

The virtual element method in 50 lines of MATLAB

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Received: 20 April 2016 / Accepted: 10 November 2016 / Published online: 3 December 2016
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Abstract We present a 50-line MATLAB implementation of the lowest order virtual element method for the two-dimensional Poisson problem on general polygonal meshes. The matrix formulation of the method is discussed, along with the structure of the overall algorithm for computing with a virtual element method. The purpose of this software is primarily educational, to demonstrate how the key components of the method can be translated into code.

Keywords Virtual element method · Polygonal meshes · MATLAB implementation

1 Introduction

The virtual element method, introduced in [6], is a generalisation of the standard conforming finite element method for the approximation of solutions to partial differential equations. The method is designed in such a way as to enable the construction of high order approximation spaces which may include an arbitrary degree of global regularity [10] on meshes consisting of very general polygonal (or polyhedral) elements. This cocktail of desirable features has attracted the method a lot of attention (see, for example, [1, 3, 4, 11, 12, 14, 21, 30]) and is made possible through the *virtual element space* of trial and test functions, which is implicitly defined on each mesh element through local PDE problems. These local problems are designed in such a way that the virtual element space includes a

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subspace of polynomials of some prescribed degree (referred to as the *degree* of the method) alongside other, typically unknown, *virtual* functions. In this respect, and like many other conforming approaches to polygonal meshes such as the polygonal finite element method [18, 27] or BEM-based FEM [22], the virtual element method falls within the broad class of generalised finite element methods [25]. What sets the virtual element method apart from these other approaches, however, is that the extra non-polynomial *virtual* functions never need to be determined or evaluated in practice. Instead, they are understood and used solely through certain defining properties of the virtual element space and through their degrees of freedom, which, along with the discrete bilinear form, are carefully selected to ensure that the final stiffness matrix can be directly and exactly computed.

There is a history of short, simple codes being used to demonstrate the practical implementation details of various aspects of finite element methods. We refer, for instance, to the ‘Remarks around 50 lines of MATLAB’ paper [2] which presented a simple and transparent MATLAB implementation of the conforming finite element method, the 99-line topology optimisation code presented in [23], or the mixed-FEM implementations presented by [5]. The purpose of these codes is primarily to demonstrate how the theoretical concepts can be distilled into code, although they typically fall at the opposite end of the educational spectrum from many other educational finite element libraries such as FreeFEM++ [16], FEniCS [17], or FEAP [29], amongst many others, or commercially available ‘black-box’ finite element software packages. Such software packages are typically designed to hide many of the actual implementation details of the methods they implement behind an intuitive interface which may automatically generate the necessary code directly from the weak form of the problem or simply a description of the physical system. Instead, the focus of these minimalist codes is on elucidating the ‘behind-the-scenes’ work necessary to construct and solve the matrix form of the method in a specific case. This is a tradition which we continue here, presenting a 50-line MATLAB implementation of the lowest order virtual element method for the Poisson problem in 2D on general polygonal meshes.

To the best of our knowledge, this is the first publicly available implementation of the virtual element method, although there are various references which discuss the matrix formulation of the method, see for instance [7, 14, 15]. In particular, [7] contains detailed explanations of the formulation of the terms in the matrix equations for the high order virtual element method applied to a reaction-diffusion problem. In many ways, the work here can be seen as a spiritual successor to [7] in the sense that while we restrict ourselves to just the linear virtual element method for the Poisson problem, we take the process one step further and provide a clear, useable MATLAB implementation of the method. The full code of the implementation is available from Netlib (<http://www.netlib.org/numeralgo/>) as the na45 package, and includes several examples of polygonal meshes.

The remainder of this work is structured as follows. In Section 2, we present the model problem of the Poisson problem. A very brief introduction to the virtual element framework is presented in Section 3, with a discussion of the discrete function spaces and bilinear forms. The details of the implementation of this method are given in Section 4, where we derive the matrix form of the discrete problem and show how

this may be computed in practice. Section 5 contains a brief explanation of how to run the code using MATLAB. Finally, we offer some concluding remarks and ideas for possible extensions to the code in Section 6.

2 Model problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and consider the Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

with $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. This problem can be written in variational form as: find $u \in H_g^1(\Omega) := \{w \in H^1(\Omega) : w = g \text{ on } \partial\Omega\}$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v) =: \ell(v) \quad \forall v \in H_0^1(\Omega) \tag{2.2}$$

where (\cdot, \cdot) denotes the standard $L^2(\Omega)$ inner product. This variational problem possesses a unique solution by the Lax-Milgram lemma.

3 The virtual element method

Let \mathcal{T}_h be a family of partitions of the domain Ω into non-overlapping polygonal elements with maximum diameter h whose boundaries are not self-intersecting. A theoretical analysis of the method, as given in [6], requires certain additional assumptions on the regularity of the mesh. Although we do not specifically state the assumptions here, the resulting set of requirements is general enough to include polygonal elements consisting of an arbitrary (but uniformly bounded) number of edges, which may also be non-convex. Moreover, recent work in [9] suggests that these mesh requirements may be relaxed even further to include meshes with arbitrarily small edges. To simplify the implementation, we restrict the mesh to include only elements which contain their own centroid, as defined in (4.4). We note that this class of elements includes those with co-planar edges, as commonly found in locally refined meshes with hanging nodes, and even non-convex elements.

The vertices of a polygonal element E with N^E edges are denoted by v_i for $i = 1, \dots, N^E$, and we adopt the convention that the edge e_i connects v_i and v_{i+1} , where the indices are understood to wrap within the range 1 to N^E .

3.1 Virtual element function spaces

The discrete function space is defined to be

$$V_h := \{v_h \in H_g^1(\Omega) : v_h|_E \in V_h^E \text{ for all } E \in \mathcal{T}_h\}$$

where the local space V_h^E on the element E can be understood through the following three properties:

- V_h^E includes the space \mathcal{P}_E of physical-frame polynomials on E with total degree ≤ 1 as a subspace.
- Any function in V_h^E can be uniquely identified by its values at the vertices of E , which are taken to be the degrees of freedom of the space. We note that this implies that the dimension of the space V_h^E is equal to N^E .
- Every function in V_h^E is a linear polynomial on each edge of E .

The subspace of linear polynomials provides the approximation power of the virtual element space, and is responsible for the accuracy of the method. On triangular elements, the space consists entirely of these linear polynomials, and thus, the method reduces to the standard linear finite element method. However, on more general -shaped polygonal elements, the space will also include other, implicitly defined, ‘virtual’ functions, cf. (3.3). The method is designed in such a way that these will never need to be explicitly computed or evaluated and are instead understood solely through their values at the vertices of E , which we take to be the degrees of freedom of the space V_h^E . In this respect, the virtual element space can be seen as a straightforward generalisation of the standard linear conforming finite element space on triangles to more general shaped elements.

The first observation we make about this space is that just the properties outlined above allow us to compute the Ritz projection $\Pi^E : V_h^E \rightarrow \mathcal{P}_E$ of any function in the local virtual element space V_h^E onto the subspace of linear polynomials. This projection is defined for $v_h \in V_h^E$ by the conditions

$$\begin{cases} (\nabla(\Pi^E v_h - v_h), \nabla p)_{0,E} = 0 \text{ for all } p \in \mathcal{P}_E, \\ \overline{\Pi^E v_h} = \overline{v_h}, \end{cases} \tag{3.1}$$

where $\overline{w_h} := \frac{1}{N^E} \sum_{i=1}^{N^E} w_h(v_i)$ denotes the average value of w_h at the vertices of E . This second condition is necessary to fix the constant part of $\Pi^E v_h$ and is clearly computable for any $v_h \in V_h^E$ from just its degrees of freedom.

From (3.1), the divergence theorem, and the fact that the Laplacian of a linear function is zero, we have that, for any $v_h \in V_h^E$ and $p \in \mathcal{P}_E$,

$$(\nabla \Pi^E v_h, \nabla p)_{0,E} = (\nabla v_h, \nabla p)_{0,E} = \sum_{e \in \partial E} \int_e v_h n_e \cdot \nabla p ds, \tag{3.2}$$

where n_e denotes the unit normal vector to the edge e directed out of the element E . The final expression on the right hand side here can be *exactly* evaluated since v_h is a linear polynomial on each edge of E , entirely determined by its values at the vertices, while the gradient of the linear polynomial p is a known constant. By picking a basis for the polynomial space \mathcal{P}_E , (3.2) can be written as a matrix problem which can be solved to find the coefficients of $\Pi^E v_h$ with respect to this polynomial basis. We will come back to this in Section 4, although for now we just rely on the fact that this projection is *computable*.

The actual definition of the virtual element space which we use here is the lowest order space introduced in [6], given by

$$V_h^E := \{v \in H^1(E) : \Delta v = 0, \quad v|_{\partial E} \in C^0(\partial E), \\ v|_e \in \mathcal{P}_e \text{ for each } e \in \partial E\}, \tag{3.3}$$

where \mathcal{P}_e denotes the space of linear polynomials on the edge e . The fact that the vertex values can be used as degrees of freedom to describe this space is proven in [6].

The global degrees of freedom for V_h are simply taken to be the value of the function at each vertex in the mesh, thus imposing the continuity of the ambient space. The degrees of freedom at vertices on the domain boundary are fixed in accordance with the boundary condition. The dimension of the global virtual element space V_h shall be denoted by N .

3.2 Discrete bilinear form

Define the bilinear form $a^E : H^1(E) \times H^1(E) \rightarrow \mathfrak{R}$ to be the restriction of a to the element E , i.e. $a^E(v, w) := (\nabla v, \nabla w)_{0,E}$ for any $v, w \in H^1(E)$. Following [6], we introduce the discrete counterpart $a_h^E : V_h^E \times V_h^E \rightarrow \mathfrak{R}$ of a^E which we define as

$$a_h^E(v_h, w_h) := (\nabla \Pi^E v_h, \nabla \Pi^E w_h)_{0,E} + S^E(v_h - \Pi^E v_h, w_h - \Pi^E w_h), \tag{3.4}$$

with

$$S^E(v_h, w_h) := \sum_{r=1}^{N^E} \text{dof}_r(v_h) \text{dof}_r(w_h),$$

where $\text{dof}_r(v_h)$ denotes the value of the r th local degree of freedom defining v_h in V_h^E with respect to some arbitrary (but fixed) ordering¹. This means that S^E is simply the Euclidean inner product between vectors of degrees of freedom. Finally, we define

$$a_h(v_h, w_h) := \sum_{E \in \mathcal{T}_h} a_h^E(v_h, w_h),$$

to be the discrete counterpart of a .

Crucial to the method is the observation that the local discrete bilinear forms satisfy the following two properties [6]:

- *Polynomial consistency:* for any $v_h \in V_h^E$ and $p \in \mathcal{P}_E$,

$$a_h^E(v_h, p) = a^E(v_h, p).$$

- *Stability:* there exists a constant C_{stab} independent of h and E such that

$$C_{stab}^{-1} a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq C_{stab} a^E(v_h, v_h),$$

for any $v_h \in V_h^E$.

The requirement of *polynomial consistency* implies that the method satisfies the patch test commonly used in the engineering literature, expressing the fact that the method

¹For instance, this could be achieved simply by numbering the vertices of the polygon E .

is *exact* when the solution is a piecewise linear polynomial with respect to the mesh \mathcal{T}_h and provides the accuracy of the method. The *stability* property, on the other hand, ensures that the discrete bilinear form inherits the continuity and coercivity of the original variational form a , as proven in [6]. In the final matrix formulation of the problem, this property can be viewed as ensuring that the problem stiffness matrix is of the correct rank.

In light of these two properties, the two terms of a_h^E are referred to as the *consistency* and *stabilising* terms respectively since only the first term is non-zero when either v_h or w_h is a polynomial, and thus single-handedly ensures that the polynomial consistency property is satisfied, while the second term ensures that the stability property is satisfied even when v_h or w_h are in the kernel of Π^E . For a proof that this choice of stabilising term S^E does indeed satisfy the stability property, we refer to [9, 14].

Moreover, both terms of a_h^E in (3.4) are computable using just the degrees of freedom of the virtual element space (to compute the projector Π^E and to evaluate the stabilising term) and knowledge of the polynomial subspace \mathcal{P}_E (to evaluate the consistency term of a_h^E , which is made of integrals of polynomials, just like in a standard finite element method). This will be further demonstrated in Section 4, where it will also become apparent that this particular virtual element method can be implemented *without requiring any quadrature* to compute the stiffness matrix, although a very simple barycentric quadrature scheme is used to evaluate the forcing term.

Still following [6], the linear form ℓ on the right-hand side of the variational problem (2.2) is discretised by $\ell_h : V_h^E \rightarrow \mathfrak{R}$ such that

$$\ell_h(v_h) := \sum_{E \in \mathcal{T}_h} (\Pi_0^E f, \overline{v_h})_{0,E}, \tag{3.5}$$

where $\Pi_0^E : V_h^E \rightarrow \mathfrak{R}$ denotes the $L^2(E)$ -orthogonal projection onto constants, defined for any $w_h \in V_h^E$ to be such that

$$\int_E (w_h - \Pi_0^E w_h) dx = 0.$$

The discrete problem which we solve can then be written as: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \ell_h(v_h), \tag{3.6}$$

for all $v_h \in V_h^E$.

4 Implementation

As with a typical finite element method, we start by introducing the Lagrangian basis $\{\varphi_i\}_{i=1}^N$ of V_h with respect to the global set of degrees of freedom, defined by the property that $\varphi_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We also introduce the Lagrangian basis of the local virtual element space V_h^E on the element E as $\{\varphi_i^E\}_{i=1}^{N_E}$, defined by the local equivalent of the same property.

We will also need a basis for the space \mathcal{P}_E of local physical frame linear polynomials on each element E . Many choices are possible here, although in keeping with the convention commonly adopted in the literature on virtual element methods, we choose the set of *scaled monomials* of degree 1. These are defined on the element E as

$$\mathcal{M}_E := \left\{ m_1(x, y) := 1, \quad m_2(x, y) := \frac{x - x_E}{h_E}, \quad m_3(x, y) := \frac{y - y_E}{h_E} \right\}, \quad (4.1)$$

where x_E and y_E respectively denote the x and y coordinates of the centroid of the element in the standard Cartesian coordinate system, and h_E is the diameter of the element E . We denote by $N_{\mathcal{P}} = 3$ the cardinality of this basis and therefore the dimension of \mathcal{P}_E

In the hope of avoiding confusion, we adopt the convention of indexing coefficients and basis functions in the basis of V_h^E using Latin letters, while those of \mathcal{P}_E will be indexed using Greek letters.

With these two bases at our disposal, we can now derive the matrix form of the discrete problem (3.6). Expanding the virtual element solution u_h as

$$u_h = \sum_{i=1}^N U_i \varphi_i,$$

problem (3.6) can be rewritten using the definitions (3.4) of a_h^E and (3.5) of ℓ_h as: find $U \in \mathfrak{R}^N$ such that

$$\sum_{i=1}^N U_i \sum_{E \in \mathcal{T}_h} \left((\nabla \Pi^E \varphi_i, \nabla \Pi^E \varphi_j)_{0,E} + S^E(\varphi_i - \Pi^E \varphi_i, \varphi_j - \Pi^E \varphi_j) \right) = \sum_E (\Pi_0^E f, \overline{\varphi_j})_{0,E}$$

for $j = 1, \dots, N$. This may be expressed in matrix form as $KU = F$ where

$$K_{j,i} = \sum_{E \in \mathcal{T}_h} \left((\nabla \Pi^E \varphi_i, \nabla \Pi^E \varphi_j)_{0,E} + S^E(\varphi_i - \Pi^E \varphi_i, \varphi_j - \Pi^E \varphi_j) \right), \quad F_j = \sum_E (\Pi_0^E f, \overline{\varphi_j})_{0,E},$$

for $i, j = 1, \dots, N$. Since both these terms are defined through sums over elements, the obvious way to compute the entries of K and F is by computing the non-zero local contributions from each element E in the form of the *local stiffness matrix* $K^E \in \mathfrak{R}^{N^E \times N^E}$ and *local forcing vector* $F^E \in \mathbb{R}^{N^E}$, given by

$$K_{j,i}^E = (\nabla \Pi^E \varphi_i^E, \nabla \Pi^E \varphi_j^E)_{0,E} + S^E(\varphi_i^E - \Pi^E \varphi_i^E, \varphi_j^E - \Pi^E \varphi_j^E), \quad F_j^E := (\Pi_0^E f, \overline{\varphi_j^E})_{0,E}, \quad (4.2)$$

for $i, j = 1, \dots, N^E$, and adding them into the corresponding entries of K and F .

This, of course, dictates that the overall structure of a virtual element method implementation will be much the same as for a standard finite element method, as outlined in Algorithm 1. The key point of departure from the standard finite element method is in *how* the element stiffness matrices should be calculated. Firstly, the computation of the local stiffness matrices relies on first computing the local

Ritz projector Π^E on each element. Secondly, where the implementation of a conventional finite element might rely on a mapping to a reference element, no such equivalent process is possible here because the mesh elements are allowed to be general (possibly non-convex) polygons.

Algorithm 1 A typical finite/virtual element method implementation

Input: Mesh

- 1: Initialise stiffness matrix and forcing vector
- 2: **for** each element in the mesh **do**
- 3: Compute local stiffness matrix
- 4: Compute local forcing contributions
- 5: Assemble local matrices into the correct places in the global matrices
- 6: Condense boundary conditions from matrix system
- 7: Solve the resulting matrix problem
- 8: Replace boundary conditions in the solution vector

Output: Vector containing the degrees of freedom of the discrete solution

The main code of the implementation is shown in Listing 1. In the remainder of this section, we dissect the code to highlight how the various steps outlined in Algorithm 1 can be implemented in code. Much of the matrix formulation presented in this section is similar to that in [7], although here we focus specifically on the case of the lowest order method and take the process a step further to include the details of how each step is accomplished in the code. The full code also relies on several auxiliary functions:

- `square_domain_rhs.m`: contains the definition of the forcing function

$$f(x, y) = 15 \sin(\pi x) \sin(\pi y),$$

- `square_domain_boundary_condition.m`: contains the definition of the Dirichlet boundary condition

$$g(x, y) = xy \sin(\pi x),$$

- `L_domain_rhs.m`: contains the definition of the forcing function

$$f(x, y) = 0,$$

- `L_domain_boundary_condition.m`: contains the definition of the Dirichlet boundary condition

$$g(r, \theta) = r^{2/3} \sin\left(\frac{2\theta - \pi}{3}\right),$$

where r and θ are the standard polar coordinates centred at the origin

- `plot_solution.m`: produces a MATLAB figure containing a plot of the approximate solution u_h . Note that this plot is generated using the values of u_h at the vertices of the mesh, which are interpolated by the MATLAB `patch` function to produce a surface.

The implementations of these functions are available as the package `na45` in the NUMERALGO library from Netlib, along with the `vem.m` file shown in Listing 1


```

1 function u = vem(mesh_filepath, rhs, boundary_condition)
2 % Computes the virtual element solution of the Poisson problem on the specified mesh.
3 mesh = load(mesh_filepath); % Load the mesh from a .mat file
4 n_dofs = size(mesh.vertices, 1); n_polys = 3; % Method has 1 degree of freedom per vertex
5 K = sparse(n_dofs, n_dofs); % Stiffness matrix
6 F = zeros(n_dofs, 1); % Forcing vector
7 u = zeros(n_dofs, 1); % Degrees of freedom of the virtual element solution
8 linear_polynomials = {[0,0], [1,0], [0,1]}; % Impose an ordering on the linear polynomials
9 mod_wrap = @(x, a) mod(x-1, a) + 1; % A utility function for wrapping around a vector
10 for el_id = 1:length(mesh.elements)
11     vert_ids = mesh.elements{el_id}; % Global IDs of the vertices of this element
12     verts = mesh.vertices(vert_ids, :); % Coordinates of the vertices of this element
13     n_sides = length(vert_ids); % Start computing the geometric information
14     area_components = verts(:,1) .* verts([2:end,1],2) - verts([2:end,1],1) .* verts(:,2);
15     area = 0.5 * abs(sum(area_components));
16     centroid = sum((verts + verts([2:end,1],:)) .* repmat(area_components,1,2)) / (6*area);
17     diameter = 0; % Compute the diameter by looking at every pair of vertices
18     for i = 1:(n_sides-1)
19         for j = (i+1):n_sides
20             diameter = max(diameter, norm(verts(i, :) - verts(j, :)));
21         end
22     end
23     D = zeros(n_sides, n_polys); D(:, 1) = 1;
24     B = zeros(n_polys, n_sides); B(1, :) = 1/n_sides;
25     for vertex_id = 1:n_sides
26         vert = verts(vertex_id, :); % This vertex and its neighbours
27         prev = verts(mod_wrap(vertex_id - 1, n_sides), :);
28         next = verts(mod_wrap(vertex_id + 1, n_sides), :);
29         vertex_normal = [next(2) - prev(2), prev(1) - next(1)]; % Average of edge normals
30         for poly_id = 2:n_polys % Only need to loop over non-constant polynomials
31             poly_degree = linear_polynomials{poly_id};
32             monomial_grad = poly_degree / diameter; % Gradient of a linear is constant
33             D(vertex_id, poly_id) = dot(vert - centroid, poly_degree) / diameter;
34             B(poly_id, vertex_id) = 0.5 * dot(monomial_grad, vertex_normal);
35         end
36     end
37     projector = (B*D) \ B; % Compute the local Ritz projector to polynomials
38     stabilising_term = (eye(n_sides) - D * projector)' * (eye(n_sides) - D * projector);
39     G = B*D; G(1, :) = 0;
40     local_stiffness = projector' * G * projector + stabilising_term;
41     K(vert_ids,vert_ids) = K(vert_ids,vert_ids) + local_stiffness; % Copy local to global
42     F(vert_ids) = F(vert_ids) + rhs(centroid) * area / n_sides;
43 end
44 boundary_vals = boundary_condition(mesh.vertices(mesh.boundary, :));
45 internal_dofs = ~ismember(1:n_dofs, mesh.boundary); % Vertices which aren't on the boundary
46 F = F - K(:, mesh.boundary) * boundary_vals; % Apply the boundary condition
47 u(internal_dofs) = K(internal_dofs, internal_dofs) \ F(internal_dofs); % Solve
48 u(mesh.boundary) = boundary_vals; % Set the boundary values
49 plot_solution(mesh, u)
50 end

```

Listing 1 The file vem.m, implementing the virtual element method

and various sample mesh files, illustrated in Fig. 1. Examples of how to use the code are given in Section 5.

4.1 The polygonal mesh

The mesh is loaded into a structure named `mesh` from a binary MATLAB `.mat` file (the path to which is specified by the input parameter `mesh_filepath`) containing a matrix named `vertices` which specifies the coordinates of a mesh vertex on each row, a cell array named `elements` containing vectors of indices indicating the

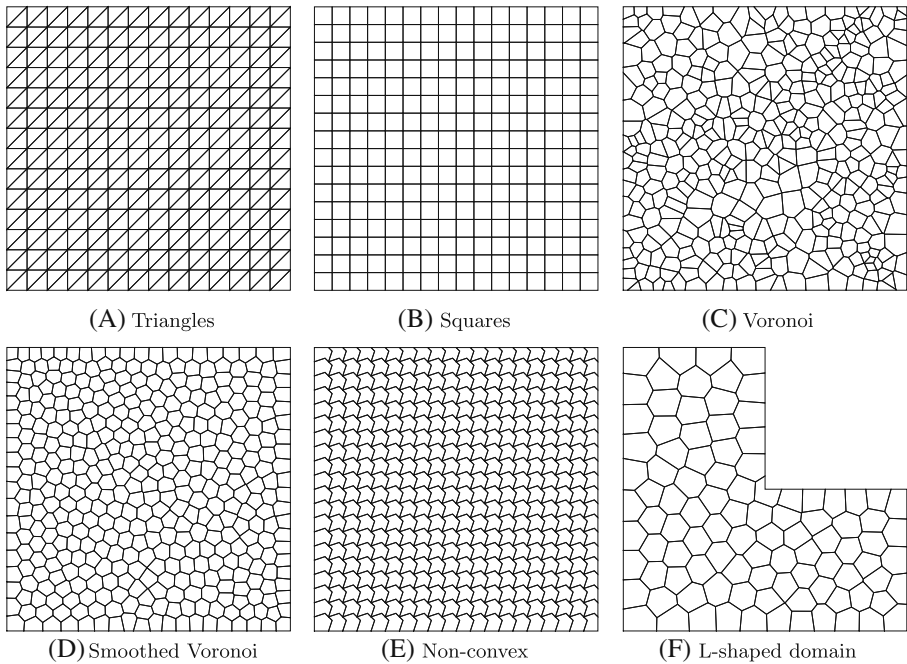


Fig. 1 The sample meshes available along with the code

vertices which make up each element in an anti-clockwise order around the element, and a vector named `boundary` containing the indices of the vertices which lie on the boundary of the mesh. Illustrated in Fig. 1 are several examples of such meshes, available alongside the code from NETLIB. This information can also be generated in the same format using the Voronoi mesh generator PolyMesher [28], also written in MATLAB.

4.2 Initialisation

The input arguments to the code are `mesh_filepath` which contains the path to the mesh file, as described above, the function handle `rhs` which is a handle to a function implementing the forcing function of the PDE, and the function handle `boundary_condition`, which should be a handle to a function implementing the boundary condition of the PDE. Examples of such input arguments are given in Section 5.

The initialisation step of the code (lines 1–22) simply sets up various variables which will be useful to us later. In the interests of efficiency, we use a sparse matrix `K` to represent the stiffness matrix. In this step, we also define the cell array `linear_polynomials` containing three pairs of numbers indicating the degree

of the associated polynomial in the x and y directions. Thus, the index of a specific polynomial in this array is taken to be the index of the polynomial in the basis \mathcal{M}_E . We note that the ordering imposed in the code coincides with the ordering in (4.1), although this choice is arbitrary.

Some extra element-specific initialisation also takes place within the loop over all mesh elements to compute various geometric properties of each element. The vector `vert_ids` contains the global indices of the vertices forming the element E with an anti-clockwise ordering. As well as providing us with a means of looking up the coordinates of the vertices using the `mesh` structure, this also provides us with a very simple way to identify the global index of a particular local degree of freedom. This is possible because the indices of the degrees of freedom of the *global* virtual element space can be taken to be just the *global* indices of their associated vertices. Similarly, the *local* indices of the vertices of the element E dictate the *local* index of the associated degree of freedom. Therefore, the i th entry in the vector `vert_ids` provides the *global* index of the i th *local* vertex and therefore also the *global* index of its associated *local* degree of freedom. Having access to this ‘local to global’ mapping is absolutely crucial when trying to assemble the global stiffness matrix and forcing vector from their local counterparts. When implementing more complex methods this sort of bookkeeping can quickly become very cumbersome, although here we are able to exploit the very simple arrangement of the degrees of freedom. Because of this property, we use the variable `n_sides` to represent both the number of sides of E and equivalently the number of local degrees of freedom of V_h^E .

The variable `area` denotes $|E|$ and is computed using the formula

$$|E| = \frac{1}{2} \left| \sum_{i=1}^{N^E} x_i y_{i+1} - x_{i+1} y_i \right|, \tag{4.3}$$

where (x_i, y_i) are the coordinates of the vertex v_i , and the indexing is understood to wrap within the range 1 to N^E . In the code this is accomplished using the utility function `mod_wrap`, which modifies the standard `mod` function to produce output in the range 1 to N^E rather than the range 0 to $N^E - 1$. This modification to the `mod` function is necessary because arrays in MATLAB start at index 1, not 0. The centroid (x_E, y_E) of the element is stored in the vector `centroid` and calculated using the usual formula:

$$\begin{aligned} x_E &= \frac{1}{6|E|} \sum_{i=1}^{N^E} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \\ y_E &= \frac{1}{6|E|} \sum_{i=1}^{N^E} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \end{aligned} \tag{4.4}$$

where the indices are again to be understood to wrap within the range 1 to N^E . In the code, we are able to combine some of the calculations which are necessary to find the area and centroid by storing the terms of the sum (4.3) in the vector `area_components`.

4.3 The Ritz projection and local stiffness matrix

We focus initially on computing $\Pi^E \varphi_i^E$ for a single basis function φ_i^E . Since $\Pi^E \varphi_i^E \in \mathcal{P}^E \subset V_h^E$, we have two different possible expansions for the projection $\Pi^E \varphi_i^E$, either in the basis of \mathcal{P}^E or in that of V_h^E :

$$\Pi^E \varphi_i^E = \sum_{\alpha=1}^{N_{\mathcal{P}}} a_{i,\alpha} m_{\alpha} = \sum_{j=1}^{N^E} s_{i,j} \varphi_j^E. \tag{4.5}$$

Recalling (3.2) and using the fact that ∇m_{β} is a constant vector for the linear polynomial m_{β} , we find that

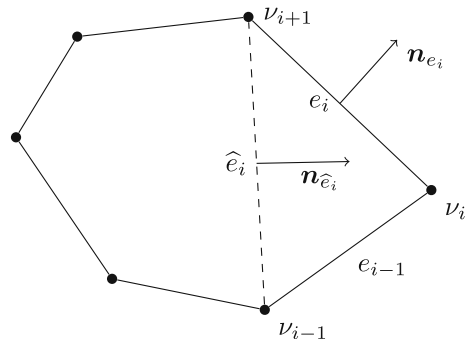
$$\begin{aligned} \sum_{\alpha=1}^{N_{\mathcal{P}}} a_{i,\alpha} (\nabla m_{\alpha}, \nabla m_{\beta})_{0,E} &= \sum_{j=1}^{N^E} \int_{e_j} v_h n_{e_j} \cdot \nabla m_{\beta} d_s \\ &= \sum_{j=1}^{N^E} \frac{|e_j|}{2} (\varphi_i^E(v_j) + \varphi_i^E(v_{j+1})) n_{e_j} \cdot \nabla m_{\beta} \\ &= \frac{|e_i| n_{e_i} + |e_{i-1}| n_{e_{i-1}}}{2} \cdot \nabla m_{\beta}, \end{aligned}$$

for any $m_{\beta} \in \mathcal{M}_E$, where in the last equality we have used the Lagrangian property of the basis functions φ_i^E at the vertices of E to determine that φ_i^E is only non-zero on the two edges e_i and e_{i-1} which meet at the vertex v_i . As shown in [13], this can be further simplified to

$$\sum_{\alpha=1}^{N_{\mathcal{P}}} a_{i,\alpha} (\nabla m_{\alpha}, \nabla m_{\beta})_{0,E} = \frac{1}{2} |\widehat{e}_i| n_{\widehat{e}_i} \cdot \nabla m_{\beta},$$

where we have denoted by \widehat{e}_i the line segment connecting the vertices v_{i-1} and v_{i+1} , and $n_{\widehat{e}_i}$ is the unit normal to \widehat{e}_i such that $n_{\widehat{e}_i} \cdot n_{e_j} \geq 0$ for $j = i, i - 1$ (Fig. 2).

Fig. 2 An illustration of the labelling of the various geometric attributes on each element. Vertices are labelled as v_i , the edge connecting v_i to v_{i+1} is denoted by e_i , and \widehat{e}_i denotes the line segment connecting v_{i-1} and v_{i+1} . The outward unit normal in each case is denoted by n with the appropriate subscript



In view of this, and following [7], we introduce the matrix $G \in \mathfrak{R}^{N_{\mathcal{P}} \times N_{\mathcal{P}}}$ and the vector $B_i \in \mathfrak{R}^{N_{\mathcal{P}}}$ such that

$$G_{\beta,\alpha} = (\nabla m_{\alpha}, \nabla m_{\beta})_{0,E}, \quad B_{\beta,i} = \frac{1}{2} |\widehat{e}_i| n_{\widehat{e}_i} \cdot \nabla m_{\beta},$$

to encode the conditions above. The problem here is that the first row (and column) of G and B_i are zero, since the gradient of a constant function is zero, and therefore, G is rank deficient. This is overcome by using the second condition in the definition (3.1) of Π^E , defining

$$\widetilde{G}_{\beta,\alpha} = \begin{cases} \frac{1}{N^E} \sum_{j=1}^{N^E} m_{\alpha}(v_j) & \text{if } \beta = 1, \\ G_{\beta,\alpha} & \text{otherwise,} \end{cases} \quad \widetilde{B}_{\beta,i} = \begin{cases} \frac{1}{N^E} & \text{if } \beta = 1, \\ B_{\beta,i} & \text{otherwise.} \end{cases} \tag{4.6}$$

Therefore, the coefficients $a_{i,\alpha}$ can be calculated by solving the (full rank) matrix equation

$$\widetilde{G}a_i = \widetilde{B}_i.$$

Defining the matrix $\widetilde{B} \in \mathfrak{R}^{N_{\mathcal{P}} \times N^E}$ such that its i th column is the vector $\widetilde{B}_i \in \mathfrak{R}^{N_{\mathcal{P}}}$, we obtain the matrix equation

$$\widetilde{G}\Pi = \widetilde{B},$$

where $\Pi \in \mathfrak{R}^{N_{\mathcal{P}} \times N^E}$ is the matrix representation of the Ritz projector Π^E , taking a vector of coefficients of a function expressed in terms of the basis of V_h^E to a vector of coefficients of the basis of \mathcal{P}_E , and has a_i as its i th column.

We also introduce the matrix $D \in \mathfrak{R}^{N^E \times N_{\mathcal{P}}}$ with $D_{i,\alpha} := \text{dof}_i(m_{\alpha})$ as a one-way ‘change of basis’ matrix for re-expressing polynomials in terms of the basis of V_h^E . It is easy to check from (3.2) that we may use D as a shortcut to compute G and \widetilde{G} , since

$$G = BD, \quad \widetilde{G} = \widetilde{B}D,$$

meaning we can compute the projection matrix as

$$\Pi = (\widetilde{B}D)^{-1}\widetilde{B}. \tag{4.7}$$

Since Π transforms a vector of coefficients representing a function expressed in the basis of V_h^E into a vector of coefficients of the basis of \mathcal{P}_E , the matrix $D\Pi$ then corresponds the same projector although with the resultant polynomial expressed in terms of the basis of V_h^E .

Finally, we can compute the local stiffness matrix as

$$K^E = \Pi^{\top}G\Pi + (I - D\Pi)^{\top}(I - D\Pi), \tag{4.8}$$

where $I \in \mathfrak{R}^{N^E \times N^E}$ denotes the identity matrix. The first term of this sum corresponds to the consistency term of the discrete bilinear form, and the second term corresponds to the stabilising term.

The code which computes the matrix form of the local Ritz projector and the local stiffness matrix is given in lines 23–41 of Listing 1. The first task (lines 23–24 of Listing 1) is to initialise the two matrices D and B representing their namesakes $D \in \mathfrak{R}^{N^E \times N_{\mathcal{P}}}$ and $B \in \mathfrak{R}^{N_{\mathcal{P}} \times N^E}$. For each of these matrices, we can immediately calculate the elements corresponding to the constant polynomial basis function m_1 . In the case of D , every element of the first column contains the value 1 since the

constant function is 1 everywhere, while it may be observed from (4.6) that every element in the first row of B is equal to $N^E - 1$.

However, the remaining elements of D and B must be computed separately for each of the basis polynomials with total degree equal to 1, and for each local degree of freedom. Computing the entries of D is a straightforward task, since it just involves evaluating the basis monomials at the vertices of the polygon.

For the entries of B , however, we must evaluate the second expression in (4.6). The quantity $|\widehat{e}_i|n_{\widehat{e}_i}$ is simple to calculate, since it is just the vector

$$|\widehat{e}_i|n_{\widehat{e}_i} = (y_{i+1} - y_{i-1}, x_{i-1} - x_{i+1})^\top$$

due to the anti-clockwise orientation of the vertices around E . In the code, this result is stored in the variable `vertex_normal`, so named because it can be interpreted as a weighted normal vector at the vertex v_i . Again, the indices here are understood to wrap within the range 1 to N^E .

To compute the entries of B we also need to evaluate ∇m_β . Since m_β is a linear polynomial, its gradient is simply a constant vector, and from the definition of \mathcal{M}_E it is clear that

$$\nabla m_2 = (h_E^{-1}, 0)^\top, \quad \nabla m_3 = (0, h_E^{-1})^\top,$$

and hence by representing the polynomial degree of m_β in the x and y directions using a vector with one entry 1 and one entry 0, as in the cell array `linear_polynomials`, the gradient can be very simply calculated by just dividing by h_E .

With the matrices D and B computed, we are in a position to use (4.7) to calculate the matrix Π representing the projector Π^E , as shown on line 37 of Listing 1, and stored in the variable `projector`. Consequently, with D , B and Π at our disposal, the local stiffness matrix can be computed as in (4.8).

The final step of this section is to add the elements of the local stiffness matrix to the positions in the global stiffness matrix corresponding to the associated global degrees of freedom. This is accomplished on line 41 through the local to global mapping discussed in Section 4.2.

4.4 The local forcing vector

To calculate the local forcing vector given in (4.2), we must first compute the projection $\Pi_0^E f$. By definition, this satisfies

$$\int_E \Pi_0^E f dx = \int_E f dx,$$

which, because we are projecting to constants, can be simplified to

$$\Pi_0^E f = \frac{1}{|E|} \int_E f dx \approx f(x_E, y_E),$$

where in the last relation we have used the barycentric quadrature rule on the polygon to approximate the integral. Since we are only considering the linear virtual element method, this is sufficiently accurate to ensure the optimal order of convergence in

the $H^1(\Omega)$ norm. It is the use of this quadrature which produces the requirement in Section 3 that the element must contain its own centroid. Clearly, more general integration methods are possible (for example by triangulating the element or using more advanced techniques such as [19, 20, 24, 26]), although for the sake of simplicity this is not something we pursue here. Since each basis function of V_h^E is defined to be 1 at a single vertex and 0 at the others, we can express

$$F_j^E = \int_E \overline{\varphi_j^E} \Pi_0^E f dx \approx \int_E \frac{f(x_E, y_E)}{NE} dx = \frac{|E|}{NE} f(x_E, y_E).$$

The code to compute this and store the result in the appropriate positions in the global forcing vector is on line 42 of Listing 1.

4.5 Applying the boundary conditions

The final step involves condensing the degrees of freedom associated with the boundary of the domain out of the linear system using the boundary condition, solving the resulting matrix equation, and re-applying the boundary data to the computed solution. This part of the procedure is exactly the same as for a standard finite element method, but for completeness we briefly review the process here.

Using the subscript B to denote the indices of degrees of freedom on $\partial\Omega$ and I to denote those in the interior of Ω , the matrix problem $KU = F$ can be expressed as

$$\begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} U_I \\ U_B \end{bmatrix} = \begin{bmatrix} F_I \\ F_B \end{bmatrix}$$

where $K_{IB} = K_{BI}^\top$ by the symmetry of the bilinear form. Therefore, since U_B is known, we find U_I by solving the problem

$$K_{II}U_I = F_I - K_{IB}U_B.$$

This is realised on lines 46–47 of the code, where we only store the result of solving the matrix system to the positions of the vector u which correspond to internal degrees of freedom while the values of the boundary degrees of freedom are set separately on line 48.

The final line of the code uses the auxiliary function `plot_solution`, also provided in the NETLIB package, to plot the vertex values of the virtual element solution and the mesh using MATLAB’s `patch` function.

Remark 1 (More general boundary conditions) Problems involving more general boundary conditions can also be implemented in a similar manner as for the finite element method. Suppose that the domain boundary is split into two components Γ_D and Γ_N such that $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Consider a problem posed with mixed type boundary conditions such that, for sufficiently regular Neumann boundary data g_N ,

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ n \cdot \nabla u &= g_N && \text{on } \Gamma_N. \end{aligned} \tag{4.9}$$

The case of mixed boundary conditions involving non-homogeneous Dirichlet data can also be handled as shown above. The weak form of this problem then reads: find $u \in H_{\Gamma_D}^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) - \int_{\Gamma_N} g_N v ds \quad \forall v \in H_{\Gamma_D}^1(\Omega),$$

where $H_{\Gamma_D}^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ consisting of functions with vanishing trace on Γ_D . A virtual element method for this problem can then be written as: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \ell_h(v_h) - \int_{\Gamma_N} g_N v_h ds \quad \forall v_h \in V_h,$$

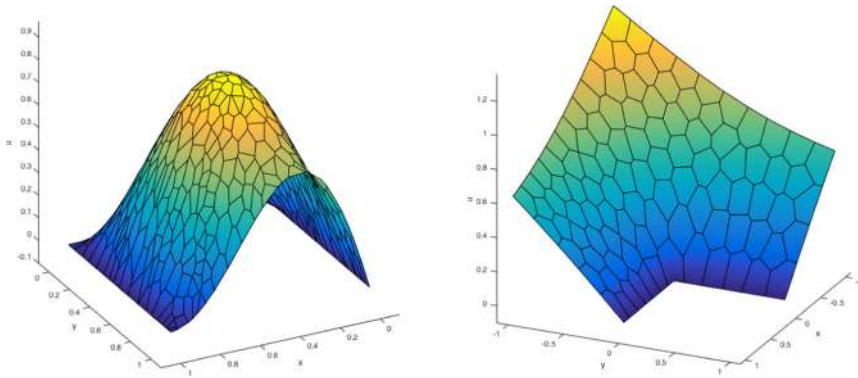
where a_h and ℓ_h are as before and V_h denotes a virtual element space incorporating the homogeneous Dirichlet boundary data on Γ_D . The left and right-hand sides of this problem can still be evaluated exactly since v_h is a known linear polynomial on each edge of the mesh and g_N is a known function on Γ_N . Consequently, only the extra vector corresponding to the term on the Neumann part of the boundary would need to be computed in the code to implement mixed boundary conditions.

Remark 2 (Extension to three spatial dimensions) Implementing the virtual element method for problems in three spatial dimensions on polyhedral meshes requires the use of the so-called enhanced virtual element spaces introduced in [1]. In these modified local function spaces, it becomes possible to compute the local L^2 -orthogonal projection of virtual element functions onto the polynomial subspace on each mesh element and face. The matrix formulation of this method and the algorithm for computing the L^2 -orthogonal projection operator is discussed in [7]. These projection operators are required to evaluate the right hand side of (3.2) when computing the local Ritz projection. This is because a virtual element function in three spatial dimensions is not a polynomial on each (in general polygonal) face of the element, but rather belongs to a virtual element space defined on that planar face.

5 Sample usage

The full code is available as the na45 package in the NUMERALGO library from Netlib. The software is distributed as a compressed archive file, and can be installed by decompressing the file to produce the directory “vem_50lines”, which contains the code. More information is provided in the README.txt file, contained in the package.

As described in Section 4, two different implementations of the right hand side forcing function and boundary condition are provided with the code. The first of these is designed for use on a square-shaped domain, while the second is designed for use on an L-shaped domain. The sample solution on the square-shaped domain is shown in Fig. 3a on the Voronoi mesh of Fig. 1c. This figure can be produced by navigating to the directory containing the file vem.m within MATLAB and executing



(A) The sample solution on a square domain, using the Voronoi mesh in Figure 1(c). (B) The sample solution on an L-shaped domain, using the mesh in Figure 1(f).

Fig. 3 The virtual element approximations to the solutions of the two sample problems supplied with the code

```
vem('meshes/voronoi.mat', @square_domain_rhs,
@square_domain_boundary_condition);
```

to provide the code with an appropriate mesh, forcing function, and boundary condition. The other sample meshes of the square domain could equivalently be used here by simply changing the first argument in the function call to be the path to the desired mesh. For instance, replacing the first argument with 'meshes/triangles.mat', 'meshes/squares.mat', 'meshes/smoothed-voronoi.mat', or 'meshes/non-convex.mat' would use the meshes shown in Fig. 1a, b, d, e, respectively.

Similarly, the sample solution for the L-shaped domain is shown in Fig. 3a. This plot is produced by calling the `vem` function with the parameters:

```
vem('meshes/L-domain.mat', @L_domain_rhs,
@L_domain_boundary_condition);
```

which corresponds to the standard problem on an L-shaped domain.

6 Conclusions and extensions

We have presented a 50-line MATLAB implementation of the linear virtual element method introduced in [6] for solving the Poisson problem on polygonal meshes in two spatial dimensions. The code for this is provided with this paper alongside several example polygonal meshes. To the best of our knowledge, this is the first publicly available implementation of the virtual element method. It is clear from the literature surrounding the method that its capabilities extend far beyond what is presented here, although the intention behind this work is to exemplify how the method can be implemented in practice, in the simplest possible setting. The ideas we present here

can, however, be generalised to much more complicated situations by applying similar processes to compute the various required terms. The possible extensions of this code are endless: the implementation of higher order methods, more general elliptic operators including lower order terms and non-constant coefficients [8, 14], basis functions with higher global regularity properties [10], mesh adaptation driven by a posteriori error indicators [12], or the consideration of time dependent problems [30] to name but a few.

Acknowledgments The support of the EPSRC whilst producing this work is gratefully acknowledged.

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