SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

Technical Report No. 932

January 1990
Revised June 1992
Revised December 1993

THE WAREHOUSE SCHEDULING PROBLEM: FORMULATION AND A LGORITHMS

by

Moncer. A. Hariga[1] and Peter. L. Jackson

[1]King Saud University, College of Engineering, P.O.Box 800, Riyadh 11421, Saudi Arabia,
BITNET: F45M036@SAKSU00

This research was supported in part by NSF grants DDM-8819542 and DMC-85-09131

ABSTRACT

The Warehouse Scheduling Problem is a deterministic multi-item inventory problem with a restriction on warehouse floor space available. We formulate a mixed integer nonlinear programming problem for the objective of minimizing long run inventory holding and order costs per unit of time. We integrate algorithms for staggering orders, described in companion papers, with a heuristic to choose the order sequences. The result is called Sequenced Staggering. We describe a new algorithm to generate order frequencies, called the powers-of-two-factor-of-three technique, as a generalization of Roundy's roundoff technique for powers-of-two policies. We report on a computational study of four hybrid algorithms for solving the warehouse scheduling problem, including the competing algorithm of Gallego, Queyranne, and Simchi-Levi. Based on these results, we recommend the combination of powers-of-two frequencies with Sequenced Staggering.

1. INTRODUCTION

Consider a warehouse supplying parts to a high-volume assembly plant for consumer durable goods such as refrigerators or television sets. One of the warehouse manager's concerns is to ensure that there is enough space to accommodate all the parts upon their delivery. In this paper we explore the problem of setting parts ordering rules to satisfy the assembly schedule as well as to make efficient use of both the limited warehouse space and the parts order and delivery system. The focus of this paper is on optimizing the space utilization.

There are two complementary techniques available for optimizing space utilization: order sizing and delivery scheduling. Reducing the size of orders has the effect of reducing the space required to hold the cycle stock. A tradeoff arises in that smaller order sizes necessitate a greater order frequency which in turn places a greater burden on the parts order and delivery system.

Delivery scheduling is the technique of coordinating the delivery times of different orders to optimize space utilization. The greatest demand for warehouse space occurs if all parts deliveries arrive at the warehouse simultaneously. By time phasing or staggering these deliveries, the peak demand for warehouse space can be moderated.

We refer to the combined problem of order sizing and delivery scheduling as the Warehouse Scheduling Problem, WSP.

The following assumptions are common in the literature of the WSP:

(a) The demand for parts occurs continuously in time at known constant demand rates. Backorders are not allowed.

(b) The delivery rate for each part is infinite and lead times are zero. Consequently, in what follows an order is synonymous with a delivery. Constant lead times would not complicate the analysis. Problems associated with supply uncertainty are not considered. See the recent paper by Ibrahim and Thomas [1991] for a consideration of finite delivery rates for the two-product problem.

(c) Delivery schedules are cyclic. The schedule of deliveries is exactly repeated in every cycle. The quantity delivered in a cycle exactly matches the demand during the cycle.

(d) Schedules are evaluated over an infinite time horizon. The performance measures of interest are the maximum space used, the long-run average ordering cost per unit time, and the long-run average inventory holding cost per unit of time.

Many authors have considered the WSP with additional assumptions. Churchman et al [1957], Holt [1958], Buchan and Koenigsberg [1963], Hadley and Whitin [1963], Parsons [1966], and Johnson and Montgomery [1974] ignored the scheduling aspect of the problem and considered order sizing exclusively. They used the *Lagrangian Multipliers Technique* (LMT) to solve the problem of minimizing the long run inventory holding and order costs per unit of time subject to a space constraint that allows for the possibility of receiving simultaneously all parts at one point in time.

Homer [1966], Page and Paul [1976], Zoller [1977], and Hall [1988] considered the scheduling aspect of the problem but restricted the parts to share an identical reorder interval: the so-called *common cycle* (CC) approach. Homer, Page and Paul, Zoller, and Hall each independently derived an approach to optimally time-phase the deliveries within the common cycle. Page and Paul further refined the CC approach by developing a grouping heuristic in which parts within the same group share an identical reorder interval. After computing the maximum space used for each group, they used the LMT to adjust the cycle length of each group under the assumption that the maximum space used across all groups occurs simultaneously. Goyal [1978] demonstrated that the Page and Paul refined approach should be further improved by allowing parts to be ordered more than once. Hartley and Thomas [1982] solved the WSP to optimality for the case of two products with multiple orders per cycle. They did not impose the restriction that the interval between successive orders of the same product be equal.

The WSP bears many similarities to the Economic Lot Scheduling Problem (ELSP) in which production runs of different products must be sized and sequenced in a single time-constrained facility. These two areas are of considerable academic interest because they serve as paradigms for the general problem of integrating lot sizing procedures with procedures to coordinate the detailed timing of operations. Dobson [1987] advanced the heuristics for this problem by restricting attention to powers-of-two frequencies and by relaxing the restriction that successive order intervals for the same product be equal.

We believe the dissertation by Hariga [1988], from which this paper and two companion papers are derived, broke new ground for the WSP in several respects. Hariga noted that the *Zero Switch Rule* (ZSR) commonly used to simplify the ELSP is optimal for the WSP. Hariga abandoned the CC approach and developed algorithms for scheduling multiple deliveries of each product during a single overall cycle. The algorithms are similar in spirit to those developed by Dobson and others for the ELSP. Hariga's method of selecting frequencies, called the *convergent frequency algorithm*, is based on an observation by Muckstadt and Singer [1978] (see also Schweitzer and Silver [1983]) that when order frequencies in the Economic Order Quantity model are constrained to be integers, the cost per unit time converges to the unconstrained optimum as the cycle length increases. Hariga's sequencing method, reported in this paper, is an extension of the Dobson (and Haessler and Hogue [1976]) powers-of-two bin packing procedure to consider arbitrary integer frequencies.

Hariga developed two algorithms for determining the detailed timing of orders in the schedule, given an order sequence. The first, motivated by Dobson's ELSP work, relaxed the restriction that the timing intervals between successive orders of the same product be equal. Analysis of this relaxation revealed several insights. For example, a feasible solution for any sequence is guaranteed to exist. In an optimal solution, every order is timed to exactly fill the warehouse. (This result generalized Hall's [1988] observation for the CC approach.) Hariga developed a lower bound on the peak space utilization that is independent of the order sequence. Hariga also established sufficient conditions under which the solution to the time variant case would naturally exhibit equal intervals. These and other results are presented in Hariga and Jackson [1991].

Recognizing the managerial advantages of the equal interval assumption, Hariga developed a method that, for a given sequence of deliveries, either found the optimal timing of equal interval deliveries or suggested a change in the sequence. This algorithm is presented in Jackson and Hariga [1991] and is shown to be sequence improving.

Anily [1991] conducted formal worst-case analysis for the CC approach. She developed a CC heuristic that generates a solution within 41% of the optimal solution in the worst case. Gallego, Shaw, and Simchi-Levi [1990] address the problem of staggering orders (choosing sequence and timing for given frequencies) under the equal interval restriction and prove that the problem is NP-complete even if only one product is ordered more than once per cycle. This result justifies the attention given to heuristics for this problem.

The paper by Gallego, Queyranne, and Simchi-Levi [1990] is a major contribution to this literature. They show that the lower bound on peak space, derived independently by Hariga and by Anily for certain policy subsets is, in fact, a lower bound for any feasible policy. They review many of the heuristics proposed for the WSP and derive worst case performance bounds for them. They develop a heuristic algorithm for the staggering problem that does not require a separate sequencing algorithm to initialize it, as is the case in Hariga [1988] or Jackson and Hariga [1991]. The algorithm is motivated by a theorem that partially characterizes optimal solutions to the staggering problem. They combine this staggering algorithm with powers-of-two frequency policies using Roundy's roundoff algorithm [1989] to obtain efficient heuristics for the WSP, called the tactical model, and a relaxed version of the WSP, called the strategic model (see also Hodgson and Howe [1982]). They compare their algorithm with Hariga [1988] and others and report similar performance, but note that the problems considered in the literature to date have been relatively easy.

In experiments not reported here we have compared Hariga's algorithm for the WSP under the equal order interval restriction with the algorithm of Gallego, Queyranne, and Simchi-Levi on problems that they have identified as difficult. We found that Hariga's algorithm is inferior to the approach of Gallego *et al.* We have traced the source of the latter's advantage to the use of the powers-of-two frequency technique. Accordingly, we have abandoned the convergent frequencies technique of generating frequencies. In this paper we demonstrate the robust nature of the powers-of-two frequency technique and show empirically that our staggering algorithm is superior to that of Gallego *et al.* To do this, we contrast the powers-of-two frequency technique with what we call a powers-of-two-factor-of-three technique.

This paper is organized as follows. In Section 2, the warehouse scheduling problem is formulated as a mixed integer, nonlinear programming problem. In Section 3, the frequency and sequence variables are assumed to be given, giving rise to a lot scheduling problem which is the subject of two companion papers. The results of these papers are summarized. In Section 4, we present a heuristic for generating order sequences given arbitrary integer frequencies. In Section 5, we describe the powers-of-two-factor-of-three technique as a generalization of Roundy's roundoff algorithm. In Section 6, we consider four hybrid algorithms, one of them being the algorithm of Gallego *et al*, and report on the results of a computational study of these algorithms. Section 7 concludes the paper.

2. PROBLEM FORMULATION

We formulate the warehouse scheduling problem as the problem of choosing a cyclic schedule of time-dated orders that exactly meets demand during a cycle and does not exceed available warehouse space in order to minimize long-run average ordering and holding costs per unit time. For given order frequencies, the problem can be cast as a mixed integer nonlinear programming problem.

Notation

Hereafter, the indices i and r are part designators: i, r=1,...,n, where n is the number of parts. The indices j, k, and l designate orders: j, k, l=1,...,m, where m is the number of orders placed in one cycle.

Parameters

The following parameters are assumed to be given:

V= warehouse space available;

m_i= number of orders for product i, i=1,...,n;

λ_i= demand rate for part i in unit space/unit time, i=1,...,n;

h_i= holding cost rate for part i in \$/unit space/unit time, i=1,...,n;

K_i= setup cost(ordering cost) in \$/order, i=1,...,n.

Note that demand rates and holding cost rates have been expressed using units of space. Conversion from units of parts to units of space is a trivial matter given the space required per unit of parts.

Decision Variables :

The decision variables are:

Z^b_{ij}= inventory of part i immediately prior to delivery of the jth order;

Z^a_{ij}= inventory of part i immediately after delivery of the jth order;

$Z_{ij} = Z^a_{ij} - Z^b_{ij}$, lot size of part i on the jth order (equals zero if part i is not ordered on the jth order);

Z_j= total inventory for all parts immediately after delivery of the jth order;

W= maximum space used (equivalently, maximum inventory, since the demand rate is expressed in unit space per unit time);

U_j= time interval between jth and (j+1)st order, interpreted cyclically;

T_j= reorder interval of the part ordered on the jth order;

τ = cycle length;

$$\delta_{ij} = \begin{cases} 1 \text{ if part i is ordered on the jth order,} \\ 0 \text{ otherwise; and} \end{cases}$$

$$\delta_{ijk} = \begin{cases} 1 \text{ if the next order of part i after the jth order is on the kth order,} \\ 0 \text{ otherwise.} \end{cases}$$

Figure 1 illustrates the inventory variables for a single part, part i, for which orders are placed on the 2nd, 5th, and 9th orders. The upper graph is the inventory pattern for a schedule in which $Z^b_{ij} > 0$ for each order j. The lower graph is the inventory pattern for a schedule in which $Z^b_{ij} = 0$. Schedules satisfying the latter property are said to obey the Zero Switch Rule (ZSR). For the WSP, in which the objective is to minimize long run average inventory holding and order costs, it is trivial to see that there exists an optimal schedule satisfying the ZSR. The inventory holding cost savings for part i from employing the ZSR in the example of Figure 1 are proportional to the shaded area of the figure.

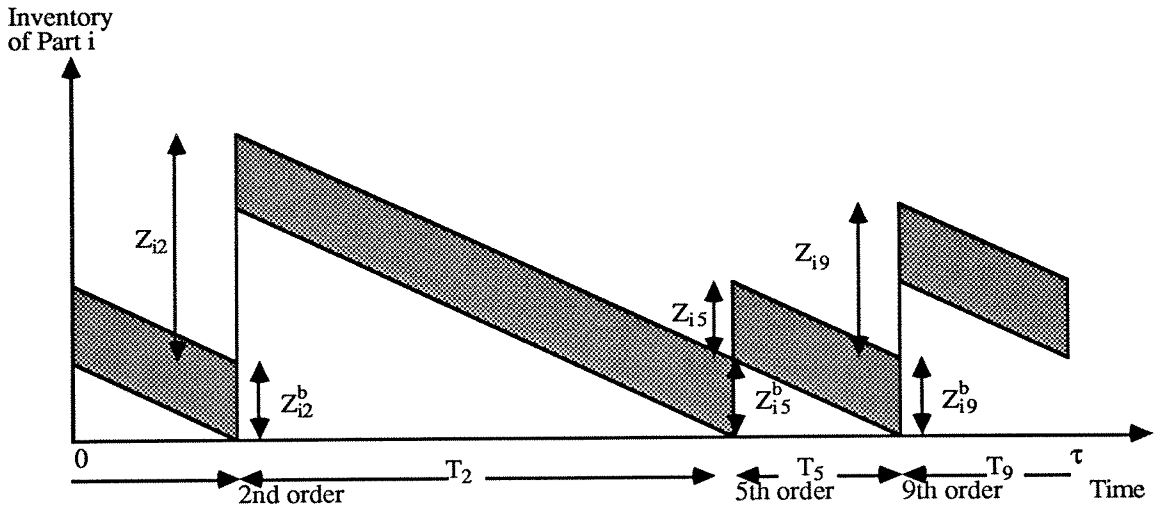

Figure 1: Consistency of the ZSR with the WSP

Under the ZSR, the average holding cost of part i becomes the sum of the area of the inventory triangles pictured in the lower graph of Figure 1.

Note that an order of part i corresponds to the jth order in the overall sequence if and only if $\delta_{ij}=1$. Thus, the long run average holding cost per unit time is:

$$\frac{1}{2\tau}\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\, h_i\, \delta_{ij}\, T_j^2 ,$$

and the long run average inventory cost per unit of time is:

$$\frac{1}{\tau}\left\{ \sum_{i=1}^{n} K_i\, m_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\, h_i\, \delta_{ij}\, T_j^2 \right\}, \tag{1}$$

The maximum inventory must occur on one of the replenishment points when the orders are received. Elsewhere in time, the inventory is depleted with rate equal to the sum, over all parts, of the demand rates. Therefore, the maximum inventory, W, must satisfy:

$$W \geq Z_j, \qquad j=1,\ldots m,$$

and the warehouse space availability constraint is:

$W \le V$.

Each order is for a single part but orders may occur simultaneously. That is, U_j may take on the value of zero, for any order j.

Let S_{ij} = time interval until part i is ordered again after the jth order, and $\Delta_{ijk} = \sum_{l=j+1}^{k} \delta_{ijl}$.

Throughout this paper, such summations are to be interpreted in a cyclic fashion. That is, if $k \le j$, then $\Delta_{ijk} = \sum_{l=j+1}^{m} \delta_{ijl} + \sum_{l=1}^{k} \delta_{ijl}$.

Note that $\Delta_{ijk}=1$ if and only if the next order of part i after the jth order is one of $\{j+1, j+2,..., k\}$, interpreted cyclically if $k \le j$. Otherwise $\Delta_{ijk}=0$. Using this fact, S_{ij} can be expressed as:

$$S_{ij} = U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk})\, U_k \ .$$

To see this, suppose $\delta_{ijj'}=1$; ie, the first time part i is delivered after the jth order is on the j'th order. In this case, $\Delta_{ijk}=0$ for $k=j+1,j+2,...,j'-1$ and $\Delta_{ijk}=1$ for $k=j',j'+1,.....,j$. Hence, $S_{ij} = U_j+U_{j+1}+...+U_{j'-1}$, as desired.

By the ZSR, the amount of inventory of part i on the jth order should be equal to the amount needed to satisfy the demand until the next order of part i. Therefore,

$$Z^a_{ij} = \lambda_i\, S_{ij} \ , \qquad j=1,...,m \text{ and } i=1,...,n,$$

and

$$Z_j = \sum_{i=1}^{n} \lambda_i\, S_{ij} \ , \qquad j=1,...,m.$$

Now, to determine T_j, the time interval until the order on which the part ordered on the jth order is reordered, it is necessary to know which part is ordered on the jth order. Using the definition of δ_{ij} and S_{ij}, T_j can be written as:

$$T_j = \sum_{i=1}^{n} \delta_{ij}\, S_{ij} \ , \qquad j=1,...,m.$$

We simplify the above relationships by eliminating the variable S_{ij} and propose the following model for the warehouse scheduling problem for fixed frequencies

(FFWSP):

$$\text{Min } \frac{1}{\tau} \left\{ \sum_{i=1}^{n} m_i K_i + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} T_j^2 \right\}$$

s.t.

$$W \le V, \tag{2}$$

$$Z_j \le W, \qquad j=1,\dots,m, \tag{3}$$

$$Z_j = \sum_{i=1}^{n} \lambda_i \left(U_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk}) U_k \right), \qquad j=1,\dots,m, \tag{4}$$

$$T_j = \sum_{i=1}^{n} \delta_{ij} \left(U_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk}) U_k \right), \qquad j=1,\dots,m, \tag{5}$$

$$\sum_{j=1}^{m} U_j = \tau, \tag{6}$$

$$\sum_{j=1}^{m} \delta_{ij} = m_i, \qquad i=1,\dots,n, \tag{7}$$

$$\sum_{i=1}^{n} \delta_{ij} = 1, \qquad j=1,\dots,m, \tag{8}$$

$$\sum_{k=1}^{m} \delta_{ijk} = 1, \qquad j=1,\dots,m \text{ and } i=1,\dots,n, \tag{9}$$

$$\delta_{ijk} - \delta_{ik} \le 0, \qquad j,k=1,\dots,m \text{ and } i=1,\dots,n, \tag{10}$$

$$\sum_{l=j+1}^{k} \delta_{il} \le \sum_{l=j+1}^{k} \delta_{ijl}, \qquad j=1,\dots,m,\ k=j+1,\dots j \text{ and } i=1,\dots,n, \tag{11}$$

$$\Delta_{ijk} = \sum_{l=j+1}^{k} \delta_{ijl}, \qquad j,k=1,\dots,m, \text{ and } i=1,\dots,n, \tag{12}$$

$$U_j \ge 0, \qquad j=1,\dots,m,$$

$$\delta_{ijk}, \delta_{ij} \in \{0,1\}, \qquad j,k=1,\dots,m, \text{ and } i=1,\dots,n.$$

The decision variables for this mixed integer nonlinear programming model are W, τ, T_j, Z_j, U_j, δ_{ij}, δ_{ijk}, and Δ_{ijk}. Constraint (6) defines the cycle length τ. Constraints (7) require that each part i should be ordered exactly m_i times. Constraints (8) state that exactly one part should be replenished on each order. Constraints (9) indicate that there has to be precisely one order on which the next order of part i after the jth order is placed. Constraints (10) ensure that if $\delta_{ijk} = 1$, then necessarily $\delta_{ik} = 1$. These are called contingency constraints. Finally, constraints (11), together with constraints (9), ensure that no order of part i is placed between the jth order and the first order of part i after the jth order.

The general warehouse scheduling problem is the fixed frequency warehouse scheduling problem (FFWSP) optimized over integer frequencies:

(WSP):

Min FFWSP

s.t.

$m_i \in \{1, 2, 3, \ldots\}, \; i = 1, \ldots, n,$

where $m = \sum_{i=1}^{n} m_i$.

3. The Lot Scheduling Problem

Because of the complexity of the general model formulated for WSP, we decompose WSP by progressively fixing decision variables. As noted in the previous section, we first fix the order frequencies to obtain the subproblem FFWSP. In this section, we fix the sequence variables to obtain a subproblem referred to as the lot scheduling problem. In particular, we fix the variables δ_{ij} and Δ_{ijk} for all i, j, and k.

Let:

$t_j = T_j / \tau$ relative reorder interval;

$u_j = U_j / \tau$ relative time interval;

$w = W / \tau$ relative maximum inventory.

With the above assumptions and transformations, WSP reduces to a nonlinear programming problem with w, τ, t_j, and u_j as decision variables.

(NLP):

$$\text{Min } \frac{1}{\tau} \sum_{i=1}^{n} m_i K_i + 0.5\, \tau \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} t_j^2 \tag{13}$$

s.t.

$$w\, \tau \leq V, \tag{14}$$

$$\sum_{i=1}^{n} \lambda_i \left(u_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk})\, u_k \right) \le w, \qquad j=1,\ldots,m, \tag{15}$$

$$t_j - \sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk})\, u_k \right) = 0, \qquad j=1,\ldots,m, \tag{16}$$

$$\sum_{j=1}^{m} u_j = 1, \tag{17}$$

$$u_j \ge 0, \qquad j=1,\ldots,m. \tag{18}$$

The lot scheduling problem, NLP, is the subject of two companion papers, Hariga and Jackson[1991] and Jackson and Hariga[1991]. The solution techniques developed in those papers relate to a relaxed version of NLP with the same decision variables,

(RNLP):

$$\text{Min } \frac{1}{\tau} \sum_{i=1}^{n} m_i K_i + \tau \left(\pi\, w + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} t_j^2 \right) - \pi V$$

subject to (15)-(18), where π is the dual variable associated with (14) in the original NLP.

Observe that RNLP is separable in terms of the cycle length and the remaining decision variables. Consequently, the optimal cycle length, τ^*, is given by

$$\tau^*(u^*, t^*, w^*, \pi) = \left(\frac{\sum_{i=1}^{n} m_i K_i}{\tau\, w^* + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} (t_j^*)^2} \right)^{1/2}, \tag{19}$$

where (u^*, t^*, w^*) is determined by the solution to the following quadratic programming problem with w, t_j, and u_j as decision variables

(QP):

$$\text{Min } \pi\, w + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} t_j^2$$

subject to (15)-(18), for a given value of π. Then, the solution to NLP is obtained by searching for a value of π to satisfy the omitted constraint $\tau^* w^* \le V$.

It is possible that the space constraint (14) is not binding in the optimal solution to the lot scheduling problem NLP. This corresponds to the situation $\pi = 0$ in RNLP and QP. It is easily seen that the optimal solution to QP when $\pi = 0$ is given by equal reorder intervals:

$$t_j^* = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j = 1,\ldots,m.$$

Let τ_0^* denote the optimal cycle length for RNLP when $\pi = 0$. It is given by (19) with $\pi = 0$ and equal reorder intervals.

For large values of π, that is, for situations in which the cost penalty of the space constraint has a high value, the solution to QP should be well approximated by the solution to a linear programming problem with the same decision variables as QP.

(LP):

Min w

subject to (15)-(18).

Several properties of (LP) are discussed in the first companion paper. In particular, the optimal solution is shown to be characterized by the situation in which the maximum inventory is achieved immediately after each order. For this reason, (LP) can be solved with a single matrix inversion. Note also that the solution to LP is independent of the value of π. Consequently, a heuristic solution to NLP is simply (τ', w', t', u') where $\tau' = \text{Min}(\tau_0^*, V/w')$ and (w',t', u') solves LP. Because of the potential appeal of the use of LP in heuristics, the first companion paper establishes a bound on the holding cost penalty of using the solution to LP rather than an optimal solution to QP.

An alternative approach to the lot scheduling problem is to restrict attention to time invariant lot sizes: the quantity replenished for each part is restricted to be the same whenever the part is ordered. There are two motivations for such a restriction. The first is that equal lot sizes together with the ZSR ensure that inventory holding costs are minimized for a given order frequency and cycle length. The second is that a schedule of equal lot sizes, and therefore, equal reorder intervals, is managerially easier to implement.

Under the restriction of equal order intervals, the lot scheduling problem NLP reduces to a linear programming problem with the same decision variables as QP and LP:

(RLP):

Min w

subject to (15)-(18) and

$$t_j = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j=1,\ldots,m. \tag{20}$$

The optimal cycle length is given by $\tau^* = \text{Min}\ (\tau_0^*, V/w^*)$, where w^* is the solution to RLP.

Problem RLP is the subject of the second companion paper. The algorithm developed in that paper either finds an optimal solution to RLP or it suggests an improvement in the sequence used to generate RLP. Since there are only a finite number of sequences, the equal interval algorithm (algorithm 4 of Jackson and Hariga [1991]) converges to an improved sequence for which RLP is solved optimally.

4. SEQUENCING HEURISTICS

The complexity of FFWSP is due to the combinatorial aspect of the problem. When solving the ELSP, the same difficulty has been circumvented by developing heuristics to generate promising sequences (Haessler and Hogue [1976], Delporte and Thomas [1977], Dobson [1987]). These heuristics are similar in a sense that they try to evenly space the orders of each part. Similarly, in this section a sequencing heuristic is constructed to develop sequences for arbitrary ordering frequencies to be used for WSP.

In uncapacitated lot sizing problems, powers-of-two policies are known to yield solutions within 6% of a lower bound on inventory ordering and holding cost (Roundy [1989]). For capacitated problems such as the WSP, the cost bounds are less encouraging. Recently, Gallego, Queyranne, and Simchi-Levi [1990] have established a cost bound of 112% for a particular use of power-of-two policies in the WSP. To see this difficulty, consider the case of a single part which is ordered 11 times at equally spaced intervals within an overall cycle of length one time unit. If the frequency of this part is rounded to 8, the nearest power-of-two integer, the cost increase is less than $((1+(11/8)^2)/(2*11/8) - 1) = 5.1\%$, assuming that the frequency of 11 minimized the inventory order and holding cost for that part. In the capacity dimension, however, the maximum inventory for that part will increase by $(11/8 - 1) = 37.5\%$. Of course, maximum total inventory will not increase by as much in a relative sense and the effect on capacity can be moderated by adjusting the cycle length. The need to adjust the cycle length to satisfy the capacity constraint is precisely why the 6% bounds do not extend to capacitated problems. We acknowledge the managerial and scheduling advantages of powers-of-two policies that we and other authors have frequently noted but the example suggests that there may be situations in which general integer-ratio policies offer a significant advantage over powers-of two policies. Accordingly, we have focused our algorithmic development on integer frequency policies. These policies include powers-of-two policies as a special case.

One difficulty in working with arbitrary frequencies lies in generating sequences for which the orders for a part can be placed at equal intervals. For example the sequence (1, 2, 2, 1, 1) results in an infeasible set of equations (20) within RLP since part 2 is ordered twice between two orders of part 1 but has a longer reorder interval than part 1. The arbitrary frequencies bin packing heuristic, presented below, is an adaptation of Dobson's sequencing heuristic to the WSP and extended to the case of arbitrary integer order frequencies with a guarantee of feasibility.

There are two essential ideas to bin packing heuristics as applied to the WSP. The first idea behind this heuristic is to produce an ordering sequence for which the orders of each part can be scheduled at equally spaced points in time over the cycle. Equal intervals are desirable from a holding cost perspective. The second idea behind the Arbitrary Frequencies Bin Packing Heuristic is to spread the peak demand for space evenly over the cycle. This is approximated by grouping orders to bins so that the maximum load across all the groups, called <u>bins</u>, is minimized. The <u>load</u> for a bin is defined to be the total space size of the orders assigned to that bin, where the space size of each order of part i is given by $\psi_i = \lambda_i/m_i$. This approximation implicitly assumes that (a) all orders for the same part have the same lot size, (b) all orders within a bin occur simultaneously, and (c) orders in other bins have no effect on the peak demand for space within a given bin. Assumption (a) is consistent with the first objective of the bin packing heuristic but assumptions (b) and (c) are almost certainly violated by the resulting schedule. Nevertheless, Hariga and Jackson[1991], indicate that, under certain conditions, this approximation is consistent with the objective of minimizing the maximum space required per cycle.

Let m^+ denote the largest frequency. Divide the time scale [0,1) into m^+ equal sized intervals. Each interval corresponds to a bin. Number the bins consecutively: 1, 2, ..., m^+. Let the pair (i, k) index the kth order of part i : $k = 1,2,\ldots,m_i$. Further, let:

n^+ = number of parts with frequency m^+.

y_{ik} = trial relative order time of the kth order of part i. (It is called <u>relative</u> because $y_{ik} \in [0,1)$),

b_{ik} = bin number of the kth order of part i,

$p(b)$ = number of part orders in bin $b = |\{(i, k): b_{ik} = b\}|$

$Bin(b)$ = ordered list of part orders in bin $b = ((i_1,k_1) , (i_2,k_2),\ldots,(i_{p(b)},k_{p(b)}))$.

$$\text{Load}(b) = \sum_{r=1}^{p(b)} \psi_{i_r} \quad \text{space load of bin b.}$$

Bin(b) is said to be a time part ordering if it is ordered lexicographically increasing in (y_{ik}, i).

If $(i,k) \notin$ Bin(b) but $b_{ik} = b$, then we define an operation INCLUDE on a time-part ordered Bin(b) as follows:

INCLUDE ((i,k),b)

1. Load(b) $\leftarrow$Load(b)+ ψ_i.
2. p(b) $\leftarrow$p(b)+1.
3. INSERT ((i,k),Bin(b)),

where the INSERT opertion inserts (i,k) into the ordered list Bin(b) while preserving the time part ordering property. The computational complexity of the INCLUDE operation is O(log(n)) since, for any bin b, the maximum number of elements in Bin(b) is n (Aho et al [1974]).

Next, the different steps of the Arbitrary Frequencies Bin Packing Heuristic are presented in an algorithmic form.

1. Renumber the parts so that they are lexicographically decreasing in (m_i, ψ_i) for i = 1,..., n.
2. Initialization:

For k =1 to m^+ do:

   2-1. For i = 1,..., n^+ do: $y_{ik} = (k-1)/m^+$,

   2-2. Bin(k) = ((1, k), (2, k),...,(n^+, k)),

   2-3. p(k) = n^+,

   2-4. Load(k) = $\psi_1 + \psi_2 + ... + \psi_{n^+}$.

3. For i = (n^+ +1) to n do:

   3-1. b* $\leftarrow$ argmin{Load(b): b = 1,..., m^+}

   3-2. $y_{i1} \leftarrow y_{i_{p(b^*)}\, k_{p(b^*)}}$,

   3-3. INCLUDE ((i, 1), b*),

   3-4. For k = 2 to m_i do:

      3-4-1. $y_{ik} \leftarrow (y_{i\,k-1} + (1/m_i))$ Modulo 1,

      3-4-2. $b_{ik} \leftarrow$ Min {integer b: $y_{ik}\, m^+ < b$},

      3-4-3. INCLUDE ((i, k), b_{ik}).

4. Concatenate the ordered lists (Bin(b): b = 1,...,m^+) to obtain the sequence P.

The computational complexity of the heuristic is no greater than $O(n\ m^+ \log(n))$. Step 1 of the heuristic needs at most $n \log(n)$ computer operations. The loop beginning with step 3-1 is evaluated once per part, the loop beginning with step 3-4-1 is evaluated at most m^+ times for each part and the most time-consuming step, step 3-4-3, requires $O(\log(n))$ operations.

The sequence obtained by the Arbitrary Frequencies Bin Packing Heuristic is feasible for the fixed order quantity model since a feasible solution to RLP can be derived from the trial relative order times y_{ik}.

If the frequencies are powers-of-two integers, then the Arbitrary Frequencies Bin Packing Heuristic reduces to a simpler heuristic described as the Powers-of-Two Bin Packing Heuristic in Hariga [1988]. Note that once b* is determined for each part i (step 3-1), then all its subsequent orders are placed in every m^+/m_i bins. By step 3-4-1, $y_{ik}=y_{ik-1}+1/m_i$ (without loss of generality, suppose that $y_{ik}<1$), then $y_{ik}\ m^+ = y_{ik-1}\ m^+ + m^+/m_i$ or $b_{ik} = b_{ik-1} + m^+/m_i$. Consequently, the computation of trial order times (steps 2-1, 3-2, and 3-4-1) can be eliminated and step 3-4-2 can be simplified. Moreover, in the Power of Two Bin Packing Heuristic the operation INCLUDE is trivial: each order assigned to bin b is appended to the end of the ordered list Bin(b).

More elaborate sequencing heuristics could easily be constructed. For example the selection of the bin to which to assign the first order of part i is currently made on the basis of a simple comparison (step 3-1). Instead, we could try assigning the first order of part i to different bins, compare the vector of bin loads that would result and select the best one (the bin load vector with the minimum maximum load). Such a comparison is unnecessary in the Power-of-Two Bin Packing Heuristic because subsequent orders of a part are always placed in bins that have the same load as the first order of the part; but it could prove valuable in the context of arbitrary frequencies. The simple comparison in step 3-1 appears to work well so experimentation with alternative schemes is left for further research.

5. THE POWERS-OF-TWO-FACTOR-OF-THREE FREQUENCY ALGORITHM

In this section, we develop a new algorithm for generating frequencies for the WSP. The starting point for the presentation is an economic order interval problem. Let $T^* = (T_i^*;\ i = 1, 2, .., n)$ denote the optimal order intervals to the singly-constrained economic order interval problem (EOP):

$$\min C(T) = \sum_{i=1}^{n} (K_i/T_i + g_i T_i) \tag{21}$$

such that

$$\sum_{i=1}^{n} \lambda_i T_i \leq W, \tag{22}$$

where $g_i = \lambda_i h_i/2$. The left hand side of (22) is an overestimate of the space required by a good solution to the WSP because it assumes simultaneous deliveries of all items whereas a good solution will stagger the deliveries to avoid this. Nevertheless, EOP serves as practical guide for setting frequencies in the WSP.

Following standard arguments, the solution to EOP is of the form

$$T_i^* = \left(\frac{K_i}{g_i + \alpha \lambda_i}\right)^{1/2},$$

$i = 1, 2, ..., n$, where α is the Lagrange multiplier for constraint (22). Let $\alpha = \alpha_W$ denote the solution to the equation

$$\sum_{i=1}^{n} \lambda_i \left(\frac{K_i}{g_i + \alpha \lambda_i}\right)^{1/2} = W.$$

Then, the optimal value of the Lagrange multiplier is $\alpha = \text{Max}(\alpha_W, 0)$.

Roundy [1989] describes an algorithm for rounding off the order intervals T_i^* to powers of two times a common factor, β ($\beta > 0$). Gallego *et al.* [1990a] recommended using these powers of two multiples to determine the frequencies in the WSP. They report that, in combination with their method to stagger the order times, their algorithm achieves a total cost that is close to the lower bound in many numerical tests. They conjecture that much of the remaining optimality gap is due to the powers-of-two roundoff. In this paper, we investigate that conjecture by considering a wider class of integer multiple solutions to EOP in order to generate a richer set of frequencies for the WSP.

Roundy's roundoff algorithm is easily generalized to a wide class of policies which employ multiples of a common factor. Let N denote the ordered countable set of allowable multiples of β. Denote the element of N by n_k, for k an integer. We assume $\lim_{k \to -\infty} n_k = 0$ and $\lim_{k \to +\infty} n_k = +\infty$, but only minor modifications in what follows are required to incorporate positive lower bounds and finite upper bounds. We assume that $0 < T_i^* < \infty$ so that the following two candidates for roundoff are well-defined:

$$n_{li} = \max\{n_k;\ n_k \le T_i^*,\ n_k \in N\}, \tag{23}$$

$$n_{ui} = \min\{n_k;\ n_k \ge T_i^*,\ n_k \in N\}. \tag{24}$$

Generalized Roundoff Algorithm

(1) Compute the roundoff candidate n_{li} and n_{ui}, for each i = 1, 2, ..., n.

(2) Renumber the products in increasing order of the ratio T_i^*/n_{li}.

(3) Generate *n* candidate roundoff vectors, $\{n^r, r = 1, 2, ..., n\}$, according to the rule

$$n_i^r = \begin{cases} n_{li}, & \text{if } i \le r, \\ n_{ui}, & \text{if } i > r, \end{cases}$$

for i = 1, 2, ..., n.

(4) For each candidate vector, n^r, generate a solution to EOP of the form

$$T_i^r = n_i^r \beta^r,$$

i = 1, 2, , n. In each case, let $T^r = (T_i^r)$ with β^r set to the value that optimizes T^r in EOP:

$$\beta^{*r} = \min\left\{\left(\frac{\sum_{i=1}^{n} K_i/n_i^r}{\sum_{i=1}^{n} g_i n_i^r}\right)^{1/2}, \frac{W}{\sum_{i=1}^{n} \lambda_i n_i^r}\right\}.$$

(5) Choose the candidate vector that minimizes $C(T^r)$. Denote the index of the optimal roundoff vector by r*.

Finally, any such roundoff solution can be used to generate the vector of frequencies for the WSP. To see this, let $M = \text{l.c.m.}\{n_i^{r*};\ i = 1, 2, ..., n\}$ and let

$$m_i = \frac{M}{n_i^{r*}}, \tag{25}$$

for i = 1, 2, ..., n. By construction, m_i and M/m_i are both integers. The vector (m_i) is a feasible vector of frequencies for the WSP.

Powers-of-Two-Factor-of-Three Frequencies

One class of policies of interest restricts allowable multiples to the form

$$N_{23} = \{2^{p_1} 3^{p_2};\ p_1, p_2 \text{ integer}\}.$$

The disadvantage of such multiples compared to a pure set such as $\{2^{p_1}\}$ or $\{3^{p_2}\}$ is that the least common multiple of any finite collection of such multiples can be considerably larger than the largest multiple. We limit this disadvantage in this paper by restricting attention to at most one factor of three; i.e. $p_2 \in \{0, 1\}$. We refer to the resulting class of multiples as *powers-of-two-factor-of-three* (P2F3).

To order the P2F3 class, we note the pattern

$$2^{p-1} < 2^{p-2}\,3 < 2^{p} < 2^{p-1}\,3 < 2^{p+1}$$

for integer p. Therefore, letting $\underline{p} = \lfloor \log_2 T_i^* \rfloor$ and $\bar{p} = \lceil \log_2 T_i^* \rceil$, we can write (23) and (24) as

$$n_{li} = \begin{cases} 2^{\underline{p}-1}\,3, \text{ if } 2^{\underline{p}-1}\,3 \le T_i^*, \\ 2^{\underline{p}}, \text{ otherwise,} \end{cases}$$

and

$$n_{ui} = \begin{cases} 2^{\bar{p}-1}\,3, \text{ if } 2^{\bar{p}-1}\,3 \le T_i^*, \\ 2^{\bar{p}}, \text{ otherwise,} \end{cases}$$

respectively. The Generalized Roundoff Algorithm can now be applied to this class. We refer to this as the *Powers-of-Two-Factor-of-Three Frequency Algorithm.* The resulting vector of frequencies (m_i), computed using (25), has the property that the least common multiple of the frequencies is at most three times the largest frequency.

6. COMPUTATIONAL RESULTS

Our research as surveyed in this paper has resulted in a collection of algorithms for the WSP. Power-of-two frequencies can be generated using Roundy's Roundoff Algorithm, as suggested by Gallego, Queyranne, and Simchi-Levi [1990, hereafter referred to as GQS]. Alternatively, a richer set of frequencies, powers-of-two-factor-of three, can be generated using the Generalized Roundoff Algorithm described in the previous section. The frequency generating algorithms are referred to as (P2) and (P2F3), respectively. Either set of frequencies can be passed to the Arbitrary Bin Packing Algorithm to generate a sequence of orders. In the former case, the sequencing algorithm reduces to Dobson's Powers-of-Two-Bin Packing Algorithm. Finally, the detailed schedule of staggered ordering times can be generated by either the time variant order interval technique of Hariga and Jackson [1991], if equal order intervals are not required, or by the equal order interval technique of Jackson and Hariga

[1991], if they are. For our computational study, we restrict attention to the equal order interval technique, since the equal order interval requirement is the basis of competing algorithms. The combination of the Arbitrary Bin Packing Algorithm with the equal order interval technique will be referred to as Sequenced Staggering (SS).

Another possibility arises in either set of frequencies (P2 or P2F3) could be passed to the Tactical Staggering Heuristic of GQS. This option will be referred to as Tactical Staggering (TS). Note that Tactical Staggering does not involve a sequencing step.

Thus, there are four hybrid algorithms to the WSP that we wish to compare: Powers-of-two with Tactical Staggering (P2 with TS), Powers-of-two with Sequenced Staggering (P2 with SS), Powers-of-two-factor-of-three with Tactical Staggering (P2F3 with TS), and Powers-of-two-factor-of-three with Sequenced Staggering (P2F3 with SS). The combination of P2 with TS is the algorithm proposed by GQS.

The algorithms have been coded in Mathematica. In the case of Tactical Staggering, Guillermo Gallego graciously provided the code for this algorithm in the Gauss language. This was easily converted to Mathematica. That code included some undocumented refinements to the frequency generating algorithm which we removed. (We note that these refinements are effective in improving the powers-of two frequencies but they would obscure the comparison between P2 and P2F3). We also implemented a perturbation scheme as GQS suggested in their Proposition 15 because we encountered several problems in which the unperturbed TS solution exhibited simultaneous orders. We believe the code fairly represents their algorithms. When comparing computation time of the TS-based algorithms with SS-based algorithms, the SS algorithm was run immediately after the TS algorithm for each data set. This ensured comparable efficiency during the Mathematica session.

The experimental setup is identical to that explored by GQS, with one exception. In particular, we generated problem sets consisting of 5, 10, 15, and 20 items for both a moderate capacity and a limited capacity situation. For each problem set, the demand rates, the setup costs, the holding costs, and the space requirements for each item were chosen from a uniform distribution over the range [0, 100]. Demand rates and holding costs were converted to units space per unit of time as suggested in section 2, above. For each item, we computed an ideal reorder interval:

$$T_i^* = \left(\frac{2\,K_i}{\lambda_i\, h_i} \right)^{1/2},$$

and computed the space required, R, under the worst case of simultaneous reordering:

$$R = \sum_{i=1}^{n} \lambda_i T_i^* .$$

For the moderate capacity cases, the warehouse space available, V, was chosen from a uniform distribution over the range [0.5 R, R]. For the limited capacity case, GQS chose V from a uniform distribution over the range [0.25 R, 0.5 R]. They noted correctly that previous studies (Hariga [1988] included) had not chosen V to be sufficiently restrictive. However, we found that even in their so-called limited capacity case, there were many problem instances in which the solutions did not fully utilize warehouse capacity. Accordingly, for our computational study, we chose V from a uniform distribution over the interval [0.025 R, 0.05 R] for the limited capacity cases. The result was that the solution to each of these problem exhibited 100% utilization of warehouse space. In summary, we generated eight problem sets of twenty-five problems each, representing both moderate capacity and limited capacity situations and a range of problem sizes from 5 to 20 items.

The problems were solved under the four hybrid algorithms on a 486 class, 66 Mhz PS/2 computer. All but two of the problems were solved successfully by all algorithms. The combination of P2F3 with TS was unable to solve two of the 20-items problems. We are confident that the Tactical Staggering code could be modified to solve these, but we are reluctant to modify this competing code further. Instead, we dropped these two problems from the test bed.

The results of the tests are summarized in Tables 1-6. Column 1 in each table indicates the number of items in the set of test problems used. In the remaining cells, the first number in each cell is the average value over all values in the set of test problems. The number in parentheses is the standard deviation. Columns 2 and 3 list the cost performance of two competing algorithms on the same problem sets. The optimality gap percentage is the difference between the cost of the hybrid algorithm solution and the lower bound, expressed as a percentage of the lower bound. The lower bound for each problem set was computed according to Theorem 5 of GQS. Column 4 lists the optimality gap of the second algorithm as a percentage of the optimality gap of the first algorithm. Similarly, column 5 lists the space utilization of the second algorithm as a percentage of the space utilization of the first. Column 6 lists the computation time of the second algorithm as a percentage of the computation time of the first.

	TS	SS	SS/TS		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	1.0 (0.4)	1.0 (0.4)	100 (0.0)	98.2 (3.0)	49.1 (20.1)
10	1.0 (0.3)	1.0 (0.3)	100 (0.0)	96.6 (3.8)	26.8 (14.0)
15	1.3 (0.2)	1.3 (0.2)	100 (0.0)	95.9 (3.8)	17.0 (13.2)
20	1.4 (0.2)	1.4 (0.2)	100 (0.0)	96.3 (3.6)	36.9 (58.2)

Table 1. Moderate Capacity Restriction, Powers-of-Two Frequencies, Tactical Staggering vs. Sequenced Staggering Algorithms

	TS	SS	SS/TS		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	0.3 (0.1)	0.3 (0.1)	100 (0.0)	100.1 (2.4)	43.6 (19.6)
10	0.3 (0.1)	0.3 (0.1)	100 (0.0)	97.2 (2.6)	22.1 (10.9)
15	0.3 (0.1)	0.3 (0.1)	100 (0.0)	96.9 (2.9)	16.3 (6.3)
20	0.3 (0.1)	0.3 (0.1)	100 (0.0)	95.3 (2.6)	13.5 (12.9)

Table 2. Moderate Capacity Restriction, Powers-of-Two-Factor-of-Three Frequencies, Tactical Staggering vs. Sequenced Staggering Algorithms

	P2F3	P2	P2/P2F3		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	0.3 (0.1)	1.0 (0.4)	450.8 (316.4)	96.8 (10.5)	63.6 (27.5)
10	0.3 (0.1)	1.0 (0.3)	391.4 (191.2)	98.4 (9.2)	55.8 (35.1)
15	0.3 (0.1)	1.3 (0.2)	423.8 (145.7)	96.3 (5.4)	58.8 (57.9)
20	0.3 (0.1)	1.4 (0.2)	414.3 (92.8)	97.4 (5.9)	97.6 (73.3)

Table 3. Moderate Capacity Restriction, Sequenced Staggering Algorithm, Powers-of-Two-Factor-of-Three vs. Powers-of-Two Frequencies

	TS	SS	SS/TS		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	8.6 (4.5)	6.8 (4.0)	80.2 (21.8)	100 (0.0)	57.6 (12.7)
10	7.6 (3.2)	4.8 (2.0)	66.8 (17.5)	100 (0.0)	24.3 (18.0)
15	5.8 (2.1)	3.6 (1.3)	63.6 (19.2)	100 (0.0)	16.6 (9.5)
20	4.5 (1.2)	3.2 (0.7)	74.2 (22.6)	100 (0.0)	8.4 (4.8)

Table 4. Limited Capacity Restriction, Powers-of-Two Frequencies, Tactical Staggering vs. Sequenced Staggering Algorithms

	TS	SS	SS/TS		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	11.7 (4.5)	10.9 (4.7)	94.2 (19.4)	100 (0.0)	47.3 (14.3)
10	9.2 (3.3)	7.5 (3.0)	84.9 (25.2)	100 (0.0)	26.1 (10.4)
15	7.5 (2.3)	5.0 (1.7)	68.3 (19.2)	100 (0.0)	18.5 (10.6)
20	7.1 (2.2)	4.1 (1.3)	59.5 (16.9)	100 (0.0)	11.6 (7.7)

Table 5. Limited Capacity Restriction, Powers-of-Two-Factor-of-Three Frequencies, Tactical Staggering vs. Sequenced Staggering Algorithms

	P2F3	P2	P2/P2F3		
Number of Items	Optimality Gap (%)	Optimality Gap (%)	Optimality Gap Ratio (%)	Utilization Ratio (%)	Time Ratio (%)
5	10.9 (4.7)	6.8 (4.0)	63.6 (26)	100 (0.0)	67.3 (18.3)
10	7.5 (3.0)	4.8 (2.0)	67.9 (25.1)	100 (0.0)	62.7 (32.8)
15	5.0 (1.7)	3.6 (1.3)	76.4 (27.9)	100 (0.0)	52.3 (51.9)
20	4.1 (1.3)	3.2 (0.7)	84.7 (25.9)	100 (0.0)	51.4 (47.8)

Table 6. Limited Capacity Restriction, Sequenced Staggering Algorithm, Powers-of-Two-Factor-of-Three vs. Powers-of-Two Frequencies

Tables 1-3 summarize the results for the moderate capacity situations. Interpreting Table 1, we note that P2 with TS has a very similar performance to P2 with SS. Both algorithms achieve virtually identical cost performance and are within 1.4% of the lower bound. The Sequenced Staggering algorithms appears to give slightly lower space utilization and are significantly faster in execution than the Tactical Staggering algorithm. Likewise, in Table 2, we note that P2F3 with TS has a very similar performance to P2F3 with SS. Again the Sequenced Staggering has a marginal advantage over the Tactical Staggering in terms of space utilization and a significant advantage in terms of computation time.

Table 3 reveals P2F3 has a significant advantage, at least in relative terms, over P2 in terms of cost in the moderate capacity case. The optimality gaps of both P2 and P2F3 are very small (no greater than 1.4%) but the P2 gap is on the order of four times larger than the P2F3 gap. The comparison in Table 3 uses Sequenced Staggering as the scheduling tool but the same conclusion holds in the case of Tactical Staggering. This would appear to confirm the conjecture of GQS that most of the optimality gap can be ascribed to the frequency roundoff technique. The P2 frequency technique does result in marginally better utilization and significantly faster execution. The faster execution clearly comes from the simplicity of the frequencies.

Tables 4-6 summarize the results of the limited capacity case. Recall that these problem sets were generated to ensure that the capacity would be 100% utilized by all solutions. As we would expect, we find that the optimality gaps are higher for these limited capacity problem sets than for the moderate capacity problems. The gaps range from 3.2% (Table 4, 20 items, SS) to 11.7% (Table 5, 5 items, TS). Tables 4 and 5 show that the Sequenced Staggering algorithm has a significant advantage over the Tactical Staggering algorithm both in terms of cost and computation time. This is true under both the P2 and the P2F3 frequency generating techniques.

In Table 6, we see a reversal of the performance observed in Table 3. For these limited capacity problems, the P2 frequency technique achieves a significantly better cost performance than the P2F3 technique. This is counter to the conjecture that the optimality gap can be explained by the roundoff to powers-of-two. As in Table 3, this conclusion does not depend on the choice of the Tactical Staggering or Sequenced Staggering as the scheduling tool. Intuitively, it appears that it is easier to satisfy a space constraint when the order frequencies are powers-of-two than when another factor is included. Additional insight is provided in Theorem 4 of Hariga and Jackson [1991]. There we show that powers-of-two frequencies are one of a set of sufficient conditions that result in the most efficient schedule possible: the timing of every order being scheduled to completely fill the available space.

Finally, Figure 2 graphs the computation time of P2 with TS algorithm compared to that of the P2 with SS algorithm for the four sets of test problems in the limited capacity case. For small problems, the algorithms differ only by a few seconds. However, the data show that computational requirements increase rapidly for both algorithms as the number of items increases. Therefore coding optimization may become an implementation issue, in which case the Sequenced Staggering technique appears to have an initial advantage.

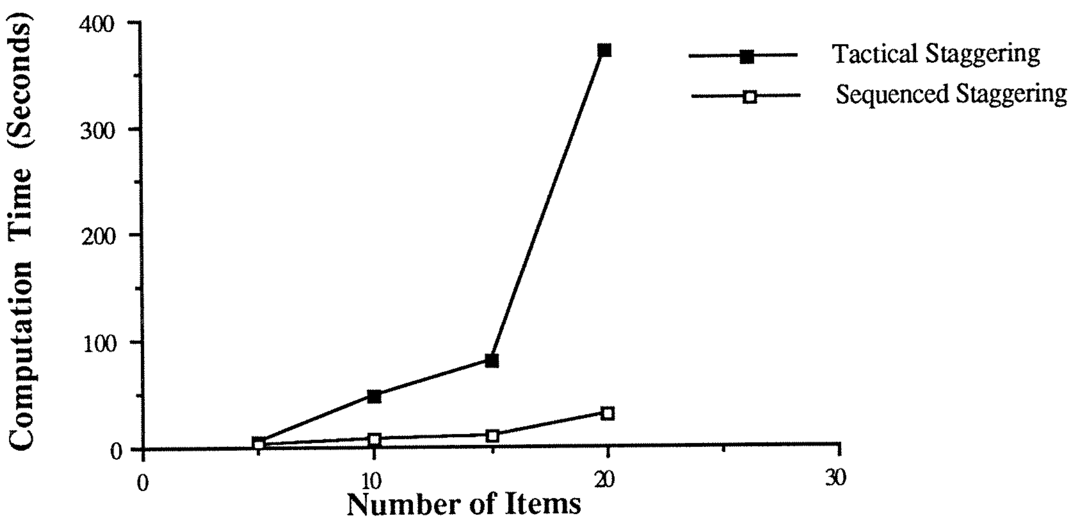

Figure 2. Limited Capacity Restriction, Powers-of-Two Frequencies, Tactical Staggering vs. Sequenced Staggering Computation Times

We conclude from this study that the GQS technique of generating powers-of-two frequencies using Roundy's Roundoff Algorithm is effective. P2 yields a small optimality gap when the capacity restriction is moderate and it dominates the powers-of-two-factor-of-three frequency algorithm when capacity is severely limited. The latter technique, P2F3, yields a relative cost advantage in moderate capacity problems but this advantage is likely to be outweighed by the cost of managing the complexity of the resulting schedules. We also conclude that the technique of Sequenced Staggering described in this paper and Jackson and Hariga [1991] is generally superior to the Tactical Staggering technique of GQS. It yields comparable cost performance in moderate capacity problems and a significant cost advantage in severely limited capacity problems. Furthermore, SS is significantly faster than TS in execution, at least in the current implementations.

7. CONCLUSION

The goal of this research has been to develop a method of generating a cyclic schedule that minimizes the long run average inventory and ordering costs per unit of time without violating a warehouse space capacity constraint. The combination of Roundy's powers-of-two roundoff technique with our so-called Sequenced Staggering algorithm has been shown to be robust and efficient in empirical tests of several competing algorithms.

ACKNOWLEDGMENTS

The authors would like to thank the department editor, Prof. Hau L. Lee, and the anonymous referees for their contributions to improving this paper. We are also grateful to Prof. Guillermo Gallego for his encouragement and his facilitation of the computational study reported here.

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