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The Viscosity of the Mantle

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Summary

Creep under low stresses is by diffusion and has a linear relation between stress and strain rate; it also obeys the Navier–Stokes equation. Therefore the viscosity of the mantle may be calculated from solid state theory and also from the slow deformations of the Earth. The viscosities derived by these methods are in reasonable agreement, and both show that the viscosity of the lower mantle is $\sim 10^5$ greater than that of the upper. This high viscosity prevents polar wandering and lower mantle convection. Some suggested modifications of the viscosity depth calculations from post glacial uplift may improve their accuracy considerably.

1. The non-hydrostatic equatorial bulge

All calculations concerned with post-glacial uplift (Haskell 1935, McConnell 1965) and with convection within the mantle (Pekeris 1935) have used a linear relation between stress and strain rate. Creep in the material then obeys Stokes' equation and therefore may be described by a viscosity which is independent of stress. Such creep only takes place at stresses below the yield stress $< \sim 10^{-4}\,\mu$, where μ is the shear modulus; at higher stresses the relation between stress and strain rate is non-linear (see Pratt, above). At stresses below the yield point diffusion creep alone can operate, and it has a linear relation between stress and strain rate. The diffusion creep rate also depends exponentially on temperature (Gordon 1965), and therefore is not sufficiently rapid to be measurable under ordinary laboratory conditions. For this reason solids were believed to be unable to deform for stresses below the yield stress, a concept called finite strength.

The stresses caused by the melting of ice caps, by the non-hydrostatic bulge, and by regional temperature differences are probably less than the yield stress. Thus the viscosity will be stress independent, and the equation of a Maxwell solid should describe the flow. However, there are regions where earthquakes occur, and where this argument is clearly false.

The viscosity of the mantle between the crust and a depth of 1000 km may be found from the uplift of formerly glaciated areas. McConnell (1965) finds the uplift of Fennoscandia can be explained if the viscosity is least at a depth of about 300 km, increasing both upwards and downwards (Fig. 2). The viscosity of the lower mantle, which is of considerable importance to theories of convection and polar wandering, unfortunately cannot be calculated from the uplift for two reasons. The first is that the depth to which the flow penetrates in a homogeneous half space is about equal to the radius of the applied load, or about 1000 km for Fennoscandia. This difficulty can be overcome when the uplift of the Canadian Shield has been measured. If, however, McConnell is correct and there is a viscosity minimum at a depth of about 300 km a second difficulty arises because the flow in the low viscosity layer will shield the lower mantle from the applied force. It is therefore unlikely that any surface load can be used to estimate the viscosity of the lower mantle.

MacDonald (1963) pointed out that the viscosity of the lower mantle may be calculated from the size of the non-hydrostatic equatorial bulge, if this is caused by the bulge being unable to keep pace with the Earth's deceleration. The rotation produces a body force throughout the Earth, and cannot be shielded by the low viscosity layer. His argument depends on the bulge having a different origin from the other harmonics of the external gravity field. These are best compared by calculating the gravitational potential energy contained in each of them. There is a unique relation between the harmonic coefficients C_l^m and S_l^m and the energy E_l^m only if the Earth is homogeneous:

$$E_l^m = \frac{2\pi gaM}{3}(l-1)(2l+1)[(C_l^m)^2+(S_l^m)^2], \quad m \neq 0$$

$$= \frac{4\pi gaM}{3}(l-1)(2l+1)(C_l)^2, \quad m=0. \qquad (1.1)$$

M is the mass of the Earth, and the coefficients are for spherical harmonics normalized to make:

$$\int_0^\pi \int_0^{2\pi} X_l^m (X_l^m)^* \sin\theta \, d\theta \, d\phi = 4\pi.$$

The energies are shown in Table 1. The hydrostatic coefficients calculated by Jeffreys (1963) have been subtracted from the observed values before calculating the energy. It is clear that the non-hydrostatic bulge contains an order of magnitude more energy than any other coefficient. However, the non-hydrostatic bulge is the difference between two large numbers and if either were in error by 1% the energy would be greatly different. There is no reason to suspect the observations, since C_2^0 is one of the best determined coefficients (King-Hele 1965); nor are Jeffreys' calculations likely to be in error by much more than 0·04%, which is unimportant here.

Various explanations of the size of the non-hydrostatic bulge have been suggested. Jeffreys believes that it is supported by the Earth's finite strength. However, a solid can deform by diffusion creep under any stress, however small. Wang (1966) believes that the polar ice which melted at the end of the last glaciation has left behind an equatorial bulge. An approximate calculation shows that this effect would be just

Table 1

The gravitational energy in the external gravity field

Energy in units of $1{\cdot}56 \times 10^{28}$ ergs, upper values Guier & Newton (1965) lower values Izsak (1964).

	l				
m	2	3	4	5	6
0	112	14·5	6·8	0·0	0·7
	108	13·1	3·6	0·0	0·8
1		24·1	6·9	1·0	0·3
		5·0	1·3	1·0	0·5
2	17·8	13·6	5·0	4·1	1·6
	5·6	0·8	0·9	7·6	5·3
3		9·8	9·7	0·4	9·2
		7·8	1·1	11·6	1·6
4			1·1	6·7	11·4
			1·2	6·4	6·8
5				9·9	9·5
				4·6	7·8
6					1·7
					14·6

large enough to explain the observations if uplift had not taken place, but probably all but one-tenth has now been removed by flow within the mantle. Another argument against the non-hydrostatic bulge being caused by the last glaciation comes from the observations of the change in the length of day due to causes other than tidal friction (Munk & MacDonald 1960). If C and A are the time dependent moments of inertia about axes passing through the pole and the equator, differentiation of MacCullagh's formula gives:

$$\frac{dC}{dt}-\frac{dA}{dt}=Ma^2\frac{dJ_2}{dt}. \tag{1.2}$$

The mean moment of inertia is approximately independent of time:

$$\frac{dC}{dt}+2\frac{dA}{dt}=0. \tag{1.3}$$

Apart from external forces, the angular momentum of the Earth must remain constant:

$$\frac{1}{C}\frac{dC}{dt}+\frac{1}{\omega}\dot{\omega}_i=0, \tag{1.4}$$

where $\dot{\omega}_i$ is the angular acceleration produced by processes in and on the Earth. Since $C \simeq 0{\cdot}33Ma^2$ for the Earth, equations (1.2)–(1.4) give:

$$\dot{\omega}_i=-2{\cdot}0\,\dot{J}_2\,\omega. \tag{1.5}$$

Both Wang and McConnell believe that the time constant, τ, for the decay of a $X_2{}^0(\theta)$ surface disturbance is $\sim$7000 years, when:

$$J_2=H_2+U_2(0)\exp(-t/\tau).$$

H_2 is the hydrostatic value of J_2 whose time variation can be neglected in this problem. The second term describes the time variation of the non-hydrostatic bulge. Equation (1.5) then becomes:

$$\dot{\omega}_i=\frac{2{\cdot}0\omega U_2(t)}{\tau}. \tag{1.6}$$

Substituting $U_2(t)=10^{-5}$ gives $\dot{\omega}_i=7\times10^{-21}$. The observed value for $\dot{\omega}_i$ is 3×10^{-22} therefore only one-twentieth of the present non-hydrostatic bulge can be caused by the ice. This value agrees with that calculated from the change in sea level (McKenzie 1966), and suggests the observed acceleration is indeed caused by isostatic response.

Dicke (personal communication) suggested that the effect of temperature on the surfaces of constant density might be sufficient to produce the observed gravity field, but this effect was also found to be an order of magnitude too small. If the bulge were due to a world-wide convection pattern of the required shape, it is hard to understand why it should be aligned along the rotational axis. Thus the only mechanism that has been suggested which is capable of producing the non-hydrostatic bulge is the tidal deceleration.

2. Viscosities and relaxation times

Munk & MacDonald (1960) show that the bulge would have been hydrostatic $1{\cdot}8\times10^7$ years ago if the Earth were uniform. MacDonald (1963) neglects gravitational energy and interprets this time as the elastic relaxation time τ_E of a Maxwell body by viscous flow:

$$\tau_E=\eta/\mu=\frac{\rho\nu}{\mu}. \tag{2.1}$$

The mean rigidity μ of the Earth is $\sim 1{\cdot}5\times 10^{12}$ dyn/cm^2, and hence the dynamic viscosity $\upsilon\sim 1{\cdot}5\times 10^{26}$ stokes. This argument is not correct because the gravitational energy is large compared to the elastic, and therefore cannot be neglected. Darwin (1879) showed that the relaxation time for the equatorial bulge of a homogeneous viscous sphere is:

$$\tau=\frac{19\nu}{2ga}. \tag{2.2}$$

The corresponding equation for a Maxwell solid may be obtained by replacing ν by:

$$\frac{\mu\nu}{\mu-(\nu\rho/\tau)} \tag{2.3}$$

(Bland 1960). The relaxation time for an $X_2^m(\theta)$ surface disturbance is:

$$\tau=\frac{19\nu}{2ga}+\frac{\nu\rho}{\mu} \tag{2.4}$$

or for any harmonic of degree l:

$$\tau=\frac{[2(l+1)^2+1]\nu}{lga}+\frac{\nu\rho}{\mu}. \tag{2.5}$$

In the case of the bulge the first term on the right of (2.4) is five times as big as the second, which may therefore be neglected. To this approximation the viscous equations, rather than those for a Maxwell body, may be used. For Fennoscandia $l\gg 1$, and equation (2.5) becomes

$$\tau=\frac{2l\nu}{ga}+\frac{\nu\rho}{\mu}. \tag{2.6}$$

The radius of the area $\sim$1000 km, which corresponds to $l\sim 20$. In the upper mantle $\mu\sim 10^{11}$ and the two terms on the right of (2.6) are approximately equal and neither can be neglected. Thus all estimates of the viscosity from uplift with this radius will be double the correct value unless the full equations are used. In the case of Lake Bonneville (Crittenden 1963) the first term is five times the second, and therefore elastic effects can be ignored. Thus McConnell's viscosities are systematically too high, but the error is only important at wavelengths $\sim$4000 km.

Though it is mathematically straightforward to prove that elasticity increases the relaxation time, the physics is harder to understand; thus the argument which follows is an illustration, rather than an alternative, to that above. An ice cap on a Maxwell body will be completely compensated if sufficient time is allowed, and no elastic stresses will remain. When the ice melts there will be some elastic uplift immediately, followed by a slow flow to remove the deformation. The force producing both the elastic stress and the flow is the surface deformation, and this force is reduced by the elastic response. However, the total volume of material which must flow to remove the surface deformation and the elastic stress is not changed, and therefore the flow takes longer. This argument suggests that the change in relaxation time due to elastic effects is only important if the elastic uplift on removal of the load is comparable to the remaining uplift required to restore the original shape. An ice cap 2500 m thick over Fennoscandia would produce an elastic downwarp of $\sim$150 m (Slichter & Caputo 1960) at the centre; the uplift since the ice melted is $\sim$250 m. Thus the relaxation time should be significantly increased by the elastic effects, in agreement with (2.5). There is similar agreement in the case of Lake Bonneville and non-hydrostatic bulge. If only those beaches which have been formed after all the ice in Fennoscandia melted are used to determine the shape and size of the uplift, the immediate elastic uplift is not important, though the increase in the relaxation time is.

Though these arguments show that the equations for a Maxwell solid, rather than a viscous liquid, must be used for post glacial uplift, they do not apply to the non-hydrostatic bulge.

In the case of both Fennoscandia and the Canadian Shield, the uplift is measured by the height of a raised beach above the present sea level, and then a correction is applied to allow for the rise in sea level since the last ice age. This procedure assumes that the geoid is not altered by the uplift. Unfortunately the size of both these ice caps was sufficiently large to cause gravity anomalies and distort the geoid by about 40 m in the case of Fennoscandia. Therefore this effect, though probably small, may be important to viscosity depth calculations.

Substitution into (2.4) shows that a viscosity of $\sim 4\times 10^{25}$ stokes is required throughout the Earth to support the bulge. Therefore it is clear that a homogeneous Earth is too simple a model; a complete model must contain an upper mantle with a viscosity $\sim 10^{21}$ overlying a lower mantle, viscosity $\sim 10^{25}$. The core is an inviscid liquid over these time scales. It is likely that the increase in density caused by phase changes between the upper and lower mantle will increase the activation energy and hence the viscosity. Therefore the boundary between the upper and lower mantle was taken as a pressure dependent phase change. Such a boundary condition allows matter to flow across the boundary. Since the viscosity contrast across this boundary is so great ($\sim 10^4$), the tangential stress on the lower mantle due to flow in the upper mantle can be neglected. The lower mantle viscosity required to support the bulge is 6×10^{26} stokes. Other models are possible, but that of Takeuchi & Hasegawa (1965) neglects the gravity field due to the ellipticity of the core–mantle boundary; which governs the external gravity field in the model above. The main conclusion is that the size of the bulge requires a considerable part of the mantle to have a viscosity of $>4\times 10^{25}$ stokes.

3. Diffusion creep, convection and polar wandering

The linear relation between stress and strain rate is true only if the deformation is produced by diffusion creep. It is therefore of interest to discover whether the viscosities found from glacial uplift and the bulge are consistent with the present knowledge of the mantle.

Creep by diffusion of ions or vacancies through a crystal lattice must preserve charge neutrality. Therefore creep will be rate limited by the ionic species which diffuses least rapidly and hence has the highest activation energy. In close packed silicates the oxygen ion is the most difficult to move and governs the deformation rate.

If the activation enthalpy for self diffusion of oxygen, H^*, and the temperature are known the viscosity can be calculated:

$$\nu = \frac{10kTR^2}{Dm_a}, \tag{3.1}$$

where R is the mean crystal radius, D the diffusion coefficient, and m_a the mass of an oxygen ion. D is an exponential function of temperature:

$$D = D_0 \exp\left(-\frac{H^*}{kT}\right). \tag{3.2}$$

D_0 is a constant which varies little with temperature. H^* may be expressed in terms of the activation energy E^*, and volume V^*:

$$H^* = E^* + PV^*. \tag{3.3}$$

The most reliable estimates of temperature come from the electrical conductivity, σ, of the mantle:

$$\sigma = \sigma_0 \exp\left(-\frac{H}{kT}\right), \tag{3.4}$$

where σ_0 is a constant. Tozer (1959) used this equation, but because there were no measurements of σ_0 and H for the high density phases of the lower mantle, he was forced to assume both constants were the same throughout the mantle. Bradley *et al.* (1963), and Akimoto & Fujisawa (1965) synthesized the spinel phase of fayalite and discovered it had a considerably lower electron activation energy, E_{sp}, than the low pressure phase. In order to obtain some idea of how Tozer's temperature distribution is affected by this change in activation energy it was assumed that:

$$H_{sp} = \frac{E_{sp}}{E_{fay}} H_{ol}.$$

H_{sp} is the activation enthalpy for electrons in the spinel phase of olivine. σ_0 was taken to be the same for both phases. The corrected temperature, T_c, may then be calculated for Tozer's, T_u:

$$T_c = \frac{E_{sp}}{E_{fay}} T_u. \tag{3.5}$$

This correction is required in the lower mantle only. Fig. 1 shows that T_c is less than T_u, and does not have a sharp change in gradient at 400 km depth.

The electrical conductivity in the mantle is caused by the movement of electrons and positive holes, and not by the movement of charged ions. Therefore there is no

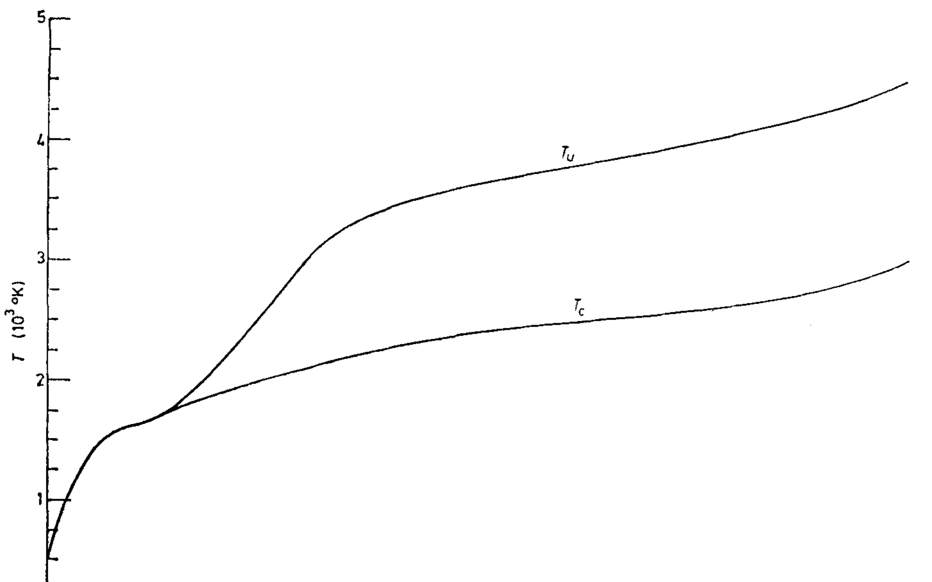

FIG. 1. Temperature within the mantle. T_u, Tozer (1960); T_c, corrected for phase change.

relation between H and H^*, and H^* must be estimated in a different way. Gordon (1965) discusses measurements on aluminium oxide by Oishi and Kingery and believes 6eV for E^* and 10 Å^3 for V^* are reasonable values for oxygen ion diffusion in the lower mantle, where the close packed phases are probably similar to this oxide. However, the phases in the upper mantle are less dense, and therefore have a lower value for E^*. Keyes (1963) has developed a semiempirical theory which relates bulk properties of a solid to the activation energy. This theory suggests a value of 4eV for E^* in the upper mantle. Equations (3.1)–(3.3) can now be used to calculate the viscosity if the grain size R can be estimated. There is at present no means of finding R anywhere within the mantle, and therefore any value is a guess. Gordon uses $R = 0{\cdot}05$ cm and $D_0 = 5\,\mathrm{cm^2/s}$. Fig. 2 compares the viscosity calculated from the theory of diffusion creep with that derived from post glacial uplift and the non-hydrostatic bulge. The agreement is reasonable considering the uncertainties in the estimates of the parameters.

The lower mantle was previously believed to have a viscosity of $\sim 10^{21}$. The 10^5 increase required to produce the bulge has two effects: convection involving the whole mantle is prevented; and the rotational axis is fixed to the lower mantle.

Most discussions of convection within the mantle assume that there is a solution to the equations of motion with the relevant boundary conditions when the fluid is not in motion. In the upper mantle there is no such solution because of the regional temperature differences between oceans and continents (see discussion following Knopoff, above). In the lower mantle the isotherms are not the same as the surfaces of constant density because the Earth is rotating (Eddington 1926). Since in neither case is there a solution without fluid flow, the stability analysis should be made on the convecting system. However, only the static problem of stability has been solved, and therefore convection in the mantle can only be discussed from this point of view.

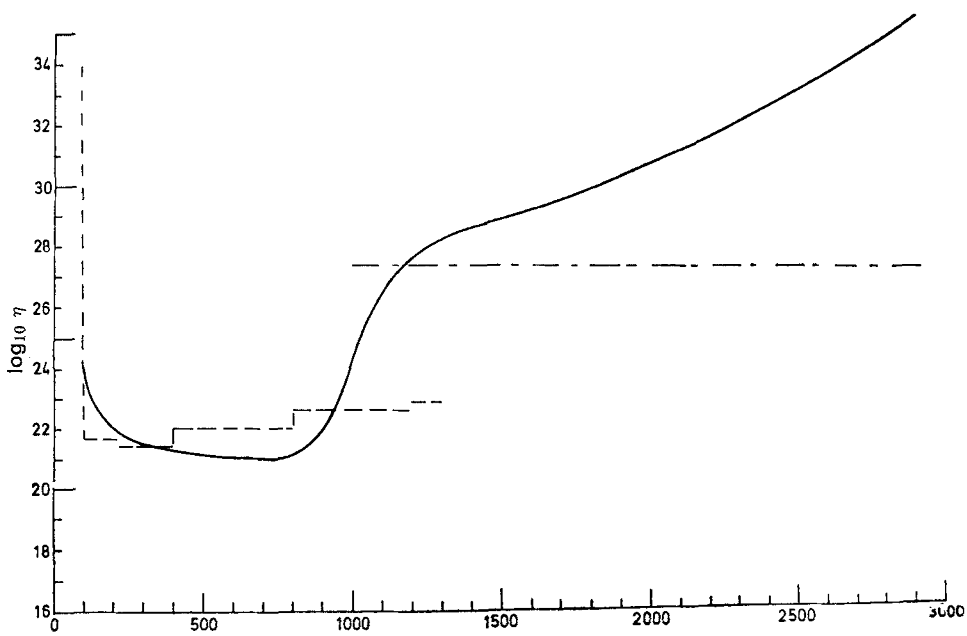

FIG. 2. The viscosity within the mantle. ----, post glacial uplift; —·—·—, equatorial bulge; ——, calculated from T_c, using $E^* = 4$ eV, $V_a = 10$ Å^3 above 1000 km; $E^* = 6$ eV, $V_a = 10$ Å^3 below 1000 km.

The Rayleigh criterion for convection in a horizontal layer (Chandrasekhar 1961) of thickness d uniformly heated from below requires a temperature gradient β in excess of the adiabatic:

$$\beta = \frac{\kappa \nu R_c}{g \alpha d^4}.$$

R_c is the critical Rayleigh number, $\sim 2 \times 10^3$, κ the thermal conductivity $\sim 0{\cdot}01$ cal/°C/s, and α is the thermal expansion coefficient $\sim 2 \times 10^{-5}$/°C. If $\nu \sim 6 \times 10^{26}$ the value of β required before convection can take place is ~ 10 °C/km, or a temperature difference of $\sim 20\,000$ °C across the lower mantle. The actual temperature difference is probably between 1000 °C and 2000 °C and is far too small to cause convection. The adiabatic gradient is 0·5 °C/km, or a temperature difference across the lower mantle of 1000 °C; thus the actual temperature gradient may not even exceed the adiabatic.

Convection in the upper mantle is not affected by these calculations, nor is there any difficulty in convecting through the phase change region if this is spread over ~ 500 km (Verhoogen 1965).

Gold (1955) showed that the equatorial bulge reduced but did not prevent polar wandering. The decay time of the Chandler wobble for a homogeneous Earth is the same as the relaxation time of the equatorial bulge. The Chandler wobble is probably damped in ~ 10 years and therefore Gold had no difficulty in producing polar wandering in $\sim 10^5$ years, caused by the present distribution of continents. However, the Earth is not homogeneous, and whereas the wobble will be damped by the less viscous layers, the rate of polar wandering, like the non-hydrostatic bulge, will be governed by the most viscous layer. A relaxation time of $\sim 10^8$ years is therefore more likely for the equatorial bulge, and then the continents require $\sim 10^{12}$ years to move the pole. Even though the density variations in the mantle are probably greater than those caused by the continents, it is unlikely that they are sufficiently great to produce polar wandering in geological time.

Conclusion

It is likely that the form of the Earth's surface is determined by processes in the upper mantle, extending to a depth of ~ 1000 km. The lower mantle below this behaves as a rigid core to any movements within the upper mantle. Thus a good approximation is to neglect the Earth's curvature in convective and tectonic problems. The calculation of the viscosity of the lower mantle is explained more fully in another paper (McKenzie 1966).

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