

The W_2 -curvature tensor on warped product manifolds and applications

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The purpose of this paper is to study the W_2 -curvature tensor on (singly) warped product manifolds as well as on generalized Robertson–Walker and standard static space-times. Some different expressions of the W_2 -curvature tensor on a warped product manifold in terms of its relation with W_2 -curvature tensor on the base and fiber manifolds are obtained. Furthermore, we investigate W_2 -curvature flat warped product manifolds. Many interesting results describing the geometry of the base and fiber manifolds of a W_2 -curvature flat warped product manifold are derived. Finally, we study the W_2 -curvature tensor on generalized Robertson–Walker and standard static space-times; we explore the geometry of the fiber of these warped product space-time models that are W_2 -curvature flat.

Keywords: W_2 -curvature; standard static space-time; generalized Robertson–Walker space-time; warped products.

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1. Introduction

In [1], Pokhariyal and Mishra first defined the W_2 -curvature tensor and they studied its physical and geometrical properties. Since then the concept of the W_2 -curvature tensor has been studied as a research topic by mathematicians and physicists (see [2–5]). Pokhariyal defined many symmetric and skew-symmetric curvature tensors on the same line of the W_2 -curvature tensor and studied various geometrical and physical properties of manifolds admitting these tensors in [3]. Among many of his results, we would like to mention that he proved that the vanishing of one of

these curvature tensors in an electromagnetic field implies a purely electric field. Another study to establish applications of the W_2 -curvature in the theory of general relativity was carried in [6] where the authors particularly prove that a space-time with vanishing W_2 -curvature tensor is an Einstein manifold. They also consider the case of vanishing W_2 -curvature tensor in relation with a perfect fluid space-time. In [2, 5], the authors study the properties of flat space-time under some conditions regarding the W_2 -curvature tensor and W_2 -flat space-times. Moreover, there are many studies regarding the geometrical meaning of the W_2 -curvature tensor in different types of manifolds (see [7–10] and references therein).

The main aim of this paper is to study and explore the W_2 -curvature tensor on warped product manifolds as well as on well-known warped product space-times. The concept of the W_2 -curvature tensor has never been studied on warped products before this paper in which we intent to fill this gap in the literature by providing a complete study of the W_2 -curvature tensor on such spaces.

This paper is organized as follows. In Sec. 2, we state well-known curvature related formulas of warped product manifolds and the W_2 -curvature tensor properties on pseudo-Riemannian manifolds. We also define and study a new curvature tensor, $K(X, Y)Z$, that will be used in the characterization of the W_2 -curvature tensor on pseudo-Riemannian manifolds. In Sec. 3, we explore the relation between the W_2 -curvature tensor of a warped product manifold and that of the fiber and base manifolds. Section 4 is devoted to the study of the W_2 -curvature tensor on generalized Robertson–Walker space-time and standard static space-time.

2. Preliminaries

In this section, we will provide basic definitions and curvature formulas about warped product manifolds.

Suppose that (M_1, g_1, D_1) and (M_2, g_2, D_2) are two C^∞ -pseudo-Riemannian manifolds equipped with pseudo-Riemannian metric tensors g_i where D_i is the Levi-Civita connection of the metric g_i for $i = 1, 2$. Further suppose that $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ are the natural projection maps of the Cartesian product $M_1 \times M_2$ onto M_1 and M_2 , respectively. If $f : M_1 \rightarrow (0, \infty)$ is a positive real-valued smooth function, then the warped product manifold $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ defined by

$$g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^*(g_2),$$

where $*$ denotes the pull-back operator on tensors [11, 12]. The function f is called the warping function of the warped product manifold $M_1 \times_f M_2$. In particular, if $f = 1$, then $M_1 \times_1 M_2 = M_1 \times M_2$ is the usual Cartesian product manifold. It is clear that the submanifold $M_1 \times \{q\}$ is isometric to M_1 for every $q \in M_2$. Moreover, $\{p\} \times M_2$ is homothetic to M_2 . Throughout this paper we use the same notation for a vector field and for its lift to the product manifold. Let D , R and Ric be the

Levi–Civita connection, curvature tensor and Ricci curvature of the metric tensor g . Their formulas are well-known (see [11, 12]).

The W_2 -curvature tensor on a pseudo-Riemannian manifold (M, g, D) is defined as follows [1]. Let $X, Y, Z, T \in \mathfrak{X}(M)$, then

$$W_2(X, Y, Z, T) = g(R(X, Y)Z, T) + \frac{1}{n-1}[g(X, Z)\text{Ric}(Y, T) - g(Y, Z)\text{Ric}(X, T)],$$

where $R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X, Y]}Z$ is the Riemann curvature tensor. It is clear that $W_2(X, Y, Z, T)$ is skew-symmetric in the first two positions. More explicitly, $W_2(X, Y, Z, T) = -W_2(Y, X, Z, T)$.

Now we redefine W_2 -curvature tensor as follows. The W_2 -curvature tensor, as shown above, is also given by

$$W_2(X, Y, Z, T) = g(K(X, Y)T, Z),$$

where

$$K(X, Y)T := -R(X, Y)T + \frac{1}{n-1}[\text{Ric}(Y, T)X - \text{Ric}(X, T)Y].$$

The study of the W_2 -curvature tensor on warped product manifolds contains large formulas and a huge amount of computations. Thus, this new tool will enable us to minimize computations in our study.

Remark 1. Let M be a pseudo-Riemannian manifold. Then

$$K(X, Y)T + K(T, X)Y + K(Y, T)X = 0$$

for any vector fields $X, Y, T \in \mathfrak{X}(M)$.

The following proposition is a direct consequence of the new definition of the W_2 -curvature tensor.

Proposition 2. *Let M be a pseudo-Riemannian manifold. Then the W_2 -curvature tensor vanishes if and only if the tensor K vanishes.*

Now, we will note that the tensor K can be simplified if the last position is a concurrent field. First, recall that a vector field ζ is called a concurrent vector field if

$$D_X \zeta = X,$$

for any vector field X . It is clear that a concurrent vector field is a conformal vector field with factor 2. Let ζ be a concurrent vector field, then

$$R(X, Y)\zeta = 0.$$

Now suppose that ζ is a concurrent vector field. Then

$$K(X, Y)\zeta = \frac{1}{n-1}[\text{Ric}(Y, \zeta)X - \text{Ric}(X, \zeta)Y].$$

Finally, a Riemannian metric g on a manifold M is said to be of Hessian type metric if there are two smooth functions k and σ such that $H^\sigma = kg$ where

H^σ is the Hessian of σ . This topic is closely related to the research of Shima on Hessian manifolds (see [13, 14]) and its extension to pseudo-Riemannian manifolds in [15, 16].

3. W_2 -Curvature Tensor on Warped Product Manifolds

In this section, we provide an extensive study of W_2 -curvature tensor on (singly) warped product manifolds. Throughout the section, (M, g, D) is a (singly) warped product manifold of $(M_i, g_i, D_i), i = 1, 2$ with dimensions $n_i \neq 1$ where $n = n_1 + n_2$. R, R^i denote the curvature tensor and Ric, Ric^i denote the Ricci curvature tensor on M, M^i , respectively. Moreover, ∇f denotes the gradient and Δf denotes Laplacian of f on M_1 , and also the Hessian of f on M_1 is denoted by H^f . The sharp of f is given by $f^\sharp = f\Delta f + (n_2 - 1)g_1(\nabla f, \nabla f)$. Finally, W_2 -curvature tensor and the tensor K on M and M_i are denoted by W_2, K and W_2^i, K^i , respectively for $i = 1, 2$.

The following theorem provides a full description of the W_2 -curvature tensor on (singly) warped product manifolds. The proof contains long computations that can be done using previous results on warped product manifolds (see Appendix A).

Theorem 3. *Let $M = M_1 \times_f M_2$ be a singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. If $X_i, Y_i, T_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$, then*

$$\begin{aligned}
 K(X_1, Y_1)T_1 &= K^1(X_1, Y_1)T_1 \\
 &\quad - \frac{n_2}{(n-1)(n_1-1)} [\text{Ric}^1(Y_1, T_1)X_1 - \text{Ric}^1(X_1, T_1)Y_1] \\
 &\quad - \frac{1}{n-1} \left[\frac{n_2}{f} H^f(Y_1, T_1)X_1 - \frac{n_2}{f} H^f(X_1, T_1)Y_1 \right], \tag{1}
 \end{aligned}$$

$$K(X_1, Y_1)T_2 = K(X_2, Y_2)T_1 = 0, \tag{2}$$

$$K(X_1, Y_2)T_1 = - \left[\frac{1}{n-1} \text{Ric}^1(X_1, T_1) - \frac{n+n_2-1}{(n-1)f} H^f(X_1, T_1) \right] Y_2, \tag{3}$$

$$\begin{aligned}
 K(X_1, Y_2)T_2 &= -f g_2(Y_2, T_2) D_{X_1}^1 \nabla f + \frac{1}{n-1} \text{Ric}^2(Y_2, T_2)X_1 \\
 &\quad - \frac{f^\sharp}{n-1} g_2(Y_2, T_2)X_1, \tag{4}
 \end{aligned}$$

$$\begin{aligned}
 K(X_2, Y_2)T_2 &= K^2(X_2, Y_2)T_2 \\
 &\quad - \frac{n_1}{(n-1)(n_2-1)} [\text{Ric}^2(Y_2, T_2)X_2 - \text{Ric}^2(X_2, T_2)Y_2] \\
 &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2]. \tag{5}
 \end{aligned}$$

In the following part we investigate the geometry of the base factor of the warped product when the product is W_2 -curvature flat.

Theorem 4. *Let $M = M_1 \times_f M_2$ be a W_2 -curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Then*

$$W_2^1(X_1, Y_1, Z_1, T_1) = \frac{2n_2}{(n_1 - 1)f} [H^f(Y_1, T_1)g_1(X_1, Z_1) - H^f(X_1, T_1)g_1(Y_1, Z_1)] \tag{6}$$

for any vector fields $X_1, Y_1, Z_1, T_1 \in \mathfrak{X}(M_1)$.

Proof. Suppose that M is W_2 -curvature flat. Then Eqs. (1) and (3) imply that

$$\begin{aligned} 0 &= K^1(X_1, Y_1)T_1 - \frac{n_2}{(n - 1)(n_1 - 1)} [\text{Ric}^1(Y_1, T_1)X_1 - \text{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n - 1} \left[\frac{n_2}{f} H^f(Y_1, T_1)X_1 - \frac{n_2}{f} H^f(X_1, T_1)Y_1 \right], \\ 0 &= \frac{1}{n - 1} \text{Ric}^1(X_1, T_1) - \frac{n_1 + 2n_2 - 1}{(n - 1)f} H^f(X_1, T_1). \end{aligned}$$

Now, from the second equation we have

$$\text{Ric}^1(X_1, T_1) = \frac{n_1 + 2n_2 - 1}{f} H^f(X_1, T_1). \tag{7}$$

Using this identity in the first equation which eventually turns out to be:

$$\begin{aligned} K^1(X_1, Y_1)T_1 &= \frac{n_2}{(n - 1)(n_1 - 1)} \left[\frac{n_1 + 2n_2 - 1}{f} H^f(Y_1, T_1)X_1 \right. \\ &\quad \left. - \frac{n_1 + 2n_2 - 1}{f} H^f(X_1, T_1)Y_1 \right] \\ &\quad + \frac{n_2}{n - 1} \left[\frac{1}{f} H^f(Y_1, T_1)X_1 - \frac{1}{f} H^f(X_1, T_1)Y_1 \right] \\ &= \frac{2n_2^2}{(n - 1)(n_1 - 1)f} [H^f(Y_1, T_1)X_1 - H^f(X_1, T_1)Y_1]. \end{aligned}$$

Thus

$$W_2^1(X_1, Y_1, Z_1, T_1) = \frac{2n_2}{(n_1 - 1)f} [H^f(Y_1, T_1)g_1(X_1, Z_1) - H^f(X_1, T_1)g_1(Y_1, Z_1)]. \quad \square$$

Theorem 5. *Let $M = M_1 \times_f M_2$ be a W_2 -curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Then:*

- (1) M_1 is W_2 -curvature flat if and only if $H^f(X_1, Y_1) = 0$ for any vector fields $X_1, Y_1 \in \mathfrak{X}(M_1)$.
- (2) the scalar curvature S_1 of M_1 is given by

$$S_1 = \frac{n_1 + 2n_2 - 1}{f} \Delta f.$$

- (3) the scalar curvature of M_1 vanishes if M_1 is W_2 -curvature flat.

Proof. The proof just follows from Eqs. (6) and (7). □

Now, we study the geometry of the fiber factor of a warped product admitting flat W_2 -curvature.

Theorem 6. *Let $M = M_1 \times_f M_2$ be a singly warped product manifold with the metric tensor $g = g_1 \oplus f^2 g_2$. Assume that f satisfies $H^f = 0$. Then, M is W_2 -curvature flat if and only if both M_1 and M_2 are flat and $\nabla f = 0$.*

Proof. Suppose that M is W_2 -curvature flat, then M_1 is flat due to Eq. (7) and the first item of Theorem 5. Moreover, from Theorem 3 we have

$$\begin{aligned} 0 &= -f g_2(Y_2, T_2) D_{X_1}^1 \nabla f + \frac{1}{n-1} \text{Ric}^2(Y_2, T_2) X_1 - \frac{f^\#}{n-1} g_2(Y_2, T_2) X_1, \\ 0 &= K^2(X_2, Y_2) T_2 - \frac{n_1}{(n-1)(n_2-1)} [\text{Ric}^2(Y_2, T_2) X_2 - \text{Ric}^2(X_2, T_2) Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\#}{n-1} \right) [g_2(X_2, T_2) Y_2 - g_2(Y_2, T_2) X_2]. \end{aligned}$$

Since $H^f(X_1, Y_1) = 0$, the first equation becomes

$$\text{Ric}^2(Y_2, T_2) = f^\# g_2(Y_2, T_2),$$

where $f^\# = f \Delta f + (n_2 - 1) g_1(\nabla f, \nabla f) = (n_2 - 1) c^2$ where $c^2 = g_1(\nabla f, \nabla f)$, i.e. M_2 is Einstein with factor $\mu = (n_2 - 1) c^2$ and

$$\text{Ric}^2(Y_2, T_2) = (n_2 - 1) c^2 g_2(Y_2, T_2).$$

The second equation becomes

$$K^2(X_2, Y_2) T_2 = \frac{2(n_2 - 1) c^2}{(n - 1)} [g_2(Y_2, T_2) X_2 - g_2(X_2, T_2) Y_2].$$

Thus the W_2 -curvature tensor of M_2 is given by

$$W_2^2(X_2, Y_2, Z_2, T_2) = \frac{2(n_2 - 1) c^2}{(n - 1)} [g_2(Y_2, T_2) g_2(X_2, Z_2) - g_2(X_2, T_2) g_2(Y_2, Z_2)].$$

But

$$\begin{aligned} W_2^2(X_2, Y_2, Z_2, T_2) &= R^2(X_2, Y_2, Z_2, T_2) \\ &\quad + \frac{1}{n_2 - 1} [g_2(X_2, Z_2) \text{Ric}^2(Y_2, T_2) - g_2(Y_2, Z_2) \text{Ric}^2(X_2, T_2)] \\ &= R^2(X_2, Y_2, Z_2, T_2) \\ &\quad + c^2 [g_2(X_2, Z_2) g_2(Y_2, T_2) - g_2(Y_2, Z_2) g_2(X_2, T_2)]. \end{aligned}$$

Therefore,

$$R^2(X_2, Y_2, Z_2, T_2) = \frac{(n_2 - n_1 - 1) c^2}{(n - 1)} [g_2(X_2, Z_2) g_2(Y_2, T_2) - g_2(Y_2, Z_2) g_2(X_2, T_2)],$$

i.e. M_2 has a constant sectional curvature

$$\kappa_2 = \frac{(n_2 - n_1 - 1)c^2}{(n - 1)}.$$

But the Einstein factor should be $(n_2 - 1)\kappa_2$ and hence

$$n_1(n_2 - 1)c^2 = 0.$$

Thus M_2 is flat. The converse is straightforward. □

Theorem 7. *Let $M = M_1 \times_f M_2$ be a W_2 -curvature flat singly warped product manifold with the metric tensor $g = g_1 \oplus f^2g_2$. If M_2 is Ricci flat, then the W_2 -curvature of M_2 is given by*

$$\begin{aligned} W_2^2(X_2, Y_2, T_2, Z_2) &= \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)g_2(Y_2, Z_2) - g_2(Y_2, T_2)g_2(X_2, Z_2)] \end{aligned}$$

and M_1 is of Hessian type. Moreover, M_2 is flat if $n_2 \geq 3$.

Proof. Suppose that M is W_2 -curvature flat, then from Theorem 3 we have

$$\begin{aligned} 0 &= -fg_2(Y_2, T_2)D_{X_1}^1 \nabla f + \frac{1}{n-1} \text{Ric}^2(Y_2, T_2)X_1 - \frac{f^\sharp}{n-1}g_2(Y_2, T_2)X_1, \\ 0 &= K^2(X_2, Y_2)T_2 - \frac{n_1}{(n-1)(n_2-1)}[\text{Ric}^2(Y_2, T_2)X_2 - \text{Ric}^2(X_2, T_2)Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2]. \end{aligned}$$

Now suppose that M_2 is Ricci flat, then the first equation implies that

$$D_{X_1}^1 \nabla f = \frac{-f^\sharp}{(n-1)f}X_1$$

and so

$$H^f = \frac{-f^\sharp}{(n-1)f}g_1,$$

i.e. M_1 is of Hessian type. The second equation implies that

$$K^2(X_2, Y_2)T_2 = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2]$$

and hence

$$\begin{aligned} W_2^2(X_2, Y_2, T_2, Z_2) &= \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)g_2(Y_2, Z_2) - g_2(Y_2, T_2)g_2(X_2, Z_2)]. \end{aligned}$$

Moreover,

$$R^2(X_2, Y_2, T_2, Z_2) = \left(\|\nabla f\|_1^2 + \frac{f^\sharp}{n-1} \right) [g_2(X_2, T_2)g_2(Y_2, Z_2) - g_2(Y_2, T_2)g_2(X_2, Z_2)].$$

Thus M_2 has a pointwise constant sectional curvature given by

$$\kappa_2 = \|\nabla f\|_1^2 + \frac{f^\sharp}{n-1}.$$

If $n_2 \geq 3$, then by Schur’s Lemma, M_2 has a vanishing constant sectional curvature $\kappa_2 = 0$ since M_2 is Ricci flat. □

4. W_2 -Curvature on Space-Times

The study of W_2 -curvature tensor on space-times is of great interest since this concept provides an access to several geometrical and physical properties of space-times. Among such applications, we want to mention that a W_2 -curvature flat 4-dimensional space-time is an Einstein manifold [2, 5]. This section is subsequently devoted to the study of the W_2 -curvature tensor on generalized Robertson–Walker space-times and standard static space-times. We will first consider some classical space-times. Obtaining the W_2 -curvature tensor for these space-times contains long computations, and hence we omitted them.

- The Minkowski space-time is W_2 -curvature flat since it is flat.
- The Friedman–Robertson–Walker with metric

$$ds^2 = -c^2 dt^2 + a(t) \left[\frac{d\eta^2}{1 - k\eta^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

is W_2 -curvature flat if $\dot{a}(t) = k = 0$.

- The de Sitter space-time metric with cosmological constant $\Lambda > 0$ in conformally flat coordinates reads

$$ds^2 = \frac{\alpha^2}{\tau^2} [-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \tag{8}$$

where $\alpha^2 = (3/\Lambda)$. This metric is Einstein with factor $\frac{3}{\alpha^2}$ and has a constant sectional curvature $\frac{1}{\alpha^2}$. The non-vanishing components of the W_2 -curvature tensor are

$$\begin{aligned} W_2(\partial_i, \partial_j, \partial_i, \partial_j) &= R(\partial_i, \partial_j, \partial_i, \partial_j) + \frac{1}{3}(g(\partial_i, \partial_i)\text{Ric}(\partial_j, \partial_j)) \\ &= R(\partial_i, \partial_j, \partial_i, \partial_j) + \frac{1}{\alpha^2}(g(\partial_i, \partial_i)g(\partial_j, \partial_j)) \\ &= 2R(\partial_i, \partial_j, \partial_i, \partial_j), \end{aligned}$$

$$W_2(\partial_i, \partial_j, \partial_j, \partial_i) = -W_2(\partial_i, \partial_j, \partial_i, \partial_j),$$

where $i \neq j$. Direct computations show that the de Sitter space-time with metric (8) is not W_2 -curvature flat. Similarly, the anti-de Sitter is not W_2 -curvature flat.

- Kasner space-time in (t, x, y, z) coordinates is given by

$$ds^2 = -dt^2 + t^{2\lambda_1} dx^2 + t^{2\lambda_2} dy^2 + t^{2\lambda_3} dz^2,$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. This space-time is W_2 -curvature flat if $\lambda_1 = 1$.

- The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(\frac{1}{1 - \frac{r_s}{r}}\right) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where r_s is the Schwarzschild radius and c is the speed of light. The Ricci curvatures are all identically zero and so the W_2 -curvature tensor is equal to the Riemann tensor.

- A cylindrically symmetric static space-time in (t, r, θ, ϕ) coordinates can be given by

$$ds^2 = -e^v dt^2 + dr^2 + e^v d\theta^2 + e^v d\phi^2,$$

where v is a function of r . A cylindrically symmetric static space-time is W_2 -curvature flat if and only if v is constant. If v is a nontrivial function of r, θ, ϕ the situation is more complicated.

4.1. W_2 -curvature on generalized Robertson–Walker space-times

We first define generalized Robertson–Walker space-times. Let (M, g) be an n -dimensional Riemannian manifold and $f : I \rightarrow (0, \infty)$ be a smooth function. Then $(n + 1)$ -dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -dt^2 \oplus f^2 g$$

is called a generalized Robertson–Walker space-time and is denoted by $\bar{M} = I \times_f M$ where I is an open, connected subinterval of \mathbb{R} and dt^2 is the Euclidean metric tensor on I . This structure was introduced to the literature to extend Robertson–Walker space-times [17–20].

From now on, we will denote $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ by ∂_t to state our results in simpler forms.

Theorem 8. *Let $\bar{M} = I \times_f M$ be a generalized Robertson–Walker space-time equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2 g$. Then the curvature tensor \bar{K} on \bar{M} is given by*

- (1) $\bar{K}(\partial_t, \partial_t)\partial_t = \bar{K}(\partial_t, \partial_t)X = \bar{K}(X, Y)\partial_t = 0,$
- (2) $\bar{K}(\partial_t, X)\partial_t = -\frac{\ddot{f}}{f}X,$
- (3) $\bar{K}(X, \partial_t)Y = \left[\frac{n-1}{n}g(X, Y)(f\ddot{f} - \dot{f}^2) - \frac{1}{n}\text{Ric}(X, Y)\right]\partial_t,$

$$(4) \quad \bar{K}(X, Y)Z = -R(X, Y)Z + \dot{f}^2[g(Y, Z)X - g(X, Z)Y] + \frac{1}{n}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y] + \frac{1}{n}[g(Y, Z)X - g(X, Z)Y](f\ddot{f} + (n-1)\dot{f}^2)$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Now we investigate the implications of a W_2 -curvature flat generalized Robertson–Walker space-time to its fiber.

Theorem 9. *Let $\bar{M} = I \times_f M$ be a generalized Robertson–Walker space-time equipped with the metric tensor $\bar{g} = -dt^2 \oplus f^2g$. Then, \bar{M} is W_2 -curvature flat if and only if M has a constant sectional curvature $\kappa = -\dot{f}^2$.*

Proof. Assume that $\bar{M} = I \times_f M$ is W_2 -curvature flat, then

$$\begin{aligned} 0 &= -f\ddot{f}g(X, Y), \\ 0 &= \frac{1}{n}\text{Ric}(X, Y) - \frac{n-1}{n}g(X, Y)(f\ddot{f} - \dot{f}^2), \\ 0 &= -f^2R(X, Y, Z, T) + f^2\dot{f}^2[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] \\ &\quad + \frac{f^2}{n}[\text{Ric}(Y, Z)g(X, T) - \text{Ric}(X, Z)g(Y, T)] \\ &\quad + \frac{f^2}{n}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)](f\ddot{f} + (n-1)\dot{f}^2). \end{aligned}$$

The first equation implies that $\ddot{f} = 0$, i.e. $f = \mu t + \lambda$ and so the second equation yields

$$\text{Ric}(X, Y) = -\mu^2(n-1)g(X, Y).$$

The third equation implies that

$$R(X, Y, Z, T) = \mu^2[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)].$$

Thus the sectional curvature of M is

$$\kappa = -\mu^2.$$

The converse is direct by using the fact that \bar{M} is Einstein with factor $(n-1)\kappa$. □

A 4-dimensional space-time is called Petrov type O if the Weyl conformal tensor vanishes. There are many generalizations of Petrov classification for higher dimensions (see for instance [21]) but type O still has the same definition. From the above theorem, we conclude that \bar{M} is flat and hence the Weyl conformal tensor vanishes.

4.2. W_2 -curvature tensor on standard static space-times

We begin by defining standard static space-times. Let (M, g) be an n -dimensional Riemannian manifold and $f : M \rightarrow (0, \infty)$ be a smooth function. Then

$(n + 1)$ -dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 dt^2 \oplus g$$

is called a standard static space-time and is denoted by $\bar{M} = I_f \times M$ where I is an open, connected subinterval of \mathbb{R} and dt^2 is the Euclidean metric tensor on I .

Note that standard static space-times can be considered as a generalization of the Einstein static universe[22–25].

Now, we are ready to study both K and W_2 tensors on $\bar{M} =_f I \times M$. The following two theorems describe both tensors on $\bar{M} =_f I \times M$.

Theorem 10. *Let $\bar{M} =_f I \times M$ be a standard static space-time with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. If $\partial_t \in \mathfrak{X}(I)$ and $X, Y, Z \in \mathfrak{X}(M)$, then*

- (1) $\bar{K}(\partial_t, \partial_t)\partial_t = \bar{K}(\partial_t, \partial_t)X = \bar{K}(X, Y)\partial_t = 0$,
- (2) $\bar{K}(\partial_t, X)\partial_t = -f(D_X \nabla f + \frac{\Delta f}{n} X)$,
- (3) $\bar{K}(\partial_t, X)Y = \frac{1}{n}(\text{Ric}(X, Y) - (n + 1)\frac{H^f(X, Y)}{f})\partial_t$,
- (4) $\bar{K}(X, Y)Z = -R(X, Y)Z + \frac{1}{n}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y] + \frac{1}{nf}[-H^f(Y, Z)X + H^f(X, Z)Y]$.

Theorem 11. *Let $\bar{M} =_f I \times M$ be a standard static space-time with the metric tensor $\bar{g} = -f^2 dt^2 \oplus g$. Then, \bar{M} is W_2 -curvature flat if and only if M is flat and $H^f = -\frac{\Delta f}{n}g$.*

Proof. Suppose that $\bar{M} =_f I \times M$ is W_2 -curvature flat, then the second item of Theorem 10 implies that

$$D_X \nabla f = -\frac{\Delta f}{n} X, \quad H^f = -\frac{\Delta f}{n} g.$$

Taking the trace of both sides implies $\Delta f = 0$ and consequently $H^f = 0$. The third item implies that

$$\text{Ric}(X, Y) = 0$$

and so M is Ricci flat. The last item of Theorem 10 implies that

$$R(X, Y)Z = \frac{1}{n}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y] + \frac{1}{nf}[-H^f(Y, Z)X + H^f(X, Z)Y],$$

$$R(X, Y)Z = 0.$$

Thus M is flat. The converse is straightforward. □

Appendix A. A Proof of Theorem 3

Let $M = M_1 \times_f M_2$ be a warped product manifold equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ where $\dim(M_i) = n_i, i = 1, 2$ and $n = n_1 + n_2$. Let $X_i, Y_i, Z_i, T_i \in$

$\mathfrak{X}(M_i)$ for $i = 1, 2$. Then

$$\begin{aligned} K(X_1, Y_1)T_1 &= -R(X_1, Y_1)T_1 + \frac{1}{n-1}[\text{Ric}(Y_1, T_1)X_1 - \text{Ric}(X_1, T_1)Y_1] \\ &= -R^1(X_1, Y_1)T_1 + \frac{1}{n-1} \left(\text{Ric}^1(Y_1, T_1) - \frac{n_2}{f}H^f(Y_1, T_1) \right) X_1 \\ &\quad - \frac{1}{n-1} \left(\text{Ric}^1(X_1, T_1) - \frac{n_2}{f}H^f(X_1, T_1) \right) Y_1 \\ &= -R^1(X_1, Y_1)T_1 + \frac{1}{n-1}[\text{Ric}^1(Y_1, T_1)X_1 - \text{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n-1} \left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1 \right] \\ &= K^1(X_1, Y_1)T_1 \\ &\quad - \frac{n_2}{(n-1)(n_1-1)}[\text{Ric}^1(Y_1, T_1)X_1 - \text{Ric}^1(X_1, T_1)Y_1] \\ &\quad - \frac{1}{n-1} \left[\frac{n_2}{f}H^f(Y_1, T_1)X_1 - \frac{n_2}{f}H^f(X_1, T_1)Y_1 \right]. \end{aligned}$$

The second case is

$$\begin{aligned} K(X_1, Y_1)T_2 &= -R(X_1, Y_1)T_2 + \frac{1}{n-1}[\text{Ric}(Y_1, T_2)X_1 - \text{Ric}(X_1, T_2)Y_1] \\ &= 0. \end{aligned}$$

The third case is

$$\begin{aligned} K(X_1, Y_2)T_1 &= -R(X_1, Y_2)T_1 + \frac{1}{n-1}[\text{Ric}(Y_2, T_1)X_1 - \text{Ric}(X_1, T_1)Y_2] \\ &= \frac{1}{f}H^f(X_1, T_1)Y_2 - \frac{1}{n-1}\text{Ric}^1(X_1, T_1)Y_2 + \frac{n_2}{(n-1)f}H^f(X_1, T_1)Y_2 \\ &= - \left[\frac{1}{n-1}\text{Ric}^1(X_1, T_1) - \frac{n+n_2-1}{(n-1)f}H^f(X_1, T_1) \right] Y_2. \end{aligned}$$

The next case is

$$\begin{aligned} K(X_1, Y_2)T_2 &= -R(X_1, Y_2)T_2 + \frac{1}{n-1}[\text{Ric}(Y_2, T_2)X_1 - \text{Ric}(X_1, T_2)Y_2] \\ &= -fg_2(Y_2, T_2)D_{X_1}^1 \nabla f + \frac{1}{n-1}\text{Ric}^2(Y_2, T_2)X_1 \\ &\quad - \frac{f^\sharp}{n-1}g_2(Y_2, T_2)X_1. \end{aligned}$$

Also,

$$\begin{aligned} K(X_2, Y_2)T_1 &= -R(X_2, Y_2)T_1 + \frac{1}{n-1}[\text{Ric}(Y_2, T_1)X_2 - \text{Ric}(X_2, T_1)Y_2] \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} K(X_2, Y_2)T_2 &= -R(X_2, Y_2)T_2 + \frac{1}{n-1}[\text{Ric}(Y_2, T_2)X_2 - \text{Ric}(X_2, T_2)Y_2] \\ &= -R^2(X_2, Y_2)T_2 + \|\nabla f\|_1^2[g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2] \\ &\quad + \frac{1}{n-1}[\text{Ric}^2(Y_2, T_2) - f^\#g_2(Y_2, T_2)]X_2 \\ &\quad - \frac{1}{n-1}[\text{Ric}^2(X_2, T_2) - f^\#g_2(X_2, T_2)]Y_2. \end{aligned}$$

Then

$$\begin{aligned} K(X_2, Y_2)T_2 &= -R^2(X_2, Y_2)T_2 + \frac{1}{n-1}\text{Ric}^2(Y_2, T_2)X_2 - \frac{1}{n-1}\text{Ric}^2(X_2, T_2)Y_2 \\ &\quad + \|\nabla f\|_1^2[g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2] \\ &\quad - \frac{f^\#}{n-1}(g_2(Y_2, T_2)X_2 - g_2(X_2, T_2)Y_2) \end{aligned}$$

and so

$$\begin{aligned} K(X_2, Y_2)T_2 &= -R^2(X_2, Y_2)T_2 + \frac{1}{n-1}[\text{Ric}^2(Y_2, T_2)X_2 - \text{Ric}^2(X_2, T_2)Y_2] \\ &\quad - \left(\|\nabla f\|_1^2 + \frac{f^\#}{n-1} \right) [g_2(Y_2, T_2)X_2 - g_2(X_2, T_2)Y_2] \\ &= -R^2(X_2, Y_2)T_2 + \frac{1}{n-1}[\text{Ric}^2(Y_2, T_2)X_2 - \text{Ric}^2(X_2, T_2)Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\#}{n-1} \right) [g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2]. \end{aligned}$$

Thus

$$\begin{aligned} K(X_2, Y_2)T_2 &= K^2(X_2, Y_2)T_2 - \frac{n_1}{(n-1)(n_2-1)}[\text{Ric}^2(Y_2, T_2)X_2 - \text{Ric}^2(X_2, T_2)Y_2] \\ &\quad + \left(\|\nabla f\|_1^2 + \frac{f^\#}{n-1} \right) [g_2(X_2, T_2)Y_2 - g_2(Y_2, T_2)X_2] \end{aligned}$$

and the proof is now complete.

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