Coloring (BCP) and the Bandwidth Multicoloring (BMCP) Problems. The Algorithm was designed for BCP, and applied to BMCP instances as well (after their transformation to BCP instances by means of a simple procedure). The Evolutionary Algorithm proposed to solve the BCP combines an effective Tabu Search approach with a specialized crossover operator.

Extensive computational experiments on 66 BCP and BMCP instances from literature benchmark graphs were performed with the Tabu Search approach and the overall Evolutionary Algorithm. The first is able to produce very good solutions in acceptable computing times. Improved solutions can be obtained by using the overall Evolutionary Algorithm. The comparison of the latter with the most effective algorithms from the literature shows that the proposed approach represents, on the considered set of instances, the new state of the art heuristic algorithm for BCP, and equals the most effective algorithm for BMCP.

Instances of BMCP corresponding to Frequency Assignment problems were also considered, where our approach performed very poorly. This behavior can be explained by considering the structure of these instances, where the transformation of BMCP to BCP was very ineffective, because it introduced much extra symmetry into the model.

Thus, in order to overcome this limit of our approach, in future work we intend to tackle the Multicoloring Problems solving directly a Multicoloring model.

Table 3.3: Performance of the Evolutionary Algorithm and comparison with the most effective heuristic algorithms for the Bandwidth Coloring Problem.

		Lim et al. 2003 [82]		Phan and Skiena 2002 [93]	Prestwich 2005 [96]		Evolutionary			
Instance name	best	k	time	k	k	time	k	iter	time	diff.
GEOM20	20	21	0	20	21	0	21	0	0	1
GEOM20a	20	22	0	20	20	0	20	1730	0	0
GEOM20b	13	14	0	13	13	0	13	6	0	0
GEOM30	27	29	0	27	28	0	28	0	0	1
GEOM30a	27	32	0	27	27	0	27	13064	0	0
GEOM30b	26	26	0	26	26	0	26	0	0	0
GEOM40	27	28	1	27	28	0	28	479	0	1
GEOM40a	37	38	1	38	37	0	37	24428	0	0
GEOM40b	33	34	1	36	33	0	33	3750	0	0
GEOM50	28	28	1	29	28	0	28	214	0	0
GEOM50a	50	52	1	54	50	2	50	36707	0	0
GEOM50b	35	38	2	40	35	0	35	326548	0	0
GEOM60	33	34	0	34	33	0	33	3364	0	0
GEOM60a	50	53	2	54	50	1	50	23812	0	0
GEOM60b	43	46	1	47	43	0	41	18739807	29	-2
GEOM70	38	38	0	40	38	0	38	2036	0	0
GEOM70a	62	63	0	64	62	2	61	6746180	12	-1
GEOM70b	48	54	0	54	48	1	48	30608071	52	0
GEOM80	41	42	2	44	41	0	41	76891	0	0
GEOM80a	63	66	0	69	63	12	63	75341752	150	0
GEOM80b	61	65	8	70	61	0	60	69418472	145	-1
GEOM90	46	46	0	48	46	3	46	10755	0	0
GEOM90a	64	69	2	74	64	2	63	66256099	150	-1
GEOM90b	72	77	6	83	72	2	70	376194388	1031	-2
GEOM100	50	51	9	55	50	0	50	1296142	2	0
GEOM100a	68	76	6	84	68	9	68	125814038	273	0
GEOM100b	73	83	2	87	73	15	73	235521605	597	0
GEOM110	50	53	1	59	50	4	50	1824578	3	0
GEOM110a	73	82	11	88	73	7	72	72749430	171	-1
GEOM110b	79	88	5	87	79	2	78	246378762	676	-1
GEOM120	60	62	0	67	60	4	59	293669	0	-1
GEOM120a	84	92	1	101	84	4	84	263039830	614	0
GEOM120b	86	98	1	103	86	9	84	304671189	857	-2
avg. ratio (k/best)		1.0646		1.0906	1.0054		0.9980			
avg. time			2			2.5			144	

Table 3.4: Performance of the Evolutionary Algorithm and comparison with the most effective heuristic algorithms for the Bandwidth Multicoloring Problem.

		Lim et al. 2003[82]		Lim et al. 2005[81]		Prestwich 2005[96]		Evolutionary			
Instance name	best	k	time	k	time	k	time	k	iter	time	diff.
GEOM20	149	149	0	149	17	149	4	149	5293853	18	0
GEOM20a	169	169	4	169	16	170	2	169	2400000	9	0
GEOM20b	44	44	0	44	2	44	0	44	3203748	5	0
GEOM30	160	160	0	160	23	160	0	160	195203	1	0
GEOM30a	209	211	3	209	40	214	11	210	166234965	954	1
GEOM30b	77	77	0	77	7	77	0	77	112224	0	0
GEOM40	167	167	1	167	47	167	1	167	4969719	20	0
GEOM40a	213	214	99	213	54	217	299	214	75188430	393	1
GEOM40b	74	76	2	74	10	74	4	74	160125	1	0
GEOM50	224	224	11	224	77	224	1	224	247423944	1197	0
GEOM50a	318	326	27	318	12	323	51	316	600574823	4675	-2
GEOM50b	86	87	15	87	15	86	1	83	72424865	197	-3
GEOM60	258	258	13	258	96	258	77	258	27935150	139	0
GEOM60a	358	368	1037	358	162	373	10	357	1161910138	8706	-1
GEOM60b	116	119	83	116	23	116	12	115	134011555	460	-1
GEOM70	273	279	7	273	138	277	641	272	220981847	1413	-1
GEOM70a	469	478	115	469	188	482	315	473	97340773	988	4
GEOM70b	119	124	38	121	30	119	55	117	222025798	897	-2
GEOM80	383	394	1118	383	204	398	361	388	16559905	132	5
GEOM80a	379	379	187	379	190	380	109	363	1042840998	8583	-16
GEOM80b	141	145	894	141	39	141	37	141	443911693	1856	0
GEOM90	332	335	1133	332	248	339	44	332	579637236	4160	0
GEOM90a	377	382	2879	377	245	382	13	382	631825295	5334	5
GEOM90b	147	157	179	157	46	147	303	144	417466073	1750	-3
GEOM100	404	413	175	404	311	424	7	410	422527570	3283	6
GEOM100a	453	462	48	459	334	461	26	444	1259264110	12526	-9
GEOM100b	159	172	1354	170	55	159	367	156	755603823	3699	-3
GEOM110	383	389	160	383	368	392	43	383	240024033	2344	0
GEOM110a	494	501	1292	494	441	500	29	490	186931545	2318	-4
GEOM110b	206	210	3	206	68	208	5	206	82617492	480	0
GEOM120	402	409	505	402	408	417	9	396	349961343	2867	-6
GEOM120a	556	564	1476	556	633	565	41	559	327849778	3873	3
GEOM120b	195	201	240	199	97	196	3	191	613183196	3292	-4
avg. ratio (k/best)		1.0181		1.0060		1.0120		0.9954			
avg. time			397		141		87			2320	

Chapter 4

Models and Algorithms for a Weighted Vertex Coloring Problem

[1]

We consider a weighted version of the well-known Vertex Coloring Problem (VCP) in which each vertex i of a graph G has associated a positive weight w_i. Like in VCP, one is required to assign a color to each vertex in such a way that colors on adjacent vertices are different, and the objective is to minimize the sum of the costs of the colors used. The difference with respect to VCP is that the cost of each color is given by the maximum weight of the vertices assigned to that color. The corresponding problem is known to be NP-hard and arises in practical scheduling applications, where it is also known as Scheduling on a Batch Machine with Job Compatibilities. We propose a straightforward formulation for WVCP, and two alternative Integer Linear Programming (ILP) models: the first one is used to derive, dropping integrality requirement for the variables, a tight lower bound on the solution value, while the second one is used to derive a 2-phase heuristic algorithm, also embedding fast refinement procedures aimed at improving the quality of the solutions found. Computational results on a large set of instances from the literature are reported.

4.1 Introduction

We approach a weighted version of the well known *Vertex Coloring Problem* (VCP), where an undirected graph $G = (V, E)$ is given (with V denoting the vertex set, and E the edge set) and each vertex $i \in V$ has associated a positive weight w_i. Like in the classical VCP, one has to assign a color to each vertex i in such a way that colors on adjacent vertices are different. The objective of this *Weighted Vertex Coloring Problem* (WVCP) is to minimize the sum of the costs of the colors used, where the cost of each color is the maximum weight of the vertices assigned to the color (while in the classical VCP the cost of each color is equal to one, thus the total number of colors has to be minimized).

WVCP is known to be strongly NP-hard since it generalizes VCP (see Garey and Johnson [53] for complexity results on VCP). De Werra, Demange, Monnot and Paschos [40] analyzed some properties of the optimal solutions and discussed complexity and approximability results for this problem, Escoffier, Monnot and Paschos [45] continued the investigation of

[1] Preliminary results of this chapter appear in [85].

the complexity and the approximability of WVCP, while Boudhar and Finke [19] studied its complexity for several classes of graphs. These results were extended to different classes of graphs by Finke, Jost, Queyranne and Sebö [48].

WVCP has many real world applications in different industrial fields. In the scheduling literature it is also known as *Scheduling on a Batch Machine with Job Compatibilities*, and it generalizes some applications where a compatibility graph limits the simultaneous access to a resource by the users. A typical case, concerning thermic treatment or mechanical processing, arises when some jobs cannot be processed within the same time slot because of geometric constraints (see Gavranovich and Finke [54]), and the duration of each time slot corresponds to the maximum processing time of the associated jobs. This kind of temporal constraint was studied by Hochbaum and Landy [63], who considered an application dealing with semiconductor burn-in operations in which the capacity of the oven limits the number of jobs that can be processed within the same time slot.

WVCP is also a generalization of the *Matrix Decomposition Problem in Time Division Multiple Access Traffic Assignment* (see Ribeiro, Minoux and Penna [99] and Prais and Ribeiro [94]), where a traffic matrix has to be decomposed into mode matrices (i.e., matrices such that no more than a non zero element is present in each row and column), and the transmission time of each mode matrix equals the maximum of its non zero elements. An exact approach to the Matrix Decomposition Problem, based on column generation, was proposed in [99], while in [94] a heuristic approach based on a Greedy Randomized Adaptive Search Procedure (GRASP) is presented.

Some definitions will be used in the following: let n and m be the cardinalities of vertex set V and edge set E, respectively.A *clique* of a graph G is a complete subgraph of G. A subset of V is called an *independent set* if no two adjacent vertices belong to it. The stability number α is the cardinality of the largest independent set of the graph. Note that each coloring of a graph is a partition of the vertex set into independent sets; we will call these sets *color classes*. An independent set (resp. clique) is maximal if no vertex can be added still having an independent set (resp. clique). Given an independent set $s \subseteq V$ of weighted vertices, we define its *cost* c_s as the maximum weight of its vertices. We say that independent set s *dominates* independent set t if $t \subset s$ and $c_s = c_t$. An independent set t is said to be *non dominated* if no independent set s exists that dominates t. Given a (possible partial) coloring of G, the *chromatic degree* $\delta_\chi(i)$ of a vertex $i \in V$ is the number of different colors assigned to its adjacent vertices.

In the following we will assume, without loss of generality, that all weights w_i are positive integers, and that vertices are numbered in such a way that

$$w_1 \geq w_2 \geq \ldots \geq w_n. \tag{4.1}$$

The paper is organized as follows: in the next Section we propose different Integer Linear Programming (ILP) models for the problem: Section 4.2.1 presents a straightforward ILP model, while Section 4.2.2 introduces a new ILP formulation, whose LP relaxation dominates the LP relaxation of the first model (see Section 4.2.3). Finally, Section 4.2.4 presents a third ILP model, involving an exponential number of variables, which is used to derive a 2-phase heuristic approach, outlined in Section 4.3: constructive heuristics used in the first phase are described in Section 4.3.1, Section 4.3.2 describes the second phase of the algorithm, and Section 4.4 presents post-optimization procedures aimed at improving the quality of the solutions found. Computational experiments on a wide set of instances from the literature are reported in Section 4.5. Concluding remarks are discussed in Section 4.6.

4.2 ILP models

We first describe two alternative ILP formulations for WVCP, involving a polynomial number of variables and constraints, and analyze the relation between the optimal solution values of the LP relaxations of the models, since such values are valid lower bounds for WVCP. Further, we present another ILP formulation that involves an exponential number of variables, and that is used for designing a heuristic algorithm.

4.2.1 Model M1

Noting that the number of colors in any feasible solution cannot be larger than n, we can restrict our attention to solutions involving up to n colors. Hence, a straightforward ILP model for WVCP can be obtained by defining the following two sets of variables: binary variables x_{ih} ($i \in V,\ h = 1, \ldots, n$), with $x_{ih} = 1$ iff vertex i is assigned to color h, and (continuous) variables z_h ($h = 1, \ldots, n$) denoting the cost of color h in the solution.

The corresponding model for WVCP is:

$$\text{(M1)} \qquad \min \sum_{h=1}^{n} z_h \tag{4.2}$$

$$z_h \geq w_i\ x_{ih}, \qquad i \in V,\ h = 1, \ldots, n, \tag{4.3}$$

$$\sum_{h=1}^{n} x_{ih} = 1, \qquad i \in V, \tag{4.4}$$

$$x_{ih} + x_{jh} \leq 1, \qquad (i, j) \in E, h = 1, \ldots, n, \tag{4.5}$$

$$x_{ih} \in \{0, 1\}, \qquad i \in V, h = 1, \ldots, n. \tag{4.6}$$

Objective function (4.2) minimizes the sum of the costs of the colors, which are defined by constraints (4.3). Constraints (4.4) require that every vertex is given a color, while (4.5) impose that adjacent vertices cannot receive the same color. Finally, constraints (4.6) require x variables to be binary.

This formulation is the most natural way to model problems in which one is required to partition a given set of items into subsets, the feasibility and cost of each subset depending only on the items assigned to the subset itself. However, this formulation tends to perform poorly in practice, mainly for two reasons. The first drawback of this model arises when one wants to solve the ILP above, since the solution space contains many optimal solutions due to symmetries. Indeed, given a feasible solution to (4.2)–(4.6), another feasible solution having the same cost is obtained by permuting the selected color classes. This drawback can be eliminated by imposing additional constraints and by adopting appropriate branching rules in the branch-and-bound exploration. The second drawback is that the LP relaxation of the model usually produces weak lower bounds, and this is true even if constraints (4.5) are strengthened by writing the corresponding clique constraints (see below).

In the next section we present a more sophisticated ILP model, which overcomes both the drawbacks above, and turns out to be effective in practice.

4.2.2 Model M2

Model M2 is based on the simple observation that, for any solution to WVCP, an equivalent solution exists in which each independent set $s = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ having p vertices is associated to color i_1, i.e., the color corresponding to the first (lowest index) vertex belonging to s.

Hence, we consider only feasible solutions in which each color h, if used, is *initialized* by vertex h, has cost equal to w_h and can be assigned to vertices $i \geq h$. A similar idea was proposed by Lodi, Martello and Vigo [83] to model a class of two dimensional packing problems.

Moving from the observation above, we can introduce binary variables y_{ih} taking value 1 iff vertex i is assigned to color h ($i \in V, h \leq i$). Hence, model M2 is as follows:

$$\text{(M2)} \qquad \min \sum_{h=1}^{n} w_h \, y_{hh} \tag{4.7}$$

$$\sum_{h=1}^{i} y_{ih} = 1, \qquad i = 1, \ldots, n, \tag{4.8}$$

$$y_{ih} + y_{jh} \leq y_{hh}, \qquad (i,j) \in E, h = 1, \ldots, \min\{i,j\}, \tag{4.9}$$

$$y_{ih} \leq y_{hh}, \qquad h = 1, \ldots, n, \; i > h, \tag{4.10}$$

$$y_{ih} \in \{0,1\}, \qquad h = 1, \ldots, n, \; i \geq h. \tag{4.11}$$

Objective function (4.7) minimizes the sum of the costs of the colors used. Constraints (4.8) and (4.9) are the counterpart of (4.4) and (4.5), respectively. Constraints (4.10) impose that a vertex i can receive a color $h \neq i$, among those available for the vertex, only if color h is used. Finally, constraints (4.11) require y to be binary.

A first observation is that, given a color h and a vertex $i > h$, the corresponding constraint (4.10) is not imposed if at least one vertex $j > h$ such that $(i,j) \in E$ exists (since (4.10) would be dominated by the corresponding constraint (4.9)).

A second observation is that the model above can be strengthened as follows. For each $h = 1, \ldots, n$, let $\mathcal{K}(h)$ be the family of the (exponentially many) maximal cliques of vertices having index larger or equal to h. Then, we can replace constraints (4.9) and (4.10) by:

$$\sum_{i \in K} y_{ih} \leq y_{hh}, \qquad h = 1, \ldots, n, \; K \in \mathcal{K}(h), \tag{4.12}$$

imposing that, for each maximal clique $K \in \mathcal{K}(h)$, at most one vertex $i \in K$ can receive color h, if such color is used.

4.2.3 Comparing Models M1 and M2

First note that model M2, defined by (4.7)-(4.11), involves a smaller number of variables than that required by model M1, defined by (4.2)-(4.6). Moreover, since each color is associated with a vertex, M2 implicitly distinguishes between different colors, breaking most of the symmetries which deteriorate M1. Hence, we may expect an enumerative algorithm based on model M2 to perform better than an enumerative algorithm based on model M1.

The main result of this section is given by the following theorem, motivating the fact that, when facing with WVCP, it is better to use model M2 than model M1, even if only a lower bound has to be computed.

Theorem 4.1. *Let z_1 be the value of optimal solution of the continuous relaxation C(M1) of model M1 and z_2 the value of the optimal solution of the continuous relaxation C(M2) of model M2. Then, $z_2 \geq z_1$.*

Proof. The proof will be given in two steps:

(a) given any optimal solution y to C(M2) we derive a feasible solution (x, z) to C(M1) having the same cost;

(b) given the solution (x, z) found in step (a) we show that in certain situations such a solution is not optimal for C(M1).

Step (a)
Let y be an optimal solution to C(M2), and define a feasible solution (x, z) for C(M1) as follows:

for $i \in V$
  for $h := 1$ **to** i
    $x_{ih} := y_{ih}$
  endfor
  for $h := i + 1$ **to** n
    $x_{ih} := 0$
  endfor
endfor
for $h := 1$ **to** n
  $z_h := \max\{w_i\ x_{ih}, i \in V\}$
endfor

Such a solution (x, z) satisfies constraints (4.3) by construction, and constraints (4.4) and (4.5) because the corresponding constraints (4.8) and (4.9) were satisfied by y as well. As to the solution value, note that, for a given color h, constraints (4.10) imply that $w_h\ y_{hh} \geq w_h\ y_{ih}$ for each $i \geq h$. Moreover, since vertices are sorted by non increasing weights, we have that $w_h\ y_{ih} \geq w_i\ y_{ih}$ for each $i \geq h$, thus $w_h\ y_{hh} = \max\{w_i\ y_{ih}, i \geq h\}$. Finally, noting that $x_{ih} = y_{ih}$ if $i \geq h$ and $x_{ih} = 0$ otherwise, we have that

$$w_h\ y_{hh} = \max\{w_i\ x_{ih}, i \in V\} = z_h \tag{4.13}$$

i.e., the contribution of color h to the objective function in C(M1) and C(M2) is the same, i.e., the cost z_2 of solution y for C(M2) is equal to the cost of solution (x, z) for C(M1).

Note that (4.13) implies that, in any feasible solution to C(M1) derived from an optimal solution to C(M2) using the procedure above, the cost of color h is due only to vertex h and, vice-versa, vertex h gives a contribution to the objective function which is equal to $w_h\ y_{hh}$.

Step (b)
Given a solution (x, z) to C(M1) derived from an optimal solution to C(M2), let j and k be two distinct colors used in this solution, i.e., such that $x_{jj} > 0$ and $x_{kk} > 0$. According to (4.13), the contributions of the two vertices to the objective function are equal to $z_j = w_j\ x_{jj}$ and to $z_k = w_k\ x_{kk}$, respectively.

Suppose that $z_j > w_k\ x_{kj}$ i.e., constraint (4.3) associated with vertex k and color j is not tight, and that all constraints (4.5) involving variable x_{kj} are not tight as well. In this case there exists $\varepsilon > 0$ such that the new solution

$$\widetilde{x}_{ih} = \begin{cases} x_{kj} + \varepsilon & \text{for } i = k \text{ and } h = j \\ x_{kk} - \varepsilon & \text{for } i = h = k \\ x_{ih} & \text{otherwise} \end{cases} \qquad (i \in V;\ h = 1, \ldots, n)$$

and

$$\widetilde{z}_h = \max\{w_i\ \widetilde{x}_{ih}, i \in V\} \quad (h = 1, \ldots, n)$$

is feasible for C(M1). Noting that $\widetilde{z}_j = z_j$ and that $\widetilde{x}_{kk} < x_{kk}$ we have that the cost of solution $(\widetilde{x}, \widetilde{z})$ is smaller than the cost of solution (x, z), when $z_k > w_i x_{ik}\ \forall i \neq k$. □

The following example shows an instance for which $z_2 > z_1$.

Example 1.
We are given the simple graph of Figure 4.1 in which the weight of each vertex is indicated in the circle after the vertex index.

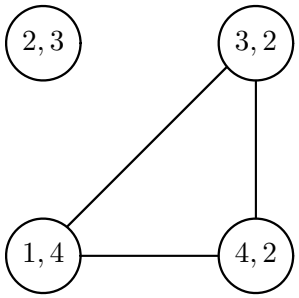


Figure 4.1: Simple graph for which the LP relaxations of M1 and M2 have different values.

The optimal solution to C(M2) is given by $y_{11} = 1, y_{21} = y_{22} = y_{32} = y_{33} = y_{42} = y_{43} = 1/2$ (the remaining variables being equal to 0), and this solution has cost equal to 13/2.

However, the optimal solution for C(M1) is $x_{11} = 1/2, x_{12} = x_{31} = x_{32} = x_{41} = x_{42} = 1/2, x_{21} = 1/3, x_{22} = 2/3, z_1 = z_2 = 2$, and has value 4.

Note that model M1 can be strengthened by additional constraints imposing that each color h, if used, can be assigned only to vertices $i \geq h$. Even adding such constraints, the LP relaxation of M1 produces a lower bound that cannot be better than the one obtained solving the LP relaxation of M2. Indeed, for the graph of Figure 4.1 the optimal solution to the strengthened LP relaxation of M1 is $x_{11} = x_{21} = 1, x_{32} = x_{33} = x_{42} = x_{43} = 1/2, z_1 = 4, z_2 = 1, z_3 = 1$, and has value 6, which is better than 4 but still lower than 13/2. □

In our algorithm (see Section 4.3) we use the value of the LP relaxation of model M2, as a lower bound on the optimal solution value for a WVCP instance. The computation of this lower bound can usually be carried out in short computing time and, in many instances, allows us to prove optimality of the best solution found (see Section 4.5), stopping the execution of the heuristic.

In the next section we present another ILP model for WVCP. Since this formulation has an exponential number of variables, even the computation of its LP relaxation can require large computing times. Hence, we use such model only as a basis to derive our heuristic algorithm.

4.2.4 Model M3

Effective models for partitioning-like problems are inspired to the famous ILP formulation proposed by Gilmore and Gomory [56, 57] for the cutting stock problem, extended to the classical VCP by Mehrotra and Trick [90].

Let $\mathcal{S}$ denote the family of all the independent sets of G and introduce, for each independent set $s \in \mathcal{S}$, a binary variable σ_s taking value 1 iff all the vertices of s receive the same color. Moreover, for each $s \in \mathcal{S}$, let $c_s = \max\{w_i : \ i \in s\}$ be the maximum weight of the vertices belonging to s, i.e., the cost to be paid if independent set s is assigned a color. We obtain the following ILP model for WVCP

$$\min \sum_{s \in S} c_s \sigma_s \tag{4.14}$$

$$\sum_{s: i \in s} \sigma_s \geq 1, \qquad i \in V, \tag{4.15}$$

$$\sigma_s \in \{0, 1\}, \qquad s \in \mathcal{S}. \tag{4.16}$$

Objective function (4.14) minimizes the sum of the costs of the selected independent sets, while constraints (4.15) impose that each vertex is assigned to (at least) one independent set (i.e., at least to one color). Note that WVCP requires that each vertex is assigned exactly to one color; however, given an independent set s and a vertex $i \in s$, also $t := s \setminus \{i\}$ is an independent set (if $t \neq \emptyset$) and $c_t \leq c_s$. Thus, given a feasible solution to (4.14)–(4.16) which selects two or more independent sets containing a given vertex $i \in V$, a feasible solution to WVCP can be obtained by removing vertex i from all the selected independent sets but one; the cost of such new solution is not bigger than the cost of the original solution. This property allows us to consider in $\mathcal{S}$ all the non dominated independent sets of G (instead of all independent sets of G), thus reducing the number of variables.

Model (4.14)–(4.16) is a *Set Covering Problem* (SCP) having a constraint for each vertex $i \in V$; hence it is often referred to in the literature as a *Set Covering Formulation*. This formulation requires to define a set of binary variables whose number is exponential in n, hence both defining all the variables (i.e., generating all the independent sets of G) and solving the associated SCP can be very large time consuming, even when column generation techniques are used. In order to produce high quality solutions to WVCP in a reasonable amount of time, we prefer not to solve exactly the model, but we use it to derive an effective heuristic algorithm. Our approach is heuristic in nature since: (i) we do not consider all the independent sets of family $\mathcal{S}$ but only those belonging to a subfamily $\mathcal{S}' \subset \mathcal{S}$, and (ii) we do not solve the associated SCP formulation exactly but through a Lagrangian-based heuristic algorithm from the literature. In the next section we present the general structure of the algorithm and provide some details of its implementation.

4.3 The 2-phase Heuristic Algorithm

We propose a 2-phase algorithm for WVCP, where a general framework of column generation (Phase 1) and column optimization (Phase 2), designed for problems with a Set Covering formulation, is integrated with fast post-optimization procedures applicable to any solution generated during the computation.

As initialization, we compute a valid lower bound for the WVCP by solving the continuous relaxation of model M2. In the first phase of the algorithm, a very large number of independent sets (*columns*) is produced. We do not use an explicit algorithm to generate the columns, but we apply in sequence some fast greedy heuristics designed for the problem, possibly considering different parameter sets. Indeed, each independent set in any heuristic solution of the original problem corresponds to a column of $\mathcal{S}$. When optimality of the incumbent

solution is not proved, such columns are stored in the family $\mathcal{S}'$, which represents a subfamily of the family $\mathcal{S}$ of all the independent sets of the graph. The second phase considers the SCP instance associated with the columns in $\mathcal{S}'$ and heuristically solves it through the Lagrangian heuristic algorithm CFT proposed by Caprara, Fischetti and Toth [23], improving many times the best incumbent solution.

This approach was successfully used by Monaci and Toth [91] to solve Bin Packing Problems, while the second phase was used as a post-optimization procedure by Malaguti, Monaci and Toth [84] in a metaheuristic approach for the classical VCP, where columns from solutions explored during the search were stored in $\mathcal{S}'$.

Clearly, the solution using the minimum number of colors (i.e., the optimal solution of VCP) is not necessarily optimal for WVCP. This suggests that, when generating columns for WVCP, we do not have to take care only of the number of vertices belonging to the associated independent sets. Indeed, we would like to generate "balanced" independent sets, i.e., we would like that vertices having high weights belong to the same independent set, provided this is feasible. Since the solution found by a greedy algorithm for WVCP depends on the order in which the vertices are given in input, to obtain a balanced structure in the columns we generate, the vertices are initially sorted according to decreasing values of the corresponding weight, thus helping the greedy heuristics to build balanced columns. In addition, the column generation phase is aimed at generating a large family of different columns. So, in our approach, the greedy procedures are applied several times, in an iterative way, locally perturbing the initial order of the vertices, so that different columns are generated. Every time a new greedy solution is generated, it can be improved by means of two fast post-optimization procedures described in Section 4.4.

Two drawbacks are evident in this approach: i) a lot of independent sets which are dominated are generated, and ii) the same independent set can be generated more than once, thus producing redundant columns. The first problem depends on the fact that we extract independent sets from feasible colorings. This drawback is solved by using a greedy procedure that, given an independent set, "completes" it so as to obtain a not dominated independent set. As to the second problem, a hashing technique is used in order to avoid to store identical columns (see Monaci and Toth [91] for more details).

Both phases can be stopped as soon as a solution which is proven to be optimal is found, i.e., if the cost of the best solution found so far is equal to a lower bound for the original problem.

The overall 2-phase algorithm is structured as follows, where *maxiter* is a parameter denoting the maximum number of executions of the greedy algorithms:

begin
Initialization Step
1. Compute a lower bound LB by solving the continuous relaxation of model M2;
2. Set $UB = \infty$; $iter = 0$;
3. Set $\mathcal{S}' = \emptyset$;
Phase 1: Column Generation
4. **while** ($iter < maxiter$)
5. Apply the greedy heuristics, possibly updating UB;
6. Apply the post-optimization procedures to each generated solution, possibly enlarging $\mathcal{S}'$ and updating UB;
7. if $LB = UB$ **stop**;

8. Locally perturb the order of the vertices;
9. $iter = iter + 1$
10. **end while**;

Phase 2: Column Optimization

11. apply heuristic algorithm CFT to the Set Covering instance corresponding to subfamily $\mathcal{S}'$ with a given time limit (possibly updating UB);
12. apply post-optimization procedures (possibly updating UB)

end.

Every new solution that is generated (step 5), is used as starting solution for the post-optimization procedures (described in section 4.4). The associated solutions are used to enlarge family $\mathcal{S}'$ (step 6). The final solution, obtained by Phase 2 in step 11, can be improved as well through the post-optimization procedures.

4.3.1 Constructive heuristics

This first phase is aimed at generating different (and possibly high quality) solutions for the problem. We use 7 different algorithms designed for WVCP, inspired from greedy algorithms proposed for VCP. As mentioned before, in order to reduce the number of high cost independent sets in the solution, it is useful to assign the vertices with high weight to the same color class. Thus, since the greedy algorithms we implemented fill each color class in sequential order of the vertices, we sort the vertices according to decreasing values of the corresponding weight. The same ordering is used for one iteration of all the greedy algorithms. Before performing a new iteration, this ordering is locally perturbed (see Section 4.5 for details). In this way the greedy algorithms are able to produce different solutions (and hence different columns) still maintaining a global ordering of the vertices.

In the generation we use a sequential coloring algorithm called SEQ and 6 versions of the DSATUR algorithm, originally proposed for the classical VCP.

The simplest greedy algorithm proposed for VCP is the SEQ algorithm [72]. It works according to a first-fit policy: vertex 1 is assigned to the first color class and each remaining vertex is assigned to the first color class where it fits. If the vertices are sorted according to decreasing weights, the color classes are filled by vertices having similar weight.

DSATUR [20, 72] is similar to SEQ, with the difference that each vertex i has associated a score $k(i)$ and, at each iteration, the vertex having the maximum score is selected and assigned to the lowest indexed color class where it fits. Ties are broken according to the order in which the vertices are considered. In the classical VCP, the score of each vertex i is given by the number of distinctly colored adjacent vertices (i.e., the chromatic degree $\delta_\chi(i)$), thus at the beginning of the computation, when $\delta_\chi(i) = 0$ for every vertex i (no vertex is colored), the first vertex in the order is considered. This rule should force "difficult" vertices to be colored at the beginning. In the generation phase, we use the original DSATUR algorithm and 4 variants giving an increasing importance to the weight w_i of vertex i in the computation of the score $k(i)$:

- $k(i) = \delta_\chi(i)$;
- $k(i) = \delta_\chi(i) \ln(1 + w_i)$;
- $k(i) = \delta_\chi(i)\sqrt{w_i}$;

- $k(i) = \delta_\chi(i) w_i$;
- $k(i) = \delta_\chi(i) w_i^2$.

The last considered variant of the DSATUR algorithm implements the original score, but it is based on the consideration that assigning vertex i to the first indexed color class where i fits is not always the best choice. Therefore, in this variant of DSATUR, vertex i is assigned to the best color class where it fits according to the following greedy rule:

- If no color class is available with a cost greater than w_i, the algorithm selects color class h such that $w_i - c_h$ is minimal;
- If one or more color classes are available with a cost greater than w_i, the algorithm selects color class h such that $c_h - w_i$ is minimal.

Every solution generated is this phase can be improved by means of the post optimization procedures described in Section 4.4.

Columns (independent sets) can then be extracted from each greedy solution. From each column, the algorithm generates two not dominated (possibly) different columns to be inserted in $\mathcal{S}'$:

- the first one with no cost increase;
- the second one with a maximum cost increase proportional to the original cost of the column.

In both cases, fixed the cost threshold, the completion to a not dominated column is performed greedily, picking vertices according to the input ordering and trying to insert them into the column, if this is feasible and if their weight does not exceed the cost threshold on the column.

As anticipated, in order to avoid the insertion in $\mathcal{S}'$ of identical columns, all the columns are checked by means of a hashing technique.

4.3.2 Column Optimization

The second phase of the algorithm is executed when the best solution found in Phase 1 is not proved to be optimal. This phase consists of the heuristic solution of the Set Covering instance corresponding to the columns stored in $\mathcal{S}'$ through the heuristic algorithm CFT [23]. This Lagrangian algorithm can handle very large Set Covering instances, producing good (possibly optimal) solutions within short computing time. In addition, algorithm CFT computes an "internal" lower bound (not valid for WVCP) on the value of the optimal solution of the corresponding Set Covering instance and its execution can be stopped as soon as this lower bound equals the value of the best incumbent solution for WVCP. It must be noted that in our approach, even when we are able to solve the Set Covering instance to optimality, this does not imply optimality for the original problem, since we do not consider the complete family S of all the not dominated independent sets of G, but only a subfamily S'.

4.4 Post-optimization procedures

We propose two different post-optimization procedures, which can be applied to any binary matrix corresponding to the columns of a feasible solution of the Set Covering Model (4.14)–(4.16). In these matrices row i corresponds to vertex i of weight w_i, and column j corresponds

to the $j-th$ color class, whose cost c_j equals the maximum weight of the vertices assigned to that color class, i.e., to the maximum weight of the rows "covered" by column j (i.e., of the rows having an entry of value 1 in column j).

4.4.1 An ILP model for matrix post-optimization

Since the (possibly optimal) solution found by solving the Set Covering instance associated with family $\mathcal{S}'$ can cover some row more than once (i.e., can associate more than one color to some vertex $i \in V$), while WVCP requires to assign one color to each vertex, the first post-optimization procedure is aimed at determining the best way to remove the "overcovered" vertices (i.e., the vertices covered more than once) from some color classes, if this reduces the corresponding cost.

More formally, given a binary matrix where some rows are covered more than once, each row i has associated a weight w_i and the cost of column j equals the maximum weight of the rows covered by column j, the problem is to delete a set of 1-entries from the overcovered rows in such a way that the cost reduction is maximized and each row is still covered. This problem is NP-complete, as discussed in the next Section.

We propose an ILP model for the solution of the problem above. The cost of a column j can be reduced only by iteratively deleting the "heaviest" 1-entry from j. Let $\overline{M} = \{1, \ldots, \overline{m}\}$ be the set of the overcovered rows and $\overline{N} = \{1, \ldots, \overline{n}\}$ be the set of the columns covering at least one overcovered row, and such that the first row covered by the column is overcovered in the original matrix (these are the only columns whose cost can be reduced). In the following we will use indices $i \in \overline{M}$, $j \in \overline{N}$ and $k = 1, \ldots, q_j$, where q_j denotes the number of 1-entries of column j (i.e., the number of rows covered by column j). Let R_j^k be the saving that can be obtained by deleting the first k 1-entries from column j, η_i the number of times row i is covered, and a_{ik}^j a coefficient equal to 1 if row i is among the first k rows covered by column j and 0 otherwise. We consider binary variables β_k^j having value 1 if the first k 1-entries are deleted from column j and 0 otherwise, obtaining the following ILP:

$$\max \sum_{j \in \overline{N}} \sum_{k=1}^{q_j} R_k^j \beta_k^j \tag{4.17}$$

$$\sum_{j \in \overline{N}} \sum_{k=1}^{q_j} a_{ik}^j \beta_k^j \leq \eta_i - 1, \qquad i \in \overline{M}, \tag{4.18}$$

$$\sum_{k=1}^{q_j} \beta_k^j \leq 1, \qquad j \in \overline{N}, \tag{4.19}$$

$$\beta_k^j \in \{0, 1\}, \qquad j \in \overline{N};\ k = 1, \ldots, q_j. \tag{4.20}$$

Objective function (4.17) maximizes the reduction of the matrix cost, constraints (4.18) impose that each row remains covered at least once, and constraints (4.19) impose that for each column j at most one variable β_k^j is set to 1, and hence that each 1-entry can be deleted at most once. Constraints (4.20) impose β variables to be integer.

This model can be used to improve the solution obtained by Phase 2, where usually some rows are covered more than once. In the matrices representing solutions obtained by the greedy algorithms considered in Phase 1, where each vertex receives exactly one color, each row is covered exactly once. These matrices can however contain dominated columns, which

can be "completed" (i.e., additional 1-entries can be inserted in the columns with no cost increase) in order to possibly re-optimize the resulting matrix. We complete columns in the following greedy way: for each row i such that its weight is the heaviest of the corresponding column j (i.e., this 1-entry determines the cost of column j), we try to insert an additional 1-entry for row i, on a column h with $c_h \geq w_i$. The post-optimization procedure could then delete the 1-entry from column j, thus reducing its cost.

Complexity

We prove that the post-optimization problem is NP-complete by polynomial reduction to the VCP. The proof holds on the fact that the VCP of a graph with stability number $\alpha < 4$ remains NP-complete. We prove in Theorem 4.2. that reducing a covering of the vertices of G to a minimal covering is NP-complete, even if the number of columns is $O(|V|^P)$, with $P \geq 3$. In Theorem 4.3. we prove that this is NP-complete even if the the number of columns is $O(|V|)$, and finally, in Theorem 4.4., we prove that the problem corresponding to the post-optimization procedure we propose is a generalization of the reduction of a cover of cardinality $O(|V|)$ to a minimal cover.

Theorem 4.2. *Let G=(V,E) be a graph and $\mathcal{S}$ a collection of independent sets of G, such that $|\mathcal{S}| = O(|V|^P)$, with $P \geq 3$, and $\bigcup_{S\in\mathcal{S}} S = V$. The problem of finding $\mathcal{S}' \subseteq \mathcal{S}$ of minimum cardinality, such that $\bigcup_{S\in\mathcal{S}'} S = V$, is NP-Complete.*

Proof. Consider a graph G with $\alpha < 4$ and $\mathcal{S}$ the collection of all the independent sets of G. Then $|\mathcal{S}| = O(|V|^3)$, and the problem of finding $\mathcal{S}'$ is the VCP of G. □

In the binary matrices we consider, we have at most $O(V)$ columns, thus we need the following:

Theorem 4.3. *Let G=(V,E) be a graph and $\mathcal{S}$ a collection of independent sets of G, such that $|\mathcal{S}| = O(|V|)$ and $\bigcup_{S\in\mathcal{S}} S = V$. The problem of finding $\mathcal{S}' \subseteq \mathcal{S}$ of minimum cardinality, such that $\bigcup_{S\in\mathcal{S}'} S = V$, is NP-Complete.*

Proof. Consider the same graph G of the proof of Theorem 4.2., and add $|\mathcal{S}|$ isolated vertices to V, thus obtaining a new vertex set V^* and a new graph G^*. Add each isolated vertex to each set in $\mathcal{S}$, thus obtaining $\mathcal{S}^*$, which is a collection of independent sets of G^* such that $|\mathcal{S}^*| < |V^*|$ and $\bigcup_{S\in\mathcal{S}^*} S = V^*$. The problem of finding $\mathcal{S}'$ of G^* is the VCP of G^* and of G. □

Theorem 4.4. *Let G=(V,E) be a graph with positive weights c_i associated to the vertices and $\mathcal{S} = \{S_1, ..., S_k\}$ a collection of independent sets of G, such that $\bigcup_{S\in\mathcal{S}} S = V$ and, $\forall S_j \in \mathcal{S}$, $\bigcup_{S\in\mathcal{S}\setminus\{S_j\}} S \subset V$ (thus, $k \leq |V|$). The problem of finding a collection $\mathcal{S}' = \{\hat{S}_1, ..., \hat{S}_k\}$, such that $\hat{S}_j \subseteq S_j$, $j = 1, ..., k$, $\bigcup_{\hat{S}\in\mathcal{S}'} \hat{S} = V$ and $\sum_{j=1}^{k} \max_{i\in\hat{S}_j}\{w_i\}$ is minimum, is NP-complete.*

Proof. Consider the same graph G^* of the proof of Theorem 4.3., and set $w_i = 1$ for $i \in V^*$. Add a clique of $|\mathcal{S}^*|$ vertices $\{v_1, \ldots, v_{|\mathcal{S}*|}\}$, not connected to vertices of G^*, and set $c_i = 0$ for $i \in \{v_1, \ldots, v_{|\mathcal{S}*|}\}$, thus obtaining G^{**}. Starting from $\mathcal{S}^* = \{S_1^*, \ldots, S_{|\mathcal{S}*|}^*\}$, define the collection $\mathcal{S}^{**} = \{S_1^* \cup \{v_1\}, \ldots, S_{|\mathcal{S}*|}^* \cup \{v_{|\mathcal{S}*|}\}\}$. Collection $\mathcal{S}^{**}$, for graph G^{**}, satisfies the conditions of Theorem 4.4. for $\mathcal{S}$. Moreover, finding $\mathcal{S}'$ is the VCP of G^* and G. □

4.4.2 An Assignment based procedure for matrix post-optimization

The second post-optimization procedure is designed for matrices (having n rows and $\overline{n}$ columns) corresponding to feasible solutions of WVCP, where each row is covered exactly once. The idea is to randomly delete a 1-entry from each of the $\overline{n}$ columns, thus uncovering $\overline{n}$ rows. Each uncovered row s can be covered again by using the original column j or a different column h (if column h still represents an independent set of graph G after the insertion of a 1-entry in row s).

More in detail, after having deleted a 1-entry from each column, let $\overline{M} = \{r_1, \ldots, r_s, \ldots, r_{\overline{n}}\}$ be the set of the uncovered rows, and c_j be the cost of column j after the deletion. We can minimize the cost of covering the rows in $\overline{M}$ by solving an Assignment Problem (AP) on a cost matrix k_{sj}. The cost k_{sj} of covering the uncovered row r_s ($s = 1, \ldots, \overline{n}$) with column j ($j = 1, \ldots, \overline{n}$), i.e., the cost of coloring vertex r_s with color j, can be computed as follows:

- $k_{sj} = \infty$ if, with the insertion of a 1-entry in row r_s, column j does not represent an independent set of graph G;
- $k_{sj} = max\{0, w_{r_s} - c_j\}$ otherwise (note that this value represents also the insertion cost of row r_s into an empty column, whose cost is 0 by definition).

The post-optimization procedure can reduce the matrix cost only when, at least in one column, the 1-entry of maximum weight is deleted, and the new maximum weight 1-entry in that column has a strictly lower weight (otherwise the cost of the column will remain unchanged). When each column has more than 1-entry of maximum weight, the matrix cost cannot be reduced by means of post-optimization procedures which remove at most one vertex at a time.

The post-optimization procedure takes in input graph $G = (V, E)$, the vertex weights (w_i) and a $(n\mathtt{x}\overline{n})$ binary matrix $(\mathcal{M}_{ij})$, corresponding to a feasible solution of WVCP, and works as follows:

begin
1. randomly delete a 1-entry r_j from each column j ($j = 1, \ldots, \overline{n}$) of matrix $\mathcal{M}_{ij}$ (let c_j be the cost of column j after the deletion);
2. compute the cost reduction *c_red* of the new matrix with respect to the original one;
3. **for** $s = 1, \ldots, \overline{n}$
4. **for** $j = 1, \ldots, \overline{n}$
5. **if** $\exists l \in V : \mathcal{M}_{lj} = 1$ **and** $(r_s, l) \in E$ **then** $k_{sj} := \infty$
6. **else** $k_{sj} := max\{0, w_{r_s} - c_j\}$
7. **end for**
8. **end for**
9. **for** $j = 1, \ldots, \overline{n} + 1$
10. $k_{n+1,j} := 0$
11. **end for**
12. **for** $s = 1, \ldots, \overline{n}$
13. $k_{s,n+1} := w_{r_s}$
14. **end for**
15. solve the Assignment Problem $AP(K)$ corresponding to the $(\overline{n}+1)\mathtt{x}(\overline{n}+1)$ cost matrix k_{sj};
16. **if** $c\_red - cost(AP(K)) > 0$ then update the solution matrix according to the solution of $AP(K)$

end

On line 5 we set the cost k_{sj} to ∞ when the assignment of row r_s to column j cannot be performed without violating any adjacency constraint on G (i.e., a constraint imposing that the columns must represent independent sets of G), otherwise we compute the correct cost on line 6. We wish to allow a 1-entry to move to a new empty column having current cost 0, when this can reduce the global cost of the solution matrix, so we consider an AP associated with an $(\overline{n}+1)\mathtt{x}(\overline{n}+1)$ cost matrix where the last column of the cost matrix equals the row weights (cost to be paid for the use of a new empty column) and the last row contains values equal to 0 (cost corresponding to the assignment of no 1-entry to a column, line 10). The AP has always a trivial solution of value *c_red*, consisting of the reassignment of the deleted 1-entries to their original columns, but it can have better optimal solutions.

This post-optimization procedure can be used to improve all the greedy solutions generated in Phase 1, where each row is covered exactly once. It can also be applied to the solution matrix found by Phase 2, after the application of the post-optimization procedure described in Section 4.4.1.

4.5 Computational Analysis

The post-optimization procedures were coded in ANSI C and compiled with full optimization option; all the other procedures, including algorithm CFT [23], were coded in ANSI FORTRAN77 and compiled with full optimization option. The LP relaxation of the ILP model M2 proposed in Section 4.2.2 was solved with CPLEX 9.0. The programs were run on a PIV 2.4MHz with 512MB RAM under Windows XP.

We tested our codes on two different sets of instances: the first set was proposed during a computational symposium on VCP and its generalizations [106], held in 2002, when weights were added to vertices of the original *DIMACS benchmark graph instances* [73]. These instances correspond to different graph types and have been used for evaluating the performance of VCP algorithms. In our experiments we considered all the random graphs (DSJC$n.x$), a subset of geometric random graphs (Geom$n$) (the smallest ones were disregarded) and all the "quasi-random" graphs (Rn_x), where n denotes the number of vertices and $x/10$ is the density of the graph. Our second set of instances is composed by problems from Traffic Matrices associated to TDMA Traffic Assignment, proposed by Ribeiro, Minoux and Penna [99] and by Prais and Ribeiro [94], and available for download at [3].

All the computing times of the experiments reported in this Section are expressed in seconds of a PIV 2.4GHz. To allow a meaningful - although approximate - comparison of the computing times obtained on coloring problems with different machines, a benchmark program (dfmax), together with a benchmark instance (r500.5), are available. Computing times obtained on different machines can be scaled w.r.t. the performance obtained on this program (our machine spent 7 seconds "user time").

4.5.1 Weighted Vertex Coloring Instances

In this Section we report the results obtained on 46 instances from [106]. No computational experience on these instances is reported in the literature.

In order to derive a lower bound on the optimal value of each instance, we solve the LP relaxation of ILP model M2 proposed in Section 4.2.2. We strengthen the model by replacing the incompatibility constraints (4.9) between adjacent vertices with a set of clique constraints (4.12), detecting cliques $K \in \mathcal{K}(h)$ in the following heuristic way: for each color h and for each edge (i, j) such that $i, j \geq h$, we compute a maximal clique K containing i, j and vertices having index larger or equal to h. This clique is initialized with vertices i and j, and then enlarged by iteratively adding new vertices, at each iteration considering the remaining vertices according to a non increasing degree ordering, and adding to the clique the first vertex which can fit (this simple greedy algorithm for the computation of a maximal clique is derived from the algorithm proposed by Johnson in [71]). The LP relaxation of the resulting model, involving $O(n.m)$ clique constraints, was solved with CPLEX 9.0 by using the barrier algorithm with a time limit of 30 seconds. The choice of the barrier algorithm is due to its performance on the set of instances considered, outperforming the primal and dual simplex algorithms. For 44 instances over 46 (see above) the LP relaxation could be solved within the time limit, providing a valid lower bound for the problem; for the remaining two instances, the LP relaxation was not solved to optimality within the time limit, hence no lower bound is available for these instances.

This lower bound LB, when successfully computed, is passed to the subsequent 2-phase heuristic algorithm of Section 4.3, whose execution is stopped as soon as an incumbent solution of value LB (i.e., an optimal solution) is found. All the results concerning the heuristic algorithm are obtained by applying in sequence the ILP based post-optimization and the Assignment based post-optimization procedures to all the greedy solutions generated during Phase 1, and to the final solution obtained by Phase 2. Due to the small size of the associated problems, the ILPs (4.17)-(4.20) can be solved to optimality within short computing times by CPLEX 9.0. Since the solutions found by the Assignment based procedure depend on the vertices (rows) randomly deleted from the columns corresponding to the current solution, we apply, for 10 times, the procedure to all the solutions generated during the computation, after the ILP-based procedure, by increasing at every iteration the probability that the first vertex of each column is deleted. At the first iteration the Assignment based procedure takes in input the output of the ILP-based procedure. The solution produced is then used as input for its next iteration, according to a local search paradigm. Considering the n_j vertices covered by a given column j according to non increasing weights, the probability p_i of the $i-th$ vertex to be deleted from column j is given by:

$$p_1 = (iter + 1)/(iter + n_j) \tag{4.21}$$

$$p_i = 1/(iter + n_j) \qquad for\ i = 2, \ldots, n_j \tag{4.22}$$

where $iter = 1, \ldots, 10$ represents the current iteration of the procedure. The Assignment Problem is solved at every iteration by means of the Hungarian Algorithm, with the implementation described by Carpaneto, Martello and Toth [28].

During Phase 1, several executions of each greedy algorithm are performed: at each execution, a small perturbation of the input ordering of the vertices is performed. The algorithm is able to generate different columns, and it is stopped as soon as 50 iterations are performed without generating new columns. Our experiments showed that using a stronger perturbation of the input order would allow the algorithm to generate many more different columns, but the quality of the final solution (i.e., the solution after phases 1 and 2) would be worse. The reason is that too many "bad" (i.e., not balanced) columns would be generated,

thus leading to large Set Covering instances, where it would be difficult, during Phase 2, to find a good solution. This suggests that it is better to generate few good columns than a large number of columns of low quality. The perturbation is obtained by swapping two vertices in the current ordering with a probability $p = 40\%$. As for the completion of the columns to not dominated columns, the algorithm generates, from each column, two not dominated (possibly) different columns: the first one with no cost increase, the second one with a maximum cost increase Δ_c, with $\Delta_c = 1.5\sigma$, where σ represents the standard deviation of the weights of the vertices of graph G.

Table 4.1 reports the computational results obtained by stopping Phase 1 after 500 iterations and Phase 2 after 75 seconds. For each instance, the following information are given:

- the instance name, number of vertices (n) and of edges (m);
- the lower bound (LB), corresponding to the rounded up solution value of the LP relaxation described above, and the total time (T_{LB}) required for the computation of this lower bound, including the heuristic generation of the maximal cliques for constraints (4.12) - for this reason the time limit of 30 seconds can be exceeded;
- the number of different columns ($\#col$) generated during Phase 1;
- the best solution value found during Phase 1 (z_1) and after Phase 2 (z_2), respectively (the latter is reported only when improved with respect to the former);
- the computing time of Phase 1 (T_1), corresponding to the column generation and the post-optimization of every generated solution, and the computing time of Phase 2 (T_2), including the post-optimization of the final solution;
- the improvements on the best solution values, obtained by applying the post-optimization procedures, with respect to the best solution values which can be obtained without the post-optimization procedures, after Phase 1 (Δ_1), and at the end of the computation (Δ_2).

Finally, the last row of the table ($\sum$) reports the sum of the corresponding computing times.

The table shows that, over the 46 instances of our test bed, 12 can be solved to proven optimality during Phase 1, confirming the effectiveness of the constructive heuristics proposed in Section 4.3.1 (optimal solutions are reported in bold). On the remaining 34 instances, the second phase of the algorithm was able to improve the incumbent solution 17 times, finding a proven optimal solution for 10 additional instances. In addition, we ran the ILP model M2 with a very long time limit on the 24 instances for which optimality was not proven, finding that 6 additional solutions (marked with a $*$ in Table 4.1) are actually optimal, even if the algorithm is not able to prove it. The post optimization procedures are quite effective, improving 8 times the incumbent solution after Phase 1 and 7 times the final solution after the two phases. E.g., for instance R50_9gb, $\Delta_1 = 7$ and $\Delta_2 = 1$ means that without the post optimization procedures, the output of Phase 1 would have been 269 (instead of 262), and that the output after Phase 2 would have been 263 (instead of 262). In other words, Phase 2 would have improved the solution from 269 to 263, while the use of the post optimizations leads to a solution of 262 after Phase 1, without further improvement on Phase 2. The only instance for which the post-optimization procedures worsen the solution is DSJC125_5gb,

Table 4.1: Results on instances derived from DIMACS instances.

Instance	n	m	LB	T_{LB}	$\#col$	z_1	z_2	T_1	T_2	Δ_1	Δ_2
DSJC125_1g	125	736	19	27	27471	24		82	70		
DSJC125_1gb	125	736	74	26	15381	95		100	70		
DSJC125_5g	125	3891		34	19540	76		110	70		
DSJC125_5gb	125	3891		35	14540	260	251	112	70		-1
DSJC125_9g	125	6961	152	55	665	173	*169	157	5	1	
DSJC125_9gb	125	6961	568	42	744	628	605	167	70	5	
GEOM30b	30	111	12	6	12	**12**		0	0		
GEOM40b	40	197	16	0	166	**16**		1	0		
GEOM50b	50	299	18	0	26	**18**		0	0		
GEOM60b	60	426	23	0	52	**23**		0	0		
GEOM70	70	337	47	1	11502	48	**47**	96	0		
GEOM70a	70	529	73	0	1325	**73**		3	0		
GEOM70b	70	558	24	1	2459	**24**		6	0		
GEOM80	80	429	66	2	75	**66**		0	0		
GEOM80a	80	692	76	3	16232	78	**76**	100	2		
GEOM80b	80	743	27	3	24005	28	**27**	88	2		
GEOM90	90	531	61	3	18009	62	**61**	97	69		
GEOM90a	90	879	73	4	17094	75	**73**	101	56		1
GEOM90b	90	950	30	5	5424	**30**		11	0		
GEOM100	100	647	65	4	20907	66	**65**	100	31		
GEOM100a	100	1092	89	7	23982	91	**89**	104	8		
GEOM100b	100	1150	32	7	7767	**32**		15	0		
GEOM110	110	748	66	7	23641	69		102	70		
GEOM110a	110	1317	97	12	25172	102	**97**	106	5		2
GEOM110b	110	1366	37	19	3024	**37**		5	0		
GEOM120	120	893	72	8	32198	73	**72**	104	53		1
GEOM120a	120	1554	105	10	34597	107	**105**	108	28		
GEOM120b	120	1611	35	28	9615	**35**		14	0	1	
R50_1g	50	108	14	0	49	**14**		0	0		
R50_1gb	50	108	52	0	759	54	*53	94	1		
R50_5g	50	612	35	1	984	*37		97	70		
R50_5gb	50	612	126	1	922	137		75	70	1	
R50_9g	50	1092	73	1	134	*74		36	0	1	
R50_9gb	50	1092	257	1	148	*262		33	0	7	1
R75_1g	70	251	17	4	10800	19		84	70		
R75_1gb	75	251	63	4	6509	72		96	70	1	1
R75_5g	75	1407	43	6	4024	53		102	70		
R75_5gb	75	1407	160	6	3653	199	190	103	70		6
R75_9g	75	2513	108	3	249	*110		79	0		
R75_9gb	75	2513	393	3	295	399		50	0	12	1
R100_1g	100	509	18	13	19957	22		85	70		
R100_1gb	100	509	70	16	9135	84		101	70		
R100_5g	100	2456	48	28	11270	62		109	70		
R100_5gb	100	2456	182	32	9332	234		109	70		
R100_9g	100	4438	138	11	408	143	142	53	70		
R100_9gb	100	4438	499	11	501	529	520	120	7		
$\sum$				490				3322	1458		

where the final solution after Phase 2 is 251, while it could be 250 without the use of the procedures. This can be explained by the different columns that are considered for the Set Covering instance, leading the heuristic search of algorithm CFT to a slightly worse solution.

4.5.2 Traffic Decomposition Matrix Instances

In this Section we report the results obtained on instances of the Matrix Decomposition Problem in TDMA Traffic Assignment.

These traffic matrices have to be decomposed into mode matrices (i.e., matrices where no more than one non zero element is present in each row and column), in such a way that the sum of the costs of all the mode matrices is minimized, where the cost of each mode matrix equals the maximum of its non zero elements. We can transform each instance of the problem into a WVCP instance, by constructing the corresponding graph G, where every non zero element corresponds to a vertex of the same weight and vertices in each row and column are grouped into cliques of G. We take no advantage from the knowledge of the particular structure of the corresponding graph, and apply the general 2-phase algorithm to these special structured graphs, with a proper tuning of the parameters.

Two subsets of instances were proposed in the literature: the p.n instances (having up to 138 vertices and 1186 edges) that were solved to optimality by Ribeiro, Minoux and Penna [99], and heuristically by means of a GRASP algorithm by Prais and Ribeiro [94], and the R.n instances (having up to 301 vertices and 4122 edges) that were heuristically solved by means of the GRASP algorithm [94].

For each of the 35 p.n instances, Table 4.2 reports information similar to those given in Table 4.1; in addition, the last two columns of the table give the solution value (z_{PR}) found by Prais and Ribeiro [94] and the corresponding computing time (T_{PR}) on a IBM 9672 model R34 mainframe computer (note that instances p37 and p39 are missing from the instance set). The results of Table 4.2 have been obtained without imposing a time limit in the computation of the LP relaxation of model M2 (the corresponding computing time was always negligible), using no post optimization procedures, performing at most 50 iterations of each greedy algorithm and stopping Phase 2 after a time limit of 1 second.

Further, we applied the Assignment based post optimization procedure, as explained in the previous Section, to the two instances not solved to optimality, namely instances p13 and p42. The last two rows of Table 4.2 report the corresponding results. The optimal solution of instance p13 was obtained by using the standard set up of the parameters for this set of instances, while the optimal solution of instance p42 was obtained by performing, during Phase 1, 500 iterations of the greedy algorithms with a different perturbation of the vertices, and a longer time limit (6 sec.) in Phase 2, thus it is marked with a star $\star$.

The 2-phase Algorithm without the post-optimization procedures is able to find the optimal solution on 33 over 35 instances, outperforming on this set of instances the approach proposed in [94], that finds the optimal solution on 31 instances over 35. The computing times are roughly comparable if we consider that our machine is from 40 to 60 times faster [42]. In addition, our approach is more general since it was not designed by considering the special structure of the p.n instances. When we add the Assignment based post-optimization procedure, the 2-phase Algorithm is able to find the optimal solution on 34 instances over 35, and on all the 35 with an ad hoc setting of the parameters for instance p42.

We also considered the 30 R.n instances, proposed in [94]. These instances have the same structure as the p.n instances, but are larger. On these instances our approach does not

Table 4.2: Results on Traffic Decomposition Matrix Instances.

Instance	n	m	LB	T_{LB}	$\#col$	z_1	z_2	T	z_{PR}	T_{PR}
p06	16	38	565	0.0	61	**565**		0.4	**565**	1.1
p07	24	92	3771	0.0	79	**3771**		0.1	**3771**	4.0
p08	24	92	4049	0.0	179	4074	**4049**	1.3	**4049**	1.3
p09	25	100	3388	0.0	36	**3388**		0.1	**3388**	3.0
p10	16	32	3983	0.0	0	**3983**		0.1	**3983**	4.5
p11	18	48	3380	0.0	0	**3380**		0.1	**3380**	4.7
p12	26	90	657	0.0	0	**657**		0.1	**657**	3.8
p13	34	160	3220	0.1	673	3255	3225	2.3	3230	7.8
p14	31	110	3157	0.0	85	**3157**		0.1	**3157**	10.1
p15	34	136	341	0.1	101	**341**		0.1	**341**	4.7
p16	34	134	2343	0.1	414	**2343**		0.5	**2343**	14.5
p17	37	161	3281	0.1	876	3334	**3281**	1.4	**3281**	5.5
p18	35	143	3228	0.1	160	**3228**		0.2	**3228**	10.4
p19	36	156	3710	0.1	0	**3710**		0.1	**3710**	14.6
p20	37	142	1830	0.1	914	1860	**1830**	1.3	1860	20.0
p21	38	155	3660	0.1	164	**3660**		0.2	**3660**	18.4
p22	38	154	1912	0.1	232	**1912**		0.2	**1912**	20.0
p23	44	204	3770	0.1	1357	3790	**3770**	1.4	3810	21.4
p24	34	104	661	0.1	0	**661**		0.1	**661**	27.9
p25	36	120	504	0.1	0	**504**		0.1	**504**	23.9
p26	37	131	520	0.1	0	**520**		0.1	**520**	28.3
p27	44	174	216	0.1	259	**216**		0.2	**216**	7.8
p28	44	164	1729	0.1	0	**1729**		0.1	**1729**	44.5
p29	53	254	3470	0.1	0	**3470**		0.1	**3470**	65.7
p30	60	317	4891	0.2	2594	4902	**4891**	2.1	**4891**	56.6
p31	47	179	620	0.1	0	**620**		0.1	**620**	70.9
p32	51	211	2480	0.1	0	**2480**		0.0	**2480**	70.9
p33	56	258	3018	0.2	0	**3018**		0.1	**3018**	62.3
p34	74	421	1980	0.3	171	**1980**		0.1	**1980**	131.9
p35	86	566	2140	0.5	183	**2140**		0.1	**2140**	135.0
p36	101	798	7210	0.9	48	**7210**		0.1	**7210**	163.1
p38	94	537	2130	0.6	972	**2130**		0.4	**2130**	70.5
p40	86	497	4879	0.5	439	**4984**		0.2	**4984**	224.2
p41	116	900	2688	1.3	351	**2688**		0.1	**2688**	313.7
p42	138	1186	2466	2.1	5314	2509		2.8	2480	405.8
p13.col	34	160	3220	0.1	664	3255	**3220**	8.7	3230	7.8
⋆p42.col	138	1186	2466	2.1	20191	2503	**2466**	99.8	2480	405.8

work well, because too many columns must be generated during Phase 1 in order to set up a "good" subfamily $\mathcal{S}'$. In addition, Phase 2 is able to improve on the best greedy solutions only in 3 cases over 30 (the Assignment based post-optimization procedure being used). So we were able to find (in comparable computing times) only 3 times a solution better than the one reported in [94] (and two of these improvements correspond to the instances where Phase 2 is effective), 9 times a solution of the same value and in the remaining 18 cases a worse solution.

4.6 Conclusions

This paper presents 3 ILP models and a 2-phase heuristic algorithm for a Weighted Vertex Coloring Problem, for which there is theoretical interest but lack of computational work. In particular, the second model is aimed at breaking symmetries appearing into classical VCP models, and its continuous relaxation yields a good lower bound on the optimal solution of the problem. The heuristic algorithm is based on the Set Covering formulation of the problem, in which variables correspond to independent sets. In the first phase a large number of feasible solutions is generated by means of fast greedy heuristics designed for the problem; every generated solution can be improved by using two post optimization procedures, the first based on the optimal solution of an ILP model, the second on the solution of an Assignment Problem. In the second phase an associated set-covering instance is solved by using a Lagrangian-based heuristic algorithm from the literature.

Extensive computational results on two sets of instances from the literature are reported, the first set being composed of general graphs and the second set of graphs obtained from Traffic Matrices from TDMA Traffic Assignment. The reported computational experiments show the effectiveness of the approach, where each component of the Algorithm is able to improve the best incumbent solution, and the lower bound is able to prove the optimality of many solutions. On a class of the second set of instances, the 2-phase algorithm outperforms a specialized algorithm, that was specifically designed by considering the particular structure of the instances.

The two main contributions of this paper are represented by the definition of model M2 and by the two post optimization procedures. In particular, these procedures, as well as the overall 2-phase approach, can be applied to combinatorial optimization problems that can be formulated as a Set Covering Problem, and where the costs have the special structure of the considered problem, see e.g. the *Level Strip Packing Problem with Shelves* (Lodi, Martello and Vigo [83]).

In future work, we plan to better explore the properties of model M2, and to implement the LP relaxation of the Set Covering model M3 in a column generation framework, in order to design effective exact algorithms. In addition, we would like to apply the 2-phase algorithm to different problems with a Set Covering formulation and the same cost structure of the columns. It would be also interesting to solve the second phase (column optimization) by using a different approach to the Set Covering Problem, in order to improve the results on some instances where the Lagrangian heuristic algorithm we have used has a not satisfactory performance.

Chapter 5

Lower and Upper Bounds for the Bounded Vertex Coloring Problem

[1]

We address a particular Vertex Coloring Problem, where each vertex i has associated a positive weight w_i, and the sum of the weights of the vertices assigned to the same color cannot exceed a given capacity C. The problem, known as the Bounded Vertex Coloring Problem or Bin Packing Problem with Conflicts, is of practical and theoretical interest, because of its many real-world applications and because it generalizes both the Bin Packing Problem and the Vertex Coloring Problem. We present new lower and upper bounds and investigate their behavior by means of computational results on benchmark instances.

5.1 Introduction

This paper considers a *Bounded Vertex Coloring Problem*, where, given a graph $G = (V, E)$, each vertex i has associated a positive weight w_i. As in VCP, one is required to assign a color to each vertex in such a way that adjacent vertices have different colors, and the total number of colors is minimized. The problem asks to satisfy an additional constraint: the total capacity of colors is bounded, i.e. the sum of the weights of the vertices assigned to the same color cannot exceed a given capacity C, which is the same for all colors.

The same problem can be interpreted as a *Bin Packing Problem with Conflicts* (BPPC), in which we are given n items i with weight w_i and an infinite number of identical bins of capacity c. Two items i and j are said to be conflicting if and only if they cannot be assigned to the same bin. The aim of the problem is to assign all items in the minimum number of bins, while ensuring that the total weight of all items assigned to a bin does not exceed the weight capacity and that no bin contains conflicting items. Since in the following we will use and extend some results from the Bin Packing literature, we will use the Bin Packing terminology, thus considering "items and bins", and V will be the set of items. However, this terms are perfectly interchangeable with "vertices and colors", if the reader is more comfortable in a VCP setting.

Thus, the problem we tackle is a BPP with incompatibilities described by a conflicting graph $G = (V, E)$, where each item corresponds to a vertex, and two vertices are connected

[1] Preliminary results of this chapter appear in [47].

by an edge when the corresponding items are pairwise incompatible. Note that this graph does describe the incompatibilities due to the capacity constraints. In other words, if two items, when considered together, exceed the bin capacity, there is not necessarily an edge connecting them in G. In addition to graph G, we consider graph $G' = (V, E')$, with $E' = E \cup \{(i,j) \in E : w_i + w_j > C\}$. Incompatibility graph G' describes all the incompatibilities of the problem.

This problem generalizes both the classical VCP, which is a special case of BPPC where $w_j = 0$ for $j = 1, \ldots, n$; as well as the *Bin Packing Problem* (BPP, see Martello and Toth [88]), which is a special case of the BPPC where no item is in conflict with another item, i.e., $E = \emptyset$;

Some well-known applications of the problem concern examination scheduling (see, Laporte and Desroches [78]), the assignment of processes to processors and the load balancing of tasks in parallel computing (see Jansen [66]), and particular delivery problems where some substances cannot be placed in the same vehicle (see Christofides, Mingozzi and Toth [31]). A similar problem, where a number of conflicting examinations have to be scheduled in the smallest number of periods, under capacity restrictions, was considered by Carter, Laporte and Lee in [29], where greedy heuristics are reported.

The Bounded Vertex Coloring Problem with all weights equal to 1 was considered by Hansen, Hertz and Kuplinsky in [60], where upper and lower bounds, as well as complexity results are given. Jansen and Oehring [67] and Jansen [66] have shown that, for general graph, no polynomial time approximation schemes can be derived, while they derive polynomial time approximation schemes for special classes of graphs. Baker and Coffman [11] derived similar results for the problem where vertices weights can be larger that one, while Bodlaender and Jansen [17] considered the complexity of the problem for different conflict graph classes.

The most relevant computational work on BPPC was provided by Gendreau, Laporte and Semet [55]. In [55] the authors survey the previous results in the literature of the BPPC, present new lower and upper bounds and introduce benchmark instances used to test the behavior of the algorithms.

This paper is aimed at providing new lower and upper bounds for the BPPC and to provide their efficacy be extensive computational results. In Section 5.3 we survey lower bounds from the literature and introduce new ones. In Section 5.4 we present several upper bounds, both new and from the literature, and adapt to the BPPC the metaheuristic developed for the VCP by Malaguti, Monaci and Toth [84]. In Section 5.5 we give the computational results obtained.

5.2 ILP model

The BPPC can be modelled (see Gendreau, Laporte and Semet [55]) by introducing the following variables:

$$y_h = \begin{cases} 1 & \text{if bin } h \text{ is used} \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

for $h = 1, \ldots, n$, and

$$x_{ih} = \begin{cases} 1 & \text{if item } i \text{ is assigned to bin } h \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

for $i = 1, \ldots, n$, $h = 1, \ldots, n$. We obtain the following model:

$$\min \sum_{h=1}^{n} y_h \tag{5.3}$$

$$\sum_{h=1}^{n} x_{ih} = 1 \qquad i = 1, \ldots, n \tag{5.4}$$

$$\sum_{i=1}^{n} w_i x_{ih} \leq C y_h \qquad h = 1, \ldots, n \tag{5.5}$$

$$x_{ih} + x_{jh} \leq 1 \quad \forall (i,j) \in E,\ h = 1, \ldots, n \tag{5.6}$$

$$y_h \in \{0,1\} \qquad h = 1, \ldots, n \tag{5.7}$$

$$x_{ih} \in \{0,1\} \quad i = 1, \ldots, n,\ h = 1, \ldots, n \tag{5.8}$$

Objective function (5.3) minimized the number of bins (resp. colors) used. Constraints (5.4) require that each item is assigned to a bin (resp., each vertex is colored). Constraint (5.5) are the classical capacity constraints of Bin Packing, while (5.6) are the classical incompatibility constraints of VCP, which are here reported as edge constraints, but can be strengthened to clique constraints, like discussed, e.g., in [85]. Finally, (5.7) and (5.8) impose the integrality of used variables.

5.3 Lower Bounds

We first note that any lower bound for the BPP is a valid lower bound also for BPPC. Analogously, any lower bound for the VCP is also a lower bound for the BPPC. We thus implemented the following procedures derived from the BPP literature:

- L^0_{BPP}: continuous lower bound for the BPP, computed as the rounding up of the sum of the weights divided by the bin capacity:

$$L^0_{BPP} = \lceil \sum_{i=1}^{n} w_i / C \rceil \tag{5.9}$$

- L^1_{BPP}: improved bound on the BPP, denoted as L_2 in Martello and Toth [88], based on the partitioning of items into sets according to weight thresholds: given $0 < \alpha < C/2$ and $I_1 = \{i : w_i > C - \alpha\}$, $I_2 = \{i : C/2 < w_i \leq C - \alpha\}$, $I_3 = \{i : \alpha < w_i \leq C/2\}$,

$$L^1_{BPP} = |I_1| + |I_2| + \max(0, \lceil \frac{\sum_{i \in I_3} w_i - (C|I_2| - \sum_{i \in I_2} w_i)}{C} \rceil) \tag{5.10}$$

In addition, since the problem is a generalization of the VCP, we implemented the following lower bound based on the computation of a maximal clique:

- L_{MC}: compute a maximal clique on the incompatibility graph $G = (V, E)$ through the greedy heuristic of Johnson [71]. Initialize the clique with the vertex i of maximum cardinality. Then enlarge it by iteratively adding new vertices, at each iteration considering remaining vertices according to a non increasing degree ordering, and adding to the clique the first which can fit. In [55] Gendreau et al. propose to compute the clique on graph G' instead of graph G. Even if any maximum clique of G is by construction included in a maximum clique of G', computing a maximal clique of G' by means of the Johnson's algorithm can result in disappointing results. Actually, the drawback of this strategy is that vertices of high weight result systematically in high degree vertices, even when they are not included in large cliques of G'.

  We propose the following improvement to the computation of L_{MC}, and call it L'_{MC}: compute a maximal clique K on G by means of Johnson's algorithm, and then add to E all edges (i, j) such that $w_i + w_j > C$, thus obtaining graph G'. Then, expand K to K', which is a maximal clique of G'. By construction, cardinality of K' is larger or equal than cardinality of K, and, on average, (see Section 5.5), this strategy leads to larger cliques than the use of Johnson's algorithm directly on graph G'.

Now let us consider a lower bound explicitly developed for the BPPC.

- The *constrained packing lower bound* (L_{CP}) was introduced by Gendreau, Laporte and Semet in [55]. A maximal clique of graph G' is computed by means of Johnson's algorithm. A bin is used for each of the clique vertices, and then items that can fit in these bins are inserted, possibly in a fractional way, but by satisfying compatibility constraints (this requires the solution of a transportation problem, see [55] for further details). All items (integer or fractional) than can not fit in the clique bins are stored in a different set N_2, on which a continuous relaxation based bound L^0_{BPP} is computed. A valid lower bound for the problem is then $LB_{CP} = |K| + L^0_{BPP}(N_2)$.

  We improved this bound in the maximal clique computation, computing L'_{MC} instead of L_{MC}, and obtaining bound L'_{CP}.

In the next Sections we present new lower bounds for the BPPC.

5.3.1 A surrogate relaxation

Let us consider model (5.3)–(5.8). We first note that constraints (5.6) can be lifted in the following way:

$$x_{ih} + x_{jh} \leq y_h \quad \forall (i, j) \in E,\ h = 1, \ldots, n \tag{5.11}$$

since if bin h is not used, then both variables take value 0. We now study a possible surrogate relaxation of constraints (5.11). We sum the original $n|E|$ constraints over the edges, with all multipliers equal to 1, and we obtain the following n constraints, where δ_i denotes the degree of vertex i:

$$\sum_{i=1}^{n} \delta_i x_{ih} \leq |E| y_h \quad \forall h = 1, \ldots, n \tag{5.12}$$

Now note that model (5.3), (5.4), (5.5), (5.12), (5.7) and (5.8) represents what is known in the literature as *Two-Vector Packing Problem* (TVPP), which is a generalization of the BPP in which two dimensions and two capacity constraints are present. The TVPP is NP-complete, and was solved to optimality, with Branch and Bound techniques by Spieksma [104]. However, it is not our objective to solve it to optimality, but refer to lower bounds from the literature. We thus obtain the following procedure:

- L_{TVPP}: relax in a surrogate way the constraints of the BPPC so as to obtain a TVPP. Apply procedure L_c defined in Caprara and Toth [24] to the TVPP instance obtained so as to get to a valid BPPC lower bound.

5.3.2 A Lower Bound based on Matching

Consider an instance of BPPC with 5 items, having the following weights: $\{1, 1, 5, 5, 5\}$, and bins of capacity 10. The only edge of the incompatibility graph G' connects the two items of weight 1. For this instance, $L^0_{BPP} = L^1_{BPP} = LB_{CP} = LB'_{CP} = L_{TVPP} = 2$, when clearly 3 bins are needed. Actually this is a set of items that are pairwise compatible (with the exception of the two items of weight 1), but no subset of cardinality tree of the items can fit in one bin. We say that these items are incompatible 3 by 3. In general, given a set S of items with the property that at most two of them can fit in one bin, the best way to solve the corresponding BPPC is actually to match as many items as possible, by satisfying the incompatibilities of graph G', and use 1 bin for all matched pairs, and 1 bin for items that could not be matched. In other words, we can compute a valid lower bound as the cardinality of a maximum cardinality matching M on the complement graph $\overline{G'}$ of G', plus the number of unmatched items, as follows:

$$L_{match} = |M| + (|S| - 2|M|) = |S| - |M| \tag{5.13}$$

The computation of the set S, which is in general a subset of the n items, having the property that its elements are incompatible 3 by 3, is far from trivial. We want the set S to be inclusion maximal, i.e. such that, $\forall$ *item* $j \notin S$, the property does not hold for $j \cup S$. In addition, we may want S to be the maximum cardinality set of items incompatible 3 by 3. The computation of the maximum cardinality set with the desired property, on a general graph, is NP-hard, and however there is not guarantee that the maximum cardinality set S would lead to the largest value of the bound L_{match}, which depends on the cardinality of the matching.

Thus, we compute set S by combining items of large weight and items which are included in a maximal clique K in a greedy way. We define a parameter β such that $0 \leq \beta \leq 1/2C$ and it exists at least one item having weight β, and for each value of β we compute a corresponding set S_β. Then, we compute the lower bound L_{match} for each S_β and we keep the largest:

1. $L_{match} = 0$;

2. order items i according to not increasing weight;
3. **for each** β such that: $\exists i : w_i = \beta,\ 0 \leq \beta \leq 1/2$
4. $S_\beta := \{i \in K,\ w_i \geq \beta\}$;
5. **for** $i = 1, \ldots, n,\ i \notin S_\beta$
6. if S_β is a set of items incompatible 3 by 3 after the insertion of i
7. $S_\beta := S_\beta \cup \{i\}$.
8. **end for**
9. **if** $L_{match} < |S_\beta| - |M_\beta|$
10. $L_{match} := |S_\beta| - |M_\beta|$;
11. **end for**

5.4 Upper Bounds

We first adapt to the BPPC heuristic algorithms originally proposed for the BPP. Let us order the items according to non-increasing weight. We obtain:

- First-Fit Decreasing with Conflicts (FFC), adaptation of the FFD algorithm for Bin Packing: consider each item in turn and assign it to the first bin in which there is sufficient residual capacity and for which there is no conflict with the already assigned items;
- Best-Fit Decreasing with Conflicts (BFC): as in FFDC, but the items is assigned to the bin for which there is enough residual capacity, no conflicts, and for which the resulting total sum of the weight is maximum;
- Worst-Fit Decreasing with Conflicts (WFC): as in BFDC, but resulting total sum of the weight must be minimum.

Obviously the previous algorithms are direct adaptation to the BPPC of the well-known heuristics by Johnson [70] (see also Coffman, Garey and Johnson [33] for heuristic algorithms for the BPP). Let us define FD as the best solution value found by the previous three algorithms.

An improved class of heuristics can be obtained by considering the following surrogate weight:

$$w_i^s = \alpha \frac{w_i}{C} + (1 - \alpha) \frac{\delta_i}{|E|} \tag{5.14}$$

for $i = 1, \ldots, n$, with $0 \leq \alpha \leq 1$ and δ_i is equal to the number of incident edges in G' (i.e., the number of incompatibilities of i). Now we can order the items according to non-increasing w_i^s and execute the three previous heuristics. According to computational evidence we set $\alpha = 0, 0.1, \ldots, 1$, running 11 times each heuristic. Let us define FFC-α, BFC-α, WFC-α the solutions obtained by running respectively FFC, BFC and WFC. Let us also define FD-α as the best solution among these three.

So far we implemented six different heuristics. We finally include in the list the heuristic H6, which is the best among the upper bounds proposed by Gendreau, Laporte and Semet [55].

We refer to Section 5.5 for the computational results obtained by these heuristics. If the solution found is not equal to the lower bound of the previous section, then it is possibly improved by means of the following metaheuristic procedure.

5.4.1 Evolutionary Algorithm

To find high quality solutions on difficult instances, we use a variation of the metaheuristic Evolutionary Algorithm described in [84], which combines an effective tabu search with a specialized diversification strategy.

Tabu Search Algorithm

In [84] a Tabu Search procedure is proposed which moves between *partial feasible solutions*, i.e., in a VCP framework, colorings where all incompatibility and capacity constraints are satisfied, but not all vertices are colored, or, with regards to a BPPC framework, solutions in which each bin satisfies capacity and incompatibility constraints, but not all items are assigned to a bin. We modify this procedure in order to solve the BPPC. In [92] Morgenstern defines the *Impasse Class Neighborhood*, a structure used to improve a partial solution of cost k to a complete solution of the same value. The *Impasse Class* requires a target value k for the number of bins (resp. colors in the VCP) to be used. A solution S is a partition of V in $k+1$ bins $\{V_1, ..., V_k, V_{k+1}\}$ in which all bins, but possibly the last one, are feasible bins. This means that the first k bins constitute what we define as a *partial feasible k partitioning*, while all items that do not fit in the first k bins are in the last one. Making this last bin empty gives a complete feasible *k partitioning*. To move from a solution S to a new solution $S' \in N(S)$ one can randomly choose an unassigned item $i \in V_{k+1}$, assign i to a different bin, say h, and move to bin $k+1$ all items j in bin h that are incompatible to i. This assures that bin h remains feasible with respect to incompatibility constraints. If the capacity constraint is not satisfied, extra items have to be moved from bin h to $k+1$, till the capacity of bin h is exceeded. We considered four possibilities:

- move items according to decreasing weight ordering;
- move items according to increasing weight ordering;
- move items according to a random ordering;
- move items by minimizing the score (defined below) of moved items. If B is the actual bin weight and C the capacity, we want to select a subset of the items in bin h, to be moved to bin $k+1$, such that the sum of the corresponding scores is minimized and the total weight is larger than $B - C$. This is equivalent to maximize the sum of the scores of items that are kept in h, without exceeding the bin capacity, i.e. it is a Knapsack Problem or a Subset Sum Problem if the score of each item equals its weight (see Martello and Toth [88]).

The Class h is chosen by comparing different target classes by means of a function $f(S)$ for evaluating the corresponding solutions S. Rather than simply minimizing $\mid V_{k+1} \mid$ it seems a better idea to minimize the value:

$$f(S) = \sum_{j \in V_{k+1}} \delta_j \tag{5.15}$$

if we want to give more importance to the compatibility constraints, and

$$f(S) = \sum_{j \in V_{k+1}} w_j \tag{5.16}$$

if we want to give more importance to the capacity constraints. Moreover, we can use the surrogate weight w_j^s to combine weight and degree of the vertices:

$$f(S) = \sum_{j \in V_{k+1}} w_j^s \tag{5.17}$$

At every iteration we move from a solution S to the best solution $S' \in N(S)$ (even if $f(S) < f(S')$). To avoid cycling, we use the following tabu rule: an item i cannot enter the same bin h it entered during one of the last T iterations; for this purpose we store in a tabu list the pair (i, h). While pair (i, h) remains in the tabu list, item i cannot be assigned to bin h. We also use an *Aspiration Criterion*: a tabu move can be performed if it improves on the best solution encountered so far. To summarize, the Tabu Search algorithm takes in input:

- graph $G(V, E)$;
- the weights of the items and the bin capacity C;
- the target value k for the number of bins;
- a feasible partial k *partitioning*;
- the maximum number L of iterations to be performed ;
- the tabu tenure T.

If the algorithm solves the problem within L iterations it gives on output a feasible solution of value k, otherwise it gives on output the best scored infeasible solution found during the search.

The Tabu Search algorithm is very simple and requires as parameter to be experimentally tuned only the tabu tenure T. At the same time it has a good experimental behavior, since it is often able to find good solutions in very short computing times. The main drawback of this approach is that it seems unable to efficiently explore the whole solution space, thus suggesting the integrations of the Tabu Search with a suitable diversification strategy.

Evolutionary Diversification

To improve its performance, the Tabu Search can be used together with a diversification operator, as in the Evolutionary algorithm proposed in [84].

In our algorithm we start with an initial pool of partial feasible solutions of value k (in the following simply *solutions*) obtained by using the Tabu Search procedure initialized with different random seeds. Then we apply the Tabu Search algorithm to improve these solutions during the local search phase. We use a crossover operator similar to the one used in [84], which is a variation of the specialized crossover operator *Greedy Partition Crossover* proposed by Galinier and Hao [51] for Vertex Coloring Problems. Our purpose is to extend a feasible partial k *partitioning* to a complete partitioning. The general procedure is summarized as follows: given a pool of solutions, we randomly choose two parents from the pool and generate

an offspring, which is improved by means of the Tabu Search algorithm and finally inserted in the pool, deleting the worst parent. After the initialization, the generation-improvement-insertion procedure is iterated until the problem is solved (i.e. a partial feasible solution is extended to a complete solution) or a time limit is reached.

The crossover operator has the same structure as the one described in [84] for the classical VCP: given two parents $S^1 = \{V_1^1, ..., V_k^1, V_{k+1}^1\}$ and $S^2 = \{V_1^2, ..., V_k^2, V_{k+1}^2\}$ the crossover operator outputs the offspring $S^3 = \{V_1^3, ..., V_k^3, V_{k+1}^3\}$ by considering iteratively parent S^1 and S^2, picking up the bin maximizing a given score in the actual parent, copying it to the offspring, and deleting items contained in the chosen bin from both parents. The only difference concerning the operator is that we consider different possible scores, when picking up the next bin to be copied to the offspring:

- the cardinality of the bin;
- the total weight of the bin;
- the sum of the degrees of the items in the bin;
- the total surrogate weight of the bin.

Solution evaluation

The quality (fitness) of every solution S in the pool is evaluated through the function $f(S)$ defined by (5.15) or (5.16) or (5.17) and used during the Tabu Search algorithm. This allows us to compare the solutions and to tune the quality of the pool during the computation. The offspring is then inserted in the pool, substituting the worst parent.

Parameter Setting

The Evolutionary Algorithm uses different parameters, whose set-up is important in order to obtain good quality solutions. Moreover, the strategy adopted when emptying bins, in order to satisfy the capacity constraints, the solutions evaluating function and the score function used by the crossover operator must be specified. Our experiments on the set of instances proposed by Gendreau et al. [55] suggested to consider two different kind of instances: low density instances (with graph density up to 0.3) and hight density instances (with graph density larger than 0.3). For low density instances, we set $L = 5000$, $poolsize = 10$, $T = 90$ and we use the solution evaluating function (5.16). For high density instances, we set $L = 5000$, $poolsize = 10$, $T = 60$ and we use the solution evaluating function (5.15). In both cases, when emptying a bin, we consider items in a random ordering and we remove them till the bin exceeds the capacity. This introduces more randomness in the algorithm than other strategies, and leads to better results on the considered instances. We initialize the algorithm with a dummy solution (all bins are empty) and a very large number of bins available. We compute then the actual number of used bins, say k, and we iterate reducing k to $k - 1$ until the algorithm in able to solve the problem within a given time limit, which is set to 50 seconds. In the crossover operator, we use as score for the bin to be copied from the parent to the offspring, the cardinality of the bin.

5.5 Computational Experiments

The algorithms were coded in C and run on a Pentium IV 3 GHz. Their behavior was tested on the instances proposed by Gendreau, Laporte and Semet [55]. They considered 8 classes of instances from Falkenauer [46], each class containing 10 bin packing instances. By adding, for the instances of each class, random incompatibility graphs G, with densities varying from 0 to 0.9, they obtained 800 instances (100 for each class). We refer the reader to the original publication for more details.

The results of the upper bounds are summarized in Tables 5.1 and 5.2, where Ev. stands for the metaheuristic of Subsection 5.4.1. In each column we report the average value found by each algorithm on the ten instances of each class, for different values of the density *dens*. of the associated graph G. It can be seen as Ev., whose computing times are comparable with those of $GLS6$, clearly outperforms previous techniques.

The results of the lower bounds are summarized in Tables 5.3 and 5.4. We report results on the bounding procedures presented on this paper, with the exception of L_{TVPP} and L_{match}, for which the interest is only theoretical since they have a non satisfactory performance from the computational viewpoint. L^1_{BPP}, by definition, cannot perform worse than L^0_{BPP}, hence the results of the latter are not reported. Also in this case we can see that there is a lower bound that clearly outperforms the others, namely L^1_{BPPC}.

Finally in Table 5.5 we compare upper and lower bounds. In the table, #opt stands for the number of proven optimal solutions, and %gap to the percentage gap between lower and upper bound. 487 instances over 800 were solved to optimality, with a very small percentage gap between lower and upper bound, being 0.5% on average.

5.6 Conclusions

We considered a bounded version of the Vertex Coloring Problem (BVCP), combining both VCP and the Bin Packing Problem. We report lower bounds from the VCP and BPP literature, which are valid for the considered problem as well, we improve a lower bound from the literature and present new lower bounds, exploiting the double nature of the problem. Similarly, we report from the literature upper bounds for the BVCP problem, we propose new heuristic upper bounds adapted from the BPP literature, and we adapt an effective metaheuristic approach, proposed for VCP, to the problem.

We implemented and compared original and literature algorithm on a set of 800 instances from a benchmark set. By using a combination of heuristic and metaheuristic approaches, we are able to solve to proven optimality 383 out of 800 instances, while previous approaches could solve to optimality only 179 out of 800, with comparable computational effort.

Table 5.1: Upper bounds results for classes 1–4.

Class	*dens.*	FFC	BFC	WFC	FFCα	BFCα	WFCα	H6	Ev.	min
1	0	49.1	49.1	49.5	49.1	49.1	49.5	49.4	48.3	48.3
	1	49.3	49.3	49.6	49.1	49.1	49.3	49.7	48.3	48.3
	2	51.1	51.1	51.0	49.1	49.1	49.7	50.4	48.3	48.3
	3	55.4	55.7	54.6	49.3	49.3	50.3	50.6	48.5	48.5
	4	63.2	63.4	59.4	53.5	53.6	54.1	54.9	53.0	53.0
	5	72.8	72.8	69.5	64.6	65.1	64.6	65.3	64.3	64.3
	6	83.0	83.1	79.8	76.0	76.5	76.3	76.9	75.8	75.8
	7	93.2	93.5	91.1	87.8	88.1	88.0	87.9	87.5	87.5
	8	102.2	102.3	100.4	98.5	98.6	98.5	99.0	98.3	98.3
	9	110.9	110.9	110.6	109.8	109.8	109.8	110.0	109.8	109.8
2	0	102.9	102.9	103.2	102.9	102.9	103.2	103.8	101.7	101.7
	1	103.6	103.6	103.4	103.0	102.9	103.1	104.0	101.7	101.7
	2	105.6	105.6	105.0	103.0	103.0	103.8	105.1	101.7	101.7
	3	114.3	114.5	110.4	103.4	103.2	104.7	106.1	101.7	101.7
	4	129.6	130.2	119.7	106.4	106.2	108.3	109.4	105.3	105.3
	5	148.1	148.3	138.0	126.7	127.5	127.5	129.2	126.0	126.0
	6	168.2	168.6	162.1	153.0	153.4	153.6	154.3	152.3	152.3
	7	188.1	188.5	184.5	177.4	178.3	178.1	179.2	177.1	177.1
	8	209.2	209.4	207.4	202.8	203.2	203.1	203.9	202.7	202.7
	9	230.3	230.3	229.2	226.8	227.0	227.2	227.9	226.8	226.8
3	0	205.3	205.3	205.7	205.3	205.3	205.7	206.5	204.0	204.0
	1	205.9	206.0	205.9	205.4	205.4	205.6	207.8	204.0	204.0
	2	211.2	211.4	210.0	205.7	205.4	206.9	209.4	203.9	203.9
	3	225.6	226.7	222.8	206.1	205.7	210.1	211.4	203.8	203.8
	4	256.3	256.8	241.5	211.7	212.3	216.1	216.4	212.1	210.8
	5	294.2	295.0	275.4	254.1	255.0	255.5	257.6	252.5	252.5
	6	335.6	336.3	320.4	305.2	306.0	305.2	307.1	304.1	304.1
	7	374.6	374.9	363.8	351.8	352.5	352.9	354.2	350.4	350.4
	8	416.7	416.9	409.9	399.9	400.4	400.6	402.2	399.1	399.1
	9	460.8	460.9	458.4	452.8	453.0	453.5	454.0	452.4	452.4
4	0	406.7	406.7	407.1	406.7	406.7	407.1	408.4	406.5	406.2
	1	407.4	407.5	407.6	406.5	406.5	407.1	411.1	406.3	406.1
	2	418.2	418.7	414.3	407.1	407.0	409.6	413.6	405.7	405.7
	3	444.6	446.4	441.0	407.7	407.6	416.8	416.0	405.5	405.5
	4	498.9	501.0	475.1	415.6	416.8	424.3	424.4	423.8	414.9
	5	579.8	580.8	543.3	509.3	511.0	510.8	513.3	505.3	505.3
	6	665.9	666.9	637.5	610.0	611.8	610.8	613.2	607.5	607.5
	7	752.5	753.0	731.3	710.2	710.9	710.9	712.9	707.6	707.6
	8	837.8	838.5	825.1	808.5	809.6	809.7	810.3	807.3	807.3
	9	922.3	922.4	916.2	906.4	906.8	907.2	908.1	905.7	905.7
Average		278.8	279.1	272.3	262.0	262.3	263.2	264.4	261.2	260.9

Table 5.2: Upper bounds results for classes 5–8.

Class	*dens.*	FFC	BFC	WFC	FFCα	BFCα	WFCα	H6	Ev.	min
5	0	23.1	23.1	23.1	23.1	23.1	23.1	20.0	21.0	20.0
	1	23.1	23.1	23.1	22.0	21.9	22.5	22.1	21.0	21.0
	2	24.0	24.0	23.7	22.2	22.1	22.3	22.5	21.0	21.0
	3	25.8	25.9	25.5	22.9	22.9	22.9	23.2	21.7	21.4
	4	29.2	29.4	28.8	26.2	26.2	26.2	26.4	24.8	24.4
	5	33.3	33.6	33.5	30.9	30.9	30.8	31.0	30.7	29.6
	6	39.0	39.3	39.1	37.2	37.2	37.2	37.1	36.3	35.7
	7	44.5	44.5	44.3	42.5	42.5	42.5	42.5	42.6	41.6
	8	50.5	50.7	50.5	49.6	49.6	49.6	49.5	49.0	48.0
	9	55.0	55.0	55.1	54.6	54.6	54.5	54.7	54.4	53.2
6	0	45.5	45.5	45.5	45.5	45.5	45.5	40.0	41.0	40.0
	1	45.8	45.8	45.7	44.5	44.0	44.2	44.5	41.0	41.0
	2	47.1	47.1	46.7	44.1	44.2	44.2	44.4	41.0	41.0
	3	50.7	50.9	49.6	44.1	44.1	43.8	44.6	41.2	41.2
	4	57.7	58.2	57.6	50.8	50.8	50.9	50.9	48.3	47.7
	5	67.9	68.3	67.6	62.6	62.6	62.5	62.6	60.9	59.2
	6	78.2	78.6	78.3	73.3	73.2	73.3	73.4	74.4	70.7
	7	89.1	89.4	89.2	85.9	85.8	85.9	86.0	84.9	82.9
	8	101.1	101.2	100.9	98.3	98.3	98.3	98.4	96.6	95.3
	9	112.7	112.8	112.7	111.8	111.8	111.8	111.8	109.0	109.0
7	0	94.9	94.9	94.9	94.9	94.9	94.9	83.0	84.0	83.0
	1	95.0	95.0	95.0	90.5	90.3	91.4	90.3	84.0	84.0
	2	95.1	95.1	95.1	91.3	91.5	91.6	91.5	84.0	84.0
	3	98.8	99.3	99.1	91.8	92.5	90.4	92.7	84.2	84.2
	4	112.1	113.1	113.3	101.9	101.9	102.0	101.9	98.4	97.4
	5	132.3	133.8	135.0	125.3	125.2	125.3	125.3	124.8	122.1
	6	155.5	156.4	156.5	148.8	148.8	148.9	148.8	150.1	146.2
	7	178.7	179.2	180.6	173.1	173.2	173.2	173.1	174.7	169.6
	8	204.3	204.6	204.2	200.1	200.1	200.1	200.1	200.2	197.7
	9	227.3	227.4	227.4	224.4	224.3	224.5	224.3	225.1	223.0
8	0	190.2	190.2	190.2	190.2	190.2	190.2	167.0	169.0	167.0
	1	190.3	190.3	190.3	181.1	180.5	183.2	180.9	168.8	168.8
	2	191.1	191.1	191.0	183.8	183.3	183.1	184.1	168.4	168.4
	3	197.1	198.4	196.7	183.4	183.4	179.3	184.7	169.3	169.3
	4	226.8	229.0	228.6	204.9	204.9	205.1	204.9	204.9	204.9
	5	268.2	270.5	274.5	255.6	255.4	255.8	255.5	255.3	255.3
	6	316.8	318.2	320.5	305.4	305.3	305.6	305.4	305.3	305.3
	7	364.9	366.4	367.4	355.3	355.2	355.7	355.4	355.1	355.1
	8	412.3	413.4	413.8	405.4	405.3	405.6	405.5	405.3	405.3
	9	458.5	459.0	458.7	454.1	454.0	454.2	454.1	453.8	453.8
Average		131.3	131.8	131.8	126.3	126.3	126.3	125.4	123.1	122.2

Table 5.3: Lower bounds results for classes 1–4.

Class	*dens.*	L^1_{BPP}	L'_{MC}	L'_{CP}	max
1	0	48.3	36.7	48.3	48.3
	1	48.3	12.7	48.3	48.3
	2	48.3	25.1	48.3	48.3
	3	48.3	37.8	48.3	48.3
	4	48.3	51.5	52.8	52.8
	5	48.3	63.3	64.3	64.3
	6	48.3	74.0	75.6	75.6
	7	48.3	87.0	87.4	87.4
	8	48.3	97.3	98.0	98.0
	9	48.3	109.0	109.5	109.5
2	0	101.4	79.6	101.4	101.4
	1	101.4	26.6	101.4	101.4
	2	101.4	50.6	101.4	101.4
	3	101.4	75.9	101.4	101.4
	4	101.4	99.3	104.5	104.5
	5	101.4	124.9	125.8	125.8
	6	101.4	150.8	151.8	151.8
	7	101.4	174.5	176.1	176.1
	8	101.4	201.4	202.2	202.2
	9	101.4	225.5	226.6	226.6
3	0	202.6	158.2	202.6	202.6
	1	202.6	50.4	202.6	202.6
	2	202.6	101.5	202.6	202.6
	3	202.6	155.0	202.6	202.6
	4	202.6	204.3	208.5	208.5
	5	202.6	250.8	251.9	251.9
	6	202.6	302.7	303.8	303.8
	7	202.6	347.9	349.6	349.6
	8	202.6	397.0	398.7	398.7
	9	202.6	451.9	452.2	452.2
4	0	401.8	311.0	401.8	401.8
	1	401.8	102.5	401.8	401.8
	2	401.8	204.9	401.8	401.8
	3	401.8	307.1	401.8	401.8
	4	401.8	403.0	409.2	409.2
	5	401.8	503.4	504.5	504.5
	6	401.8	606.3	606.9	606.9
	7	401.8	705.6	706.8	706.8
	8	401.8	805.6	806.7	806.7
	9	401.8	904.7	905.2	905.2
Average		188.5	226.9	259.9	259.9

Table 5.4: Lower bounds results for classes 5–8.

Class	*dens.*	L^1_{BPP}	L'_{MC}	L'_{CP}	max
5	0	20.0	1.0	20.0	20.0
	1	20.0	6.8	20.0	20.0
	2	20.0	13.3	20.0	20.0
	3	20.0	19.9	21.8	21.8
	4	20.0	25.4	26.0	26.0
	5	20.0	30.3	30.7	30.7
	6	20.0	36.8	36.9	36.9
	7	20.0	41.9	42.4	42.4
	8	20.0	49.0	49.4	49.4
	9	20.0	54.2	54.5	54.5
6	0	40.0	1.0	40.0	40.0
	1	40.0	13.3	40.0	40.0
	2	40.0	25.9	40.0	40.0
	3	40.0	37.7	40.9	40.9
	4	40.0	50.2	50.7	50.7
	5	40.0	61.6	62.4	62.4
	6	40.0	72.4	72.9	72.9
	7	40.0	85.2	85.6	85.6
	8	40.0	97.4	98.1	98.1
	9	40.0	111.5	111.5	111.5
7	0	83.0	1.0	83.0	83.0
	1	83.0	25.8	83.0	83.0
	2	83.0	51.5	83.0	83.0
	3	83.0	75.8	83.5	83.5
	4	83.0	100.2	100.7	100.7
	5	83.0	124.3	125.0	125.0
	6	83.0	148.2	148.5	148.5
	7	83.0	172.9	173.0	173.0
	8	83.0	199.9	200.0	200.0
	9	83.0	223.9	224.2	224.2
8	0	167.0	1.0	167.0	167.0
	1	167.0	51.1	167.0	167.0
	2	167.0	100.4	167.0	167.0
	3	167.0	152.2	169.9	169.9
	4	167.0	204.1	205.6	205.6
	5	167.0	254.7	255.1	255.1
	6	167.0	304.8	305.1	305.1
	7	167.0	354.5	355.0	355.0
	8	167.0	404.8	405.2	405.2
	9	167.0	453.5	453.8	453.8
Average		77.5	106.0	123.0	123.0

Table 5.5: Comparison between lower and upper bounds.

Class	max LB	min UB	#opt	%gap
1	68.1	68.2	87	0.2
2	139.3	139.7	61	0.3
3	255.0	255.9	36	0.4
4	554.7	557.2	30	0.6
5	32.2	32.6	74	1.2
6	64.2	64.1	66	0.8
7	130.4	131.1	64	0.5
8	265.1	265.3	69	0.2
Average	188.6	189.4	487	0.5

Part II

Fair Routing

Chapter 6

On the Fairness of ad-hoc Telecommunication Networks

We consider a network of nodes communicating through wireless connections, which are feed by a node owned energy source. The problem of routing packets through the network should consider both the efficiency of the routing and its fairness, i.e. the contribution that each node gives to the network with respect to the benefit it obtains from being in the network. This requires the definition of the fairness of a routing, and leads us to the design of centralized and decentralized fair routing protocols.

6.1 Introduction

We consider the problems of nodes communicating through a wireless ad hoc network, i.e. a network formed by a set of (possibly mobile) nodes not making use of any preexisting infrastructure (see Tonguz and Ferrari [105] for an introduction to ad-hoc networks). Examples are temporary networks present during meeting or happenings, or installed for disaster recovery or created for geographic surveys. In such kind of networks, nodes communicate by using wireless radio connections, which are feed by a node owned energy source, normally a battery. Nodes can communicate and perform their internal operations (measure, computing, etc.) since they have enough energy in their batteries. Each node can participate to the network by sending or receiving information it is interested in, and, in addition, it can forward packets for other nodes, in order to ensure the connectivity and improve the efficiency of the network. This is without a tangible benefit for the node; of course, the node has a benefit if other nodes forward too. Since sending and receiving packets has a cost in terms of energy consumption, a trade-off can exist between a routing of the packets which is efficient for the system as a whole, and a routing which is fair for the single node.

The absence of immediate benefit for nodes contributing to the network, raises the problem of nodes that take advantage from the network but, acting selfishly, do not forward other node's packets. The problem has encountered a large interest in the literature, and has been considered mainly from the game theoretical point of view.

A topic widely considered in the literature is the study of the efficiency of equilibria found by ad-hoc networks, where nodes act selfishly, when there is no centralized control. Kesselman, Kowalski and Segal [76] study the efficiency of equilibria found by nodes with respect to the efficiency of a centralized solution, in a network where the transmitting cost is shared among

sender and receiver nodes. Ji and Huang [68] consider the problem of many nodes who want to communicate with a single destination node, and can adjust their transmitting power in a non cooperative way. A similar problem, but with different utility functions for the nodes, is considered by Lee, Mazumdar and Shroff [79]. Another problem of ransmitting power selection, where nodes want to ensure a desired connectivity to the network, is considered by Eidenbenz, Kumar and Zust in [44]. Chun, Fonseca, Stoica, Kubiatowicz [32] study the structure of routing networks generated by selfishly nodes. In the different domain of routing in a capacitated network, Correa, Schulz and Stier Moses considered the efficiency of equilibria where users choose their own routing by acting selfishly [35].

When the equilibria found by nodes acting selfishly are close to the system optimum, i.e. the equilibrium which could be imposed by a central control, there is no need to act on the system. However, the nodes' equilibria can be very inefficient, in this case the design of mechanisms to increase cooperation and to impose desired behaviors are needed. Anderegg and Eidenbenz [8] consider routing in ad-hoc networks, and design a mechanism where the best strategy for each node leads to the system optimum. Marti, Giuli, Lai and Baker [89] design a protocol to improve the routing of an ad-hoc network where some nodes misbehave, because they are broken, overloaded or selfish. Alvin [7] considers a punishment mechanism in order to impose a desired behavior to nodes. Buchegger and Le Boudec [22, 21] propose protocols aimed at detecting and isolating misbehaving nodes, thus making unattractive to deny cooperation. A similar ides is studied by Mahajan, Rodrig, Wetherall and Zahorjan in [97]. Differently, Zhong,Yang and Chen [113] propose a credit base system for stimulating cooperation.

Even if a wide literature is available on ad-hoc networks, all previous works are devoted to the study of the efficiency of networks, and to the study of mechanism to obtain a desired behavior. The study of the fairness of ad hoc networks, and the study of fair routing protocols in such networks, have been rarely considered in the literature, and, to the best of our knowledge, never in a deep way. Conversely, the fairness of routing problems have been studied by Kleinberg, Rabani and Tardos in [77] and by Goel, Meyerson and Plotkin in [58]; Afek, Mansour and Ostfeld considered fair bandwidth allocation in [6, 5]; Goel, Meyerson and Plotkin in [59] consider the problem of fair balancing.

This work is intended as a first step in the study of fairness in ad-hoc networks. More in detail, we aim at studying the relation existing between an efficient packet routing protocol and a fair one, in term of battery consumption and contribution given to the network by each node. This requires, first of all, the definition of a measure of fairness and a study of how this affects the desired behavior of the system. We suppose that nodes participate to the network because the network is fair, i.e. the routing protocol will charge nodes proportionally to their activity in the network. It is not in the scope of this paper to discuss in detail why fairness is desirable.

In order to keep the analysis general enough to be application or protocol independent, the problem is considered from a network flow/optimization perspective, by avoiding strong application related assumptions.

This paper is structured as follows: the next Section introduces some preliminary assumptions, and proposes a fair routing model and two alternative measure of fairness for wireless ad hoc networks. Computational experiments on randomly generated networks are reported in Section 6.3. A distributed fair routing algorithm, and related computational experiments are reported in Section 6.4.

6.2 A Model for Fair Routing

6.2.1 Preliminaries

We assume that all nodes know, in every time slot, the position of the other nodes in the network (for example, because they are equipped with GPS, or because the routing protocol returns this information), or at least the set of nodes in visibility for each terminal in the network.

The energy available for each node (battery), that is used for routing the traffic originating or going to the node, is a scarce resource. This energy is also used for retransmitting others nodes' traffic, and thus a local cost increase can result in a benefit (connectivity or efficiency) at system level. The energy consumed by a node in communication activities is given by the sum of three main contributes: energy used during transmission (TX), during reception (RX) and during IDLE. Furthermore, some protocols put the terminals in a sleep state when possible to save energy: in this state the energy consumed is not involving antenna activities but only processing ones. In our approach we simplify the behavior by considering only the energy consumed during transmission and reception: if nodes with high traffic generation and forwarding are considered, we can assume it as a major contribute. Regarding the transmit power, it can be fixed or can be tunable by following a power control technique.

- There are OD pairs that want to communicate by exchanging information (multi-commodity flow), and an origin/destination (OD) matrix represents, at each entry, the quantity of information to be transmitted between pairs of nodes. The OD matrix is referred to a single period, and the routing of the information through the network is computed period by period.

  The first assumptions for the demand matrix is that all nodes have a non zero amount of traffic to send or receive (which is a realistic assumption if a sensor network, e.g. for geographical survey, is considered; but can be acceptably realistic even for individuals using laptops during a meeting, that are likely to switch on their wireless connections only if they have something to transmit or receive).

  However, it could be the case that some nodes participate to the network only by forwarding other node's traffic, and thus being, by definition (regardless how fairness is defined), in a unfair condition.

- for any OD pair, the information to be routed can be split into multiple paths (splittable),

- the routing is constrained by the capacities given by the residual battery life associated with the nodes (constrained).

In other words, possible routings are given by the feasible solution to a *capacity-constrained splittable multi-commodity* flow problem.

A routing protocol should consider two conflicting objectives: fairness and efficiency.

Fairness: We assume that nodes have a utility from participating in the network, because they can reach other nodes, which can be too far or to expensive to be reached directly, through the network. They contribute their power to forward packets, because without contributing nodes the ad-hoc network doesn't exist. There must be a balance between the utility nodes get by participating in the network, and the cost they incur when they contribute their power to

the network itself, otherwise nodes could decide to leave the network. This means, generally speaking, that the network should not route all the flow through the same nodes and drain their batteries, while other nodes do not contribute their energy.

Efficiency: The routing protocol must be efficient, which means that a global cost function should be minimized. Different efficiency measures can be considered in wireless networks; in this paper we consider a routing to be efficient if the total energy consumption is minimized. We approximate the energy consumption by considering the energies spent when receiving packets (independent from the origin of the packets) and the energies spent when sending packets, which depend on the distance of the receiver and on the length of the packet.

The system can optimize one and use the other as a constraint, or the problem can be formulated as multicriteria and all the Pareto optimal solutions can be computed.

From a feasibility viewpoint, the capacity-constrained splittable multi-commodity flow problem, whose solutions represent possible routings, is an LP. But it becomes strongly NP-hard if the simultaneous maximization of both efficiency and fairness is considered. If instead one of the objective functions is used as a constraint, the two problems remain tractable (can be LPs depending on how the fairness is defined). In this paper the problem is approached in the latter way, i.e. we study how the cost of the best routing, in terms of efficiency, varies when different levels of fairness are imposed. This leads us compute how fairness affects efficiency in different network configurations.

The relation between efficiency and fairness in routing problems have been studied, in a multi-commodity flow framework, by Jahn, Möhring, Schulz and Stier Moses in [65], by Schulz and Stier Moses [102] and by Correa, Schulz, and Stier Moses in [34].

6.2.2 A Proposed Model

We would like our analysis to be independent from the specific adopted transmitting protocol, and then we use a fluid model representation, i.e. we describe the traffic in the network as a flow (of bits). Bits will be grouped into packets, but how will depend on the chosen protocol. The problem can be formulated as a routing problem in which the graph is fixed (determined by the maximum transmitting power) and the transmitting power can be either independent on the two nodes that are communicating, or the minimum needed to reach the receiving node (power control). In the following we will call bit the elementary "piece" of information. If the transmission time is assumed to be the same for every bit, the energy spent can be linked directly to the transmit power.

Given two nodes i and j at distance $p(i,j)$, it is a common assumption in telecommunications [98] that the power they need to communicate directly is $p(i,j)^\alpha$, where $\alpha \in [2,8]$ is a constant depending on the channel. Thus, if $pMax$ is the maximum transmitting power, they can communicate directly if $p(i,j)^\alpha \leq pMax$. In this case (i,j) is an arc in the graph underlying the network. The network is then fixed and given by a weighted graph $G = (V, A)$, where the weight δ_{ij} of arc $ij \in A$ represents the energy needed to send a bit on ij, i.e. the power $p(i,j)^\alpha$ needed to connect i and j multiplied by the transmission time for a bit (let set this time equal to 1). When power control is not performed $\delta_{ij} = const.(= 1\ w.l.o.g.)$ $\forall ij \in A$.

The demand between every pair of nodes s and t, corresponding to a commodity st, is given by d_{st}. The set of all commodities is K. The paths between those nodes are denoted by $\mathcal{P}_{st}$ and the set of all simple paths is $\mathcal{P}$. The decision variables are represented by the flow variables x_{ij}^{st}, i.e. the flow on arc $ij \in A$ due to commodity st, and x_P, i.e. the flow along

path P. A node $i \in V$ has a battery life p_i which bounds the maximum energy it can use to transmit to (and receive from) other nodes. If ρ is defined as the relative power needed to receive, the set of feasible flows is given by

$$\sum_{P:ij\in P;P\in\mathcal{P}_{st}} x_P = x_{ij}^{st} \quad \forall st \in K,\ \forall ij \in A \tag{6.1}$$

$$\sum_{st\in K} x_{ij}^{st} = x_{ij} \quad \forall ij \in A \tag{6.2}$$

$$\sum_{P\in\mathcal{P}_{st}} x_P = d_{st} \quad \forall st \in K \tag{6.3}$$

$$\rho \sum_{j\in V:\ ij\in A} x_{ji} + \sum_{j\in V:\ ij\in A} \delta_{ij} x_{ij} \leq p_i \quad \forall i \in V \tag{6.4}$$

$$x_P \geq 0 \quad \forall P \in \mathcal{P} \tag{6.5}$$

To measure efficiency, the total energy cost of the system is considered, which is given by

$$\rho \sum_{ij\in A} x_{ij} + \sum_{ij\in A} \delta_{ij} x_{ij} \tag{6.6}$$

and reduces to the total number of hops performed by bits when power control is not used.

To measure fairness, two alternative definitions are proposed and discussed in the next Sections.

I Measure of Fairness

A first possibility to measure the fairness of a node is to distinguish between the energy a node is spending to send (and receive) traffic that is "useful" for itself (the traffic that it is originating or is addressed to it) and the energy it is spending for forwarding others nodes' traffic. Let $0 \leq r_i \leq 1$ be the proportion of energy that is useful. This can be computed as the following ratio:

$$r_i := \frac{\rho \sum_{st:t=i} \sum_{j\in V:ji\in A} x_{ji}^{st} + \sum_{st:s=i} \sum_{j\in V:ij\in A} \delta_{ij} x_{ij}^{st}}{\rho \sum_{j\in V:\ ij\in A} x_{ji} + \sum_{j\in V:\ ij\in A} \delta_{ij} x_{ij}} \tag{6.7}$$

which reduces, when power control is not performed, to

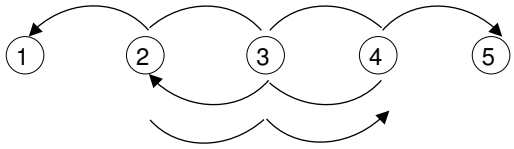
$$r_i := \frac{\sum_{j\in V} (\rho d_{ji} + d_{ij})}{\rho \sum_{j\in V:\ ij\in A} x_{ji} + \sum_{j\in V:\ ij\in A} x_{ij}} \tag{6.8}$$

A node with a high value of r_i is using its power for its benefit while a node with a low value is contributing its energy to the system benefit.

E.g., consider a node i spending a quantity 0.9 of its energy to send or receive its own traffic, i.e. traffic corresponding to a commodity st such that i is origin ($i = s$) or destination ($i = t$) of the commodity, and suppose the total energy spent by i to be equal to 1. In this case, $r_i = 0.9$, and i is spending only 0.1 of its energy to the "common good".

Consider a second example: 5 equidistant nodes are on a segment, namely nodes $1, 2, 3, 4, 5$, and the following OD traffic matrix has to be transmitted: $d_{3,1} = 1,\ d_{3,5} = 1,\ d_{2,4} = 1,\ d_{4,2} =$

Figure 6.1: An example with fairness 1/2.



1. Let us simplify the example by considering that only transmitting has a cost ($\phi = 0$), and $\alpha > 1$.

The optimal solution (w.r.t. the energy cost), as represented in the picture above, is $x_{3,2}^{3,1} = 1,\ x_{2,1}^{3,1} = 1,\ x_{3,4}^{3,5} = 1,\ x_{4,5}^{3,5} = 1,\ x_{4,3}^{4,2} = 1,\ x_{3,2}^{4,2} = 1,\ x_{2,3}^{2,4} = 1,\ x_{3,4}^{2,4} = 1$ ($x_{i,j}^{s,t}$ is the flow on arc i, j due to the commodity originated in s and with destination t). In this solution, the lowest fairness experienced by a node, when computed with definition (6.7), is 1/2, since only half of energy spent by nodes 2,3 and 4 is used to send their how traffic.

II Measure of Fairness

But is this solution really unfair? The amount of energy that nodes 2,3 and 4 are using for forwarding other node's traffic equals the amount of energy that other nodes are using to forward 2,3 and 4's traffic. Formally, if we define $e_{i,j}$ as the energy that node i is spending to forward j's traffic, in this solution $e_{i,j} = e_{j,i}\ \forall i, j \in V$. In addition, this is not only the less expensive solution for the system, but it is also the solution minimizing the maximum energy spent by a node (and thus it not clear why a more expensive solution should be more fair, or at least more desirable). This example suggests that an alternative measure of fairness can be considered, assessing the unbalance of the energy that each node i is spending to forward other nodes' traffic, and the energy that other nodes are spending to forward i's traffic.

Consider a commodity k originating at node s and having as destination node t. Let $l, m \in A$ be an arc of the network. When a flow $x_{l,m}^{s,t} > 0$ is routed on arc (l, m), node l is spending $\delta_{l,m} x_{l,m}^{s,t}$ of its energy to send the flow and node m is spending $\rho x_{l,m}^{s,t}$ of its energy to receive the flow. They are working for the origin node s and the destination node t, who are the ones having an interest in the flow. Thus, if we define as $Sq \in [0, 1]$ the quota of the energy cost that is imputed to the origin s and $Rc = 1 - Sq$ the quota of the energy cost that is imputed to the destination t, and we define the unbalance of a node i, $unb(i)$, as the difference between the energy it is spending for other nodes and the energy other nodes are spending for i, a flow $x_{l,m} > 0$ has the following effect on the unbalance of nodes $s,\ t,\ l,\ m$:

- $unb(s) = -Sq(\delta_{l,m} + \rho)x_{l,m}^{s,t}$,
- $unb(t) = -Rq(\delta_{l,m} + \rho)x_{l,m}^{s,t}$,
- $unb(l) = +\delta_{l,m} x_{l,m}^{s,t}$,
- $unb(m) = +\rho x_{l,m}^{s,t}$,

The higher the unbalance of a node, the less fair its condition is, because it is spending its energy for other nodes without advantage. Thus a possible way to measure the normalized unfairness of node i is to compute the ratio between $unb(i)$ and the total energy $en(i)$ that node i is spending for other nodes, including the receiver quota Rq when i is sender, and viceversa (which is better then simply normalizing over the total energy i is spending, otherwise i could reduce its unfairness - i.e. the ratio - by routing its packets through a more expensive arc). Briefly, given:

$$en(i) = \sum_{st:i\neq s,t} \sum_{j\in V:ij\in A} \delta_{ij} x_{ij}^{st} + \sum_{st:i\neq s,t} \sum_{j\in V:ji\in A} \rho x_{ji}^{st} + \sum_{st:i=s} \sum_{j\in V:ij\in A} Rq\delta_{ij} x_{ij}^{st} + \sum_{st:i=t} \sum_{j\in V:ji\in A} Sq\rho x_{ji}^{st} \tag{6.9}$$

we can measure the fairness (i.e. 1 - unfairness) experienced by node i as:

$$r_i := 1 - \frac{unb(i)}{en(i)} \tag{6.10}$$

Note that this definition normalizes to 0 the lowest fairness experienced by a node in the network, and that the fairness of a node can have any value in the interval $[0, \infty[$ (actually this is not a problem if we care of nodes that are in an unfair condition).

An alternative normalization (still in the interval $[0, \infty[$) could compute the ratio between $unb(i)$ and the maximum energy spent for other nodes by a node in the network [1].

6.2.3 Formulation

The network's efficiency and fairness are studied as follows: the energy cost of the system (6.11) is minimized, and a minimum fairness ϕ is imposed as a constraint (6.12) (ϕ is a constant). The idea of considering the minimum fairness as a measure of the fairness of the system was proposed by Jaffe [64] and is present Bertsekas and Gallager [13] and in the work of Afek, et al. [6, 5]. Alternative measures and a unifying framework are discussed by Bhargava, Goel and Meyerson in [14]. By proceeding with a constraint on the minimal fairness of a node, we can compute the routing maximizing system efficiency (by imposing $\phi = 0$), and we can evaluate how the routing cost increases when fairness is imposed, i.e. we can asses the cost of a fair solution in terms of efficiency.

With respect to the notation defined in 6.2.2, this requires the solution of the following LP:

$$\min \rho \sum_{ij\in A} x_{ij} + \sum_{ij\in A} \delta_{ij} x_{ij} \tag{6.11}$$

$$r_i > \phi \qquad \forall i \in V \tag{6.12}$$

$$\sum_{P:ij\in P; P\in\mathcal{P}st} x_P = x_{ij}^{st} \quad \forall st \in K,\ \forall ij \in A \tag{6.13}$$

$$\sum_{k\in K} x_{ij}^{st} = x_{ij} \qquad \forall ij \in A \tag{6.14}$$

$$\sum_{P\in\mathcal{P}_{st}} x_P = d_{st} \qquad \forall st \in K \tag{6.15}$$

[1] This requires, in the next Section, the solution of an ILP, but the problem seems to be easy for general purpose ILP solvers

$$\rho \sum_{j \in V:\ ij \in A} x_{ji} + \sum_{j \in V:\ ij \in A} \delta_{ij} x_{ij} \leq p_i \qquad \forall i \in V \tag{6.16}$$

$$x_{ij}^{st} \geq 0 \quad \forall st \in K,\ \forall ij \in A \tag{6.17}$$

the fairness r_i in constraint (6.12) can be measured by means of the first definition:

$$r_i := \frac{\rho \sum_{st:t=i} \sum_{j \in V: ji \in A} x_{ji}^{st} + \sum_{st:s=i} \sum_{j \in V: ij \in A} \delta_{ij} x_{ij}^{k}}{\rho \sum_{j \in V:\ ij \in A} x_{ji} + \sum_{j \in V:\ ij \in A} \delta_{ij} x_{ij}} \tag{6.18}$$

or by means of the second definition:

$$r_i := 1 - \frac{unb(i)}{en(i)} \tag{6.19}$$

In this formulation, the decision variables are represented by the flow variables x_{ij}^{st}, i.e. the flow on arc $ij \in A$ due to commodity st. We do not consider explicitly variables corresponding to paths x_P, which can be exponentially many w.r.t. the number of arcs in the graph.

By varying the value of ϕ (i.e., by solving the LP with different values of ϕ), the curve representing the energy cost of the system, as function of the minimum fairness, can be computed.

Cyclic Solutions

When the second definition of fairness (6.10) is used, the solution of model (6.11)–(6.17) can have cycles. Actually, suppose node i has a low fairness, i.e., $r_i < \phi$, or, in other words, $unb(i)$ is too large. The unbalance of i can be reduced (its fairness increased) if nodes forwarding a commodity it originated in i (or si with destination i) spend more energy with this commodity. This happens, for example, if i's flow is routed through a cycle in the path from i to t.

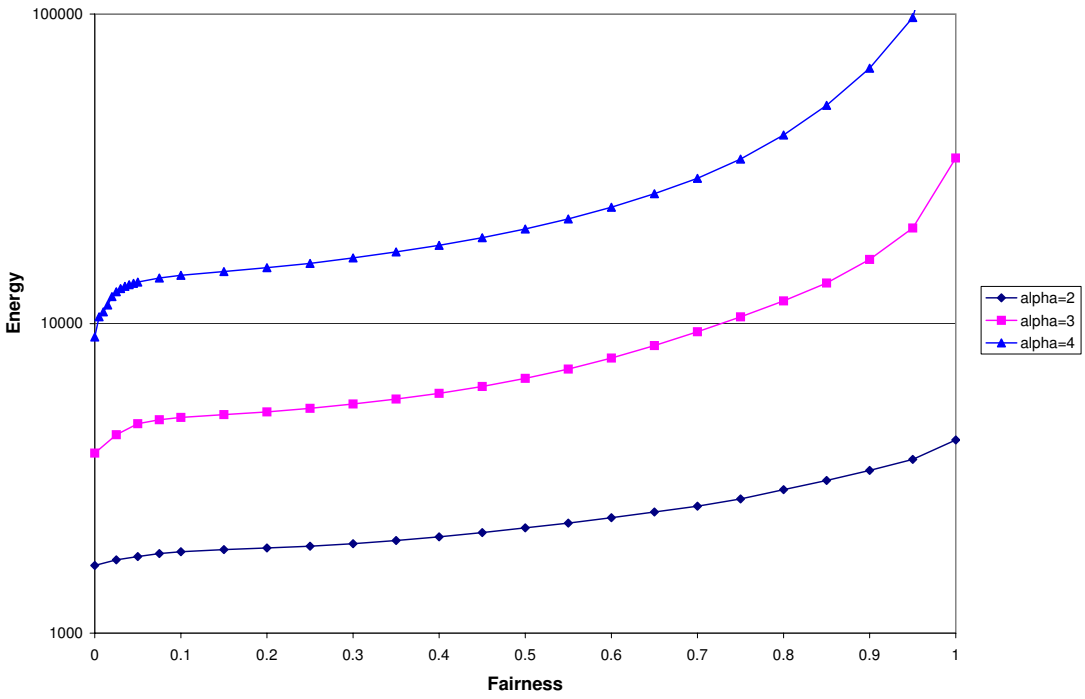
To avoid this situation it can be imposed that, for every commodity st, only arcs having a positive component is the direction defined by st can be used, thus making the "feasible" graph different for each commodity and acyclic. This has a cost in terms of efficiency, because less arcs are available for each commodity (discussed in 6.3).

6.3 Computational Experiments

This Section reports computational experiments on randomly generated networks. Nodes are randomly generated with a given probability distribution on a surface. With a given probability p_t, a node i is a transmitter. For every node $j \neq i$, there is a probability p_r that i is transmitting to j. The quantity of bits transmitted has uniform distribution between 0 and a maximum value (actually it is normalized to 1). When power control is used, the power needed to transmit from i to j is $d(i,j)^{\alpha}$, where $d(i,j)$ is the Euclidean distance between the nodes and $\alpha \in [2,8]$ is a constant depending on the channel. The maximum transmitting power and the Euclidean distances between nodes define the graph underlying the network, in terms of connectivity and cost of the arcs. When power control is not used, the cost for transmitting on any (existing) arc is set to 1.

For each considered instance, we solve problem (6.11)- (6.17); i.e., we solve the LP model minimizing the energy cost of the system, with the constraint that a minimum level of fairness ϕ is assured to all nodes in the network (i.e. $\min_{i \in V} r(i) \geq \phi$). By solving the model for

Figure 6.2: Total energy consumption (log) and fairness. First definition of fairness.



different values of parameter ϕ, the behavior of the system for different values of imposed minimum fairness can be analyzed.

In particular, the feasibility and energy cost of the solution as function of the minimum fairness ϕ, and the corresponding routing configuration will be object of the analysis.

6.3.1 Computational Experiments with the First Fairness Measure

In this Section the first measure of fairness (6.7) is used. The first example considers a network of 20 nodes, generated with uniform probability on a 10X10 square, each one participating to the network (i.e. transmitting or receiving some traffic). The probabilities p_t and p_r are set to 0.6 and 0.7, respectively. The battery of nodes is not a constraint (i.e. every node has enough capacity), and the maximum transmitting power $pMax$ is large enough to let every node reach all other nodes in the network (i.e. the graph underlying the network is complete).

The graph of Figure 6.2 represents the total energy cost of the network (y axis, in logarithmic scale) as function of the imposed minimum value of fairness (x axis), when power control is used. Each curve corresponds to a different value of the constant α.

In order to have a correct comparison with the second definition of fairness, for every commodity k we allow only to use arcs having no negative component with respect to the direction defined by the commodity (i.e. we do not allow back arcs, see Section 6.2.3). This can be inefficient, in particular when α is large. However, experimentally the inefficiency

results quite small. In this example, any level of fairness can be obtained, however, the corresponding energy cost is very large, especially when high fairness is required.

The same network can be considered without the use of power control. In this case we have to reduce the maximum transmitting power (otherwise each node could talk directly to its destination, at minimum cost for the system and maximum fairness). In this case, levels of fairness up to 0.35 can be obtained, with very small energy cost increase. This means that, among the many equivalent solution of low energetic cost, there are some that are more fair. When a larger fairness is imposed, the problem (6.11)- (6.17) becomes infeasible.

It is interesting to analyze how, when increasing fairness is imposed, the flow over arcs changes. We consider the case of power control, (with $\alpha = 4$ in this example). In every one of the pictures in Figure 6.3, colored lines represent the flow between nodes due to a given commodity (i.e. OD pair in the traffic matrix). The width of each line is proportional to the amount of flow. Given two nodes i, j, a line over the segment connecting i, j is a flow from i to j, and viceversa.

Every picture corresponds to an increased minimum fairness, starting from 0 at the top left up to 1 at the bottom right. By imposing the minimum fairness to be equal to 1.0 (picture on the bottom right), every transmitter is forced to talk directly to its destination, thus this picture also gives a graphic representation of the OD matrix. If a lower levels of fairness is accepted, the flow can be routed through intermediate nodes, thus saving energy. The total cost for the system is lower and nodes have to forward other's flow. When no fairness at all is imposed (picture on the top left), the configuration of flow minimizing the energy cost of the system is obtained.

6.3.2 Computational Experiments with the Second Fairness Measure

We consider the same network discussed above ($\alpha = 4$, 20 nodes, $p_t = 0.6$ and $p_r = 0.7$), but now the minimum fairness ϕ is imposed by using the second definition (6.10).

The graph of Figure 6.4 considers the case with power control, and compares, for different levels of imposed minimum fairness, and for the two alternative measures of fairness proposed, the energetic cost of the corresponding solutions. If no fairness is imposed the solution obtained has $\phi = 0.11$ and corresponds to the one obtained by imposing $\phi \geq 0.1$. The graph shows that, when measuring the fairness experienced by each node by means of the second definition, it is possible to obtain fair solutions with acceptable energy cost increase.

Figure 6.5 represents the flow over the network, when an increasing minimum fairness is imposed, starting from 0.1 at the top left up to 1 at the bottom right. Even when a minimal fairness equal to 1.0 is imposed, part of the flow can be routed through intermediate nodes of the network, obtaining a globally less expensive solution.

It is important to point out that, when imposing fairness, we should not only consider the total energy spent by the system, but also the energies locally spent by single nodes. Actually, a fair routing has always a cost (total energy of the system) larger or equal than a less fair one. However, it is meaningful if there is at least a node which is spending less energy than in the less fair routing. The two graphs reported in Figures 6.6 and 6.7 show that this is the case in the examples previously discussed in this Section (second definition of fairness, traffic between all nodes). The graph of Figure 6.6 considers the case when power control is used ($\alpha = 4$ in this example), and reports the energies spent by the 20 nodes of the network as function of a minimum level of imposed fairness. The dashed red line describes the average energy. It can be noticed than, even if the average energy cost is increasing when

Figure 6.3: Flow distribution for minimal fairness of 0, 0.025, 0.5, 0.1, 0.3, 0.5, 0.8, 0.9, 1.0. First definition of fairness.

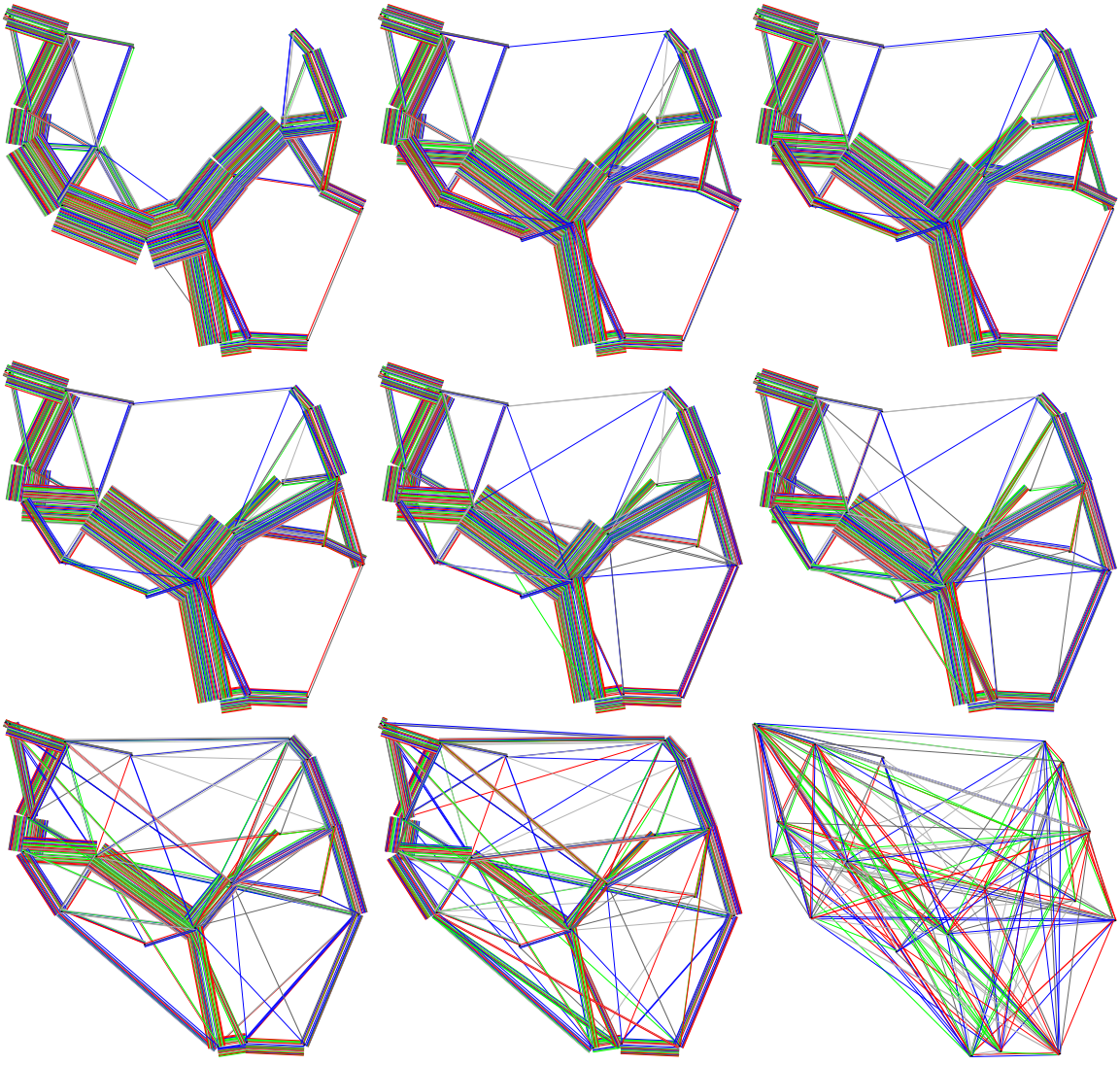
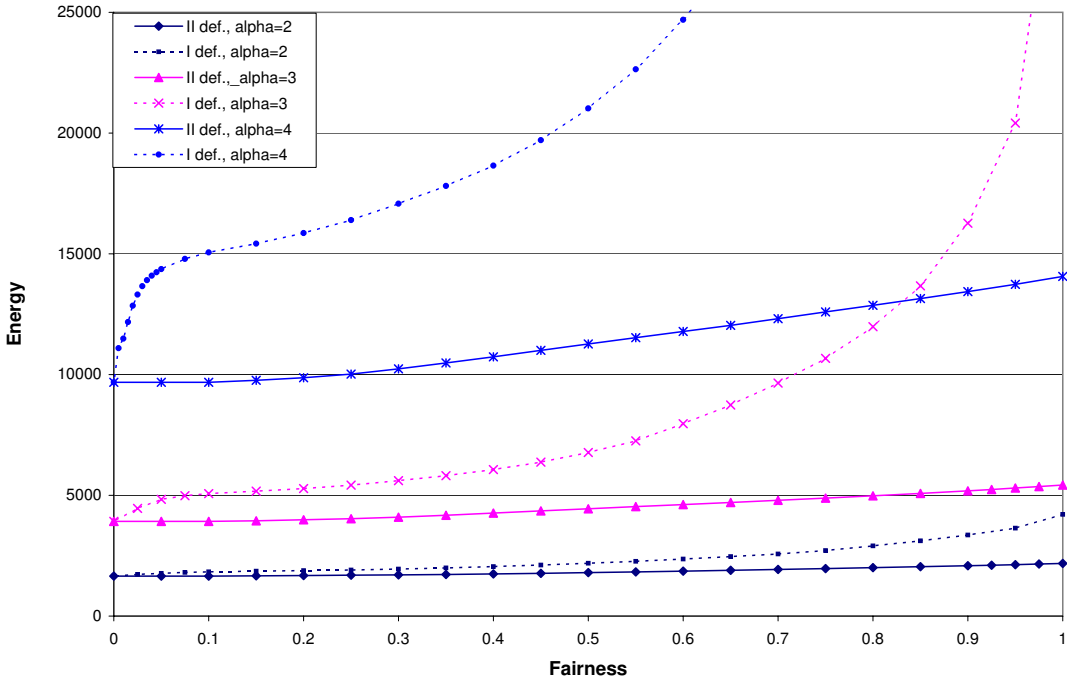


Figure 6.4: Total energy consumption and fairness. Comparison between first and second definition of fairness.



larger fairness is imposed, there are always nodes whose energy cost is decreasing, i.e. that have a real benefit when larger fairness is imposed.

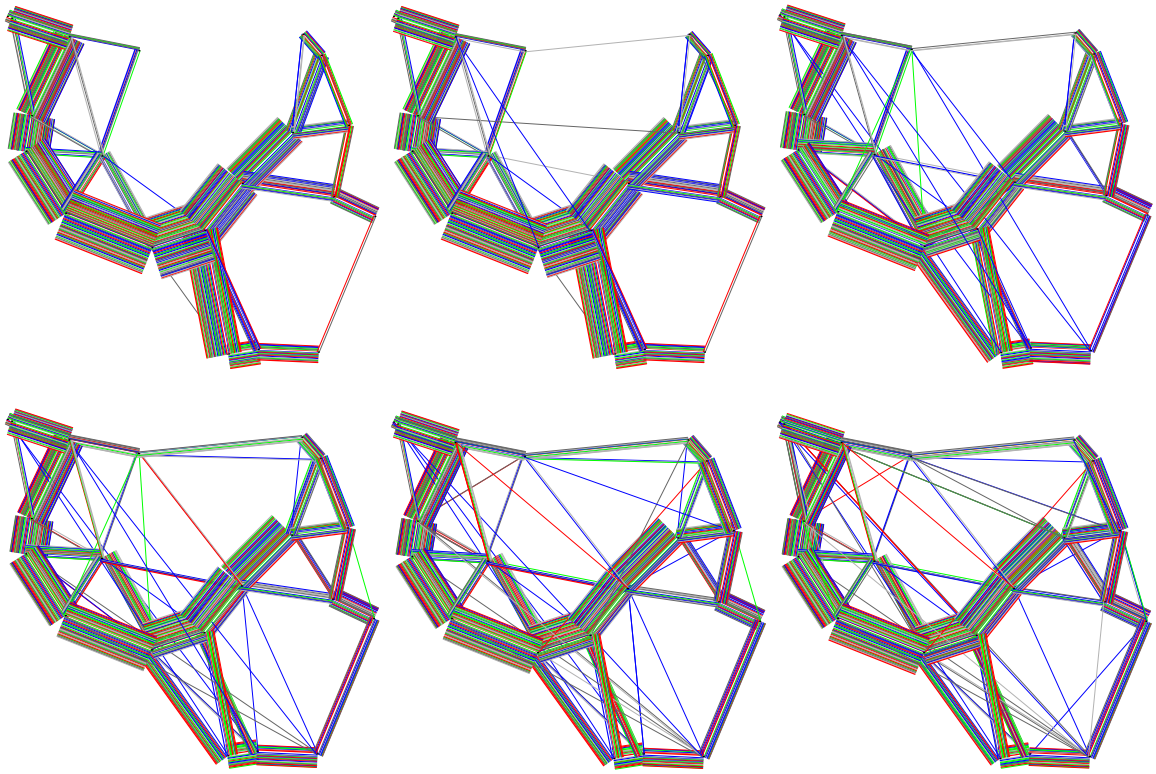
The graph of Figure 6.7 considers the same network, but now the transmitting distance equals 4/10 of the 10X10 square, and power control is not used. Thus, all existing links have the same cost (= 1 in our experiment). It is very interesting to node that a fairness of approximately 0.4 can be obtained "for free" (i.e., without energy cost increase for the system as a whole, but simply distributing the flow in a different way). This means that, among the many feasible routing of minimum cost, there is one that is quite fair.

6.3.3 Access Point Configuration

In this Section a different traffic configuration is considered, representing a network where the nodes talk only to a server, i.e. all commodities are originated from or directed to a server. This configuration can describe laptops accessing the Internet via a wireless connection through an Access Point (server). Since $\alpha \in [2, 8]$, even if a node can communicate directly to the server, it can be less expensive to forward the flow through an intermediate node. We consider a centralized control approach of this network.

Nodes are generated with a specified distribution on a surface; with a given probability p_t, node i transmits to the server, with a probability p_r, node i receives from the server (a node can both transmit and receive). The quantity of bits transmitted or received has uniform

Figure 6.5: Flow distribution for minimal fairness of 0.1, 0.3, 0.5, 0.8 and 1.0. Second definition of fairness.



distribution between 0 and a maximum value (actually it is normalized to 1). We consider the case when power control is used.

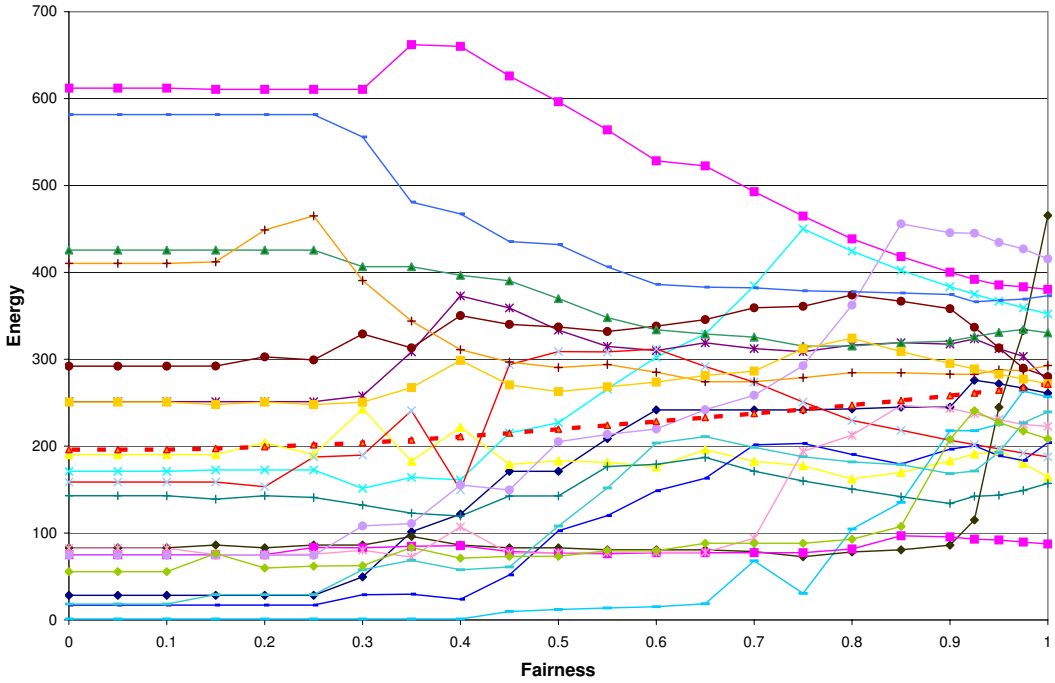
We can assume that the server is connected to a power supply system, and than the energy it spends does not represent a cost for the system. For the same reason (and because the server is always transmitting its own traffic), fairness of the server is disregarded (which is always equal to 1 when measured according to the first definition (6.7)).

In this kind of network, the second measure of fairness (6.10) is useless, actually, in no efficient solution nodes close to the server can be in a fair condition, unless they simply do not forward other node's traffic. Imposing such kind of fairness would produce useless solution, where flows having as destination nodes close to the server (which could be directly sent with no cost) would take long routes only to reduce the unbalance of such nodes.

The first definition seems more meaningful, since this configuration is unbalanced by definition (unlucky nodes is the middle have to forward). Imposing a minimum fairness would affect the quantity of flow that is routed through long arcs instead of using short connection with overloaded neighbors, and would balance the flow among forwarding nodes.

The numerical example considered has 30 nodes: 1 server in the center and 29 users with uniform distribution on a 10X10 square. The server transmits to all nodes ($p_r = 1$), while 60% of the nodes transmit to the server ($p_t = 0.6$). No constraint is imposed on the battery of the

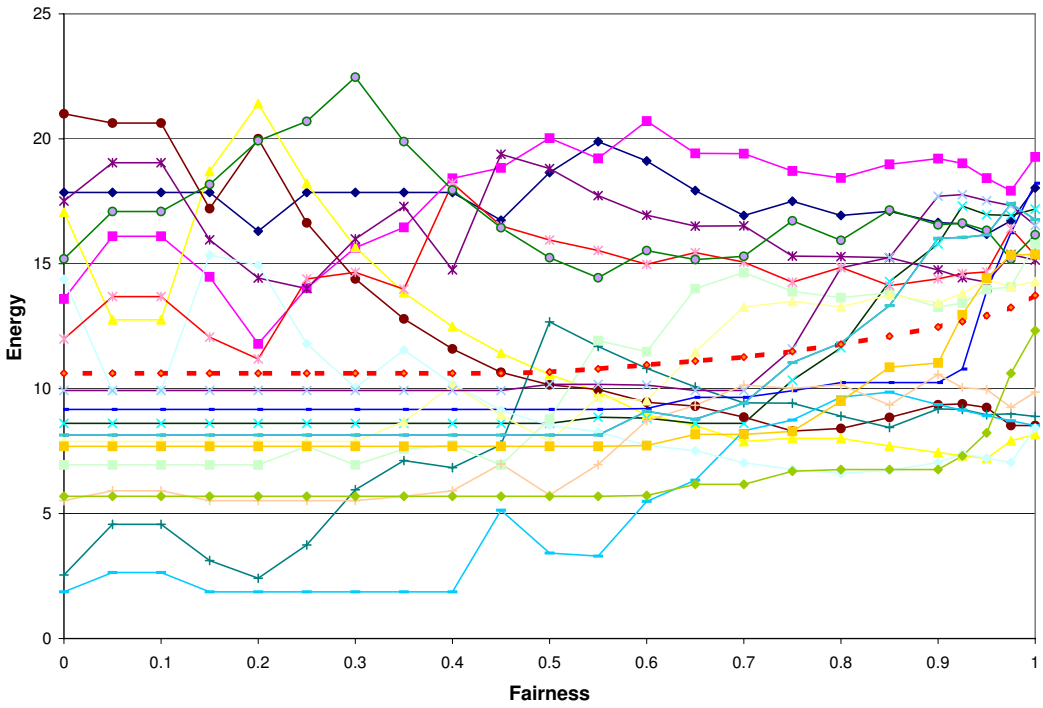
Figure 6.6: Energy consumption of the single nodes and fairness, power control. Second definition of fairness.



nodes, and power control is used. In the graph of Figure 6.8 we consider 3 different values of α, namely $\alpha = 2, 3, 4$. The maximum transmitting power is set in such a way that the maximum transmitting distance equals 3/10, 4/10 and 10/10 of the 10X10 square. The total energy cost of the solution (log) is reported for different values of fairness. In this configuration, imposing fairness has a large cost in terms of energy spent by the system, however, moderate levels of fairness can be obtained with acceptable energy cost. Not surprisingly, for a given value of α and ϕ, reducing the maximum transmitting power increases the energy cost, because less arcs are available on the network. When a high level of fairness is required, the problem becomes infeasible (and the corresponding curve is interrupted in the graph of Figure 6.8).

The pictures of Figure 6.9 represent the flow on the network, in the case of $\alpha = 4$ and maximum transmitting distance equal to 4/10 of the 10X10 square, when increasing values of fairness are imposed. The last picture ($\alpha = 1.0$), which does not represent a feasible solution because arcs that are not available in the network are used, is reported to visualize the OD pairs. It can be noticed that the flows tend to be routed on few efficient paths when fairness is disregarded, while it is split on many path when a more fair solution is imposed.

Figure 6.7: Energy consumption of the single nodes and fairness, no power control. Second definition of fairness.



6.4 A Distributed Routing Algorithm

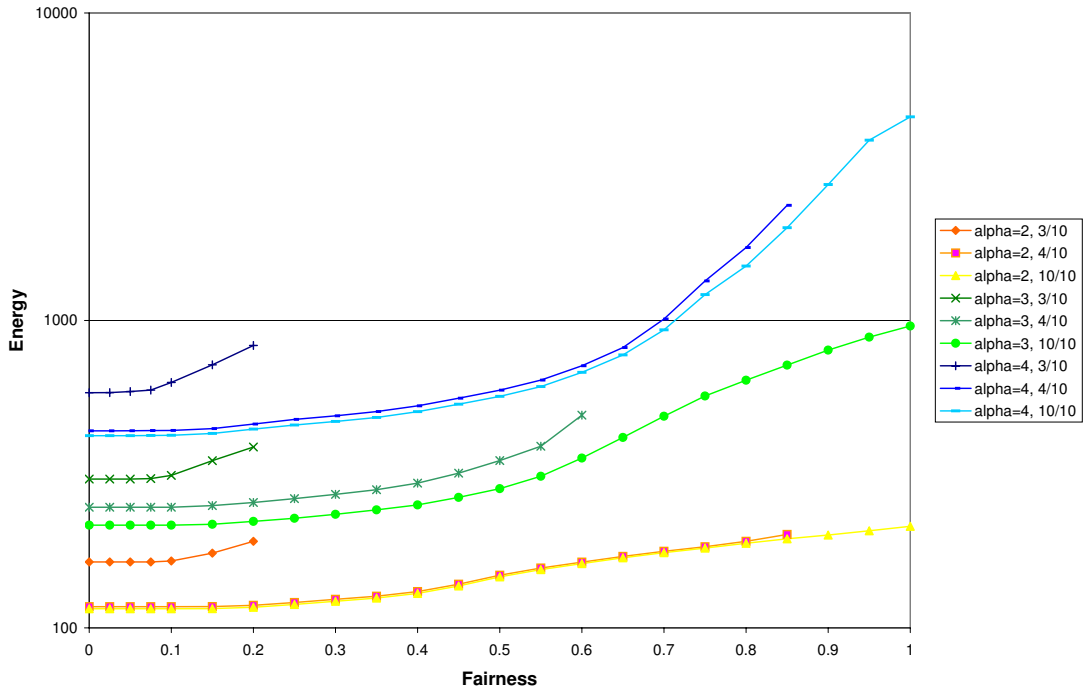
The model described in the previous Sections can be used to compute the most efficient solution for the system, under fairness constraint. It considers a static system, and requires some kind of centralized control. Moreover, it requires that all the traffic to be transmitted during a given time interval is known in advance. Conversely, ad hoc network are decentralized networks where:

- there is no centralized system control;
- only local information is available at node level;
- decisions are taken "on line", without knowledge of the future traffic.

This means that decentralized algorithms, using only local information, must be considered when routing packets through these networks. These algorithms, using only local information and decentralize control, won't possibly reach the maximal theoretical efficiency, which should be considered as a benchmark on the system performance.

In absence of node capacity (battery) and fairness constraints, the best routing for each commodity is the one using the shortest path. Our proposal is to continue using shortest path

Figure 6.8: Total energy consumption (log) and fairness. Access point configuration, first definition of fairness.



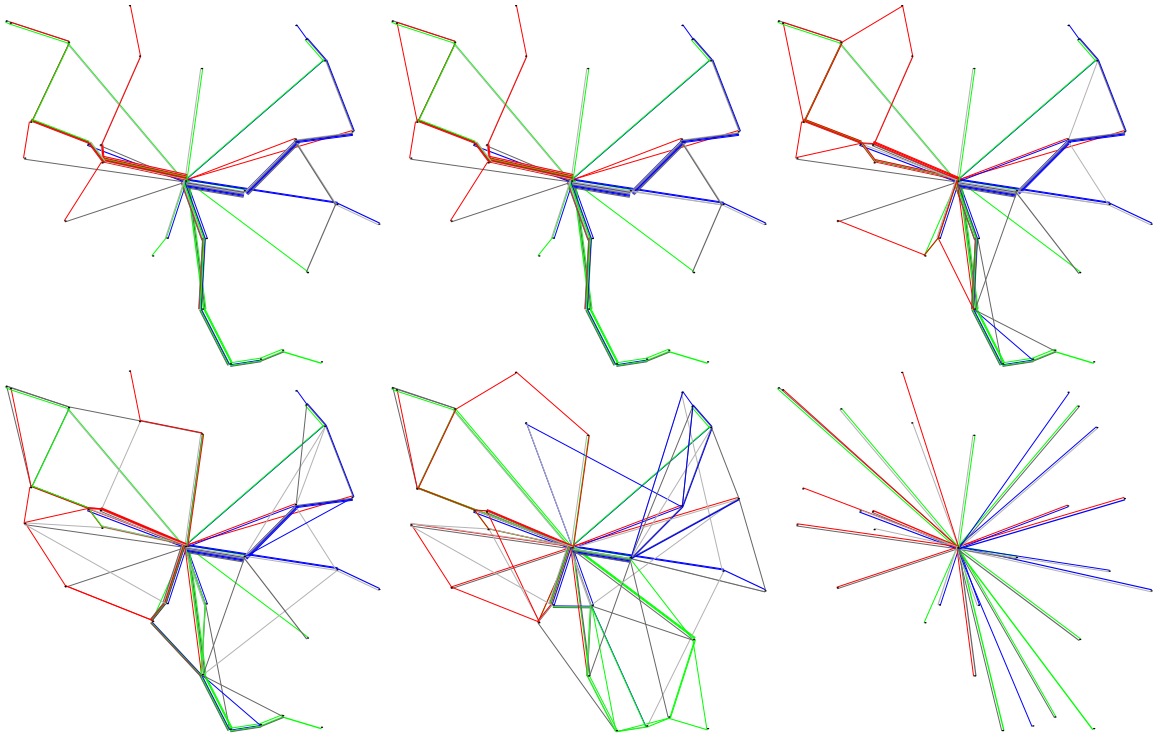
connections between origins and destinations, by changing the cost of each arc according to the unbalance and remaining batteries of its endpoints. Shortest path routing protocols are quite common in telecommunications (see, e.g., the routing protocol for ad hoc networks defined by Johnson, Maltz and Broch [69]), and we can use any of them in a transparent way, the only difference being the cost for the use of each arc.

In a dynamic setting, the flow changes over time and thus the routing should be updated at periodic intervals. When a node receives a packet to be routed to a destination, it will decide "on line", according to the actual costs of the arcs, which path to use. At the update, each node computes its current unbalance and remaining battery, and the costs of its connections are updated accordingly. Let i, j be an edge of the network, and define the original cost of this edge as the unitary energy cost: $c_{ij} = \rho + \delta_{ij}$, or $c_{ij} = \rho + 1$ when power control is not used. If remaining batteries and fairness are disregarded, these are the costs to be used to compute the shortest path routing. If we want to take into consideration also remaining capacities and fairness, the cost of each arc can be defined as: $\tilde{c}_{ij} = c_{ij}/f(r_i, r_j, p(i), p_0(i), p(j), p_0(j))$, where r_i is the fairness of node i, and $p(i)$ and $p_0(i)$ represent the remaining and starting (or maximum) batteries of node i, respectively.

The function $f(r_i, r_j, p(i), p_0(i), p(j), p_0(j))$ should have the following properties:

- $f() \geq 0$, in order to avoid the use of long inefficient paths, only to reduce the solution's cost;

Figure 6.9: Flow distribution for minimal fairness of 0.0, 0.1, 0.3, 0.5, 0.8 and 1.0. Access point configuration.



- $f(r_i, r_j, p_0(i), p_0(i), p_0(j), p_0(j)) = 1$, because we imagine the fairness to be an issue only for nodes with scarce battery;
- increasing w.r.t. r_i (a large value of r_i means that i is not contributing to the network);
- increasing w.r.t. p();

Using this kind of routing protocol requires each node to be able to know its unbalance, in order to compute its fairness and the transmitting cost of its connections. While each node knows its remaining battery and how much energy it is spending to forward other node's packets (the negative components of its unbalance), a mechanism has to be designed in order to let the node know how much energy other nodes are spending for it. Consider a commodity st: in Section 6.2.2 we defined Sq and Rq as the quota of the energy cost which are imputed to the origin s and the destination t, respectively. When all the energy cost is imputed to the receiver, forwarding nodes can simply add a "receipt" to forwarded packets (or update a string storing the total transmission cost of the packet, up to the actual node), in order to let the receiver know how much they worked for it (the receiver must know only the total amount of energy spent). When the sender too has to pay for the transmission (i.e. $Sq > 0$), the receipt can be sent back from the receiver to the sender.

6.4.1 Experimental Results

The static setting considered in this paper, where all the demand matrix is optimized at once and the traffic is sent on the less expensive paths, satisfying a required fairness level and optimizing the global cost, is the best solution for the system. It needs a central control, and in addition it requires that all the traffic to be transmitted is known in advance, which is possibly a stronger assumption. Conversely, the distributed routing algorithm that we sketched in the previous Section requires only local information, and the routing decision can be taken "on line", by considering the actual cost of the arcs (depending on the history of the system). A way to compare the performance of the decentralized algorithm with the benchmark, is to consider an OD matrix and transmit it by using the decentralized routing algorithm. However, the question is how one can imagine the transmission to be scheduled: in the static centralized setting, everything is optimized simultaneously, while the decentralized routing algorithm works "on line", the actual routing depending on the actual arcs costs, i.e. on the previous history.

Concerning the scheduling of the transmission, we can imagine to route a given fraction of all the OD matrix (the same fraction for all commodities), then update the arcs' costs, and route the next fraction, until the whole OD matrix has been routed. The advantage of this procedure is that the final routing does not depend on the order the commodities are considered, but only depends on the fraction that is routed between two consecutive updates (or, in other words, depends on the frequency of update). This procedure gives the performance of the decentralized algorithm when the traffic between each OD pair does not change over time, which is the static situation solved by the centralized approach, thus being a fair comparison. We can imagine that the OD matrix represents the traffic of a long time interval, and many updates of the arcs' cost are be performed during the transmission of the matrix.

Some numerical examples are discussed in the following. To keep the analysis simple, we do not consider the remaining batteries of the nodes. The OD matrix has to be sent in one time period, and let $freq$ be the frequency we update the arcs' costs in the time period; $update = 1/freq$ is the number of updates performed in the period. The decentralized algorithm is applied in the following way:

begin
1. iter=1;
2. **while** $(iter \leq update)$
3. **for** all commodities $st \in K$
4. $p := shortest\_path(s,t)$;
4. send $d_{st}/update$ on p;
5. **endfor**;
6. update arcs' costs;
7. **endwhile**;

In order to compute the arcs' costs $\tilde{c}_{ij} = c_{ij}/f(r_i, r_j, p(i), p_0(i), p(j), p_0(j))$, we define $f(r_i, r_j, ., ., ., .) = (\delta_{ij}\ r_i + \rho\ r_j)^\beta$ ($\delta_{ij} = 1$ when power control is not used), where β is a parameter to be experimentally tuned, which represents the importance that is given to fairness in the solution.

In the following examples, we consider the same network of Section 6.3.2 and we run the decentralized algorithms with increasing values of β. We report curves describing the cost of the solutions as function of the corresponding fairness. It's worth noting that, while in model (6.11)- (6.17) the fairness is imposed as a constraint, and thus it is know a priori, when using the decentralized algorithm we can set only the value of β, obtaining the cost and the fairness of the corresponding solution as an output. In other words, there is no guarantee on the minimum fairness of the network, but only a parameter to increase the importance given to fairness in the routing. In our experiments, we start by setting $\beta = 0$ (i.e. we route every commodity on the shortest path) and we iteratively increase it by 0.1 steps until $\beta \in [0, 1]$, and of 1.0 steps when $\beta > 1$.

In the first example, we consider the usual network of 20 nodes, generated with uniform probability on a 10X10 square, each one participating in the network (i.e. transmitting or receiving some traffic). The probabilities p_t and p_r are set to 0.6 and 0.7, respectively. The quantity of bits transmitted between each OD pair has uniform distribution between 0 and a maximum value (actually it is normalized to 1). The battery of nodes is not a constraint (i.e. every node has enough capacity), power control is used. We perform 50 updates of the arcs' cost during the transmission.

The graph of Figure 6.10 reports in purple the curve representing the total energy cost of solutions found by the decentralized algorithm, with respect to the minimal fairness experienced by a node of the network (the second definition of fairness 6.10 being used), for increasing values of the parameter β. The curve can be compared with the benchmark curve (blue), representing the solutions of model (6.11)- (6.17) when different levels of fairness are imposed (note that in this case back arcs are allowed, because they can be used by the decentralized algorithm). Even if the decentralized algorithm curve is in general monotone (by increasing β the corresponding fairness and cost are increased), this is not always true, especially for large values of β. The performance of the decentralized algorithm is close, at least for values of fairness up to 0.7, to the benchmark, showing the effectiveness of the proposed routing algorithm.

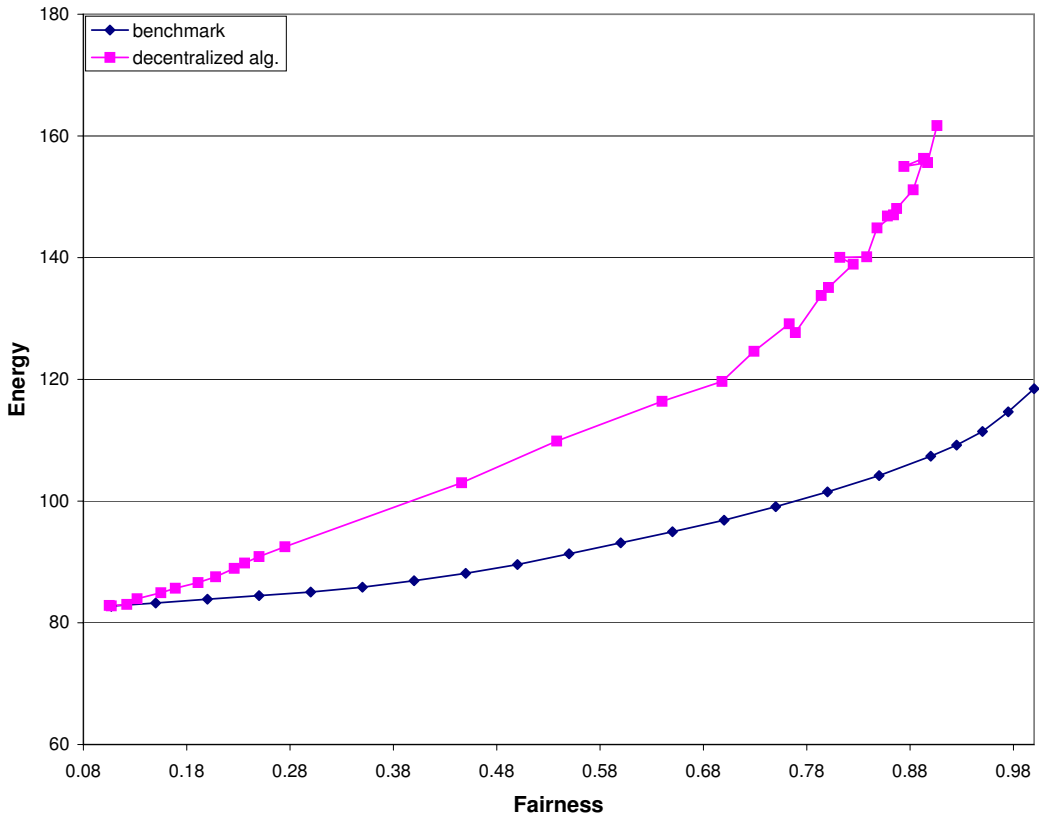
The graph of Figure 6.11 reports the distribution of energy costs among the 20 nodes of the network, with respect to the minimal fairness experienced by a node of the network. The red dashed curve reports the average cost. For quite high levels of fairness, there are still nodes whose energy is decreasing. This means that the solution is really useful, i.e. the fairness is not only obtained by making some nodes perform useless extra work, but it is really obtained by reducing the cost for nodes working too much. When β is very large, we can get solutions close to 0.9% fair. However, in this case all nodes are spending more energy that they would in less fair solutions, which are actually more desirable.

We then consider the same network, without using power control, i.e. all existing arcs have cost 1. The maximum transmitting power $pMax$ is set large enough to let every node reach all nodes within a distance equal to 4/10 of the diagonal of the square.

In the graph of Figure 6.12 the performance of the decentralized algorithm (purple curve) is compared with the benchmark. It is very interesting to node that a fairness of approximately 40% can be obtained "for free" in this example. This means that, among the many feasible routing of minimum cost, there is one that is quite fair, and the decentralized algorithm is able to find it. When we search for larger fairness, this has a cost, but the performance of the algorithm is still comparable with the benchmark.

The graph of figure 6.13 reports the distribution of energy costs among the 20 nodes of the network. The red dashed curve reports the average cost. The first "gap" in the graph,

Figure 6.10: Total energy consumption and fairness, power control. Decentralized Algorithm and Benchmark.



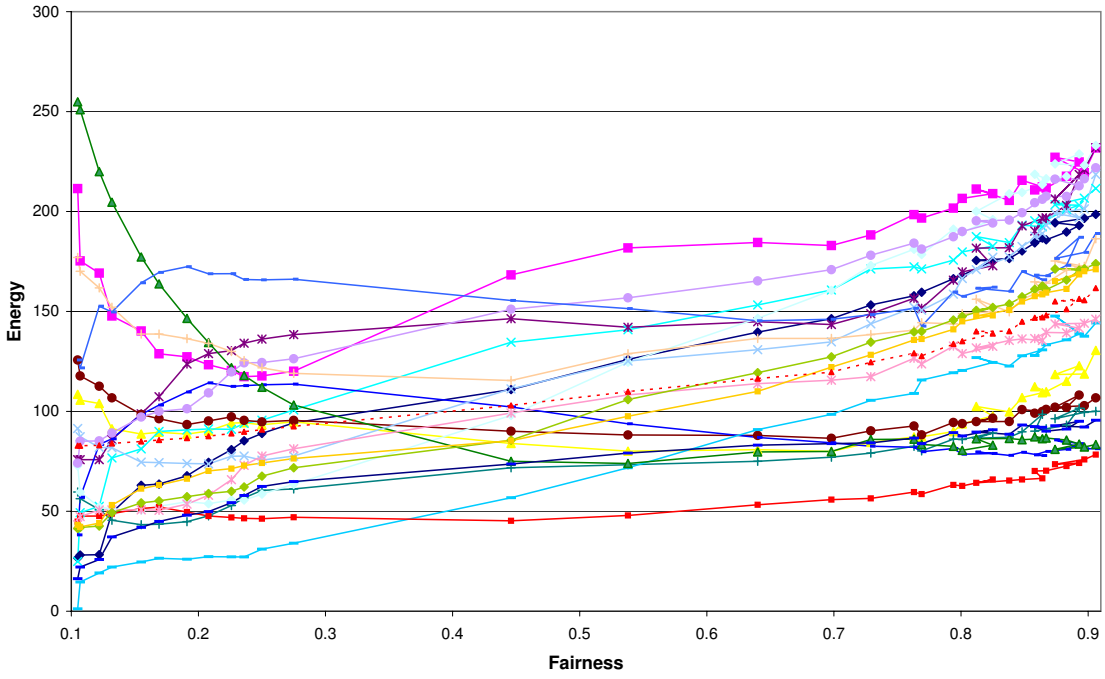
corresponding to an increase of the value β from 0 to 0.1, shows that this is enough to lead the algorithm to a fair solution, among the many equivalent solution of minimum cost.

6.5 Conclusions

Wireless ad hoc networks, where nodes communicate without making use of any preexisting infrastructure and centralized control, will be highly pervasive in the next future. A wide literature is available on ad-hoc networks, mainly devoted to the study of the efficiency of networks, and to the design of mechanisms to obtain a desired behavior from the network nodes. The efficiency of ad hoc network is highly related to the routing protocols that the network uses. However, the study of the fairness of routing protocols in ad hoc network have been rarely considered in the literature, and, to the best of our knowledge, never in a deep way.

In this paper we propose a possible model for the routing of information packets in wireless ad hoc networks, where the objective is the design of the most efficient routing, i.e. the one

Figure 6.11: Energy consumption of the single nodes and fairness, power control. Decentralized Algorithm.



of minimum energy cost. We define two alternative measures for the fairness of a routing in such networks, and discuss how an efficient routing can be computed by satisfying a minimum fairness constraint, basically through the solution of a Linear Programming Model. The computation of this routing, however, requires a set of information which is normally not available at single node level, where the routing decisions are taken. Thus, it can be considered as a benchmark on the best possible routing, and could be implemented only by a centralized control of the system, which is impossible for the intrinsic decentralized nature of ad hoc network.

So, we propose also a distributed routing algorithm, which uses only local information, available at node level. The algorithm is aimed at computing an efficient routing, while taking into consideration the fairness experienced by the nodes in the network.

Extensive computational experiments on randomly generated networks show that the measure of fairness we propose are meaningful, leading a true reduction on the energy consumption of some nodes of the network, with an acceptable loss in the network efficiency. In addition, the distributed fair routing algorithm that we propose is able to find fair routings, by using only locally available information, with a performance which is comparable to the benchmark, i.e. to the routing which could be computed by a centralized control algorithm, by solving a Linear Program.

Figure 6.12: Total energy consumption and fairness, no power control. Decentralized Algorithm and Benchmark.

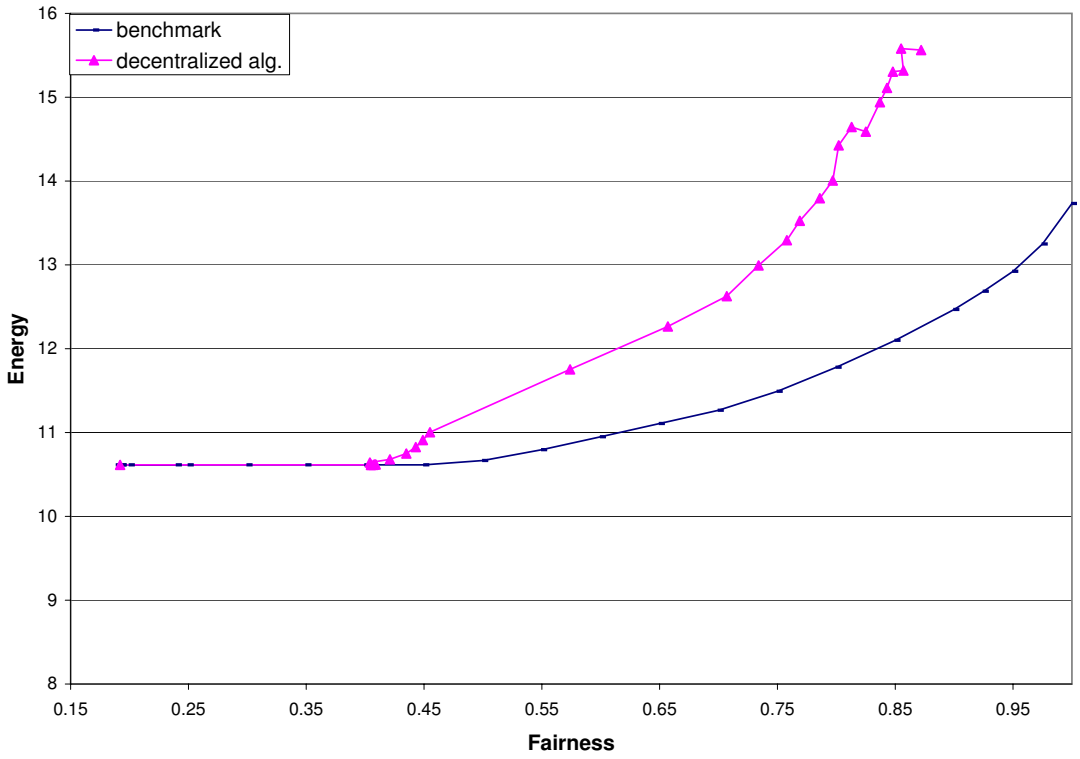
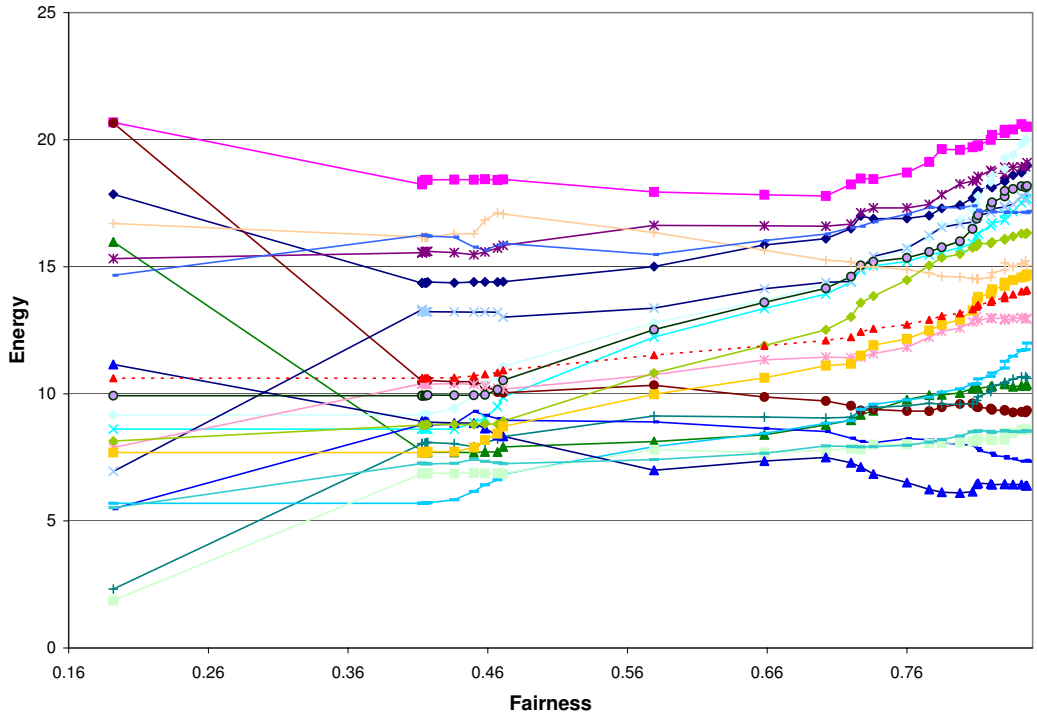


Figure 6.13: Energy consumption of the single nodes and fairness, no power control. Decentralized Algorithm.



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