THE VISCOUS DAMPING OF GRAVITY WAVES IN SHALLOW WATER

BY

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1. — Introduction

A knowledge of the rate of viscous dissipation of gravity wave energy is required in the assessment of various processes of wave generation by wind, and in particular for the calculation of the minimum critical wind speed, Miles (1957, 1959, 1962). A further application arises in wave propagation in hydraulic models which generally include some geometrical distorsion of scales, Biesel (1949), Hunt (1952). Moreover, an accurate estimate of laminar dissipation is a necessary requirement before experimental observations of wave decay can be interpreted in terms of turbulent dissipation.

To a first approximation, the rate of attenuation of wave amplitude in shallow water can be obtained from a boundary layer approximation and the associated rate of energy dissipation. Such a method does not conveniently lend itself to the calculation of higher order approximations, in inverse powers of a Reynolds number, and it appears rather more convenient to solve the linearised Navier-Stokes equations directly together with the resulting characteristic equation. Moreover, it has been supposed that rates of decay with distance and with time could be derived one from the other by consideration of the flux of wave energy. This is only correct to a first approximation. Higher order approximations become determinate only when the energy flux is specified to the same order, or alternatively when the precise mode of decay is specified in some other way. Two distinct modes of decay are considered here. The first concerns decay with time in which the wave number k is real, the frequency σ is complex, and the motion is therefore strictly periodic with distance. The second mode concerns decay with distance where k is complex, σ is real, and so the motion is strictly periodic in time. These are the simplest modes to consider and do not require further specification. Other modes involve decay with both time and distance depending on the energy flux.

Assuming small amplitude gravity waves in a constant depth of water, the exponential modulus of decay is derived for each of these modes as a power series in $\nu^{1/2}$. Explicit expressions are derived for the coefficients as far as the third order terms. Such expansions neglect certain exponentially small terms of order $e^{-1/\nu^{1/2}}$ which, although numerically negligible in the range of interest, give rise to the usual singularity at $\nu = 0$. The decay moduli are presented graphically as functions of the wave number k and the depth h, and demonstrate the transition from shallow water damping of order $\nu^{1/2}$ to deep water damping of order ν.

The damping coefficient for decay with time is used to determine the minimum critical wind speed for the initiation of wave growth in shallow water by the instability mechanism of Miles (1957), (1959). At the minimum critical wind speed, the rate of energy imput from a logarithmic mean wind profile is equated to the rate of laminar dissipation in the water due to the oscillatory wave motion, excluding any mean current. The minimum wind speed is found to increase by only 5 % at depths as small as 10 cm. The appropriate critical wavelength de-

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creases with depth, falling from ~ 27 cm in deep water to ~ 17 cm at a depth of 10 cm.

2. — Formulation

In a fluid of kinematic viscosity ν and uniform depth h, we can represent small amplitude wave motions by the velocity components:

$$u = -\varphi_x - \psi_y, \qquad v = -\varphi_y + \psi_x \tag{1}$$

where:

$$\nabla^2\varphi = 0, \qquad \nu\nabla^2\psi = \partial\psi/\partial t. \tag{2}$$

A progressive wave requires solutions of (2) to be of the form:

$$\varphi = (Ae^{ky} + Be^{-ky})\, e^{i(kx-\sigma t)} \tag{3}$$

$$\psi = (Ce^{my} + D'e^{-my})\, e^{i(kx-\sigma t)} \tag{4}$$

where the equation to the free surface $y = \eta$ is:

$$\eta = h + R_0\, e^{i(kx-\sigma t)} \tag{5}$$

From equations (2) and (4) we have:

$$m^2 = k^2 - i\sigma/\nu. \tag{6}$$

The boundary conditions at the surface $y = \eta$ are:

$$\left.\begin{aligned} \eta_t - v &= 0 \\ u_y + v_x &= 0 \\ -p + 2\,\nu\rho v_y &= 0 \end{aligned}\right\} \tag{7}$$

using suffixes to denote partial derivatives, where the pressure p is given by:

$$p/\rho = \varphi_t - g\,(y - h) \tag{8}$$

The boundary conditions on $y = 0$ are simply:

$$u = v = 0. \tag{9}$$

Substitution of (3) and (4) in (1) and the boundary conditions (7) and (9), enables the constants A, B, C, D' and A_0 to be eliminated leaving a relation between k, m and σ. This eliminant is most conveniently expressed in terms of the non-dimensional parameters:

$$kh = K, \qquad mh = M, \tag{10}$$

$$\varepsilon = \left(\frac{4\,\nu^2 k^3}{g}\right)^{1/4} \tag{11}$$

and:

$$S = \sigma/(g\,k)^{1/2}. \tag{12}$$

The resulting expression is:

$$\begin{aligned} &-(iS - \varepsilon^2)^2 \{e^{2M}(K\tanh K - M) - (K\tanh K + M)\} \\ &\quad + \{e^{2M}(M\tanh K - K) + (M\tanh K + K)\} \\ &- (M/K)\,\varepsilon^4\, \{e^{2M}(M\tanh K - K) - (M\tanh K + K)\} \\ &+ 4\,Me^{M}\varepsilon^2 (iS - \varepsilon^2)\,\mathrm{sech}\,K = 0. \end{aligned} \tag{13}$$

In this notation, equation (6) becomes:

$$2\,K^2S = i\varepsilon^2 (M^2 - K^2). \tag{14}$$

It is clear that in liquids of vanishing viscosity, as $\varepsilon \to 0$, S and K assume their inviscid significance and remain finite. Consequently from (14), as $\varepsilon \to 0$ $M \to \infty$ as ε^{-1}. The presence of terms of the type e^{2M} in addition to powers of M in (13) indicates an essential singularity at $\varepsilon = 0$ which precludes any rational process of successive approximation. For practical purposes however, we are interested in values of ε which are so small that exponential terms in M dominate the equation. Retaining then only the terms in e^{2M}, and eliminating S by means of (14), (13) reduces to:

$$\varepsilon^4\left(1 + \frac{M^2}{K^2}\right)^2 (K\tanh K - M) - 4\left(1 - \frac{M}{K}\,\varepsilon^4\right)(M\tanh K - K) = 0. \tag{15}$$

which does admit solution by successive approximation in powers of ε. The relevant order of magnitude of ε is indicated in table 1 for various wavelengths in water, with $\nu = .01$, $g = 980$ in c.g.s. units, and with k real.

TABLE 1

λ (cm)	k (cm^{-1})	ε
1	6.28	1.003×10^{-1}
10	6.28×10^{-1}	1.784×10^{-2}
10^2	6.28×10^{-2}	3.172×10^{-3}
10^3	6.28×10^{-3}	5.642×10^{-4}
10^4	6.28×10^{-4}	1.003×10^{-4}

The largest term omitted from equation (15) is of order $e^{-kh/\varepsilon}$ in deep water, and $e^{-(kh)^{5/4}/\varepsilon}$ in shallow water. The first is invariably negligible, and the second in shallow water requires $\varepsilon/(kh)^{5/4}$ to be small. This is precisely the same condition as is found in the next section for the rapid convergence of the solutions in shallow water. Within the range

$$e \ll (kh)^{5/4},$$

the approximations are therefore consistent with the derivation of equation (15). For $kh > 2.0$ for example, and $\lambda \geqslant 1$ cm, $e^{-kh/\varepsilon} < 5 \times 10^{-8}$. In very shallow water, at $kh = 0.1$ for example,

$$e^{-(kh)^{5/4}/\varepsilon} < 5 \times 10^{-8}$$

for all $\lambda > 100$ cm.

3. — Solution of the characteristic equation

We seek a solution to (15) of the form:

$$\frac{M}{K} = \frac{m_0}{\varepsilon} + m_1 + m_2\varepsilon + m_3\varepsilon^2 + \dots \tag{16}$$

Substituting in (15) and equating coefficients of powers of ε, leads to the following values for the m_i:

$$m_0 = (1 - i) \tanh^{1/4} K,$$

$$m_1 = -\frac{1}{2} \operatorname{cosech} 2 K,$$

and, writing $4 \sinh^2 K = Y$,

$$m_2 = -(1 + i) \coth^{1/4} K \frac{(Y + 1)(Y + 5)}{4 Y (Y + 4)}$$

$$m_3 = -i \tanh^{1/2} K \frac{Y^3 + 2 Y^2 - 10 Y - 10}{2 Y^2 (Y + 4)}$$

From (14), the wave frequency S (K, ε) is now given by:

$$S = s_0 + s_1 \varepsilon + s_2 \varepsilon^2 + s_3 \varepsilon^3 + \ldots \qquad (17)$$

where:

$$s_0 = \tanh^{1/2} K$$

$$s_1 = -\frac{1}{2} (1 + i) \tanh^{1/4} K \operatorname{cosech} 2 K$$

$$s_2 = -i \frac{Y^2 + 5 Y + 2}{Y (Y + 4)}$$

and:

$$s_3 = (1 - i) \tanh^{3/4} K \frac{2 Y^3 + 3 Y^2 - 26 Y - 25}{4 Y^2 (Y + 4)}$$

In deep water, as $|K| \to \infty$, we have:

$$\frac{M}{K} = \frac{1 - i}{\varepsilon} - \frac{\varepsilon}{4}(1 + i) - i \frac{\varepsilon^2}{2} \ldots$$

and:

$$S = 1 - i\varepsilon^2 + \frac{1}{2}(1 - i) \varepsilon^3 \ldots$$

In shallow water, as $|K| \to 0$, we have:

$$M = (1 - i)\left(\frac{K^{5/4}}{\varepsilon}\right) - \frac{1}{4} - \frac{5(1 + i)}{64}\left(\frac{\varepsilon}{K^{5/4}}\right) + \frac{5 i}{64}\left(\frac{\varepsilon}{K^{5/4}}\right)^2 + \ldots$$

$$\frac{S}{K^{1/2}} = 1 - \frac{1 + i}{4}\left(\frac{\varepsilon}{K^{5/4}}\right) - \frac{i}{8}\left(\frac{\varepsilon}{K^{5/4}}\right)^2 - \frac{25(1 - i)}{256}\left(\frac{\varepsilon}{K^{5/4}}\right)^3 + \ldots$$

It follows that rapid convergence of the series requires not only that:

$$|\varepsilon| \ll 1$$

but that in addition, in shallow water:

$$|\varepsilon| \ll |kh|^{5/4} \qquad (18)$$

Substituting for ε from (11), this last condition (18), merely states that the boundary layer thickness δ at the bottom satisfies the inequality $\delta \ll h$, where

$$\delta = (2 \nu/\sigma)^{1/2} \qquad \text{and } \sigma/k \approx (gh)^{1/2}.$$

The series do not therefore converge rapidly, for the very longest waves in shallow water of given depth. The higher powers in ε must then be retained in estimating the damping of waves whenever the shallow water approximation is invoked, that is when $kh \ll 1$. These are precisely the circumstances under which the linearised sinusoidal wave gives a weak first approximation, valid only for extremely small wave amplitudes given by:

$$|A_0| \ll k^2 h^3.$$

More realistic amplitudes then require the cnoidal approximation, when $|A_0| \sim k^2 h^3$. The convergence of the series is found to be quite rapid for intermediate depths and for deep water, $kh \geq 0.1$. Even in very shallow water, $kh < 0.1$ say, convergence is rapid for $\lambda > 100$ cm. It should be noted that while the method fails in shallow water for the longest waves for a given h, it is most accurate for the longest waves for a given kh.

It is now necessary to distinguish between various possible forms of wave decay. The simplest case concerns decay with time but not with distance, so that σ is complex and K real. The second case to be considered supposes the motion to be strictly periodic in time and decaying with distance, so that σ is real and K is complex.

4. — Decay with time

We return now to dimensional variables writing $K = kh$ and $S = \sigma/(gk)^{1/2}$. If we write:

$$\sigma = \sigma_r + i\sigma_i \qquad (19)$$

then σ_r denotes the wave frequency, and σ_i the exponential modulus of decay with time. Separating (17) into real and imaginary parts,

$$\frac{\sigma_r}{(gk)^{1/2}} = \tanh^{1/2} kh - \varepsilon \frac{\tanh^{1/4} kh}{2 \sinh 2 kh} + \varepsilon^3 \tanh^{3/4} kh \frac{2 Y^3 + 3 Y^2 - 26 Y - 25}{4 Y^2 (Y + 4)} \qquad (20)$$

$$\frac{\sigma_i}{(gk)^{1/2}} = -\varepsilon \frac{\tanh^{1/4} kh}{2 \sinh 2 kh} - \varepsilon^2 \frac{Y^2 + 5 Y + 2}{Y (Y + 4)} - \varepsilon^3 \tanh^{3/4} kh \frac{2 Y^3 + 3 Y^2 - 26 Y - 25}{4 Y^2 (Y + 4)} \qquad (21)$$

where:

$$\varepsilon = (4 \nu^2 k^3/g)^{1/4} \qquad \text{and } Y = 4 \sinh^2 kh.$$

The first two damping terms of σ_i, of order ε and ε^2 agree with the result given by Biesel (1949), equation (30).

In deep water, as $kh \to \infty$, (20) and (21) reduce to

$$\sigma_r = (gk)^{1/2} \left\{ 1 + \frac{1}{2} \varepsilon^3 + \ldots \right\} \qquad (22)$$

and:

$$\sigma_i = (gk)^{1/2} \left\{ -\varepsilon^2 - \frac{1}{2} \varepsilon^3 \ldots \right\} \qquad (23)$$

while in shallow water, as $kh \to 0$,

$$\frac{\sigma_r}{(gk)^{1/2}} = (kh)^{1/2} \left\{ 1 - \frac{1}{4} \frac{\varepsilon}{(kh)^{5/4}} - \frac{25}{256} \frac{\varepsilon^3}{(kh)^{15/4}} \ldots \right\} \qquad (24)$$

and:

$$\frac{\sigma_i}{(gk)^{1/2}} = (kh)^{1/2} \left\{ -\frac{1}{4}\frac{\varepsilon}{(kh)^{5/4}} - \frac{1}{8}\frac{\varepsilon^2}{(kh)^{5/2}} + \frac{25}{256}\frac{\varepsilon^3}{(kh)^{15/4}} \cdots \right\} \quad (25)$$

The modulus of decay is thus of order $\nu^{1/2}$ in shallow water and is of order ν in deep water, as is well known. The higher order terms in (21) enable us to estimate the depth as which the transition from one law to the other may be said to occur. When the series converge rapidly, the term in σ_i of order ε is dominant at small kh while the term of order ε^2 is dominant at large kh. An estimate of the critical depth may therefore be obtained by equating these terms giving:

$$(\tanh kh_{cr})^{5/4}(\cosh 2\,kh_{cr} + 1) = \varepsilon(\cosh 4\,kh_{cr} + \cosh 2\,kh_{cr} - 1).$$

The critical value of kh is shown in figure 1 as a function of ε, and may be regarded as separating regions in which shallow-water or deep-water damping dominates σ_i. Shallow water effects in inviscid wave motion are generally supposed to become important for $kh <$ about 3, for all wavelengths. The inclusion of viscous terms clearly introduces the additional parameter $\varepsilon(\lambda)$. For large kh,

$$kh_{cr} \sim -\frac{1}{2}\log_e \varepsilon,$$

while for very small kh, the convergence condition is violated, giving the critical value $\varepsilon/kh_{cr}^{5/4} \sim 2$, so that figure 1 becomes inaccurate for $\lambda < 1.0$ cm.

The damping coefficient $\sigma_i/(gk)^{1/2}$, from equation (21), has been computed as far as the term of order ε^3 for various values of kh, and the results appear in figure 2. The transition region is clearly distinguishable as that in which the curves change their slope from 1.0 in shallow water to 2.0 in deep water.

Comparison of equations (22) and (24) shows that the classical free wave period is decreased by viscous effects in deep water, but is increased in shallow water.

5. — Decay with distance

For a wave motion strictly periodic in time, $\sigma = (gk)^{1/2}S$ is real, and the complex wave number may be written :

$$k = k_0 + iD$$

where k_0 and D the modulus of decay are real. To solve for D and σ from (17) we replace K by $h(k_0 + iD)$ and ε by $\varepsilon_0 (k/k_0)^{3/4}$ where:

$$\varepsilon_0 = (4\,\nu^2 k_0^3/g)^{1/4}.$$

Separating (17) into real and imaginary parts, the wave frequency is given by:

$$\sigma = (gk_0)^{1/2}\{s'_0 + s'_1\varepsilon_0 + s'_2\varepsilon_0^2 + s'_3\varepsilon_0^3 \ldots\} \quad (26)$$

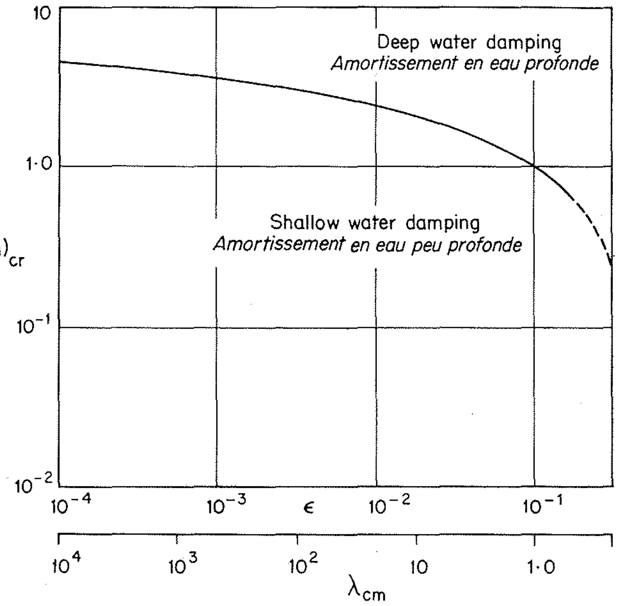
where:

$$s'_0 = T^{1/2}, \qquad s'_1 = -\frac{1}{2}T^{1/4}/S_2,$$

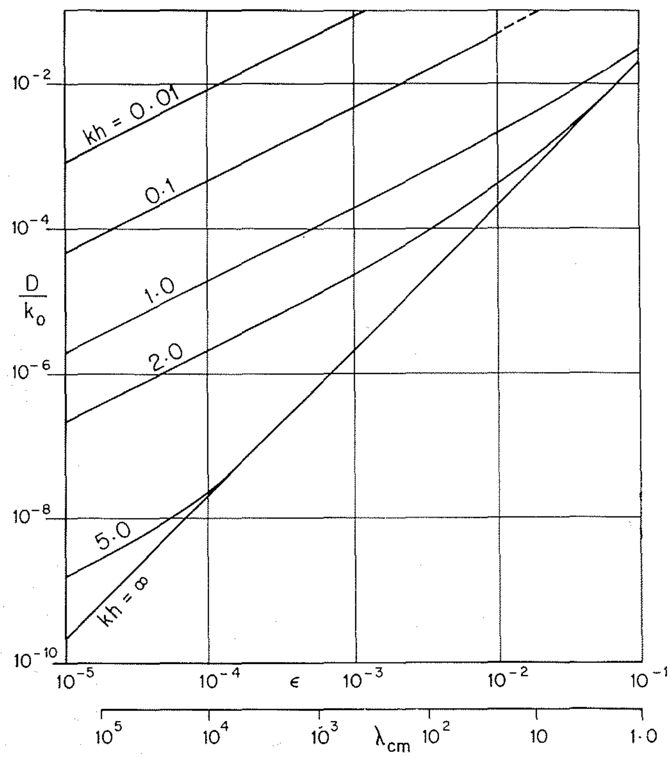
$$s'_2 = -\frac{(1 - T^4)(k_0h)^2 + 4\,T^3 k_0 h - 3\,T^2}{4\,T^2(2\,k_0h + S_2)^2}$$

$$s'_3 = S_2 T^{-17/4} \times$$

$$\times \frac{(k_0h)^3 P_1(T) + (k_0h)^2 P_2(T) + k_0 h P_3(T) + P_4(T)}{128\,(2\,k_0h + S_2)^3}$$



Critical depth for shallow water/deep water damping /1
Hauteur d'eau critique correspondant à l'amortissement en eau peu profonde/eau profonde



Modulus of decay with time /2
Coefficient d'amortissement en fonction du temps

and $T = \tanh k_0h$, $S_2 = \sinh 2\,k_0h$, while the modulus of decay is given by:

$$D = d_1\varepsilon_0 + d_2\varepsilon_0^2 + d_3\varepsilon_0^3 + \dots \qquad (27)$$

where:

$$d_1 = \frac{k_0 T^{-1/4}}{2\,k_0h + S_2}$$

$$d_2 = \frac{k_0 S_2 T^{-5/2}}{16\,(2\,k_0h + S_2)^2}\left\{\begin{matrix}2\,k_0h\,(T^4 + 30\,T^2 + 1)\\ + S_2\,(T^4 + 22\,T^2 + 9)\end{matrix}\right\}$$

$$d_3 = \frac{k_0 S_2 T^{-15/4}}{64\,(2\,k_0h + S_2)^4}\left\{\begin{matrix}(k_0h)^3\,Q_1(T) + (k_0h)^2\,Q_2(T)\\ + k_0hQ_3(T) + Q_4(T)\end{matrix}\right\}$$

The functions $P_i(T)$, $Q_i(T)$ are the following polynomials:

$$P_1(T) = 41\,T^8 - 508\,T^6 + 506\,T^4 - 108\,T^2 - 31$$

$$P_2(T) = T\,\{-35\,T^6 + 305\,T^4 + 103\,T^2 - 117\}$$

$$P_3(T) = \frac{1}{2}\,S_2T\,\{53\,T^6 - 655\,T^4 + 927\,T^2 - 69\}$$

$$P_4(T) = \frac{1}{4}\,S_2^2T\,\{47\,T^6 - 493\,T^4 + 557\,T^2 + 145\}$$

and:

$$Q_1(T) = \frac{1}{3}\,\{3174\,T^6 + 3262\,T^4 - 4162\,T^2 - 3510\}$$

$$Q_2(T) = S_2\,\{1083\,T^6 + 1191\,T^4 - 1319\,T^2 - 1219\}$$

$$Q_3(T) = \frac{1}{4}\,S_2^2\,\{1094\,T^6 + 1550\,T^4 - 958\,T^2 - 1190\}$$

$$Q_4(T) = \frac{1}{8}\,S_2^3\,\{-14\,T^6 + 298\,T^4 + 150\,T^2 + 78\}$$

In deep water, as $k_0h \to \infty$, (26) and (27) reduce to:

$$\frac{\sigma}{(gk_0)^{1/2}} = 1 + \frac{1}{2}\,\varepsilon_0^3 \dots \qquad (28)$$

and:

$$\frac{D}{k_0} = 2\,\varepsilon_0^2 + \varepsilon_0^3 \dots \qquad (29)$$

which agree with equations (20) of Hunt and Massoud (1962), where σ and D are evaluated for deep water, as far as ε^6.

In shallow water, as $k_0h \to 0$, (26) and (27) become:

$$\frac{\sigma}{(gk_0)^{1/2}} = (k_0h)^{1/2}\left\{\begin{matrix}1 - \frac{1}{4}\frac{\varepsilon_0}{(k_0h)^{5/4}}\\ + \frac{1}{32}\frac{\varepsilon_0^2}{(k_0h)^{5/2}} - \frac{19}{1024}\frac{\varepsilon_0^3}{(k_0h)^{15/4}}\dots\end{matrix}\right\} \qquad (30)$$

and:

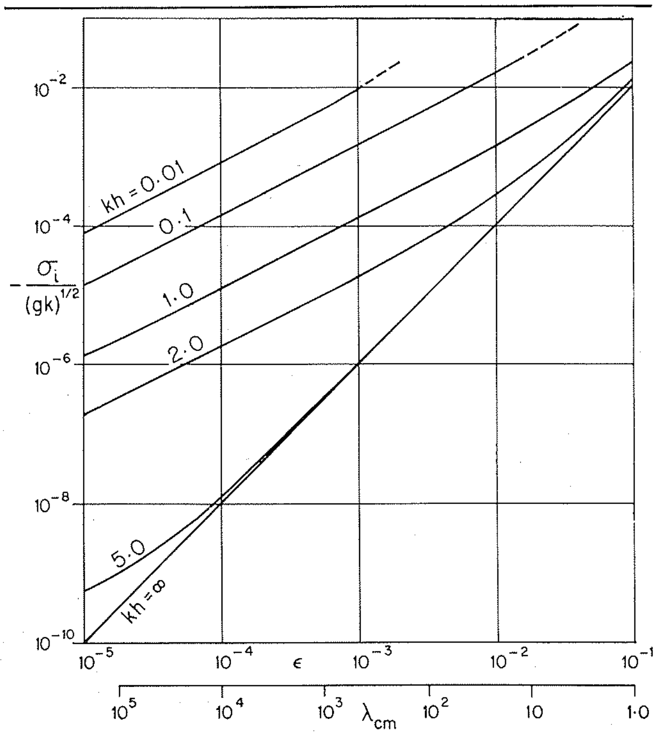
$$\frac{D}{k_0} = \frac{1}{4}\frac{\varepsilon_0}{(k_0h)^{5/4}} + \frac{5}{32}\frac{\varepsilon_0^2}{(k_0h)^{5/2}} - \frac{295}{512}\frac{\varepsilon_0^3}{(k_0h)^{15/4}} \dots \qquad (31)$$

The correction to the wave frequency of order ε_0 in equation (26) s'_1 agrees with a result obtained elsewhere (Hunt (1959) equation (25)). It is also the same as the corresponding term in equation (20) for decay with time, although higher order terms in the two modes of decay are different. Again, the free-wave period is decreased in deep water and increased in shallow water for sufficiently small $\varepsilon_0/(k_0h)^{5/4}$.

The first damping term d_1 has been obtained before by various methods, Biesel (1949), Hunt (1952) while the higher order corrections are new results. The term of order ε^2 differs from that given by Biesel, who attempted to derive it from the corresponding term for decay with time by an energy flux method. That such a procedure must be incorrect follows from the essentially different nature of time and distance decay and the different energy fluxes in the two cases, which are readily calculated from the preceding expressions.

The damping coefficient D has been computed as far as the term of order ε_0^3 for various values of kh and is shown in figure 3. The transition from shallow water damping where $D/k_0 \propto \varepsilon$, to deep water damping where $D/h_0 \propto \varepsilon^2$ again takes place not at a fixed value of kh, but depends also on $\varepsilon\,(k)$. A critical depth can be defined as in the case of decay with time, leading to a functional relation $kh_{cr}(\varepsilon)$ similar to that shown in figure 1, except that for decay with distance we find at large kh,

$$kh_{cr} \sim -\frac{1}{2}\log_e(\varepsilon_0/32).$$



Modulus of decay with distance /3
Coefficient d'amortissement en fonction de la distance

6. — Minimum wind speed for wave growth

It is of interest to examine the effect of shallow water on the instability mechanism of Miles (1957) for the generation of gravity waves. This frequency-selective process depends for energy transfer on inviscid Reynolds stresses at the critical height where wind speed and wave speed are equal, and these stresses in turn depend on the curvature of the mean wind profile.

The surface wave:

$$\eta\,(x, t) = h + A_0 e^{ik(x-ct)}$$

is maintained by perturbations of aerodynamic mean pressure:

$$p = (\alpha + i\beta)\, \rho_a U_1^2\, k\eta$$

where ρ_a is the air density and U_1 a reference wind speed. The "negative damping ratio" or growth rate is defined by:

$$\zeta_a = 2\left\{\frac{\mathcal{I}m\,(c)}{\mathcal{R}e\,(c)}\right\}_p = \frac{\rho_a}{\rho_w}\,\beta\left(\frac{U_1}{c}\right)^2,$$

Miles (1957, 1959), where:

$$\beta\left(\frac{c}{U_1},\ \Omega\right)$$

is tabulated by Conte and Miles (1959) for the mean wind profile:

$$U(y) = U_1 \log(y/y_0)$$

where:

$$\Omega = gy_0/U_1^2.$$

The minimum critical wind speed is found by equating $-\zeta_a$ to the rate of laminar dissipation ζ_w, which for deep water is given to a first approximation by:

$$\zeta_w = 2\left\{\frac{\mathcal{I}m\,(c)}{\mathcal{R}e\,(c)}\right\}_v = -4\,\nu k/c_f,$$

where c_f is the real, free wave speed.

In shallow water, the energy imput ζ_a to a given wavenumber k is modified by the use of the relation $c_f^2 = (g/k)\tanh kh$ in place of $c_f^2 = g/k$, and may be computed from the tables of β of Conte and Miles. In order to span the transition from deep water to shallow water, it is necessary to include at least two terms in the series for ζ_w in power of ε. From equations (20) and (21) we have:

$$\zeta_w = \varepsilon\,\frac{\operatorname{cosech} 2\,kh}{\tanh^{1/4} kh} + \varepsilon^2\,\frac{2\cosh 4\,kh + 2\cosh 2\,kh - 1}{\tanh^{1/2} kh\,(\cosh 4\,kh - 1)}$$

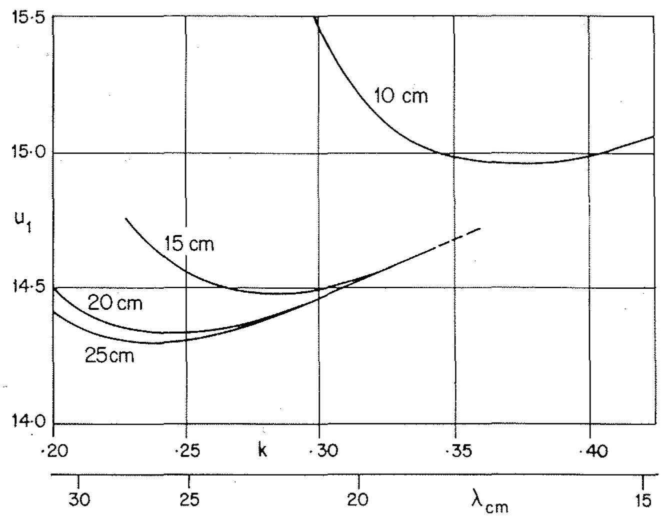
where ε is given by equation (11). The minimum reference wind speed $U_1 = U_*/0.4$, defined by:

$$\zeta_a + \zeta_w = 0$$

is shown in figure 4 as a function of wavenumber k for various depths of water using:

$$g = 980,\ \nu = 10^{-2},\ \rho_a/\rho_w = 1.2 \times 10^{-3}\ \text{c.g.s.}$$

It is evident that the minimum wind speed increases in shallow water due to the viscous damping, but that the increase is hardly significant. The critical wavelength at which waves begin to grow at the minimum wind speed decreases appreciably with depth. Values are given in table 2 for $h = 10\ (5)\ 25$ cm. Over the range of computed values, it appears that no waves can grow in shallow water which could not have grown with the same wind speed in deep water.



The minimum reference wind speed U, for wave growth in shallow water of depth h = 10, 15, 20, 25 cm **/4**

Vitesse minimale de référence du vent, U, pour la croissance de la houle en eau peu profonde; hauteurs d'eau : 10, 15, 20, 25 cm

TABLE 2

The minimum critical reference wind speed $U_{1\text{crit}}$ for wave growth shallow water by Miles's Instability Mechanism

h (cm)	k_{crit} (cm^{-1})	λ_{crit} (cm)	$U_{1\,\text{crit}}$ (cm/sec)
10	0.37	17.0	14.96
15	0.28	22.4	14.48
20	0.25	25.1	14.33
25	0.23	27.3	14.30
∞	0.23	27	14.3

It is evident that this instability mechanism is incapable of explaining the generation of very short waves, of wavelengths, less than about 10 cm, even in very shallow water, where the $c_f(k)$ relation slightly enhances their rate of energy imput from a given mean wind profile. The most probable mechanisms for short wave generation therefore appear to be résonance between surface waves and Tollmien-Schlichting oscillations in the shearing air flow, Miles (1962), and at low wind speeds, resonance between surface waves and turbulent air pressure fluctuations, Phillips (1957).

The shallow water effect on $U_{1\,\text{crit}}$ and λ_{crit} at the onset of wave generation only become apparent at depths of water less than 25 cm. Other shallow water effects are to be expected at substantially greater depths. A fully-developed wave spectrum for example will exhibit a skewness in shallow water relative to the deep water spectrum due partly to the greater damping of long waves, but also to wave breaking, Darbyshire (1959). Similarly, the Neumann spectrum, with an energy maximum at a wave period of about 7-8 sec, will experience significant distortion when propagated into water of depth less than 10 m, so that the mean tangential wave drag and averaged values of β will change.

References

BIESEL (F.). — *La Houille Blanche,* 1949, **4**, 630.
CONTE (S. D.) & MILES (J. W.). — *J. Soc. Indust. Appl. Math.,* 1959, **7**, 361.
DARBYSHIRE (J.). — *Dtsch. Hydrogr. Z.,* 1959, **12**, 153.
HUNT (J. N.). — *La Houille Blanche,* 1952, **7**, 836.
HUNT (J. N.). — *J. Geophys. Res,* 1959, **64**, 437.
HUNT (J. N.) & MASSOUD (S. K. A.). — *Geofis. Pura e Appl.,* 1962, **53**, 65.
MILES (J. W.). — *J. Fluid Mech.,* 1957, **3**, 185.
MILES (J. W.). — *J. Fluid Mech.,* 1959, **6**, 568.
MILES (J. W.). — *J. Fluid Mech.,* 1962, **13**, 433.
PHILLIPS (O. M.). — *J. Fluid Mech.,* 1957, **2**, 417 .

Résumé

L'amortissement visqueux des ondes de gravité en eau peu profonde

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Il est nécessaire de connaître le taux de dissipation visqueuse de l'énergie d'une onde de gravité, pour pouvoir évaluer les différents procédés de formation d'une onde par le vent, et en particulier, pour le calcul de la vitesse minimale critique du vent (Miles, 1957, 1959, 1962). Une autre application se présente dans le domaine des modèles hydrauliques, dans lequel intervient généralement une certaine distorsion d'échelle (Biesel 1949, Hunt 1952). En outre, l'évaluation précise de la dissipation laminaire est condition essentielle pour l'interprétation de toute observation expérimentale sur l'amortissement des ondes, en fonction de la dissipation turbulente.

Il est possible, en première approximation, d'obtenir le terme d'atténuation de l'amplitude de l'onde en eau peu profonde, à partir d'une approximation en couche limite, ainsi que le taux de dissipation d'énergie correspondant. Une telle méthode ne se prête pas aisément au calcul d'approximations d'ordre supérieur, et en des puissances inverses d'un nombre de Reynolds; il apparaît qu'un procédé plus simple consiste à résoudre directement les équations linéarisées de Navier-Stokes, ainsi que l'équation caractéristique qui en découle. Par ailleurs, il a été supposé que les taux d'amortissement, en fonction du temps et de l'espace, pouvaient être déduits, les uns des autres, en tenant compte du flux de l'énergie de l'onde. Or, ceci n'est valable qu'en première approximation : les approximations d'ordre supérieur ne deviennent déterminées que lorsque le flux énergétique est spécifié au même ordre, ou bien lorsque le mode exact d'amortissement l'est d'une autre manière quelconque. Deux modes d'amortissement bien distincts sont examinés dans la présente étude. Le premier a trait à l'amortissement en fonction du temps, le nombre d'onde k étant réel, la fréquence σ étant complexe, et le muvement étant, par conséquent, rigoureusement périodique en fonction de la distance. Le deuxième mode a trait à l'amortissement en fonction de la distance, k étant complexe, σ étant réelle, et, par conséquent, le mouvement étant rigoureusement périodique dans le temps. Ces deux modes sont les plus simples que l'on peut considérer, et il n'est pas nécessaire de les préciser davantage. D'autres modes font intervenir l'amortissement, à la fois dans le temps et l'espace, en fonction du flux énergétique.

Dans l'hypothèse que les ondes de gravité sont de faible amplitude, dans une masse d'eau présentant une hauteur constante, le module exponentiel de l'amortissement se déduit comme série exponentielle en $\nu^{1/2}$. On déduit, pour des cofficients, des expressions explicites jusqu'aux termes du troisième ordre. De tels développements négligent certains termes exponentiellement petits, d'ordre $e^{-1/\nu^{1/2}}$, lesquels, bien qu'étant négligeables, numériquement, dans la zone considérée, donnent néanmoins lieu à la singularité habituellement constatée lorsque $\nu = 0$. Les modules d'amortissement sont présentés, sous forme graphique, en fonction du nombre d'ondes k, et de la hauteur d'eau h; ils mettent en évidence la transition de l'amortissement d'ordre $\nu^{1/2}$ en eau peu profonde, à celui d'ordre ν en eau profonde.

Le coefficient d'amortissement en fonction du temps sert pour la détermination de la vitesse critique minimale du vent, correspondant au début de croissance de l'onde en eau peu profonde, suivant le mécanisme d'instabilité de Miles (1957, 1959). Compte tenu de la vitesse minimale du vent, on met en équation, d'une part le taux d'alimentation d'énergie, fournie par un profil de vent moyen logarithmique, et d'autre part le taux de dissipation laminaire au sein de la masse d'eau, due au mouvement d'onde oscillatoire, et à l'exclusion de tout courant moyen. Il se montre que la vitesse minimale du vent n'augmente que de 5 % à des profondeurs de 10 cm seulement. La longueur d'onde critique correspondante diminue en hauteur d'eau décroissante, passant de 27 cm environ en eau profonde, à l'ordre de 17 cm sous une hauteur d'eau de 10 cm.

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