THE VERTEX REINFORCED JUMP PROCESS AND A RANDOM SCHRÖDINGER OPERATOR ON FINITE GRAPHS

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We introduce a new exponential family of probability distributions, which can be viewed as a multivariate generalization of the inverse Gaussian distribution. Considered as the potential of a random Schrödinger operator, this exponential family is related to the random field that gives the mixing measure of the Vertex Reinforced Jump Process (VRJP), and hence to the mixing measure of the Edge Reinforced Random Walk (ERRW), the so-called magic formula. In particular, it yields by direct computation the value of the normalizing constants of these mixing measures, which solves a question raised by Diaconis. The results of this paper are instrumental in [Sabot and Zeng (2015)], where several properties of the VRJP and the ERRW are proved, in particular a functional central limit theorem in transient regimes, and recurrence of the 2-dimensional ERRW.

1. Introduction. In this paper, we introduce a new multivariate exponential family, which is a multivariate generalization of the inverse Gaussian law. This exponential family is associated to a network of conductances and provides a random field on the vertices of the network, the latter having the remarkable property that the marginals have inverse Gaussian law and that the field is decorrelated at distance two.

This exponential family is mainly motivated by the study of two self-interacting processes, namely the Edge Reinforced Random Walk (ERRW) and the closely related Vertex Reinforced Jump Process (VRJP), but it could also find some applications in other topics, such as Bayesian statistics for instance. Note that Diaconis and Rolles [8] introduced in 2006 a family of Bayesian priors for reversible Markov chains, similarly associated to the limit measure of the ERRW.

More precisely, we consider a nondirected finite graph $\mathcal{G} = (V, E)$ with strictly positive conductances $W_{i,j} = W_{j,i}$ on the edges. Denote by Δ^W the discrete Laplace operator associated to the conductance network $(W_{i,j})$ and write $W_i = \sum_{j:\{i,j\}\in E} W_{i,j}$. The exponential family provides a random vector of positive reals $(\beta_j)_{j\in V}$ such that

$$H_\beta := -\Delta^W + V$$

is a.s. a positive operator, where $V = 2\beta - W$ is the operator of multiplication by $(2\beta_i - W_i)$ and $2\beta - W$ is considered as a random potential. We prove in Theorem 3 that if the Green function is defined by $G = (H_\beta)^{-1}$, then the field (e^{u_j}) giving the mixing measure of the VRJP starting from i_0, cf. [14], is equal in law to $(G(i_0, j)/G(i_0, i_0))$.

This has several consequences. First, it relates the VRJP to a random Schrödinger operator with an explicit random potential with decorrelation at distance 2. Note that Anderson localization was the main motivation in the papers of Disertori, Spencer, Zirnbauer ([10, 11]): in these works, the supersymmetric field related to the mixing measure of the VRJP (cf. [14]) is viewed as a toy model for some supersymmetric fields that appears in the physics literature in connection with random band matrices. Second, it enables one to couple the mixing fields of the VRJP starting from different points. Finally, using the link between VRJP and ERRW [14, 17], it yields an answer to an old question of Diaconis about the direct computation of the normalizing constant of the "magic formula" for the mixing measure of ERRW.

Results of this paper are instrumental in [16], where the representation in terms of a random Schrödinger operator is extended to infinite graphs. Interesting new phenomena appear in the transient case, where a generalized eigenfunction of the Schrödinger operator is involved in the representation. Several consequences follow on the behavior of the VRJP and the ERRW in [16]: in particular, a functional central limit theorem is proved for the VRJP and the ERRW in dimension $d \geq 3$ at weak reinforcement, and recurrence of the 2-dimensional ERRW is shown, giving a full answer to an old question of Diaconis.

The paper is organized as follows. In Section 2, we define the new exponential family of distributions and give its first properties. In Section 3, we discuss the link between the exponential family and the Vertex reinforced jump processes. In Section 4, we consider the ERRW and answer the question of Diaconis. Sections 5 and 6 provide the proof of the two main results, namely Theorem 1 and Theorem 3.

2. A new exponential family. Let $V = \{1, \ldots, n\}$ be a finite set, and let $(W_{i,j})_{i \neq j}$ be a set of nonnegative reals with $W_{i,j} = W_{j,i} \geq 0$. Denote by E the edges associated to the positive $W_{i,j}$, that is, consider the graph $\mathcal{G} = (V, E)$ with $\{i, j\} \in E$ if and only if $W_{i,j} > 0$, and write $i \sim j$ if $\{i, j\} \in E$. Let $\mathrm{d}_{\mathcal{G}}$ be the graph distance on $\mathcal{G}$.

When A is a symmetric operator on $\mathbb{R}^V$ (also be considered as a $V \times V$ matrix), write $A > 0$ if A is positive definite, and $|A|$ for its determinant.

THEOREM 1.   *Let $P = (P_{i,j})_{1 \leq i, j \leq n}$ be the symmetric matrix given by*

$$P_{i,j} = \begin{cases} 0, & i = j, \\ W_{i,j}, & i \neq j. \end{cases}$$

For any $\theta \in \mathbb{R}_+^n$, we have

(1)
$$\left(\frac{2}{\pi}\right)^{n/2} \int \mathbb{1}_{\{2\beta - P > 0\}} e^{-\langle \theta, \beta \rangle} \frac{d\beta}{\sqrt{|2\beta - P|}}$$

$$= \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}\sqrt{\theta_i \theta_j}\right) \cdot \prod_{i=1}^n \frac{1}{\sqrt{\theta_i}},$$

where $d\beta = d\beta_1 \cdots d\beta_n$, and $2\beta - P$ is the operator on $\mathbb{R}^V$ defined by

$$[(2\beta - P)f](i) = 2\beta_i f(i) - \sum_{j:j\sim i} W_{i,j} f(j).$$

DEFINITION 1.   The exponential family of random probability measures $\nu^{W,\theta}(d\beta)$ is defined by

$$\nu^{W,\theta}(d\beta) = \mathbb{1}_{2\beta - P > 0}\left(\frac{2}{\pi}\right)^{n/2} \exp\left(-\langle \theta, \beta \rangle + \sum_{\{i,j\}\in E} W_{i,j}\sqrt{\theta_i \theta_j}\right) \frac{\prod_i \sqrt{\theta_i}}{\sqrt{|2\beta - P|}}\, d\beta,$$

where $\langle \theta, \beta \rangle = \sum_{i\in V} \theta_i \beta_i$. We will simply write ν^W for $\nu^{W,1}$ in the case where $\theta_i = 1$ for all $i \in V$.

The proof of Theorem 1 is given in Section 5. We deduce the following simple but important properties of the measure $\nu^{W,\theta}$.

PROPOSITION 1.   *The Laplace transform of $\nu^{W,\theta}$ is*

$$\int e^{-\langle \lambda, \beta \rangle} \nu^{W,\theta}(d\beta)$$

$$= \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}(\sqrt{\lambda_i + \theta_i}\sqrt{\lambda_j + \theta_j} - \sqrt{\theta_i \theta_j})\right) \cdot \prod_{i=1}^n \sqrt{\frac{\theta_i}{\lambda_i + \theta_i}}.$$

Moreover, if β is a random vector with distribution $\nu^{W,\theta}$, then:

- *The marginals β_i are such that $\frac{1}{2\beta_i \theta_i}$ is an Inverse Gaussian distribution with parameters $(\frac{1}{\sum_{j\sim i} W_{i,j}\sqrt{\theta_i \theta_j}}, 1)$.*
- *If $V_1 \subset V$, $V_2 \subset V$ are two subsets of V such that $d_{\mathcal{G}}(V_1, V_2) \geq 2$, then $(\beta_i)_{i\in V_1}$ and $(\beta_j)_{j\in V_2}$ are independent.*

PROOF.   The Laplace transform of $\nu^{W,\theta}$ can be computed directly from Theorem 1, from which we deduce independence at distance at least 2. We can also deduce, by identification of the Laplace transforms, that the marginals of this law are reciprocal inverse gaussian up to a multiplicative constant.   □

The family can be reduced to the case $\theta = 1$ by changing W, as shown in the next corollary.

COROLLARY 1. *Let $(\beta_j)_{j \in V}$ be distributed according to $\nu^{W,\theta}$. Then $(\theta\beta)$ is distributed according to ν^{W^θ}, where $W^\theta_{i,j} = W_{i,j}\sqrt{\theta_i \theta_j}$.*

It is clear from the expression of the Laplace transform that if the graph has several connected components then the random field $(\beta_j)_{j \in V}$ splits accordingly into independent random subvectors. Therefore, we will always assume in the sequel that the graph $\mathcal{G}$ is connected.

3. Link with the vertex reinforced jump process.

3.1. *Vertex reinforced jump process*: *Definition and main properties.* In this section, we explain the link between the exponential family of Section 2 and the Vertex reinforced Jump Process (VRJP), which is a linearly reinforced process in continuous time, defined in [6], investigated on trees in [3], and on general graphs by the first two authors in [14]. Consider as in the previous section a conductance network $(W_{i,j})$ and the associated graph $\mathcal{G} = (V, E)$. Fix also some positive parameters $(\phi_i)_{i \in V}$ on the vertices. Assume that the graph $\mathcal{G}$ is connected.

We call VRJP with conductances $(W_{i,j})$ and initial local time (ϕ_i) the continuous-time process $(Y_t)_{t \geq 0}$ on V, starting at time 0 at some vertex $i_0 \in V$ and such that, if Y is at a vertex $i \in V$ at time t, then, conditionally on $(Y_s, s \leq t)$, the process jumps to a neighbour j of i at rate

$$W_{i,j} L_j(t),$$

where

$$L_j(t) := \phi_j + \int_0^t \mathbb{1}_{\{Y_s = j\}} \, ds.$$

The following time change, introduced in [14], plays a central role. Let

$$(2) \qquad D(t) = \sum_{i \in V} \left(L_i^2(t) - \phi_i^2 \right),$$

define Z_t as the time changed process

$$Z_t = Y_{D^{-1}(t)}.$$

Let $(\ell_j(t))$ be the local time of Z at time t [i.e., $\ell_j(t) = \int_0^t \mathbb{1}_{Z_s = j} \, ds$]. Conditionally on the past, at time t, the process Z jumps from $Z_t = i$ to a neighbour j at rate (cf. [15], Lemma 3)

$$\frac{W_{i,j}}{2} \sqrt{\frac{\phi_j^2 + \ell_j(t)}{\phi_i^2 + \ell_i(t)}}.$$

We state below one of the main results of [14], Proposition 1 and Theorem 2. The theorem was stated in [14] in the case $\phi = 1$, this version of the theorem can be deduced by a simple change of time, details are given in Appendix B.

THEOREM 2.   *Assume that $\mathcal{G}$ is finite. Suppose that the VRJP starts at i_0. The limit*

$$U_i = \frac{1}{2} \lim_{t \to \infty} \left(\log\left(\frac{\ell_i(t) + \phi_i^2}{\ell_{i_0}(t) + \phi_{i_0}^2} \right) - \log\left(\frac{\phi_i^2}{\phi_{i_0}^2} \right) \right)$$

exists a.s. and, conditionally on U, Z is a Markov jump processes with jump rate from i to j:

$$\frac{1}{2} W_{i,j} e^{U_j - U_i}.$$

Moreover, (U_j) has the following distribution on $\{(u_i), u_{i_0} = 0\}$:

(3)
$$\mathcal{Q}_{i_0}^{W,\phi}(du) = \frac{\prod_{j \neq i_0} \phi_j}{\sqrt{2\pi}^{|V|-1}} e^{-\sum_{j \in V} u_j} e^{-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} (e^{u_i - u_j} \phi_j^2 + e^{u_j - u_i} \phi_i^2 - 2\phi_i \phi_j)}$$
$$\times \sqrt{D(W,u)}\, du,$$

with $du = \prod_{j \in V \setminus \{i_0\}} du_j$ and

$$D(W, u) = \sum_T \prod_{\{i,j\} \in T} W_{i,j} e^{u_i + u_j},$$

where the sum runs on the set of spanning trees T of $\mathcal{G}$. We simply write $\mathcal{Q}_{i_0}^W$ for $\mathcal{Q}_{i_0}^{W,1}$.

The fact that the total mass of the measure $\mathcal{Q}_{i_0}^{W,\phi}$ is 1 is both a nontrivial and a useful fact: in particular, it plays a central role in the delocalization and localization results of [10, 11]. In [14], it is a consequence of the fact that it is the probability distribution of the limit random variables U. In [11], it is proved using a sophisticated supersymmetric argument, the so-called localization principle. Theorem 3 below provides a direct "computational" proof of that result, based on the identity (1) and on the change of variable in Proposition 2 that relates the field (u_j) to the random vector (β_j) in Definition 1.

3.2. *Link with the random potential β.*   The second main result of this paper enables us to construct the mixing field e^u defined in the previous subsection from the random potential (β_j) defined in Definition 1. It gives also a natural way to couple the mixing measure of VRJP starting from different points.

Let us first state the following Proposition 2, which provides some elementary observations on the Green function.

Define

$$\mathcal{D} = \left\{ (\beta_i)_{i \in V} \in \left(\mathbb{R}_+ \setminus \{0\} \right)^V, 2\beta - P > 0 \right\}.$$

PROPOSITION 2. *Let $\beta \in \mathcal{D}$, and let G be the inverse of $(2\beta - P)$. Then $(G(i, j))$ has positive coefficients. Define $(u(i, j))_{i,j \in V}$ by*

$$e^{u(i,j)} = \frac{G(i, j)}{G(i, i)}.$$

Then for $i_0 \in V$, the function $j \to u(i_0, j)$ is the unique solution $j \mapsto u_j$ of the equation

$$(4) \qquad \begin{cases} \displaystyle\sum_{j \sim i} \frac{1}{2} W_{i,j} e^{u_j - u_i} = \beta_i, & i \neq i_0, \\ u_{i_0} = 0. \end{cases}$$

In particular, $(u(i_0, j))_{j \in V}$ is $(\beta_j)_{j \in V \setminus \{i_0\}}$ measurable. Moreover, at the site i_0 we have

$$\beta_{i_0} = \frac{1}{2G(i_0, i_0)} + \sum_{j:j \sim i_0} \frac{1}{2} W_{i_0, j} e^{u(i_0, j)}.$$

THEOREM 3. *Let β be a random potential with distribution $\nu^{W, \phi^2}(d\beta)$ as in Definition 1, and let $(u(i, j))_{i,j \in V}$ be defined as in Proposition 2. Then the following properties hold:*

(i) *The random field $(u(i_0, j))_{j \in V}$ has the distribution of the mixing measure $\mathcal{Q}_{i_0}^{W, \phi}(du)$ of the VRJP starting from i_0 with initial local time $(\phi_i)_{i \in V}$.*

(ii) *The random variable $G(i_0, i_0)$ has the distribution of $1/(2\gamma)$, where γ is a gamma random variable with parameters $(1/2, 1/\phi_{i_0}^2)$. Moreover, $G(i_0, i_0)$ is independent of $(\beta_j)_{j \neq i_0}$, and thus also of the field $(u(i_0, j))_{j \in V}$.*

The proofs of Proposition 2 and Theorem 3 are given in Section 6. The next Corollary 2 describes how to construct the random potential β from the field u of Theorem 2.

COROLLARY 2. *Consider a VRJP with edge weight $(W_{i,j})$ and initial local time $(\phi_i)_{i \in V}$, starting at i_0. Let $(u_i)_{i \in V}$ be distributed according to $\mathcal{Q}_{i_0}^{W, \phi}$ of Theorem 2. Let*

$$(5) \qquad \tilde{\beta}_i = \frac{1}{2} \sum_{j:j \sim i} W_{i,j} e^{u_j - u_i}.$$

Let γ be a Gamma distributed random variable with parameters $(\frac{1}{2}, 1/\phi_{i_0}^2)$, independent of (u_j), and let

$$(6) \qquad \beta_i = \tilde{\beta}_i + \mathbb{1}_{i_0} \gamma.$$

Then β has the law ν^{W, ϕ^2} of Definition 1.

Corollary 2 indeed follows directly from Theorem 3 and Proposition 2: the law of β in (6) is uniquely determined by the laws of $(u_i)_{i \in V}$ and γ independent from $(\beta_i)_{i \neq i_0}$. Hence it is sufficient to show that, if β has distribution $\nu^{W, \phi^2}(d\beta)$ and u is defined from (4) by Proposition 2, then $(u_i)_{i \in V}$ indeed has distribution $\mathcal{Q}_{i_0}^{W, \phi}$, and $\gamma = \beta_{i_0} - \tilde{\beta}_{i_0} = 1/(2G(i_0, i_0))$ has distribution $\Gamma(1/2, 1/\phi_{i_0}^2)$, which follows from Theorem 3.

As mentioned in the Introduction, Theorem 3 has several consequences. First, it explicitly relates the VRJP to the random Schrödinger operator $-\Delta^W + V$, where V is the random potential $V_i = 2\beta_i - W_i$. Second, it yields a natural coupling between the random fields $(u_j)_{j \in V}$ associated with the VRJP starting from different sites, since the exponential family $(\beta_i)_{i \in V}$ gives the same role to each vertex of the graph, and $(u(i, j))_{i, j \in V}$ arises from these random variables $(\beta_i)_{i \in V}$. Finally, it also gives a computational proof of the identity $\int \mathcal{Q}_{i_0}^{W, \theta}(du) = 1$, for any θ, as a consequence of Theorem 1 that allows to define $\nu^{W, \phi^2}(d\beta)$ as a probability measure.

4. Link with the edge reinforced random walk and a question of Diaconis.

4.1. *Definition and magic formula.* The Edge Reinforced Random Walk (ERRW) is a famous discrete time process introduced in 1986 by Coppersmith and Diaconis [5].

Let $(a_{i, j})_{\{i, j\} \in E}$ be a set of positive weights on the edges of the graph $\mathcal{G}$. Let $(X_n)_{n \in \mathbb{N}}$ be a random process that takes values in V, and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ be the filtration of its past. For any $e \in E$, $n \in \mathbb{N}$, let

$$(7) \qquad Z_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{\{X_{k-1}, X_k\}=e\}}$$

be the number of crosses of the (nondirected) edge e up to time n plus the initial weight a_e.

Then $(X_n)_{n \in \mathbb{N}}$ is called Edge Reinforced Random Walk (ERRW) with starting point $i_0 \in V$ and weights $(a_e)_{e \in E}$, if $X_0 = i_0$ and, for all $n \in \mathbb{N}$,

$$(8) \qquad \mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j \sim X_n\}} \frac{Z_n(\{X_n, j\})}{\sum_{k \sim X_n} Z_n(\{X_n, k\})}.$$

We denote by $\mathbb{P}_{i_0}^{\text{ERRW}, (a)}$ the law of the ERRW starting from the initial vertex i_0 and initial weights (a).

A fundamental property of the ERRW, stated in the next theorem, is that on finite graphs the ERRW is a mixture of reversible Markov chains, and the mixing measure can be determined explicitly (the so-called Coppersmith–Diaconis measure, or "magic formula"). It is a consequence of a de Finetti theorem for Markov chains due to Diaconis and Freedman [7], and the explicit determination of the law

is due to Diaconis and Coppersmith [5, 12, 13]. It has also applications in Bayesian statistics [1, 2, 8].

THEOREM 4 ([5, 12]). *Assume that $\mathcal{G} = (V, E)$ is a finite graph and set $a_i = \sum_{j:\{i,j\}\in E} a_{i,j}$ for all $i \in V$. Fix an edge e_0 incident to i_0, and define $\mathcal{H}_{e_0} = \{y : \forall e \in E, y_e > 0, y_{e_0} = 1\}$ (similarly let $y_i = \sum_{i\in e} y_e$). Consider the following positive measure defined on $\mathcal{H}_{e_0}$ defined by its density:*

$$(9) \qquad \mathcal{M}_{i_0}^{(a)}(dy) = C(a, i_0) \frac{\sqrt{y_{i_0}} \prod_{e\in E} y_e^{a_e}}{\prod_{i\in V} y_i^{\frac{1}{2}(a_i+1)}} \sqrt{D(y)} \prod_{e\neq e_0} \frac{dy_e}{y_e},$$

with

$$D(y) = \sum_T \prod_{e\in T} y_e,$$

where the sum runs on the set of spanning trees T of $\mathcal{G}$, and with

$$C(a, i_0) = \frac{2^{1-|V|+\sum_{e\in E} a_e}}{\sqrt{\pi}^{|V|-1}} \cdot \frac{\prod_{i\in V} \Gamma(\frac{1}{2}(a_i + 1 - \mathbb{1}_{i=i_0}))}{\prod_{e\in E} \Gamma(a_e)}.$$

Then $\mathcal{M}_{i_0}^{(a)}$ is a probability measure on $\mathcal{H}_{e_0}$, and it is the mixing measure of the ERRW starting from i_0, more precisely

$$\mathbb{P}_{i_0}^{\mathrm{ERRW},(a)}(\cdot) = \int_{\mathcal{H}_{i_0}} P_{i_0}^{(y)}(\cdot) \, d\mathcal{M}_{i_0}^{(a)}(y),$$

where $P_{i_0}^{(y)}$ denotes the reversible Markov chain starting at i_0 with conductances (y).

4.2. *The question of Diaconis.* The fact that $\mathcal{M}_{i_0}^{(a)}(dy)$ is a probability measure is a consequence of the fact that it is the mixing measure of the ERRW. In fact, it is obtained as the limit distribution of the normalized occupation time of the edges [12]:

$$\left(\frac{Z_n(e)}{Z_n(e_0)}\right)_{e\in E} \xrightarrow{\text{law}} \mathcal{M}_{i_0}^{(a)}.$$

One question raised by Diaconis is the following:

$$(10) \qquad \text{(Q) Prove by direct computation that } \int \mathcal{M}_{i_0}^{(a)}(dy) = 1.$$

An answer was proposed by Diaconis and Stong [9] in the case of the triangle, using a subtle change of variables. Also note that Merkl and Rolles offered in [13] analytic tools for the computation of the ratio of the normalizing constants of the magic formula for two initial weights differing by integer values, which may possibly be extended to provide the normalizing constant.

We provide below an answer to that question. A first simplification comes from [14, 17], where an explicit link was made between the VRJP and the ERRW.

THEOREM 5 (Theorem 1, [14]).   *Consider (Y_n) the discrete time process associated with the VRJP (Y_t) (i.e., taken at jump times) with conductances $(W_{i,j})$ and $\phi = 1$. Take now the conductances $(W_e)_{e \in E}$ as independent random variables with gamma distribution with parameters $(a_e)_{e \in E}$. Then the "annealed" law of Y_n [i.e., the law after taking expectation with respect to the random (W_e)] is the law of the ERRW (X_n) with initial weights $(a_e)_{e \in E}$.*

This immediately implies an identity between the mixing measures $\mathcal{M}_{i_0}^{(a)}$ and $\mathcal{Q}_{i_0}^W$: indeed, by Theorem 2, (Y_n) is a mixture of Markov jump processes with conductances $W_{i,j} e^{u_i + u_j}$, which implies that for all 0-homogeneous bounded test functions ϕ (i.e., $\phi[\lambda y] = \phi(y)$, $\forall \lambda > 0$], we have

$$
\int_{\mathcal{H}_{e_0}} \phi((y_e)) \mathcal{M}_{i_0}^{(a)}(dy)
$$

(11)
$$
= \int_{\mathbb{R}^E} \prod_{e \in E} \frac{W_e^{a_e - 1} e^{-W_e}}{\Gamma(a_e)} \left(\int \phi((W_{i,j} e^{u_i + u_j})) \mathcal{Q}_{i_0}^W(du) \right) dW
$$

with $dW = \prod_{e \in E} dW_e$. This identity was checked by direct computation in Section 5 of [14].

Now, the fact that $\int \mathcal{Q}_{i_0}^W(du) = 1$ is a consequence of the computation of the integral (1) in Theorem 1 and the change of variables in Theorem 3, as explained at the end of Section 3. Therefore,

$$
\int_{y_{e_0} = 1} d\mathcal{M}_{i_0}^a(y) = 1.
$$

Note that this fact can be used to prove directly that $\mathcal{M}_{i_0}^a(dy)$ is the mixing measure of the ERRW starting from initial condition (a) and initial vertex i_0. Indeed, for any finite path $\sigma : i_0 \to i_1 \to \cdots \to i_n$, let $N(i)$ [resp. $N(e)$] be the number of times vertex i (resp., edge e) is visited (resp., crossed):

$$
N(i) = |\{k; 0 \le k \le n - 1, i_k = i\}|,
$$
$$
N(e) = |\{k; 0 \le k \le n - 1, \{i_k, i_{k+1}\} = e\}|.
$$

The probability of σ for the reversible Markov chain of conductance y is

$$
p_{i_0}^y(\sigma) = \frac{\prod_{e \in E} y_e^{N(e)}}{\prod_{i \in V} y_i^{N(i)}}.
$$

The integration of $p_{i_0}^y(\sigma)$ w.r.t. $d\mathcal{M}_{i_0}^a(y)$ can be computed by changing the constant $\Gamma(a_e)$ to $\Gamma(a_e + N_e)$ and $\Gamma(\frac{1}{2}(a_i + 1))$ to $\Gamma(\frac{1}{2}(a_i + 1) + N_i)$. Using the property $\Gamma(x + 1) = x\Gamma(x)$ and the notation $(a, n) = \prod_{k=0}^{n-1}(a + k)$, we deduce

$$
\int p_{i_0}^y(\sigma) d\mathcal{M}_{i_0}^a(y) = \frac{\prod_e (a_e, N(e))}{\prod_i (a_i, N(i))}
$$

which is the probability of an ERRW to follow the path σ.

5. Proof of Theorem 1.

LEMMA 1. *Let $P = (P_{i,j})_{1 \leq i, j \leq n}$ be a symmetric matrix with*

$$P_{i,j} = \begin{cases} 0, & i = j, \\ W_{i,j} \in \mathbb{R}^+, & i \neq j, \end{cases}$$

and let β be a diagonal matrix with entries β_i, $i = 1, \ldots, n$, such that $M = 2\beta - P$ is positive definite.

Let L be the lower triangular $n \times n$ matrix and U be the upper unitary (with 1 on the diagonal) upper triangular matrix such that $M = LU$ (i.e., the LU decomposition of M), which exist and are unique.

Then

$$U = \begin{pmatrix} x_1 & -H_{1,2} & \cdots & -H_{1,n} \\ 0 & x_2 & \cdots & -H_{2,n} \\ & \cdots & & -H_{n-1,n} \\ 0 & \cdots & 0 & x_n \end{pmatrix},$$

where $(x_i)_{1 \leq i \leq n}$ and $(H_{i,j})_{1 \leq i < j \leq n}$ are defined recursively by

$$\begin{cases} H_{1,j} = W_{1,j}, & j > 1, \\ H_{i,j} = W_{i,j} + \sum_{k=1}^{i-1} \dfrac{H_{k,i} H_{k,j}}{x_k}, & i \geq 2, j > i, \\ x_i = 2\beta_i - \sum_{k=1}^{i-1} \dfrac{H_{k,i}^2}{x_k}, & i \geq 1. \end{cases}$$

Furthermore,

$$x_i = \frac{M(1, \ldots, i \mid 1, \ldots, i)}{M(1, \ldots, i-1 \mid 1, \ldots, i-1)},$$

where $M(I \mid J)$ is the minor of matrix M that corresponds to the rows with index in I and columns with index in J.

The result follows directly from (2.6) of [18], but we prove it in Appendix A for completeness' sake.

CLAIM 1. *For any $\theta_1 > 0$, $\theta_2 \geq 0$,*

$$\int_0^\infty \exp\left(-\frac{\theta_1 x}{2} - \frac{\theta_2}{2x}\right) \frac{1}{\sqrt{x}} \, dx = \exp(-\sqrt{\theta_1 \theta_2}) \sqrt{\frac{2\pi}{\theta_1}}.$$

PROOF.   The case $\theta_2 = 0$ corresponds to the normalisation of the $\Gamma(\frac{1}{2})$ variable. The case $\theta_2 > 0$ corresponds to the normalization of the Inverse Gaussian law $\mathrm{IG}(\frac{\theta_1}{\theta_2}, \frac{1}{\theta_2})$.   □

Let us now prove Theorem 1. In the sequel, we take the convention, given any real sequence $(a_k)_{k \in \mathbb{N}}$, that $\sum_{k=n}^m a_k = 0$ if $n > m$.

By Lemma 1,

$$\sum_{k=1}^n \theta_l \beta_l = \sum_{k=1}^n \theta_k \left(\frac{x_k}{2} + \sum_{l=1}^{k-1} \frac{H_{l,k}^2}{2x_l} \right) = \sum_{l=1}^n \left[\frac{\theta_l x_l}{2} + \frac{1}{2x_l} \left(\sum_{k=l+1}^n \theta_k H_{l,k}^2 \right) \right].$$

Define

$$\Psi : \left(\mathbb{R}_+ \setminus \{0\} \right)^n \longrightarrow \mathcal{D},$$

$$(x_i)_{1 \leq i \leq n} \longmapsto (\beta_i)_{1 \leq i \leq n} = \left(\frac{x_i}{2} + \sum_{k=1}^{i-1} \frac{H_{k,i}^2}{x_k} \right)_{1 \leq i \leq n}.$$

Then Ψ is a bijection, since a symmetric matrix is positive definite if and only if all of its diagonal minors are positive. Its Jacobian is 2^{-n}, hence it is a diffeomorphism.

Therefore,

$$I := \int \mathbb{1}_{\{2\beta - P > 0\}} \frac{\exp(-\theta\beta)}{\sqrt{|2\beta - P|}} \, d\beta$$

$$= \int_{\mathbb{R}_+^n} \exp\left(-\sum_{l=1}^n \left[\frac{\theta_l x_l}{2} + \frac{1}{2x_l} \left(\sum_{k=l+1}^n \theta_k H_{l,k}^2 \right) \right] \right) \frac{1}{\sqrt{x_1 \cdots x_n}} \frac{1}{2^n} \, dx.$$

Let, for all $1 \leq l \leq m \leq n$,

$$R_{l,m} = \left(\sum_{j=m+1}^n H_{l,j} \sqrt{\theta_j} \right)^2 + \sum_{k=l+1}^m \theta_k H_{l,k}^2,$$

$$S_{l,m} = \frac{\theta_l x_l}{2} + \frac{R_{l,m}}{2x_l}.$$

Note that $R_{l,m}$ (resp., $S_{l,m}$) only depends on $x_1, \ldots, x_{l-1}$ (resp., $x_1, \ldots, x_l$).

Let, for all $1 \leq m \leq n$,

$$I_m := \int_{\mathbb{R}_+^m} \exp\left(-\sum_{l=1}^m S_{l,m} \right) \frac{dx_1 \cdots dx_m}{\sqrt{x_1 \cdots x_m}}.$$

We will take the convention that, if $m = 0$, the integral of $dx_1 \cdots dx_m$ is 1, so that $I_0 = 1$.

Note that $I = I_n / 2^n$. We also have the following lemma.

LEMMA 2. *For all $1 \leq m \leq n$, we have*

$$I_m = \sqrt{\frac{2\pi}{\theta_m}} \exp\left(-\sum_{j=m+1}^{n} W_{m,j}\sqrt{\theta_m\theta_j}\right) I_{m-1}.$$

PROOF. Using Claim 1, we deduce

(12)
$$I_m = \int_{\mathbb{R}_+^m} \exp\left(-\left[\frac{\theta_m x_m}{2} + \frac{R_{m,m}}{2x_m} + \sum_{l=1}^{m-1} S_{l,m}\right]\right) \frac{dx_1 \cdots dx_m}{\sqrt{x_1 \cdots x_m}}$$

$$= \int_{\mathbb{R}_+^{m-1}} \exp\left(-\sqrt{R_{m,m}\theta_m} - \sum_{l=1}^{m-1} S_{l,m}\right) \frac{dx_1 \cdots dx_{m-1}}{\sqrt{x_1 \cdots x_{m-1}}}.$$

Now $R_{m,m} = (\sum_{j=m+1}^{n} H_{l,j}\sqrt{\theta_j})^2$ and

$$H_{m,j} = W_{m,j} + \sum_{l=1}^{m-1} \frac{H_{l,m} H_{l,j}}{x_l},$$

so that

$$\sqrt{R_{m,m}\theta_m} = \sum_{j=m+1}^{n} W_{m,j}\sqrt{\theta_m\theta_j} + \sum_{l=1}^{m-1} \frac{H_{l,m}\sqrt{\theta_m}}{x_l} \sum_{j=m+1}^{n} H_{l,j}\sqrt{\theta_j}.$$

On the other hand, for all $1 \leq l \leq m - 1$,

$$S_{l,m} - S_{l,m-1} = -\frac{H_{l,m}\sqrt{\theta_m}}{x_l} \sum_{j=m+1}^{n} H_{l,j}\sqrt{\theta_j}.$$

Therefore,

$$\sqrt{R_{m,m}\theta_m} + \sum_{l=1}^{m-1} S_{l,m} = \sum_{j=m+1}^{n} W_{m,j}\sqrt{\theta_m\theta_j} + \sum_{l=1}^{m-1} S_{l,m-1},$$

which enables to conclude by (12). □

We deduce from Lemma 2, by induction, that

$$I = \frac{I_n}{2^n} = \frac{1}{2^n} \sqrt{\frac{(2\pi)^n}{\theta_n \cdots \theta_1}} \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}\sqrt{\theta_i\theta_j}\right),$$

which enables us to conclude.

6. Proofs of Proposition 2 and Theorem 3.

6.1. *Useful results on M-matrices.*   We start by stating some useful results on M-matrices. We follow reference [4], Chapter 6, which provides a detailed account on the subject. We adopt the following definition which is equivalent to the more classical definition (1.2) of [4], using Theorem 2.3, property (G_{20}) of [4].

DEFINITION 2.   A real $n \times n$ matrix A is called a nonsingular M-matrix if it has nonpositive off-diagonal coefficients, that is,

$$a_{i,j} \leq 0 \qquad \forall i \neq j,$$

and if the real parts of all of its eigenvalues are positive.

It is clear from this definition that when β is distributed according to $\nu^{W,\theta}(d\beta)$ in Definition 1, then the matrix $2\beta - P$ is a.s a symmetric nonsingular M-matrix. We will need the following properties.

PROPOSITION 3 (Theorem 2.3, Chapter 6, [4]).   *Assume that A is a real $n \times n$ matrix with nonpositive off-diagonal coefficients*, *that is*, $A \in Z^{n \times n}$ *in the notation of* [4], *that is*,

$$a_{i,j} \leq 0 \qquad \forall i \neq j.$$

The assertion "A is a nonsingular M-matrix" is equivalent to each of the following assertions:

(1) (*Property (N_{38}) in* [4]) *A is invertible and A^{-1} has nonnegative coefficients. If moreover A is irreducible, this implies that A^{-1} has positive coefficients by Theorem* 2.7 [4].
(2) (*Property (L_{32}) in* [4]) *there exists a vector x with positive coefficients such that $y := Ax$ has nonnegative coefficients and such that if $y_{j_0} = 0$ for some j_0, then there exists a sequence of indices $j_1, \ldots, j_k$ with $y_{j_k} > 0$ such that $a_{j_l, j_{l+1}} \neq 0$ for all $l = 0, \ldots, k-1$.*

6.2. *Proof of Proposition 2.*   Fix $i_0 \in V$, and let $\beta \in \mathcal{D}$. Let us first justify the existence and uniqueness of $u(i_0, i)$ defined by the linear system (4). Clearly, from Definition 2, $(2\beta - P)$ is a symmetric nonsingular M-matrix since $\beta \in \mathcal{D}$. It is obviously irreducible since the graph is connected, we deduce from Proposition 3(1) that its inverse G satisfies $G(i, j) > 0$ for any $i, j \in V$. A solution (u_j) of equation (4) is necessarily of the form $e^{u_j} = 2\gamma G(i_0, j)$ for some constant $\gamma \in \mathbb{R}$. The normalization $u_{i_0} = 0$ implies $\gamma = (2G(i_0, i_0))^{-1}$. Hence, the unique solution of the system (4) is given by $u_j = u(i_0, j)$ defined in Proposition 2.

6.3. *Proof of Theorem* 3. We start by a simple lemma.

LEMMA 3. *The following map*:

(13)
$$\Phi : \mathcal{D} \to \{(u_j)_{j \in V} \in \mathbb{R}^V, u_{i_0} = 0\} \times (\mathbb{R}_+ \setminus \{0\}),$$
$$(\beta) \mapsto ((u_j), \gamma),$$

where (u_j) *is the unique solution of the system* (4) *and* $\gamma = (2G(i_0, i_0))^{-1}$, *is a diffeomorphism.*

PROOF. By Proposition 2 the map is well defined and injective. Conversely, starting from $((u_j), \gamma)$ on the right hand side, we define (β_i) by

(14)
$$\beta_i = \sum_{j \sim i} \frac{1}{2} W_{i,j} e^{u_j - u_i} + \mathbb{1}_{i=i_0} \gamma.$$

It is clear that with that definition, (u_j) is the solution of (4) with (β_j) and $\gamma = (2G(i_0, i_0))^{-1}$. It remains to prove that $A = 2\beta - P > 0$ or equivalently that A is a nonsingular M-matrix (cf. Definition 2). Now $x = e^u$ and $y = 2\gamma (\mathbb{1}_{i=i_0})_{i \in V}$ satisfy assumption (2) of Proposition 3 since G is connected and, using $e^{u_{i_0}} = 1$,

$$Ae^u = (2\beta - P)(e^u) = 2\gamma (\mathbb{1}_{i=i_0})_{i \in V} = y. \qquad \square$$

We give two proofs of Theorem 3.

FIRST PROOF OF THEOREM 3. We make the change of variable given by Φ, in (13) and we now prove that if β has distribution ν^{W, ϕ^2}, then $(u, \gamma) = \Phi(\beta)$ has distribution $\mathcal{Q}_{i_0}^{W, \phi} \otimes \Gamma(\frac{1}{2}, \frac{1}{\phi_{i_0}^2})$.

Let J be the Jacobian matrix of Φ^{-1} (i.e., $J_{i,j} = \frac{\partial \beta_i}{\partial u_j}$, $j \neq i_0$, $J_{i,i_0} = \frac{\partial \beta_i}{\partial \gamma}$), then

$$J_{i,j} = \begin{cases} \delta_{i,i_0} & \text{if } j = i_0, \\ \frac{1}{2} W_{i,j} e^{u_j - u_i} & \text{if } i \neq j, j \neq i_0, \\ -\beta_i & \text{if } i = j \neq i_0. \end{cases}$$

We can factorize the ith row of J by e^{-2u_i} for each i, then expand the resulting matrix according to the i_0th column, and we find that

$$|J| = \frac{1}{2^{|V|-1}} e^{-2\sum_i u_i} D(W, u).$$

On the other hand, by (14) we deduce

$$|2\beta - P| = 2\gamma e^{-2\sum_i u_i} D(W, u).$$

Let ψ be a positive test function. We have

$$\int \psi(u,\gamma) \nu^{W,\phi^2}(d\beta)$$

$$= \int \psi(u,\gamma) 2^{|V|/2} \frac{\prod_i \phi_i}{\pi^{|V|/2}} \frac{\exp(-\sum_i \beta_i \phi_i^2 + \sum_{\{i,j\}\in E} W_{i,j}\phi_i\phi_j)}{\sqrt{2\gamma e^{-2\sum_i u_i} D(W,u)}}$$

$$\times \frac{1}{2^{|V|-1}} e^{-2\sum_i u_i} D(W,u)\, du\, d\gamma$$

$$= \int \psi(u,\gamma) \frac{\prod_i \phi_i}{(2\pi)^{(|V|-1)/2}} e^{-\sum_i u(i_0,i)} e^{-\frac{1}{2}\sum_{i\sim j} W_{i,j}(e^{u_i-u_j}\phi_j^2 + e^{u_j-u_i}\phi_i^2 - 2\phi_i\phi_j)}$$

$$\times \sqrt{D(W,u)} \cdot \frac{e^{-\phi_{i_0}^2 \gamma}}{\sqrt{\pi\gamma}}\, du\, d\gamma$$

$$= \int \psi(u,\gamma) \mathcal{Q}_{i_0}^{W,\phi}(du) \frac{\phi_{i_0} e^{-\phi_{i_0}^2 \gamma}}{\sqrt{\pi\gamma}}\, d\gamma.$$

This concludes the proof of Theorem 3 and of Corollary 2. $\square$

SECOND PROOF OF THEOREM 3.   This proof does not make use of the explicit expression of law $\mathcal{Q}_{i_0}^{W,\phi}$ of U in (3), but rather deduces its Laplace transform from direct computation of the probability of a loop. Note that compared to the first proof, this one uses the representation of the VRJP as a mixture of Markov jump processes (cf. Theorem 2 of [14] or Theorem 2 in Section 3), and hence it uses implicitly that the measure $\mathcal{Q}_{i_0}^{W,\phi}$ is a probability measure.

We will show that, if (u,γ) has distribution $\mathcal{Q}_{i_0}^{W,\phi} \otimes \Gamma(\frac{1}{2}, \frac{1}{\phi_{i_0}^2})$, then $\beta = \Phi^{-1}(u,\gamma)$ has distribution ν^{W,ϕ^2}, which clearly implies the result.

It follows by direct computation (see [15], proof of Theorem 3) that the probability that, at time t, the VRJP Z has followed a loop $Z_0 = x_0, x_1, \ldots, Z_t = x_n = x_0$ with jump times respectively in $[t_i, t_i + dt_i]$, $i = 1, \ldots, n$, where $t_0 = 0 < t_1 < \cdots < t_n < t = t_{n+1}$, is $p_t\, dt$, where

$$p_t = \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}\left(\sqrt{\phi_i^2 + \ell_i}\sqrt{\phi_j^2 + \ell_j} - \phi_i\phi_j\right)\right) \prod_{i\neq i_0} \frac{\phi_i}{\sqrt{\phi_i^2 + \ell_i}},$$

$$dt = \prod_{i=1}^n \frac{1}{2} W_{x_{i-1}x_i}\, dt_i,$$

with $(\ell_i)_{i\in V} = (\ell_i(t))_{i\in V}$ local time at time t.

On the other hand, using that, conditionally on $U = (U_i)_{i\in V}$ in Theorem 2, Z is a Markov jump process with jump rate $W_{ij}e^{U_j-U_i}/2$ from i to j, this probability

of a loop is also $q_t\,dt$, where

$$q_t = \int e^{-\sum_{i\in V}\tilde{\beta}_i \ell_i}\, \mathcal{Q}_{i_0}^{W,\phi}(du)$$

and $\tilde{\beta}$ is defined in (5).

Let $\Gamma = \Gamma(\frac{1}{2}, \frac{1}{\phi_{i_0}^2})$. By identification of p_t and q_t we deduce that

$$\int e^{-\sum_{i\in V}\beta_i \ell_i}\, \mathcal{Q}_{i_0}^{W,\phi}(du)\Gamma(d\gamma)$$

$$= \int e^{-\sum_{i\in V}\tilde{\beta}_i \ell_i}\, \mathcal{Q}_{i_0}^{W,\phi}(du)\int e^{-\ell_{i_0}\gamma}\Gamma(d\gamma)$$

$$= \exp\left(-\sum_{\{i,j\}\in E} W_{i,j}\left(\sqrt{\phi_i^2 + \ell_i}\sqrt{\phi_j^2 + \ell_j} - \phi_i\phi_j\right)\right)$$

$$\times \left(\prod_{i\neq i_0}\frac{\phi_i}{\sqrt{\phi_i^2 + \ell_i}}\right)\frac{1}{\sqrt{1 + \ell_{i_0}/\phi_{i_0}^2}},$$

which shows that the distribution $\mathcal{Q}_{i_0}^{W,\phi}\otimes\Gamma(\frac{1}{2}, \frac{1}{\phi_{i_0}^2})$ has the same Laplace transform as ν^{W,ϕ^2} in Proposition 1. $\square$

APPENDIX A: PROOF OF LEMMA 1

We perform successive Gauss elimination on M to make it upper triangular. Denote by $l_1, \ldots, l_n$ the n rows of any $n \times n$ matrix. First, let

$$M^{(1)} = M = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & -H_{1,n}^{(1)} \\ -H_{1,2}^{(1)} & x_2^{(1)} & \cdots & -H_{2,n}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ -H_{1,n}^{(1)} & -H_{n,2}^{(1)} & \cdots & x_n^{(1)} \end{pmatrix},$$

where we set, for any $1 \le i, j \le n$, $x_i^{(1)} = 2\beta_i$ and $H_{i,j}^{(1)} = W_{i,j}$.

We define a sequence of matrices $M^{(k)}$ recursively, such that

$$M^{(k)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & -H_{1,n}^{(1)} \\ 0 & x_2^{(2)} & -H_{2,3}^{(2)} & & & & & -H_{2,n}^{(2)} \\ \vdots & 0 & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & x_{k-1}^{(k-1)} & -H_{k-1,k}^{(k-1)} & \cdots & \cdots & -H_{k-1,n}^{(k-1)} \\ \vdots & & & 0 & x_k^{(k)} & -H_{k,k+1}^{(k)} & \cdots & -H_{k,n}^{(k)} \\ \vdots & & & & -H_{k,k+1}^{(k)} & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & -H_{n-1,n}^{(k)} \\ 0 & 0 & \cdots & 0 & -H_{k,n}^{(k)} & \cdots & -H_{n-1,n}^{(k)} & -x_n^{(k)} \end{pmatrix},$$

by the following rule: $M^{(k+1)}$ is constructed from $M^{(k)}$ by addition of columns $l_{k+1} \leftarrow l_{k+1} + \frac{H_{k,k+1}^{(k)}}{x_k^{(k)}} l_k, \ldots, l_n \leftarrow l_n + \frac{H_{k,n}^{(k)}}{x_k^{(k)}} l_k$ in $M^{(k)}$. In other words,

$$T_k M^{(k)} = M^{(k+1)} \qquad \text{where } [T_k]_{i,j} = \begin{cases} 1, & i = j, \\ \dfrac{H_{k,i}^{(k)}}{x_k^{(k)}}, & i > j = k, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $(x_i^{(k)})_{i \geq k}$, $(H_{i,j}^{(k)})_{i,j \geq k}$ satisfy the following induction rule:

$$\begin{cases} H_{i,j}^{(k+1)} = H_{i,j}^{(k)} + \dfrac{H_{k,i}^{(k)} H_{k,j}^{(k)}}{x_k^{(k)}}, & i, j \geq k+1, \\ x_i^{(k+1)} = x_i^{(k)} - \dfrac{(H_{k,i}^{(k)})^2}{x_k^{(k)}}, & i \geq k+1. \end{cases}$$

At step n, we have

$$T_{n-1} \cdots T_1 M = M^{(n)} = \begin{pmatrix} x_1^{(1)} & -H_{1,2}^{(1)} & \cdots & -H_{1,n}^{(1)} \\ 0 & x_2^{(2)} & \cdots & -H_{2,n}^{(2)} \\ \vdots & \ddots & \ddots & -H_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & x_n^{(n)} \end{pmatrix}.$$

Hence, it gives the LU-decomposition of M where $L^{-1} = T = T_{n-1} T_{n-2} \cdots T_1$ and $U = M^{(n)}$. It is easy to check that

$$\begin{cases} x_i = x_i^{(i)}, & i = 1, \ldots, n \\ H_{i,j} = H_{i,j}^{(i)}, & i < j, \end{cases}$$

satisfy the recursion in the statement, and that $x_i = M(1, \ldots, i \mid 1, \ldots, i)/M(1, \ldots, i-1 \mid 1, \ldots, i-1)$.

APPENDIX B: TIME RESCALING

Let Y_s be the VRJP with conductances (W) and initial local time $(\phi_i)_{i \in V}$ defined in Section 3. Recall that $L_i(t) = \phi_i + \int_0^t \mathbb{1}_{Y_s = i} \, ds$. Consider the increasing functional $A(s) = \sum_i (\frac{L_i(s)}{\phi_i} - 1)$, and the time-changed process $\tilde{Y}_{\tilde{s}} = Y_{A^{-1}(\tilde{s})}$. Let

$$\tilde{L}_i(\tilde{s}) = 1 + \int_0^t \mathbb{1}_{\{\tilde{Y}_{\tilde{s}} = i\}} \, d\tilde{s}.$$

We always denote by $\tilde{s}$ the time scale of $\tilde{Y}$, we can write

$$\tilde{s} = A(s), \qquad d\tilde{s} = \frac{ds}{\phi_{Y_s}}, \qquad L_i(\tilde{s}) = \frac{1}{\phi_i} L_i(s).$$

Obviously, $\tilde{Y}$ is a VRJP with edge weight $W_{i,j}\phi_i\phi_j$ and initial local local time 1; that is, conditionally on $\mathcal{F}_{\tilde{s}}^{\tilde{Y}}$, $\tilde{Y}$ jumps from i to j at rate

$$W_{i,j}\phi_i\phi_j\tilde{L}_j(\tilde{s}).$$

Note for simplicity

$$W_{i,j}^{\phi} = W_{i,j}\phi_i\phi_j.$$

We can apply [14], Theorem 2 to $\tilde{Y}$. Let

$$\tilde{D}(\tilde{s}) = \sum_i \tilde{L}_i(\tilde{s})^2 - 1,$$

and set $\tilde{Z}_{\tilde{t}} = \tilde{Y}_{\tilde{D}^{-1}(\tilde{t})}$, with local time $\tilde{\ell}_i(\tilde{t}) = \int_0^{\tilde{t}} \mathbb{1}_{\tilde{X}_u = i} du$. By Proposition 1 of [14] translated in time scale L [cf. relation (2.1) of [14]], we have that $\log \tilde{L}_i(\tilde{s}) - \frac{1}{N}\sum_{j \in V} \log \tilde{L}_j(\tilde{s})$ converges a.s. when $\tilde{s} \to \infty$ to a random vector with distribution given by (3.1) of Theorem 1 of [14], where the weights $(W_{i,j})$ are replaced by $(W_{i,j}^{\phi})$. Changing to variables $u_i \to u_i - u_{i_0}$, we deduce

$$\lim_{\tilde{s}\to\infty} \log \tilde{L}_i(\tilde{s}) - \log \tilde{L}_{i_0}(\tilde{s}) = U_i$$

exists and has distribution

$$\mathcal{Q}_{i_0}^{W^{\phi}}(du) = \frac{1}{\sqrt{2\pi}^{N-1}} e^{-\sum_{j\in V} u_j} e^{-\frac{1}{2}\sum_{i\sim j} W_{i,j}^{\phi}(\cosh(u_i-u_j)-1)} \sqrt{D(W^{\phi}, u)}\, du,$$

and that $\tilde{Z}$ is a mixture of Markov Jump Process with jumping rates $\frac{1}{2}W_{i,j}^{\phi}e^{U_j-U_i}$. We now come back to (Z_t). Recall that $Z_t = Y_{D^{-1}(t)}$, where $D(t)$ is defined in (2). From this, we have

$$\tilde{t} = \tilde{D}(A(D^{-1}(t))),$$

and

$$d\tilde{t} = \frac{1}{\phi_{\tilde{Y}_{\tilde{s}}}} \frac{\tilde{L}_{\tilde{Y}_{\tilde{s}}}(\tilde{s})}{L_{Y_s}(s)} dt = \frac{1}{\phi_{Z_t}^2} dt.$$

This implies that (Z_t) is a mixture of Markov Jump processes with jumping rates $\frac{1}{2}W_{i,j}e^{U_j+\log\phi_j-U_i-\log\phi_i}$. By simple change of variables, $U_i + \log\phi_i - \log\phi_{i_0}$ has

distribution

$$\mathcal{Q}_{i_0}^{W,\phi}(du) = \frac{\prod_{j\neq i_0}\phi_j}{\sqrt{2\pi}^{N-1}} e^{-\sum_{j\in V} u_j} e^{-\frac{1}{2}\sum_{i\sim j} W_{i,j}(e^{u_i-u_j}\phi_j^2 + e^{u_j-u_i}\phi_i^2 - 2\phi_i\phi_j)}$$
$$\times \sqrt{D(W,u)}\,du.$$

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