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THE VIRTUAL THURSTON SEMINORM OF 3-MANIFOLDS

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Abstract

We show that the Thurston seminorms of all finite covers of an aspherical 3-manifold determine whether it is a graph manifold, a mixed 3-manifold or hyperbolic.

1. Introduction

Let N be a 3-manifold. (Here, and throughout the paper all 3-manifolds are understood to be compact, orientable, connected, aspherical and with empty or toroidal boundary.) Given a surface Σ with connected components $\Sigma_1, \dots, \Sigma_k$ its complexity is defined to be

$$\chi_-(\Sigma) := \sum_{i=1}^{k} \max\{-\chi(\Sigma_i), 0\}.$$

Given a 3-manifold N and $\phi \in H^1(N;\mathbb{Z})$ the Thurston norm is defined as

$$x_N(\phi) := \min\{\chi_-(\Sigma) \,|\, \Sigma \subset N \text{ is a properly embedded surface, dual to } \phi\}.$$

Thurston [20] showed that x_N is a seminorm on $H^1(N;\mathbb{Z})$. It follows from standard arguments that x_N extends to a seminorm on $H^1(N;\mathbb{R})$. If N is hyperbolic, then N is in particular atoroidal which implies easily that x_N is a norm. On the other hand, the seminorm is degenerate whenever there is a non-separating torus, e.g. if $N = S^1 \times \Sigma$ where Σ is a surface of genus $g \geq 1$. Given any seminorm x on a vector space V the set $\{v \in V \,|\, x(v) = 0\}$ is a subspace that we refer to as the *kernel* $\ker(x)$ *of* x.

In this paper we study to which degree the Thurston norm of all finite covers of a 3-manifold determines the type of the JSJ-decomposition of the 3-manifold. Hereby we distinguish the following three mutually exclusive types of JSJ-decompositions a prime 3-manifold N can have:

(1) The 3-manifold N is hyperbolic.
(2) The 3-manifold N is a graph manifold, i.e. all its JSJ-components are Seifert fibered spaces.
(3) Following [18] we say that N is *mixed* if it is if the JSJ-decomposition is non-trivial and if it contains at least one hyperbolic JSJ-component.

This question is related to the general study of properties or invariants of a 3-manifold that can be determined from its finite covers, see for example [5], [6], [16] [21], [23].

In order to state our first result we introduce a few more definitions. Given a 3-manifold N we write

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$$\begin{aligned} b_1(N) &:= \dim_{\mathbb{R}}(H_1(N;\mathbb{R})), \\ k(N) &:= \dim_{\mathbb{R}}(\ker(x_N)), \\ r(N) &:= \begin{cases} 0, & \text{if } b_1(N) = 0, \\ \frac{k(N)}{b_1(N)}, & \text{if } b_1(N) > 0. \end{cases} \end{aligned}$$

Furthermore we write

$$\mathcal{C}(N) := \text{the class of all finite regular covers } \tilde{N} \text{ of } N,$$

and

$$\widehat{r}(N) := \sup_{\widetilde{N}\in\mathcal{C}(N)} r(\widetilde{N}).$$

The following proposition is well-known to the experts.

Proposition 1.1. *Let N be an aspherical 3-manifold with empty or toroidal boundary. Then N is hyperbolic if and only if $\widehat{r}(N) = 0$.*

Proof. If N is hyperbolic, then all its finite covers are hyperbolic, and as we pointed out above, in this case the seminorm is always a norm. On the other hand, if N is not hyperbolic and aspherical, then by standard arguments, see e.g. [4, (C.10)-(C.15)] there exists a finite regular cover $\widetilde{N}$ with a homologically essential torus. In particular $r(\widetilde{N}) > 0$. □

It is harder to distinguish graph manifolds from manifolds with a non-trivial JSJ-decomposition that contain at least one hyperbolic JSJ-component. In order to distinguish these two classes of 3-manifolds, we need to consider a wider class of finite coverings, which we call subregular, since they correspond to subnormal subgroups of the fundamental groups. We say that a covering $f\colon \widehat{N} \to N$ is *subregular* if the covering f can be written as a composition of coverings $f_i\colon N_i \to N_{i-1}$, $i = 1,\dots,k$ with $N_k = \widehat{N}$ and $N_0 = N$, such that each f_i is regular.

For a 3-manifold N we define:

$$\begin{aligned} C^{sub}(N) &:= \text{the class of all finite subregular covers } \widehat{N} \text{ of } N, \\ \rho(N) &:= \inf_{\widehat{N}\in C^{sub}(N)} r(\widehat{N}), \\ \widehat{\rho}(N) &:= \sup_{\widetilde{N}\in\mathcal{C}(N)} \rho(\widetilde{N}). \end{aligned}$$

The following is the main result of this paper. It characterizes graph manifolds N in term of the invariant $\widehat{\rho}(N)$. It also gives a characterization of manifolds with non vanishing simplicial volume (i.e. with at least one hyperbolic JSJ-component). This characterization is analogous to the one for hyperbolic manifolds in Proposition 1.1, but this time we use the invariant $\rho(N)$ instead of $r(N)$.

Theorem 1.2. *Let N be an aspherical 3-manifold with empty or toroidal boundary.*

(1) *If N is a graph manifold, then $\widehat{\rho}(N) = 1$.*
(2) *If N is not a graph manifold, i.e. if N admits a hyperbolic piece in its JSJ-decomposition, then $\widehat{\rho}(N) = 0$.*

The proof of Theorem 1.2 relies on the work of Agol [1, 2], Przytycki–Wise [18] and Wise [24, 25, 26].

The next corollary is a consequence of the combination of Proposition 1.1 and Theorem 1.2:

Corollary 1.3. *Let N be an aspherical 3-manifold with empty or toroidal boundary. Then the Thurston norms of all finite subregular covers of N determine into which of the following three categories N falls:*

(1) *graph manifold if and only if $\widehat{\rho}(N) = 1$,*
(2) *mixed manifold if and only if $\widehat{r}(N) > \widehat{\rho}(N) = 0$,*
(3) *hyperbolic manifold if and only if $\widehat{r}(N) = 0$.*

Convention. Unless it says specifically otherwise, all 3-manifolds are assumed to be compact, orientable, connected, and with empty or toroidal boundary. Furthermore all surfaces are assumed to be compact and orientable. Finally, all subsurfaces of a 3-manifold are assumed to be properly embedded.

2. The calculation of ρ for graph manifolds

The following theorem immediately implies Theorem 1.2 (1).

Theorem 2.1. *Let N be an aspherical graph manifold. Then given any $\epsilon > 0$ there exists a finite regular cover $\widehat{N}$ of N such that for any finite cover $\overline{N}$ of $\widehat{N}$ we have $r(\overline{N}) > 1 - \epsilon$.*

The proof of Theorem 2.1 will require the remainder of this section. Given a compact manifold X we write

$$c(X) := \dim_{\mathbb{R}} \left(\operatorname{coker}\{H_1(\partial X; \mathbb{R}) \to H_1(X; \mathbb{R})\}\right).$$

On several occasions we will need the following lemma.

Lemma 2.2. *Let $p\colon \widetilde{X} \to X$ be a finite covering of a manifold X. Then $c(\widetilde{X}) \geq c(X)$.*

Proof. We consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
H_1(\partial\widetilde{X};\mathbb{R}) & \longrightarrow & H_1(\widetilde{X};\mathbb{R}) & \longrightarrow & \operatorname{coker}\{H_1(\partial\widetilde{X};\mathbb{R}) \to H_1(\widetilde{X};\mathbb{R})\} & \longrightarrow & 0 \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
H_1(\partial X;\mathbb{R}) & \longrightarrow & H_1(X;\mathbb{R}) & \longrightarrow & \operatorname{coker}\{H_1(\partial X;\mathbb{R}) \to H_1(X;\mathbb{R})\} & \longrightarrow & 0.
\end{array}$$

For the left two vertical maps we also have the transfer maps p^* going from the bottom to the top. These maps have the property that the compositions $p_* \circ p^*$ are multiplication by $[X : \widetilde{X}]$, in particular the transfer maps are injective. Furthermore, the transfer maps give rise to a commutative diagram on the left. A straightforward diagram chase shows that the right vertical map also has a transfer map p^* such that the composition $p_* \circ p^*$ is injective. □

The next lemma is an immediate consequence of the Künneth Theorem.

Lemma 2.3. *For any surface Σ we have $c(S^1 \times \Sigma) = c(\Sigma)$.*

We say that a graph manifold N is of *product type* if each JSJ-component N_v is a product $S^1 \times \Sigma_v$ where Σ_v is a surface with $\chi(\Sigma_v) < 0$ and with at least two boundary components.

Proposition 2.4. *Let N be a graph manifold that is not a Seifert fibered space and that is not finitely covered by a torus bundle. Let $C > 0$. Then N is covered by a graph manifold $\widehat{N}$ of product type such that for each JSJ-component N_v of $\widehat{N}$ we have $c(N_v) > C$.*

Proof. Let N be a graph manifold that is not a Seifert fibered space and that is not finitely covered by a torus bundle. Let $C > 0$. By [3, Section 4.3] (see also [4, (C.10)] and [15]) there exists a finite cover N' that is of product type.

Furthermore, by [3, Proposition 5.22] there exists a prime $p \geq C$ and a finite cover N'' of N' such that for each JSJ-component $N''_v = S^1 \times \Sigma''_v$ the map $H_1(N''_v; \mathbb{F}_p) \to H_1(N''; \mathbb{F}_p)$ is injective. We denote by $\widehat{N}$ the finite cover of N'' that corresponds to the kernel of $\pi_1(N'') \to H_1(N''; \mathbb{Z}) \to H_1(N''; \mathbb{F}_p)$. In light of Lemma 2.3 it suffices to prove the following claim.

CLAIM. Each JSJ-component of $\widehat{N}$ is of the form $S^1 \times \Sigma$ where Σ is a surface with $c(\Sigma) > C$.

By Proposition 1.9.2 and Theorem 1.9.3 of [4] the JSJ-decomposition of $\widehat{N}$ is the pull-back of the JSJ-decomposition of N''. It follows from this fact and the above discussion of the chosen group homomorphism that each JSJ-component of $\widehat{N}$ is the finite cover of a manifold of the form $S^1 \times \Sigma$, where Σ is a surface with at least two boundary components and with $\chi(\Sigma) < 0$, and where we consider the cover corresponding to the kernel of the group homomorphism $\pi_1(S^1 \times \Sigma) \to H_1(S^1 \times \Sigma; \mathbb{F}_p)$. Note that this cover is of the form $S^1 \times \widehat{\Sigma}$ where $\widehat{\Sigma}$ is the finite cover of Σ corresponding to the kernel of the group homomorphism $\pi_1(\Sigma) \to H_1(\Sigma; \mathbb{F}_p)$. We write $d = |H_1(\Sigma; \mathbb{F}_p)|$. Since $\chi(\Sigma) < 0$ we have $d \geq p^2$. We make the following observations:

(1) By definition of 'product type' the surface Σ has at least two boundary components. It follows that every boundary component of Σ has image of order precisely p in $H_1(\Sigma; \mathbb{F}_p)$. Therefore

$$b_0(\partial\widehat{\Sigma}) = \frac{d}{p} \cdot b_0(\partial\Sigma).$$

(2) By the multiplicativity of the Euler characteristic we have

$$b_1(\widehat{\Sigma}) - 1 = d \cdot (b_1(\Sigma) - 1).$$

(3) For any surface Σ we have

$$b_0(\partial\Sigma) \;=\; b_1(\partial\Sigma) \;\leq\; b_1(\Sigma) + 1.$$

We now obtain that

$$\begin{aligned}
c(\widehat{\Sigma}) &= \dim_{\mathbb{R}} \Big(\operatorname{coker}\{H_1(\partial\widehat{\Sigma}; \mathbb{R}) \to H_1(\widehat{\Sigma}; \mathbb{R})\}\Big) \\
&\geq b_1(\widehat{\Sigma}) - b_1(\partial\widehat{\Sigma}) \\
&\geq d(b_1(\Sigma) - 1) + 1 - b_0(\partial\widehat{\Sigma}) \\
&\geq d(b_1(\Sigma) - 1) + 1 - \tfrac{d}{p}(b_1(\Sigma) + 1) \\
&= d(b_1(\Sigma) - 1) + 1 - \tfrac{d}{p}(b_1(\Sigma) - 1) - \tfrac{2d}{p} \\
&\geq -(d - \tfrac{d}{p})\chi(\Sigma) \\
&\geq d - \tfrac{d}{p}.
\end{aligned}$$

Hereby the first equality is given by definition, the following inequality is obvious, the next inequality is given by (2) and the fact that the boundary components of a surface are circles,

the following equality stems from (1) and (3), the next equality is purely algebraic, the following inequality is a consequence of $\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma)$ and $d \geq p^2$, and the final inequality comes from $\chi(\Sigma) \leq -1$.

Summarizing we have shown that $c(\widehat{\Sigma}) \geq d - \frac{d}{p}$. But since $d \geq p^2$ we see that the last term is at least $p \geq C$. Thus we have shown that $c(\widehat{\Sigma}) \geq C$. □

For the record we also mention the following elementary lemma.

Lemma 2.5. *If $f\colon \widehat{N} \to N$ is a finite covering of a 3-manifold, then there exists a finite regular covering $g\colon \overline{N} \to N$ that factors through f.*

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let N be an aspherical graph manifold and let $\epsilon > 0$.

If N is covered by a torus bundle, then there exists a finite regular cover $\widetilde{N}$ with vanishing Thurston norm and with $b_1(\widetilde{N}) \geq 1$. In particular there exists a finite regular cover $\widetilde{N}$ with $r(\widetilde{N}) = 1$.

If N is Seifert fibered, then there exists a finite regular cover $\widetilde{N}$ that is an S^1-bundle over a surface Σ. (See [3, Section 4.3] and [15] for details.) Since N is aspherical we know that Σ is not a sphere. The Thurston norm evidently vanishes if Σ is a disk, or if it is an annulus, or if it is a torus, i.e. in these cases we have $r(\widetilde{N}) = 1$. Thus we can now suppose that $\chi(\Sigma) < 0$.

If $\widetilde{N}$ is a non-trivial S^1-bundle over Σ, then Σ is closed and it follows from $\chi(\Sigma) < 0$, that $b_1(\widetilde{N}) \geq 1$. Furthermore it is straightforward to see that all homology classes are represented by tori, thus $k(\widetilde{N}) = b_1(\widetilde{N})$ and we see that $r(\widetilde{N}) = 1$.

On the other hand, if $\widetilde{N}$ is a trivial S^1-bundle over Σ, then $\widetilde{N} = S^1 \times \Sigma$. In that case it is well-known that $k(S^1 \times \Sigma) = b_1(\Sigma)$. Since $\chi(\Sigma) < 0$ there exists a cover $S^1 \times \overline{\Sigma}$ of $S^1 \times \Sigma$ with $r(S^1 \times \overline{\Sigma}) > 1 - \epsilon$. Furthermore, using Lemma 2.5 we can arrange that $S^1 \times \overline{\Sigma}$ is in fact a regular cover of N.

For the remainder of the proof we can now assume that N is neither covered by a torus bundle nor is it Seifert fibered. It follows from Proposition 2.4 and Lemmas 2.2 and 2.5 that there exists a finite regular cover $\widehat{N}$ of N such that $\widehat{N}$ is of product type and such such that for each JSJ-component N_v of $\widehat{N}$ we have $c(N_v) > \frac{1}{\epsilon}$. Now let $\overline{N}$ be a finite cover of $\widehat{N}$. As above, the JSJ-decomposition of $\overline{N}$ is induced by the JSJ-decomposition of $\widehat{N}$. It is thus again of product type.

We denote the JSJ-components of $\overline{N}$ by $\overline{N}_v = S^1 \times \Sigma_v$, $v \in V$. It follows from Lemma 2.2 and from the above that for each JSJ-component $\overline{N}_v$ we have $c(\overline{N}_v) > \frac{1}{\epsilon}$. For each v we denote by $f_v \in H_1(\overline{N}; \mathbb{Z})$ the element determined by the S^1-factor.

It follows from [8, Proposition 3.5] and the standard calculation of the Thurston norm for products $S^1 \times \Sigma$ that for any $\phi \in H^1(\overline{N}; \mathbb{R})$ the Thurston norm is given by

$$x_{\overline{N}}(\phi) = \sum_{v \in V} |\phi(f_v)| \cdot \chi_-(\Sigma_v).$$

In particular, the Thurston norm vanishes if ϕ vanishes on all elements $f_v, v \in V$. We thus see that

$$k(\overline{N}) \geq b_1(\overline{N}) - |V|.$$

On the other hand, it follows from the Mayer–Vietoris sequence corresponding to the de-

composition of $\overline{N}$ along the JSJ-tori into the JSJ-components that

$$b_1(\overline{N}) \geq \sum_{v \in V} \dim_{\mathbb{R}} \left(\operatorname{coker}\{H_1(\partial \overline{N}_v; \mathbb{R}) \to H_1(\overline{N}_v; \mathbb{R})\}\right) > \frac{1}{\epsilon} \cdot |V|.$$

Putting the last two inequalities together we see that

$$1 - r(\overline{N}) \leq \frac{b_1(\overline{N}) - k(\overline{N})}{b_1(\overline{N})} \leq \frac{|V|}{\frac{1}{\epsilon}|V|} = \epsilon.$$

□

3. The calculation of ρ for non-graph manifolds

The goal of this section is to prove the following theorem, which together with Theorem 2.1 implies Theorem 1.2, since the property of being aspherical and not being a graph manifold is preserved by going to finite covers.

Theorem 3.1. *Let N be an aspherical 3-manifold with empty or toroidal boundary that is not a graph manifold. Then given any $\epsilon > 0$, there exists a finite subregular cover $\overline{N}$ of N such that $r(\overline{N}) < \epsilon$. In particular $\rho(N) = 0$.*

We introduce the following definitions:

(1) Let N be a 3-manifold. An integral class $\phi \in H^1(N; \mathbb{Z}) = \operatorname{Hom}(\pi_1(N), \mathbb{Z})$ is called *fibered* if there exists a fibration $p\colon N \to S^1$ with $\phi = p_*\colon \pi_1(N) \to \mathbb{Z}$. We say N *is fibered* if N admits a fibered class.
(2) We say that a homomorphism $\phi\colon \pi \to \mathbb{Z}$ is *large* if ϕ is non-trivial and if it factors through an epimorphism from π onto a non-cyclic free group.

In the following proofs we will several times make use of the followings facts:

(A) If $p\colon \widetilde{M} \to M$ is a finite cover and $\phi \in H^1(M; \mathbb{Z})$ is a fibered class, then $p^*\phi \in H^1(\widetilde{M}; \mathbb{Z})$ is also fibered. In particular, if M is fibered, then $\widetilde{M}$ is also fibered.
(B) If $p\colon \widetilde{M} \to M$ is a finite cover and $\phi\colon \pi_1(M) \to \mathbb{Z}$ is large, then the composition $\phi \circ p_*\colon \pi_1(\widetilde{M}) \to \mathbb{Z}$ is also large.

Here the first statement is obvious and the second statement follows from the fact that any finite-index subgroup of a non-cyclic free group is again a non-cyclic free group.

One key ingredient in the proof of Theorem 3.1 is the Virtual Fibering theorem for non-graph manifolds that is due to Agol [1, 2], Przytycki–Wise [18] and Wise [24, 25, 26]. We refer to [4] for precise references. (See also [11, 13] and [9] for alternative proofs.)

Theorem 3.2 (Virtual Fibering Theorem)**.** *Any aspherical 3-manifold that is not a graph manifold admits a finite regular cover that is fibered.*

Before we continue we want to clarify our language for the JSJ-decomposition. Let N be an aspherical 3-manifold.

(1) We refer to the collection of the JSJ-tori together with the boundary tori as the *characteristic tori of N*.
(2) Given an aspherical 3-manifold N with boundary tori $S_1, \dots, S_k$ and JSJ-tori $T_1, \dots, T_l$ we pick disjoint tubular neighborhoods $S_i \times [-1, 0]$, $i = 1, \dots, k$ and $T_i \times [-1, 1]$,

$i = 1, \dots, l$ and we refer to the components of

$$N \setminus \bigcup_{i=1}^{k} S_i \times (-\tfrac{1}{2}, 0] \setminus \bigcup_{i=1}^{l} T_i \times (-\tfrac{1}{2}, \tfrac{1}{2})$$

as the JSJ-components of N. In particular, the complement of the union of the JSJ-components consists of tubular neighborhoods of all the characteristic tori.

On two occasions we will make use of the following lemma.

Lemma 3.3. *Let N be a 3-manifold and let N_v be a JSJ-component of N. If $\widetilde{N}_v$ is a finite cover of N_v, then there exists a finite regular covering $p\colon N' \to N$ such that each component of $p^{-1}(N_v)$ is a finite covering of $\widetilde{N}_v$.*

For closed 3-manifolds this is a result of Wilton–Zalesskii [22, Theorem A]. The case of 3-manifolds with non-trivial boundary can easily be reduced to the closed case (see e.g. [4, (C.35)] for details).

We continue with the following lemma.

Lemma 3.4. *Let N be a 3-manifold that is not a graph manifold. Then N admits a finite regular cover N' such that there exists a hyperbolic JSJ-component N'_h with $c(N'_h) > 0$.*

Proof. Let N_h be a a hyperbolic JSJ-component of N. It follows from the work of Agol [2] and Wise [24, 25, 26] (see also [4, Flowchart 4] for details) that $\pi_1(N_h)$ is large, i.e. $\pi_1(N_h)$ admits a finite index subgroup that surjects onto a non-cyclic free subgroup. This implies, see e.g. [4, (C.17)], that N_h admits a finite-index cover $\widetilde{N}_h$ with $c(\widetilde{N}_h) > 0$. Thus the lemma is an immediate consequence of Lemmas 2.2 and 3.3. □

We also have the following lemma which might be of independent interest.

Lemma 3.5. *Let N be a 3-manifold and let $\phi \in H^1(N;\mathbb{Z})$ be a non-trivial non-fibered class. Then there exists a finite regular covering $p\colon N' \to N$ such that the composition $\phi \circ p_*\colon \pi_1(N') \to \mathbb{Z}$ is large.*

The proof of the lemma is closely related to the proof of the main theorems of [10] and of [7] and to [17, Proof of Theorem 3.2.4].

Proof. We start out with a simple observation. Let Σ be a surface (not necessarily connected) in a 3-manifold dual to a class $\psi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$. We denote by $\Gamma(\Sigma)$ the graph whose vertices are precisely the components of M cut along Σ and whose edges are the components of Σ with the obvious maps from the edges to the vertices. Then the map $\psi\colon \pi_1(M) \to \mathbb{Z}$ factors through the canonical epimorphism $\pi_1(M) \to \pi_1(\Gamma(\Sigma))$.

Now we turn to the proof of the lemma. It is clear that it suffices to prove the lemma for primitive classes. We pick a Thurston norm minimizing surface Σ dual to ϕ that has the minimal number of components among all Thurston norm minimizing surfaces dual to ϕ. In particular Σ has no components that are separating. It follows easily that $\chi(\Gamma(\Sigma)) \leq 0$. If $\chi(\Gamma(\Sigma)) < 0$, then we are done by the above observation.

Now suppose that $\chi(\Gamma(\Sigma)) = 0$. Since ϕ is primitive and since Σ has the minimal number of components it follows from the argument on [7, p. 73] that Σ is connected. By Przytycki–Wise [19, Theorem 1.1] the subgroup $\pi_1(\Sigma) \subset \pi_1(M)$ is separable, i.e. given any $g \notin \pi_1(\Sigma)$ there exists a homomorphism $\alpha\colon \pi_1(M) \to G$ onto a finite group such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$.

Since Σ is not a fiber there exists by [14, Theorem 10.5] a $g \in \pi_1(M \setminus \Sigma \times (0,1))$ that does not come from $\pi_1(\Sigma \times \{0\})$. It now follows from a standard argument, see e.g. [4, (C.15)] or [17, Proof of Theorem 3.2.4], that applying subgroup separability to this g allows to build an epimorphism of $\pi_1(M)$ onto a free product with amalgamation of finite groups. The fact that the target group is virtually a free group of rank two gives the desired statement. □

Lemma 3.6. *Let N be a hyperbolic 3-manifold and let $\alpha, \beta \in H^1(N;\mathbb{Z})$ be linearly independent. Then there exist $p, q \in \mathbb{Z} \setminus \{0\}$ such that $p\alpha + q\beta$ is not fibered.*

Proof. We say that a rational class $\phi \in H^1(N;\mathbb{Q})$ is fibered if some integral multiple $n\phi \in H^1(N;\mathbb{Z})$, $n \in \mathbb{N}$ is fibered. We denote by

$$B := \{\phi \in H^1(N;\mathbb{Q}) \,|\, x_N(\phi) \leq 1\}$$

the norm ball of the Thurston seminorm. Since x_N is a seminorm the set B is convex and non-degenerate, the latter meaning that it is not contained in a lower-dimensional subspace of $H^1(N;\mathbb{Q})$. By assumption N is hyperbolic, this implies that the Thurston seminorm on $H^1(N;\mathbb{Q})$ is in fact a norm, i.e. B is compact. Thurston [20] showed that B is a polyhedron with rational vertices. Furthermore he showed that the set of fibered classes is given by the union of cones on certain open top-dimensional faces of the polyhedron B.

Now we denote by V the subspace of $H^1(N;\mathbb{Q})$ spanned by α and β. By assumption V is 2-dimensional. The intersection $B \cap V$ is a compact polytope in V with rational vertices. Since the polytope $B \cap V$ is compact and non-degenerate it has at least three vertices. By the aforementioned result of Thurston any class in the cone of any of the vertices is not fibered. Since α and β are linearly independent and since there are at least three vertices, and since the vertices are rational we can find *non-zero* $p, q \in \mathbb{Z} \setminus \{0\}$ such that $p\alpha + q\beta$ lies in the cone of one of the vertices, in particular it is not fibered. □

In the following we will on several occasions make use of the following lemma which is a straightforward consequence of Proposition 1.9.2 and Theorem 1.9.3 in [4].

Lemma 3.7. *Let N be a prime 3-manifold and let N_h be a hyperbolic JSJ-component of N. Then for each finite cover $p\colon N' \to N$ all the components of $p^{-1}(N_h)$ are hyperbolic JSJ-components of N'.*

Lemma 3.8. *Let N be a mixed 3-manifold. Then there exists a finite regular cover N' of N, a hyperbolic JSJ-component N'_h and a class $\phi \in H^1(N';\mathbb{Z})$ such the restriction of ϕ to N'_h is non-fibered but such that the restriction of ϕ to $\overline{N \setminus N'_h}$ is fibered.*

Proof. By Theorem 3.2, Lemmas 2.5 and 3.4 and Observation (A) there exists a finite regular cover N' of N that admits a fibered class $\phi \in H^1(N';\mathbb{Z})$ and that admits a hyperbolic JSJ-component N'_h with the property that $c(N'_h) > 0$. This implies that there exists a non-trivial homomorphism $\psi_h\colon H_1(N'_h;\mathbb{Z}) \to \mathbb{Z}$ that is trivial on the image of any boundary component of N'_h. In particular ψ_h factors through $H_1(N', \partial N';\mathbb{Z})$. We denote the resulting homomorphism

$$H_1(N';\mathbb{Z}) \to H_1(N', \overline{N' \setminus N'_h};\mathbb{Z}) \cong H_1(N'_h, \partial N'_h;\mathbb{Z}) \xrightarrow{\psi_h} \mathbb{Z}$$

by ψ.

We denote by ϕ_h the restriction of ϕ to N'_h. The classes ϕ_h and ψ_h in $H^1(N'_h;\mathbb{Z})$ are linearly independent since the former, as a fibered class is non-trivial on each boundary component of N'_h whereas the latter is by construction trivial on each boundary component. By Lemma 3.6 there exist $p, q \in \mathbb{Z}\setminus\{0\}$ such that $p\phi_h + q\psi_h$ is a non-fibered class in $H^1(N'_h;\mathbb{Z})$. On the other hand, the restriction of $p\phi_h + q\psi_h$ to $\overline{N \setminus N'_h}$ equals the restriction of $p\phi_h$ to $\overline{N \setminus N'_h}$. Since $p \neq 0$ this is a fibered class. □

Lemma 3.9. *Let N be a mixed 3-manifold. Then there exists a finite subregular cover N' of N, hyperbolic JSJ-components $N'_1, \dots, N'_k$, $k \geq 1$ of N', and a homomorphism $\phi \in \mathrm{Hom}(H_1(N';\mathbb{Z}),\mathbb{Z}) = H^1(N';\mathbb{Z})$ such that the restriction of ϕ to each N'_i is large but such that the restriction of ϕ to $\overline{N' \setminus (N'_1 \cup \dots \cup N'_k)}$ is fibered.*

Proof. In light of Lemma 3.8 we can without loss of generality assume that there exists a hyperbolic JSJ-component N_h of N and a class $\phi \in H^1(N';\mathbb{Z})$ such the restriction of ϕ to N_h is non-fibered but such that the restriction of ϕ to $\overline{N \setminus N_h}$ is fibered.

By Lemmas 3.5 and 3.3 and Observation (B) there exists a finite regular cover $p\colon N' \to N$ such that for one (and hence all) components $N'_1, \dots, N'_k$ of $p^{-1}(N_h)$ the map $p\circ\phi\colon \pi_1(N'_i) \to \pi_1(N_h) \to \mathbb{Z}$ factors through an epimorphism onto a non-cyclic free group.

On the other hand it follows from Observation (A) that the restriction of $p^*\phi$ to $\overline{N' \setminus (N'_1 \cup \dots \cup N'_k)} = p^{-1}(\overline{N \setminus N_h})$ is fibered. □

Let N be a 3-manifold. We have the following notations:

(1) Given $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ and $n \in \mathbb{N}$ we denote by $\phi_n\colon \pi_1(N) \to \mathbb{Z}_n$ the homomorphism that is given by the composition of ϕ with the projection map $\mathbb{Z} \to \mathbb{Z}_n$.

(2) Given a homomorphism $\alpha\colon \pi_1(N) \to G$ we denote by N_α the corresponding cover. If α is not surjective, then N_α consists of $|\mathrm{coker}(\alpha)|$ copies of the finite cover of N corresponding to $\ker(\alpha)$.

We recall the following well-known lemma.

Lemma 3.10. *Let N be a 3-manifold and let $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ be a fibered class. Then for all but finitely many primes p we have*

$$b_1(N_{\phi_p}) \leq 3 + x_N(\phi).$$

Proof. Let ϕ be a fibered class. We write $\phi = d\psi$ where ψ is a primitive class and $d \in \mathbb{N}$. It is well-known that ψ is again fibered with $x_N(\phi) = dx_N(\psi)$. We denote by S the fiber of the surface bundle corresponding to ψ. Surthermore we denote by $\varphi\colon \pi_1(S) \to \pi_1(S)$ the corresponding monodromy. Also, given any automorphism γ of $\pi_1(S)$ we denote by $\mathbb{Z} \ltimes_\gamma \pi_1(S)$ the corresponding semidirect product.

Now let $n \in \mathbb{N}$. It is straightforward to see that

$$\begin{aligned} H_1(N_{\psi_n};\mathbb{Z}) &\cong H_1(n\mathbb{Z} \ltimes_\varphi \pi_1(S);\mathbb{Z}) \\ &\cong H_1(\mathbb{Z} \ltimes_{\varphi^n} \pi_1(S);\mathbb{Z}) \cong \mathbb{Z} \oplus H_1(S;\mathbb{Z})/(\varphi^n - \mathrm{id}). \end{aligned}$$

It follows that

$$b_1(N_{\psi_n}) \leq \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z} \oplus H_1(S;\mathbb{Z})/(\varphi^n - \mathrm{id})) \leq 1 + b_1(S) \leq 3 + x_N(\psi).$$

Now let p be a prime that is coprime to d. It follows that the map

$$\pi_1(N) \xrightarrow{d\cdot\psi=\phi} d\mathbb{Z} \to \mathbb{Z}_p$$

is surjective. In particular $N_{\phi_p} = N_{\psi_p}$, and we thus see from the above that

$$b_1(N_{\phi_p}) = b_1(N_{\psi_p}) \leq 3 + x_N(\psi) \leq 3 + x_N(\phi).$$

□

The following is the last lemma that we will need for the proof of Theorem 3.1.

Lemma 3.11. *Let N be a 3-manifold and let $\phi\colon \pi_1(N) \to \mathbb{Z}$ be a large homomorphism such that the restriction of ϕ to all boundary-components of N is non-trivial. Then for all but finitely many primes p we have*

$$c(N_{\phi_p}) \geq p - 1 - 2b_0(\partial N).$$

Proof. Let N be a 3-manifold and let $\phi\colon \pi_1(N) \to \mathbb{Z}$ be a non-trivial homomorphism that factors through an epimorphism $\alpha\colon \pi_1(N) \to F$ onto a non-cyclic free group F and such that the restriction of ϕ to all boundary-components of N is non-trivial. By a slight abuse of notation we denote the induced homomorphism $F \to \mathbb{Z}$ by ϕ as well.

We denote the boundary components of N by $T_1, \dots, T_k$. For each $i \in \{1, \dots, k\}$ we define $d_i \in \mathbb{N}$ by the condition that $\phi(\pi_1(T_i)) = d_i\mathbb{Z}$. Similarly we define d by $\phi(\pi_1(N)) = d\mathbb{Z}$. By our hypothesis we know that d and all the d_i are non-zero.

Now let p be any prime that is coprime to d and to $d_1, \dots, d_k$. This choice of p implies that the restriction of ϕ_p to each boundary component is surjective. Furthermore the homomorphism $\phi_p\colon F \to \mathbb{Z}_p$ is surjective. We deduce that

$$b_1(N_{\phi_p}) \;\geq\; \operatorname{rank}(\ker(\phi_p\colon F \to \mathbb{Z}_p)) \;\geq\; p - 1.$$

Since the restriction of ϕ_p to each boundary component is surjective we see that the induced covering of each boundary component is connected. Put differently, N_{ϕ_p} has precisely k boundary components, each of which is a torus. We conclude that

$$\begin{aligned} c(N_{\phi_p}) &= \operatorname{rank}\left(\operatorname{coker}\{H_1(\partial N_{\phi_p};\mathbb{Z}) \to H_1(N_{\phi_p};\mathbb{Z})\}\right) \\ &\geq b_1(N_{\phi_p}) - b_1(\partial N_{\phi_p}) \;\geq\; p - 1 - 2b_0(\partial N). \end{aligned}$$

□

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let N be an aspherical 3-manifold that is not a graph manifold. We need to show that given any $\epsilon > 0$, there exists a finite subregular cover $\overline{N}$ of N such that $r(\overline{N}) < \epsilon$.

So let N be an aspherical 3-manifold that is not a graph manifold and let $\epsilon > 0$. If N is hyperbolic then it follows from Proposition 1.1 that already $r(N) = 0$. Thus henceforth we can restrict ourselves to the case that N is not hyperbolic, i.e. N is a mixed manifold.

By Lemma 3.9 we can without loss of generality assume that there exists $k \geq 1$ hyperbolic JSJ-components $N_1, \dots, N_k$ of N and a homomorphism $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(H_1(N;\mathbb{Z}),\mathbb{Z})$ such the restriction of ϕ to each N_i, $i = 1, \dots, k$ is large but such that the restriction of ϕ to

$M := \overline{N \setminus (N_1 \cup \cdots \cup N_k)}$ is fibered.

By our definition of JSJ-components we see that M contains all characteristic tori of N. Since $\phi|_M$ is fibered it follows from [8, Section 4] that the restriction of ϕ to a tubular neighborhood of each characteristic torus is a fibered class. It follows in particular that the restriction of ϕ to each characteristic torus is non-zero. This in turn implies that for almost all primes p the restriction of ϕ_p to each characteristic torus is an epimorphism.

We write $C := 3 + x_M(\phi|_M)$. We denote by j the number of JSJ-tori of N and we denote by b the number of boundary tori of N. By the above and by Lemmas 3.10 and 3.11 there exists a prime p such that the covering map $f\colon \overline{N} \to N$ corresponding to the homomorphism $\phi_p\colon \pi_1(N) \to \mathbb{Z}_p$ has the following properties:

(1) The restriction of ϕ_p to each characteristic torus and to each JSJ-component is an epimorphism. In particular the preimages of the JSJ-tori and the JSJ-components under f are connected.
(2) For each $i \in \{1, \ldots, k\}$ we have $c(f^{-1}(N_i)) > \frac{C+6j+2b}{k\epsilon}$.
(3) We have $b_1(f^{-1}(M)) \leq C$.

We claim that $\overline{N}$ has the desired property.

It follows from the Mayer–Vietoris sequence applied to the decomposition of $\overline{N}$ along the j tori that are given by the preimages of the JSJ-tori of N and from (3) that

$$\sum_{i=1}^{k} c(f^{-1}(N_i)) \;\leq\; b_1(\overline{N}) \leq C + 2j + \sum_{i=1}^{k} b_1(f^{-1}(N_i)).$$

The union of the $f^{-1}(N_i)$, $i = 1, \ldots, k$ has at most $2j + b$ boundary tori. It follows easily that

$$\sum_{i=1}^{k} b_1(f^{-1}(N_i)) \;\leq\; 4j + 2b + \sum_{i=1}^{k} c(f^{-1}(N_i)).$$

Putting the above two inequalities together we obtain that

$$\sum_{i=1}^{k} c(f^{-1}(N_i)) \;\leq\; b_1(\overline{N}) \;\leq\; C + 6j + 2b + \sum_{i=1}^{k} c(f^{-1}(N_i)).$$

On the other hand, it follows from the same Mayer–Vietoris sequence together with the fact that the Thurston seminorm is in fact a norm on hyperbolic 3-manifolds that

$$k(\overline{N}) \leq b_1(\overline{N}) - \sum_{i=1}^{k} c(f^{-1}(N_i)).$$

The combination of the last two inequalities together with (2) shows that

$$r(\overline{N}) \;=\; \frac{k(\overline{N})}{b_1(\overline{N})} \;\leq\; \frac{C + 6j + 2b}{\sum_{i=1}^{k} c(f^{-1}(N_i))} < \epsilon.$$

□

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